A Note On G -normal Distributions

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Abstract

As is known, the convolution $\mu * \nu$ of two G -normal distributions μ, ν with different intervals of variances may not be G -normal. We shows that $\mu * \nu$ is a G -normal distribution if and only if $\frac{\bar{\sigma}_\mu}{\underline{\sigma}_\mu} = \frac{\bar{\sigma}_\nu}{\underline{\sigma}_\nu}$.

Key words: G -normal distribution, Cramer's decomposition

MSC-classification: 60E10

1 Introduction

Peng (2007) introduced the notion of G -normal distribution via the viscosity solutions of the G -heat equation below

$$\begin{aligned}\partial_t u - G(\partial_x^2 u) &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) &= \varphi(x),\end{aligned}$$

where $G(a) = \frac{1}{2}(\bar{\sigma}^2 x^+ - \underline{\sigma}^2 x^-)$, $a \in \mathbb{R}$ with $0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$, and $\varphi \in C_{b,Lip}(\mathbb{R})$, the collection of bounded Lipstchiz functions on $\mathbb{R}$.

Then the one-dimensional G -normal distribution is defined by

$$N_G[\varphi] = u^\varphi(1, 0),$$

where u^φ is the viscosity solution to the G -heat equation with the initial value φ . We denote by $\mathcal{G}^0$ the collection of functions G defined above. For $G_1, G_2 \in \mathcal{G}^0$, the convolution between N_{G_1}, N_{G_2} is defined as $N_{G_1} * N_{G_2}[\varphi] := N_{G_1}[\phi]$ with $\phi(x) := N_{G_2}[\varphi(x + \cdot)]$.

As is well-known, the convolution of two normal distributions is also normal. How about G -normal distributions? We state the question in the PDE language: For $G_1, G_2 \in \mathcal{G}^0$, set $G(t, a) = G_2(a)$, $t \in (0, \frac{1}{2}]$ and $G(t, a) = G_1(a)$, $t \in (\frac{1}{2}, 1]$. Let v^φ be the viscosity solution to the following PDE

$$\begin{aligned}\partial_t v - G(t, \partial_x^2 v) &= 0, \\ v(0, x) &= \varphi(x).\end{aligned}$$

Can we find a function $G \in \mathcal{G}^0$ such that

$$v^\varphi(1, 0) = u^\varphi(1, 0), \quad \text{for all } \varphi \in C_{b,Lip}(\mathbb{R})?$$

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Here u^φ is the viscosity solution to the G -heat equation. However, [2] gave a counterexample to show that generally $N_{G_1} * N_{G_2}$ is not G -normal any more.

We denote by $\mathcal{G}$ the subset of $\mathcal{G}^0$ consisting of the elements with $\underline{\sigma} > 0$. N_G is called non-degenerate if G belongs to $\mathcal{G}$. For $G \in \mathcal{G}$, set

$$\beta_G := \frac{\bar{\sigma}}{\underline{\sigma}} \quad \text{and} \quad \sigma_G = \frac{\bar{\sigma} + \underline{\sigma}}{2}.$$

For abbreviation, we write β, σ instead of β_G, σ_G when no confusion can arise.

We shall prove that $N_{G_1} * N_{G_2}$ is a G -normal distribution if and only if $\beta_{G_1} = \beta_{G_2}$. Besides, the convolution between G -normal distributions is not commutative. We also prove that $N_{G_1} * N_{G_2} = N_{G_2} * N_{G_1}$ if and only if $\beta_{G_1} = \beta_{G_2}$.

Therefore, in order to emphasize the importance of the ratio β , we denote the G -normal distribution N_G by $N_\beta(0, \sigma^2)$.

2 Characteristic Functions

First we shall consider the solutions of special forms to the G -heat equation

$$\partial_t u - G(\partial_x^2 u) = 0. \quad (2.1)$$

Assume that $u(t, x) = a(t)\phi(x)$ with $a(t) \geq 0$ is a solution to the G -heat equation (2.1). Then we conclude that

$$\frac{a'(t)}{a(t)} = \frac{G(\phi''(x))}{\phi(x)}$$

is a constant. Assuming that they are equal to $-\frac{\rho^2}{2}$, we have $a(t) = e^{-\frac{\rho^2}{2}t}$ and

$$G(\phi''(x)) = -\frac{\rho^2}{2}\phi(x). \quad (2.2)$$

If $\phi''(x)$ is positive, the equation (2.2) reduces to

$$\bar{\phi}''(x) = -\frac{\rho^2}{\bar{\sigma}^2}\bar{\phi}(x). \quad (2.3)$$

and the solution is

$$\bar{\phi}(x) = \bar{\lambda} \cos\left(\frac{\rho}{\bar{\sigma}}x + \bar{\theta}\right).$$

On the other hand, if $\phi''(x)$ is negative, the equation (2.2) reduces to

$$\underline{\phi}''(x) = -\frac{\rho^2}{\underline{\sigma}^2}\underline{\phi}(x). \quad (2.4)$$

and the solution is

$$\underline{\phi}(x) = \underline{\lambda} \cos\left(\frac{\rho}{\underline{\sigma}}x + \underline{\theta}\right).$$

Motivated by the arguments above, we shall construct the solutions to the equation (2.2) in the following way. Denoting by $\beta = \frac{\bar{\sigma}}{\underline{\sigma}}$, set

$$\phi_\beta(x) = \begin{cases} \frac{2}{1+\beta} \cos\left(\frac{1+\beta}{2}x\right) & \text{for } x \in \left[-\frac{\pi}{1+\beta}, \frac{\pi}{1+\beta}\right); \\ \frac{2\beta}{1+\beta} \cos\left(\frac{1+\beta}{2\beta}x + \frac{\beta-1}{2\beta}\pi\right) & \text{for } x \in \left[\frac{\pi}{1+\beta}, \frac{(2\beta+1)\pi}{1+\beta}\right). \end{cases} \quad (2.5)$$

Figure 1: $\phi_\beta(x)$

This is a variant of the trigonometric function $\cos x$ (see Figure 1).

Then extend the definition of ϕ_β to the whole real line by the property $\phi(x + 2k\pi) = \phi(x)$, $k \in \mathbb{Z}$. Clearly, $\phi_1(x) = \cos x$ and ϕ_β belongs to $C^{2,1}(\mathbb{R})$, the space of bounded functions on $\mathbb{R}$ with uniformly Lipschitz continuous second-order derivatives.

Proposition 2.1 ϕ_β is a solution to equation (2.2) with $\rho = \frac{\sigma + \bar{\sigma}}{2} =: \sigma$.

Proof. The proof follows from simple calculations. For $x \in [-\frac{\pi}{1+\beta}, \frac{\pi}{1+\beta})$,

$$\phi_\beta''(x) = -\frac{1+\beta}{2} \cos\left(\frac{1+\beta}{2}x\right) = -\frac{(1+\beta)^2}{4} \phi_\beta(x) \leq 0.$$

So

$$G(\phi_\beta''(x)) = -\frac{(1+\beta)^2 \underline{\sigma}^2}{8} \phi_\beta(x) = -\frac{\sigma^2}{2} \phi_\beta(x).$$

For $x \in [\frac{\pi}{1+\beta}, \frac{(2\beta+1)\pi}{1+\beta})$,

$$\phi_\beta''(x) = -\frac{1+\beta}{2\beta} \cos\left(\frac{1+\beta}{2\beta}x + \frac{\beta-1}{2\beta}\pi\right) = -\frac{(1+\beta)^2}{4\beta^2} \phi_\beta(x) \geq 0.$$

So

$$G(\phi_\beta''(x)) = -\frac{(1+\beta)^2 \bar{\sigma}^2}{8\beta^2} \phi_\beta(x) = -\frac{\sigma^2}{2} \phi_\beta(x).$$

□

Corollary 2.2 Let $\sigma := \frac{\sigma + \bar{\sigma}}{2}$ and $\beta := \frac{\bar{\sigma}}{\underline{\sigma}}$. $e^{-\frac{\sigma^2}{2}t} \phi_\beta(x)$ is the solution to equation (2.1) with the initial value $\phi_\beta(x)$. In other words, $N_G[\phi_\beta(x + \sqrt{t}\cdot)] = e^{-\frac{\sigma^2}{2}t} \phi_\beta(x)$.

For $\lambda > 0$ and $c, \theta \in \mathbb{R}$, set $\phi_\beta^{\lambda, c, \theta}(x) := \lambda \phi_\beta(cx + \theta)$. It's easy to check that

$$G((\phi_\beta^{\lambda, c, \theta})'') = -\frac{c^2 \sigma^2}{2} \phi_\beta^{\lambda, c, \theta}.$$

So $e^{-\frac{c^2 \sigma^2}{2}t} \phi_\beta^{\lambda, c, \theta}(x)$ is the solution to equation (2.1) with the initial value $\phi_\beta^{\lambda, c, \theta}(x)$. For any $\beta > 1$, we call

$$(\phi_\beta^{\lambda, c, \theta}(x))_{\lambda, c, \theta}$$

the characteristic functions of G -normal distributions $N_\beta(0, \sigma^2)$.

3 Cramer's Type Decomposition for G -normal Distributions

Let's introduce more properties on the characteristic functions $\{\phi_\beta(x)\}_{\beta \geq 1}$. For $1 \leq \alpha < \beta < \infty$, we have

$$\phi_\alpha(x) - \phi_\beta(x) \geq \frac{2(\beta - \alpha)}{(1 + \alpha)(1 + \beta)} =: e(\alpha, \beta) > 0.$$

Figure 2: $\phi_\alpha(x)$ and $\phi_\beta(x)$

Theorem 3.1 For $G_1, G_2 \in \mathcal{G}$, the convolution $N_{G_1} * N_{G_2}$ is a G -normal distribution if and only if $\beta_{G_1} := \frac{\bar{\sigma}_{G_1}}{\underline{\sigma}_{G_1}} = \frac{\bar{\sigma}_{G_2}}{\underline{\sigma}_{G_2}} =: \beta_{G_2}$.

Proof. The sufficiency is obvious, and we only prove the necessity. Assume $N := N_{G_1} * N_{G_2}$ is a G -normal distribution. Then $\underline{\sigma}_N^2 = \underline{\sigma}_{G_1}^2 + \underline{\sigma}_{G_2}^2$ and $\bar{\sigma}_N^2 = \bar{\sigma}_{G_1}^2 + \bar{\sigma}_{G_2}^2$. If $\beta_{G_1} \neq \beta_{G_2}$, we may assume that $\beta_{G_1} < \beta_{G_2}$, and the other case can be proved similarly. Then we have $\beta_{G_1} < \beta_N < \beta_{G_2}$. On one hand we have

$$N[\phi_{\beta_N}(\sqrt{t} \cdot)] = e^{-\frac{\sigma_N^2}{2}t} \phi_{\beta_N}(0) = \frac{2}{1 + \beta_N} e^{-\frac{\sigma_N^2}{2}t}.$$

On the other hand we have

$$\begin{aligned} & N_{G_2}[\phi_{\beta_N}(\sqrt{t}(x + \cdot))] \\ & \geq N_{G_2}[\phi_{\beta_{G_2}}(\sqrt{t}(x + \cdot))] + e(\beta_N, \beta_{G_2}) \\ & = e^{-\frac{\sigma_{G_2}^2}{2}t} \phi_{\beta_{G_2}}(\sqrt{t}x) + e(\beta_N, \beta_{G_2}) \\ & \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2}t} + e(\beta_N, \beta_{G_2}). \end{aligned}$$

Consequently,

$$N[\phi_{\beta_N}(\sqrt{t} \cdot)] = N_{G_1} * N_{G_2}[\phi_{\beta_N}(\sqrt{t} \cdot)] \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2}t} + e(\beta_N, \beta_{G_2}).$$

Combining the above arguments we have

$$\frac{2}{1 + \beta_N} e^{-\frac{\sigma_N^2}{2}t} \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2}t} + e(\beta_N, \beta_{G_2}), \text{ for any } t > 0,$$

which is a contradiction noting that $e(\beta_N, \beta_{G_2}) > 0$. $\square$

Theorem 3.2 For $G_1, G_2 \in \mathcal{G}$, $N_{G_1} * N_{G_2} = N_{G_2} * N_{G_1}$ if and only if $\beta_{G_1} = \beta_{G_2}$.

Proof. We shall only prove the necessity. Assume that $\beta_{G_1} < \beta_{G_2}$. Then

$$N_{G_1}[\phi_{\beta_{G_1}}(\sqrt{t}x + \sqrt{t}\cdot)] = e^{-\frac{\sigma_{G_1}^2}{2}t} \phi_{\beta_{G_1}}(\sqrt{t}x) \leq \frac{2}{1 + \beta_{G_1}} e^{-\frac{\sigma_{G_1}^2}{2}t}.$$

So

$$N_{G_2} * N_{G_1}[\phi_{\beta_{G_1}}(\sqrt{t}\cdot)] \leq \frac{2}{1 + \beta_{G_1}} e^{-\frac{\sigma_{G_1}^2}{2}t}.$$

On the other hand

$$\begin{aligned} & N_{G_2}[\phi_{\beta_{G_1}}(\sqrt{t}x + \sqrt{t}\cdot)] \\ & \geq N_{G_2}[\phi_{\beta_{G_2}}(\sqrt{t}x + \sqrt{t}\cdot)] + e(\beta_{G_1}, \beta_{G_2}) \\ & = e^{-\frac{\sigma_{G_2}^2}{2}t} \phi_{\beta_{G_2}}(\sqrt{t}x) + e(\beta_{G_1}, \beta_{G_2}) \\ & \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2}t} + e(\beta_{G_1}, \beta_{G_2}), \end{aligned}$$

which implies that

$$N_{G_1} * N_{G_2}[\phi_{\beta_{G_1}}(\sqrt{t}\cdot)] \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2}t} + e(\beta_{G_1}, \beta_{G_2}).$$

Noting that $e(\beta_{G_1}, \beta_{G_2}) > 0$, we have

$$N_{G_1} * N_{G_2}[\phi_{\beta_{G_1}}(\sqrt{t}\cdot)] > N_{G_2} * N_{G_1}[\phi_{\beta_{G_1}}(\sqrt{t}\cdot)] \text{ for } t \text{ large enough.}$$

□

References

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