Functional Limit Theorems for Toeplitz Quadratic Functionals of Continuous time Gaussian Stationary Processes

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Abstract

The paper establishes weak convergence in $C[0, 1]$ of normalized stochastic processes, generated by Toeplitz type quadratic functionals of a continuous time Gaussian stationary process, exhibiting long-range dependence. Both central and non-central functional limit theorems are obtained.

Key words. Stationary Gaussian process - Toeplitz-type quadratic functional - Brownian motion - Non-central limit theorem - Long memory - Wiener-Itô integral.

1 Introduction

Let $\{X(t), t \in \mathbb{R}\}$ be a centered real-valued stationary Gaussian process with spectral density $f(x)$ and covariance function $r(t)$, that is, $r(t) = \widehat{f}(t) = \int_{\mathbb{R}} e^{ixt} f(x) dx$, $t \in \mathbb{R}$. We are interested in describing the limit (as $T \rightarrow \infty$) of the following process, generated by Toeplitz type quadratic functionals of the process $X(t)$:

$$Q_T(t) = \int_0^{Tt} \int_0^{Tt} \widehat{g}(u-v) X(u) X(v) du dv, \quad t \in [0, 1], \quad (1.1)$$

where

$$\widehat{g}(t) = \int_{\mathbb{R}} e^{ixt} g(x) dx, \quad t \in \mathbb{R}, \quad (1.2)$$

is the Fourier transform of some integrable even function $g(x)$, $x \in \mathbb{R}$. We will refer to $g(x)$ and to its Fourier transform $\widehat{g}(t)$ as a generating function and generating kernel for the process $Q_T(t)$, respectively.

The limit of the process (1.1) is completely determined by the spectral density $f(x)$ (or covariance function $r(t)$) and the generating function $g(x)$ (or generating kernel $\widehat{g}(t)$), and depending on their properties, the limit can be either Gaussian (that is, $Q_T(t)$ with an appropriate normalization obeys a central limit theorem), or non-Gaussian. The following two questions arise naturally:

- (a) Under what conditions on $f(x)$ (resp. $r(t)$) and $g(x)$ (resp. $\widehat{g}(t)$) will the limit be Gaussian?
- (b) Describe the limit process, if it is non-Gaussian.

Similar questions were considered by Fox and Taqqu [6], Ginovyan and Sahakyan [9], and Terrin and Taqqu [25] in the discrete time case.

Here we work in continuous time, and establish weak convergence in $C[0, 1]$ of the process (1.1). The limit processes can be Gaussian or non-Gaussian. The limit non-Gaussian process is identical to that of in the discrete time case, obtained in [25].

But first some brief history. The question (a) goes back to the classical monograph by Grenander and Szegő [16], where the problem was considered for discrete time processes, as an application of the authors'

theory of the asymptotic behavior of the trace of products of truncated Toeplitz matrices (see [16], p. 217-219). Later the question (a) was studied by Ibragimov [17] and Rosenblatt [23], in connection to the statistical estimation of the spectral function $F(x)$ and covariance function $r(t)$, respectively. Since 1986, there has been a renewed interest in both questions (a) and (b), related to the statistical inferences for long memory processes (see, e.g., Avram [1], Fox and Taqqu [6], Ginovyan and Sahakyan [9], Ginovyan et al. [11], Giraitis et al. [12], Giraitis and Surgailis [13], Giraitis and Taqqu [15], Terrin and Taqqu [26], Taniguchi and Kakizawa [24], and references therein). In particular, Avram [1], Fox and Taqqu [6], Giraitis and Surgailis [13], Ginovyan and Sahakyan [9] have obtained sufficient conditions for the Toeplitz type quadratic forms $Q_T(1)$ to obey the central limit theorem (CLT), when the model $X(t)$ is a discrete time process.

For continuous time processes the question (a) was studied in Ibragimov [17] (in connection to the statistical estimation of the spectral function), Ginovyan [7, 8], Ginovyan and Sahakyan [10] and Ginovyan et al. [11], where sufficient conditions in terms of $f(x)$ and $g(x)$ ensuring central limit theorems for quadratic functionals $Q_T(1)$ have been obtained.

The rest of the paper is organized as follows. In Section 2 we state the main results of this paper (Theorems 2.1 - 2.4). In Section 3 we prove a number of preliminary lemmas that are used in the proofs of the main results. Section 4 contains the proofs of the main results.

Throughout the paper the letters C and c with or without indices will denote positive constants whose values can change from line to line.

2 The Main Results

In this section we state our main results. Throughout the paper we assume that $f, g \in L^1(\mathbb{R})$, and with no loss of generality, that $g \geq 0$ (see [10], [13]).

We first examine the case of *central limit theorems*, and consider the following standard normalized version of (1.1):

$$\tilde{Q}_T(t) := T^{-1/2} (Q_T(t) - \mathbb{E}[Q_T(t)]), \quad t \in [0, 1]. \quad (2.1)$$

Our first result, which is an extension of Theorem 1 of [10], involves the convergence of finite-dimensional distributions of the process $\tilde{Q}_T(t)$ to that of a standard Brownian motion.

Theorem 2.1. *Assume that the spectral density $f(x)$ and the generating function $g(x)$ satisfy the following conditions:*

$$f \cdot g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \quad (2.2)$$

and

$$\mathbb{E}[\tilde{Q}_T^2(1)] \rightarrow 16\pi^3 \int_{-\infty}^{\infty} f^2(x)g^2(x)dx \quad \text{as } T \rightarrow \infty. \quad (2.3)$$

Then we have the following convergence of finite-dimensional distributions

$$\tilde{Q}_T(t) \xrightarrow{f.d.d.} \sigma B(t),$$

where $\tilde{Q}_T(t)$ is as in (2.1), $B(t)$ is a standard Brownian motion, and

$$\sigma^2 := 16\pi^3 \int_{-\infty}^{\infty} f^2(x)g^2(x)dx. \quad (2.4)$$

To extend the convergence of finite-dimensional distributions in Theorem 2.1 to the weak convergence in the space $C[0, 1]$, we impose an additional condition on the underlying Gaussian process $X(t)$ and on the generating function g . It is convenient to impose this condition in the time domain, that is, on the covariance function $r := \hat{f}$ and the generating kernel $a := \hat{g}$. The following condition is an analog of the assumption in Theorem 2.3 of [15]:

$$r(\cdot) \in L^p(\mathbb{R}), \quad a(\cdot) \in L^q(\mathbb{R}) \quad \text{for some } p, q \geq 1, \quad \frac{1}{p} + \frac{1}{q} \geq \frac{3}{2}. \quad (2.5)$$

Remark 2.1. In fact under (2.2), the condition (2.5) is sufficient for the convergence in (2.3). Indeed, let $\bar{p} = p/(p-1)$ be the Hölder conjugate of p and let $\bar{q} = q/(q-1)$ be the Hölder conjugate of q . Since $1 \leq p, q \leq 2$, one has by the Hausdorff-Young inequality and (2.5) that $\|f\|_{\bar{p}} \leq c_p \|r\|_p$, $\|g\|_{\bar{q}} \leq c_q \|a\|_q$, and hence

$$f(\cdot) \in L^{\bar{p}}, \quad g(\cdot) \in L^{\bar{q}}, \quad \frac{1}{\bar{p}} + \frac{1}{\bar{q}} = 2 - \frac{1}{p} - \frac{1}{q} \leq 1/2.$$

Then the convergence in (2.3) follows from the proof of Theorem 3 from [10]. Note that a similar assertion in the discrete time case was established in [13].

Remark 2.2. Observe that condition (2.5) is fulfilled if the functions $r(t)$ and $a(t)$ satisfy the following: there exist constants $C > 0$, α^* and β^* , such that

$$|r(t)| \leq C(1 \wedge |t|^{\alpha^*-1}), \quad |a(t)| \leq C(1 \wedge |t|^{\beta^*-1}), \quad (2.6)$$

where $0 < \alpha^*, \beta^* < 1/2$ and $\alpha^* + \beta^* < 1/2$. Indeed, to see this, note first that $r(\cdot), a(\cdot) \in L^\infty(\mathbb{R})$. Then one can choose $p, q \geq 1$ such that $p(\alpha^* - 1) < -1$ and $q(\beta^* - 1) < -1$, which entails that $r(\cdot) \in L^p(\mathbb{R})$ and $a(\cdot) \in L^q(\mathbb{R})$. Since $1/p + 1/q < 2 - \alpha^* - \beta^*$ and $2 - \alpha^* - \beta^* > 3/2$, one can further choose p, q to satisfy $1/p + 1/q \geq 3/2$.

The next results, two functional central limit theorems, extend Theorems 1 and 5 of [10] to weak convergence in the space $C[0, 1]$ of the stochastic process $\tilde{Q}_T(t)$ to a standard Brownian motion.

Theorem 2.2. *Let the spectral density $f(x)$ and the generating function $g(x)$ satisfy condition (2.2). Let the covariance function $r(t)$ and the generating kernel $a(t)$ satisfy condition (2.5). Then we have the following weak convergence in $C[0, 1]$:*

$$\tilde{Q}_T(t) \Rightarrow \sigma B(t),$$

where $\tilde{Q}_T(t)$ is as in (2.1), σ is as in (2.4), and $B(t)$ is a standard Brownian motion.

Recall that a function $u(x)$, $x \in \mathbb{R}$, is called slowly varying at 0 if it is non-negative and for any $t > 0$

$$\lim_{x \rightarrow 0} \frac{u(xt)}{u(x)} \rightarrow 1.$$

Let $SV_0(\mathbb{R})$ be the class of slowly varying at zero functions $u(x)$, $x \in \mathbb{R}$, satisfying the following conditions: for some $a > 0$, $u(x)$ is bounded on $[-a, a]$, $\lim_{x \rightarrow 0} u(x) = 0$, $u(x) = u(-x)$ and $0 < u(x) < u(y)$ for $0 < x < y < a$. An example of a function belonging to $SV_0(\mathbb{R})$ is $u(x) = |\ln |x||^{-\gamma}$ with $\gamma > 0$ and $a = 1$.

Theorem 2.3. *Assume that the functions f and g are integrable on $\mathbb{R}$ and bounded outside any neighborhood of the origin, and satisfy for some $a > 0$*

$$f(x) \leq |x|^{-\alpha} L_1(x), \quad |g(x)| \leq |x|^{-\beta} L_2(x), \quad x \in [-a, a] \quad (2.7)$$

for some $\alpha < 1$, $\beta < 1$ with $\alpha + \beta \leq 1/2$, where $L_1(x)$ and $L_2(x)$ are slowly varying at zero functions satisfying

$$L_i \in SV_0(\mathbb{R}), \quad x^{-(\alpha+\beta)} L_i(x) \in L^2[-a, a], \quad i = 1, 2. \quad (2.8)$$

Let, in addition, the covariance function $r(t)$ and the generating kernel $a(t)$ satisfy condition (2.5). Then we have the following weak convergence in $C[0, 1]$:

$$\tilde{Q}_T(t) \Rightarrow \sigma B(t),$$

where $\tilde{Q}_T(t)$ is as in (2.1), σ is as in (2.4), and $B(t)$ is a standard Brownian motion.

Remark 2.3. The conditions $\alpha < 1$ and $\beta < 1$ ensure that the Fourier transforms of f and g are well defined. Observe that when $\alpha > 0$ the process $\{X(t), t \in \mathbb{Z}\}$ may exhibit long-range dependence. We also allow here $\alpha + \beta$ to assume the critical value $1/2$.

Remark 2.4. The assumptions $f \cdot g \in L^1(\mathbb{R})$, $f, g \in L^\infty(\mathbb{R} \setminus [-a, a])$ and (2.8) imply that $f \cdot g \in L^2(\mathbb{R})$, so that σ^2 in (2.4) is finite.

Remark 2.5. One may wonder, why, in Theorem 2.3, we suppose that $L_1(x)$ and $L_2(x)$ belong to $SV_0(\mathbb{R})$ instead of merely being slowly varying at zero. This is done in order to deal with the critical case $\alpha + \beta = 1/2$. Suppose that we are away from this critical case, namely, $f(x) = |x|^{-\alpha} l_1(x)$ and $g(x) = |x|^{-\beta} l_2(x)$, where $\alpha + \beta < 1/2$, and $l_1(x)$ and $l_2(x)$ are slowly varying at zero functions. Assume also that $f(x)$ and $g(x)$ are integrable and bounded on $(-\infty, -a) \cup (a, +\infty)$ for any $a > 0$. We claim that Theorem 2.3 applies. Indeed, choose $\alpha' > \alpha$, $\beta' > \beta$ with $\alpha' + \beta' < 1/2$. Write $f(x) = |x|^{-\alpha'} |x|^\delta l_1(x)$, where $\delta = \alpha' - \alpha > 0$. Since $l_1(x)$ is slowly varying, when $|x|$ is small enough, for some $\epsilon \in (0, \delta)$ we have $|x|^\delta l_1(x) \leq |x|^{\delta-\epsilon}$. Then one can bound $|x|^{\delta-\epsilon}$ by $c |\ln |x||^{-1} \in SV_0(\mathbb{R})$ for small $|x| < 1$. Hence one has when $|x| < 1$ is small enough, $f(x) \leq |x|^{-\alpha'} (c |\ln |x||^{-1})$. Similarly, when $|x| < 1$ is small enough, one has $g(x) \leq |x|^{-\beta'} (c |\ln |x||^{-1})$. All the assumptions in Theorem 2.3 are now readily checked with α, β replaced by α' and β' , respectively.

Now we state a *non-central limit theorem* in the continuous time case. Let the spectral density f and the generating function g satisfy

$$f(x) = |x|^{-\alpha} L_1(x) \quad \text{and} \quad g(x) = |x|^{-\beta} L_2(x), \quad x \in \mathbb{R}, \quad \alpha < 1, \beta < 1, \quad (2.9)$$

with slowly varying at zero functions $L_1(x)$ and $L_2(x)$ such that $\int_{\mathbb{R}} |x|^{-\alpha} L_1(x) dx < \infty$ and $\int_{\mathbb{R}} |x|^{-\beta} L_2(x) dx < \infty$. We assume in addition that the functions $L_1(x)$ and $L_2(x)$ satisfy the following condition, called Potter's bound (see [12], formula (2.3.5)): for any $\epsilon > 0$ there exists a constant $C > 0$ so that if T is large enough, then

$$\frac{L_i(u/T)}{L_i(1/T)} \leq C(|u|^\epsilon + |u|^{-\epsilon}), \quad i = 1, 2. \quad (2.10)$$

Note that a sufficient condition for (2.10) to hold is that $L_1(x)$ and $L_2(x)$ are bounded on intervals $[a, \infty)$ for any $a > 0$, which is the case for the slowly varying functions in Theorem 2.3.

Now we are interested in the limit process of the following normalized version of the process $Q_T(t)$ given by (1.1), with f and g as in (2.9):

$$Z_T(t) := \frac{1}{T^{\alpha+\beta} L_1(1/T) L_2(1/T)} (Q_T(t) - \mathbb{E}[Q_T(t)]). \quad (2.11)$$

Theorem 2.4. Let f and g be as in (2.9) with $\alpha < 1$, $\beta < 1$ and slowly varying at zero functions $L_1(x)$ and $L_2(x)$ satisfying (2.10), and let $Z_T(t)$ be as in (2.11). Then for $\alpha + \beta > 1/2$, we have the following weak convergence in the space $C[0, 1]$:

$$Z_T(t) \Rightarrow Z(t),$$

where the limit process $Z(t)$ is given by

$$Z(t) = \int_{\mathbb{R}^2}'' H_t(x_1, x_2) W(dx_1) W(dx_2), \quad (2.12)$$

with

$$H_t(x_1, x_2) = |x_1 x_2|^{-\alpha/2} \int_{\mathbb{R}} \left[\frac{e^{it(x_1+u)} - 1}{i(x_1+u)} \right] \cdot \left[\frac{e^{it(x_2-u)} - 1}{i(x_2-u)} \right] |u|^{-\beta} du, \quad (2.13)$$

where $W(\cdot)$ is a complex Gaussian random measure with Lebesgue control measure, and the double prime in the integral (2.12) indicates that the integration excludes the diagonals $x_1 = \pm x_2$.

Remark 2.6. Comparing Theorem 2.4 and Theorem 1 of [25], we see that the limit process $Z(t)$ is the same both for continuous and discrete time models.

Remark 2.7. Denoting by P_T and P the measures generated in $C[0, 1]$ by the processes $Z_T(t)$ and $Z(t)$ given by (2.11) and (2.12), respectively, Theorem 2.4 can be restated as follows: under the conditions of Theorem 2.4, the measure P_T converges weakly in $C[0, 1]$ to the measure P as $T \rightarrow \infty$. A similar assertion can be stated for Theorems 2.2 and 2.3.

It is worth noting that although the statement of our Theorem 2.4 is similar to that of Theorem 1 of [25], the proof is different and simpler, and does not use the hard analysis of [25], although some technical results of [25] are stated in lemmas and used in the proofs. Our approach in the CLT case (Theorems 2.1 - 2.3), uses the method developed in [10], which itself is based on an approximation of the trace of the product of truncated Toeplitz operators. For the non-CLT case (Theorem 2.4), we use the integral representation of the underlying process and properties of Wiener-Itô integrals.

3 Preliminaries

In this section we state a number of lemmas which will be used in the proof of the theorems. The following result extends Lemma 9 of [10].

Lemma 3.1. *Let $Y(t)$ be a centered stationary Gaussian process with spectral density $f_Y(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Consider the normalized process:*

$$L_T(t) := \frac{1}{T^{1/2}} \left(\int_0^{Tt} Y^2(u) du - \mathbb{E} \left[\int_0^{Tt} Y^2(u) du \right] \right). \quad (3.1)$$

Then we have the following convergence of finite-dimensional distributions:

$$L_T(t) \xrightarrow{f.d.d.} \sigma_Y B(t), \quad \sigma_Y^2 = 4\pi \int_{-\infty}^{\infty} f_Y^2(x) dx, \quad (3.2)$$

where $B(t)$ is standard Brownian motion.

Remark 3.1. Observe that the normalized processes $\tilde{Q}_T(t)$ and $L_T(t)$, given by (2.1) and (3.1), can be expressed by double Wiener-Itô integrals (see, e.g., the proof of Lemma 3.9 below). In our proofs we will use the following fact about weak convergence of multiple Wiener-Itô integrals: given the convergence of the covariance, the multivariate convergence to a Gaussian vector is implied by the univariate convergence of each component (see [21], Proposition 2).

Proof of Lemma 3.1. For a fixed t , the univariate convergence in distribution

$$L_T(t) \xrightarrow{d} N(0, t\sigma_Y^2) \quad \text{as } T \rightarrow \infty$$

follows from Lemma 9 of [10]. To show (3.2), in view of Remark 3.1 and Proposition 2 of [21], it remains to show that the covariance structure of $L_T(t)$ converges to that of $\sigma_Y B(t)$. Specifically, it suffices to show that for any $0 < s < t$,

$$\mathbb{E} \left[(L_T(t) - L_T(s))^2 \right] \rightarrow \sigma_Y^2 \cdot (t - s) \quad \text{as } T \rightarrow \infty. \quad (3.3)$$

Indeed, using the fact that for a Gaussian vector (G_1, G_2) we have $\text{Cov}(G_1^2, G_2^2) = 2[\text{Cov}(G_1, G_2)]^2$, and letting $r_Y(u) = \int_{\mathbb{R}} e^{ixu} f_Y(x) dx$ be the covariance function of $Y(t)$, we can write

$$\mathbb{E} \left[(L_T(t) - L_T(s))^2 \right] = 2(t - s) \int_{-T(t-s)}^{T(t-s)} \left(1 - \frac{|u|}{T(t-s)} \right) r_Y^2(u) du.$$

Since $f_Y(x) \in L^2(\mathbb{R})$, the Fourier transform $r_Y(u) \in L^2(\mathbb{R})$ as well. So by the Dominated Convergence Theorem and Parseval-Plancherel's identity, we have as $T \rightarrow \infty$

$$\mathbb{E} \left[(L_T(t) - L_T(s))^2 \right] \rightarrow 2(t - s) \int_{-\infty}^{\infty} r_Y^2(u) du = 4\pi(t - s) \int_{-\infty}^{\infty} f_Y^2(x) dx = \sigma_Y^2(t - s). \quad (3.4)$$

□

We now discuss some results which allow one to reduce the general quadratic functional in Theorem 2.1 to a special quadratic functional introduced in Lemma 3.1.

By Theorem 16.7.2 from [18], the underlying process $X(t)$ admits a moving average representation:

$$X(t) = \int_{-\infty}^{\infty} \hat{a}(t-s)B(ds) \quad \text{with} \quad \int_{-\infty}^{\infty} |\hat{a}(t)|^2 dt < \infty, \quad (3.5)$$

where $B(t)$ is a standard Brownian motion, and $\hat{a}(t)$ is such that its inverse Fourier transform $a(x)$ satisfies $f(x) = 2\pi|a(x)|^2$. Assuming the conditions (2.2) and (2.3), we set

$$b(x) = (2\pi)^{1/2}a(x)(g(x))^{1/2},$$

and observe that the function $b(x)$ is then in $L^2(\mathbb{R})$ due to condition (2.2). Consider the stationary process

$$Y(t) = \int_{-\infty}^{\infty} \hat{b}(t-s)B(ds) \quad (3.6)$$

constructed using the Fourier transform $\hat{b}(t)$ of $b(x)$ and the *same* Brownian motion $B(t)$ as in (3.5). The process $Y(t)$ has spectral density (see [10], equation (4.7))

$$f_Y(x) = 2\pi f(x)g(x). \quad (3.7)$$

We have the following approximation result which immediately follows from Lemma 10 of [10].

Lemma 3.2. *Let $\tilde{Q}_T(t)$ be as in (2.1) and let $L_T(t)$ be as in (3.1) with $Y(t)$ constructed as in (3.6). Then under the conditions (2.2) and (2.3), for any $t > 0$, we have*

$$\lim_{T \rightarrow \infty} \text{Var}[\tilde{Q}_T(t) - L_T(t)] = 0.$$

The following lemma is a straightforward adaptation of Lemma 4.2 of [14] for functions defined on $\mathbb{R}$.

Lemma 3.3. *If $p_j \geq 1$, $j = 1, \dots, k$, where $k \geq 2$ and $\sum_{j=1}^k \frac{1}{p_j} = k - 1$, then*

$$\int_{\mathbb{R}^{k-1}} |f_1(x_1) \dots f_{k-1}(x_{k-1}) f_k(x_1 + \dots + x_{k-1})| dx_1 \dots dx_k \leq \prod_{j=1}^k \|f_j\|_{p_j}.$$

The following lemma will be used to establish tightness in the space $C[0, 1]$ in Theorem 2.2.

Lemma 3.4. *Let the covariance function $r(t)$ and the generating kernel $a(t)$ satisfy condition (2.5), and let $\tilde{Q}_T(t)$ be as in (2.1). Then for all $0 \leq s \leq t \leq 1$ and $T > 0$, there exists a constant $C > 0$, such that*

$$\mathbb{E} \left[|\tilde{Q}_T(t) - \tilde{Q}_T(s)|^2 \right] \leq C(t-s). \quad (3.8)$$

Proof. For convenience we use the Wick product notation: $:X(u)X(v): = X(u)X(v) - \mathbb{E}[X(u)X(v)]$. So for $0 \leq s \leq t \leq 1$, we can write

$$\begin{aligned} \tilde{Q}_T(t) - \tilde{Q}_T(s) &= \frac{1}{\sqrt{T}} \left(\int_0^{Tt} \int_0^{Tt} a(u-v) :X(u)X(v): dudv - \int_0^{Ts} \int_0^{Ts} a(u-v) :X(u)X(v): dudv \right) \\ &= \frac{1}{\sqrt{T}} \int_{Ts}^{Tt} \int_{Ts}^{Tt} a(u-v) :X(u)X(v): dudv + \frac{2}{\sqrt{T}} \int_0^{Ts} \int_{Ts}^{Tt} a(u-v) :X(u)X(v): dudv \\ &:= A(s, t, T) + B(s, t, T). \end{aligned}$$

Now we estimate $B(s, t, T)$ (the function $A(s, t, T)$ can be estimated similarly). We have by Theorem 3.9 of [19] that

$$\begin{aligned}\mathbb{E}[B^2(s, t, T)] &= \frac{4}{T} \int_0^{Ts} du_1 \int_{Ts}^{Tt} dv_1 \int_0^{Ts} du_2 \int_{Ts}^{Tt} dv_2 a(u_1 - v_1) a(u_2 - v_2) \mathbb{E}(: X(u_1) X(v_1) :: X(u_2) X(v_2) :) \\ &= \frac{4}{T} \int_0^{Ts} du_1 \int_{Ts}^{Tt} dv_1 \int_0^{Ts} du_2 \int_{Ts}^{Tt} dv_2 a(u_1 - v_1) a(u_2 - v_2) [r(u_1 - u_2) r(v_1 - v_2) + r(u_1 - v_2) r(v_1 - u_2)] \\ &:= B_1(s, t, T) + B_2(s, t, T).\end{aligned}$$

By the change of variables $x_1 = u_1 - v_1$, $x_2 = v_2 - u_2$, $x_3 = u_2 - u_1$, $x_4 = v_2$, and noting that $r(\cdot)$ and $a(\cdot)$ are even functions, we have

$$B_1(s, t, T) \leq \frac{4}{T} \int_{Ts}^{Tt} dx_4 \int_{\mathbb{R}^3} |a(x_1) a(x_2) r(x_3) r(x_1 + x_2 + x_3)| dx_1 dx_2 dx_3.$$

Since $|r(t)| \leq r(0)$, we have $r(\cdot) \in L^\infty(\mathbb{R})$. We also have $r(\cdot) \in L^p(\mathbb{R})$ by condition (2.5), where $1/p + 1/q \geq 3/2$. The L^p -interpolation theorem states that if a function is in L^{p_1} and L^{p_2} with $0 < p_1 \leq p_2 \leq \infty$, then it is in $L^{p'}$, $p_1 \leq p' \leq p_2$. By the L^p -interpolation theorem, one can choose $p' \geq p$ such that $r(\cdot) \in L^{p'}(\mathbb{R})$ and

$$\frac{1}{p'} + \frac{1}{p'} + \frac{1}{q} + \frac{1}{q} = 3, \quad \text{that is,} \quad \frac{1}{p'} + \frac{1}{q} = \frac{3}{2}.$$

Then by Lemma 3.3, one has $B_1(s, t, T) \leq 4\|r\|_{p'}^2 \|a\|_q^2 (t - s)$. Similarly, one can establish the bound $B_2(s, t, T) \leq C(t - s)$, and hence $B(s, t, T) \leq C(t - s)$. So (3.8) is proved. $\square$

The lemmas that follow will be used in the proof of Theorem 2.4.

Lemma 3.5. *Define*

$$\Delta_t(x) = \int_0^t e^{isx} ds = \frac{e^{itx} - 1}{ix}, \quad (3.9)$$

Then for any $\delta \in (0, 1)$, there exists a constant $c > 0$ depending only on δ , such that

$$|\Delta_t(x)| \leq c|t|^\delta f_\delta(x), \quad t \in [0, 1], \quad x \in \mathbb{R}, \quad (3.10)$$

where

$$f_\delta(x) = \begin{cases} |x|^{\delta-1} & \text{if } |x| > 1; \\ 1 & \text{if } |x| \leq 1. \end{cases} \quad (3.11)$$

Proof. In view of (3.9), we have $|\Delta_t(x)| \leq \int_0^t |e^{isx}| ds = t$. So under the constraint $t \in [0, 1]$, we have $|\Delta_t(x)| \leq t \leq t^\delta$. On the other hand, from Lemma 2 from [25], with some constant $C > 0$, we have $|e^{ix} - 1| \leq C|x|^\delta$, $\delta \in (0, 1)$. So

$$|\Delta_t(x)| \leq \frac{|e^{itx} - 1|}{|x|} \leq C|tx|^\delta |x|^{-1} = Ct^\delta |x|^{\delta-1}.$$

Combining this with (3.11), we obtain (3.10). $\square$

We quote Lemma 1 of [25] in a special case, convenient for our purposes.

Lemma 3.6. *Let $\gamma_i < 1$, $\gamma_i + \gamma_{i+1} > 1/2$, and let δ be such that*

$$0 \leq \delta < \frac{\gamma_i + \gamma_{i+1}}{2},$$

where $i = 1, \dots, 4$ (with $\gamma_5 = \gamma_1$). Then

$$\int_{\mathbb{R}^4} f_\delta(y_1 - y_2) f_\delta(y_2 - y_3) f_\delta(y_3 - y_4) f_\delta(y_4 - y_1) |y_1|^{-\gamma_1} |y_2|^{-\gamma_2} |y_3|^{-\gamma_3} |y_4|^{-\gamma_4} d\mathbf{y} < \infty,$$

where $f_\delta(\cdot)$ is as in (3.11).

Lemma 3.6 can be used to establish the following result.

Lemma 3.7. *The function*

$$H_t^*(x_1, x_2) := |x_1|^{\alpha_1/2} |x_2|^{\alpha_2/2} \int_{\mathbb{R}} |\Delta_t(x_1 + u) \Delta_t(x_2 - u)| |u|^{-\beta} du \quad (3.12)$$

is in $L^2(\mathbb{R}^2)$ for all $(\alpha_1, \alpha_2, \beta)$ in the open region $\{(\alpha_1, \alpha_2, \beta) : \alpha_1, \alpha_2, \beta < 1, \alpha_i + \beta > 1/2, i = 1, 2\}$.

Proof. It suffices focus on the case where $t \in [0, 1]$, otherwise a change of variable can reduce it to this case. We have by suitable change of variables and Lemma 3.5 that

$$\begin{aligned} \|H_t^*\|_{L^2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^4} |\Delta_t(y_1 - y_2) \Delta_t(y_2 - y_3) \Delta_t(y_3 - y_4) \Delta_t(y_4 - y_1)| |y_1|^{-\alpha_1} |y_2|^{-\beta} |y_3|^{-\alpha_2} |y_4|^{-\beta} d\mathbf{y} \\ &\leq C \int_{\mathbb{R}^4} f_\delta(y_1 - y_2) f_\delta(y_2 - y_3) f_\delta(y_3 - y_4) f_\delta(y_4 - y_1) |y_1|^{-\alpha_1} |y_2|^{-\beta} |y_3|^{-\alpha_2} |y_4|^{-\beta} d\mathbf{y}. \end{aligned}$$

Then apply Lemma 3.6, noting that δ can be chosen arbitrarily small. $\square$

Lemma 3.8. *Define the function*

$$H_{t,T}^*(x_1, x_2) = A_{1,T}(x_1, x_2) |x_1 x_2|^{-\alpha/2} \int_{\mathbb{R}} |\Delta_t(x_1 + u) \Delta_t(x_2 - u)| |u|^{-\beta} A_{2,T}(u) du, \quad (3.13)$$

where

$$A_{1,T}(x_1, x_2) = \sqrt{\frac{L_1(x_1/T)}{L_1(1/T)} \frac{L_1(x_2/T)}{L_1(1/T)}}, \quad A_{2,T}(u) = \frac{L_2(u/T)}{L_2(1/T)}. \quad (3.14)$$

Then for large enough T , we have $H_{t,T}^*(x_1, x_2) \in L^2(\mathbb{R}^2)$.

Proof. By (2.10) and (3.14), for any $\epsilon > 0$ there exists $C > 0$, such that for T large enough,

$$|A_{1,T}(x_1, x_2)| \leq C(|x_1|^\epsilon + |x_1|^{-\epsilon})(|x_2|^\epsilon + |x_2|^{-\epsilon}) \quad (3.15)$$

and

$$|A_{2,T}(u)| \leq C(|u|^\epsilon + |u|^{-\epsilon}). \quad (3.16)$$

Hence, with some constant $C > 0$,

$$|H_{t,T}^*(x_1, x_2)| \leq C \int_{\mathbb{R}} |\Delta_t(x_1 + u) \Delta_t(x_2 - u)| |u|^{-\beta} (|u|^\epsilon + |u|^{-\epsilon}) du |x_1 x_2|^{-\alpha/2} (|x_1|^\epsilon + |x_1|^{-\epsilon})(|x_2|^\epsilon + |x_2|^{-\epsilon}). \quad (3.17)$$

Because by Lemma 3.7, the function H_t^* in (3.12) is in $L^2(\mathbb{R}^2)$ for all (α, β) in an open region $\{(\alpha, \beta) : \alpha_1, \alpha_2, \beta < 1, \alpha_i + \beta > 1/2, i = 1, 2\}$. By choosing ϵ small enough, we infer that the right-hand side of (3.17) is in $L^2(\mathbb{R}^2)$, and the result follows. $\square$

Lemma 3.9. *Let $Z_T(t)$ be as in (2.11), and let*

$$Z'_T(t) := \int_{\mathbb{R}^2}'' H_{t,T}(x_1, x_2) W(dx_1)W(dx_2), \quad (3.18)$$

where

$$H_{t,T}(x_1, x_2) = A_{1,T}(x_1, x_2)|x_1 x_2|^{-\alpha/2} \left[\int_{\mathbb{R}} \Delta_t(x_1 + u) \Delta_t(x_2 - u) |u|^{-\beta} A_{2,T}(u) du \right]. \quad (3.19)$$

Then $Z_T(t) \stackrel{f.d.d.}{=} Z'_T(t)$, that is, the processes $Z_T(t)$ and $Z'_T(t)$ have the same finite-dimensional distributions.

Proof. Using the spectral representation of $X(t)$ (see, e.g., [5], Chapter XI, Section 8): $X(t) = \int_{\mathbb{R}} e^{itx} \sqrt{f(x)} W(dx)$, where $W(\cdot)$ is a complex Gaussian measure with Lebesgue control measure, and the diagram formula (see, e.g., [20], Chapter 5), we have

$$X(u)X(v) - \mathbb{E}[X(u)X(v)] = \int_{\mathbb{R}^2}'' e^{i(u x_1 + v x_2)} \sqrt{f(x_1)f(x_2)} W(dx_1)W(dx_2).$$

By a stochastic Fubini Theorem (see [22], Theorem 2.1) and Lemma 3.8, one can change the integration order to get (note that by (1.2) we have $\hat{g}(t) = \int_{\mathbb{R}} e^{itx} g(x) dx$):

$$\begin{aligned} Z_T(t) &= \frac{1}{T^{\alpha+\beta} L_1(1/T) L_2(1/T)} \int_{\mathbb{R}^2}'' \sqrt{f(x_1)f(x_2)} \int_0^{Tt} \int_0^{Tt} \int_{\mathbb{R}} e^{i(u-v)w} g(w) dw e^{i(u x_1 + v x_2)} du dv W(dx_1)W(dx_2) \\ &= \frac{1}{T^{\alpha+\beta} L_1(1/T) L_2(1/T)} \int_{\mathbb{R}^2}'' \sqrt{f(x_1)f(x_2)} \int_{\mathbb{R}} \int_0^{Tt} e^{iu(x_1+w)} du \int_0^{Tt} e^{iv(x_2-w)} dv |w|^{-\beta} L(w) dw W(dx_1)W(dx_2) \\ &= \frac{1}{T^{\alpha+\beta} L_1(1/T) L_2(1/T)} \int_{\mathbb{R}^2}'' \sqrt{f(x_1)f(x_2)} \int_{\mathbb{R}} \Delta_{Tt}(x_1 + w) \Delta_{Tt}(x_2 - w) |w|^{-\beta} L_2(w) dw W(dx_1)W(dx_2). \end{aligned}$$

Now we use the change of variables $w \rightarrow u/T$, $x_1 \rightarrow x_1/T$, $x_2 \rightarrow x_2/T$, where the latter two change of variables are subject to the rule $W(dx/T) \stackrel{d}{=} T^{-1/2} W(dx)$ (see, e.g., [4], Proposition 4.2), to obtain

$$\begin{aligned} Z_T(t) &\stackrel{f.d.d.}{=} \frac{1}{T^{\alpha+\beta} L_1(1/T) L_2(1/T)} \times \\ &\quad \int_{\mathbb{R}^2}'' \sqrt{f(x_1/T)f(x_2/T)} \int_{\mathbb{R}} \Delta_t(x_1 + u) \Delta_t(x_2 - u) |u/T|^{-\beta} L_2(u/T) T du T^{-1} W(dx_1)W(dx_2). \end{aligned} \quad (3.20)$$

Taking into account the equality $f(x/T) = |x/T|^{-\alpha} L_1(x/T)$ and equations in (3.14), we see that the right hand side of (3.20) coincides with (3.18). This completes the proof. $\square$

The lemmas that follow will be used to establish tightness in the space $C[0, 1]$ in Theorem 2.4.

Lemma 3.10. *Let δ be a fixed number within the range $(0, (\alpha + \beta)/2)$, and let $Z_T(t)$ be as in (2.11). Then for all $0 \leq s \leq t \leq 1$ and T large enough, there exists a constant $C > 0$, such that*

$$\mathbb{E}[|Z_T(t) - Z_T(s)|^2] \leq C(t - s)^{2\delta}. \quad (3.21)$$

The same estimate also holds for the corresponding limiting process $Z(t)$ defined by (2.12), (2.13).

Proof. First, in view of Lemma 3.9, we have $\mathbb{E}[|Z_T(t) - Z_T(s)|^2] = \mathbb{E}[|Z'_T(t) - Z'_T(s)|^2]$. Next, using the linearity of the multiple stochastic integral, we can write

$$Z'_T(t) - Z'_T(s) = \int_{\mathbb{R}^2}'' H_{s,t,T}(x_1, x_2) W(dx_1)W(dx_2),$$

where

$$H_{s,t,T}(x_1, x_2) = A_{1,T}(x_1, x_2) |x_1 x_2|^{-\alpha/2} \int_{\mathbb{R}} [\Delta_t(x_1 + u) \Delta_t(x_2 - u) - \Delta_s(x_1 + u) \Delta_s(x_2 - u)] |u|^{-\beta} A_{2,T}(u) du. \quad (3.22)$$

The term in the brackets of the integrand in (3.22) can be rewritten as follows:

$$\begin{aligned} & \Delta_t(x_1 + u) \Delta_t(x_2 - u) - \Delta_s(x_1 + u) \Delta_s(x_2 - u) \\ &= \int_0^t \int_0^t e^{i w_1(x_1 + u)} e^{i w_2(x_2 - u)} dw_1 dw_2 - \int_0^s \int_0^s e^{i w_1(x_1 + u)} e^{i w_2(x_2 - u)} dw_1 dw_2 \\ &= \int_0^s dw_1 \int_s^t dw_2 \dots + \int_s^t dw_1 \int_0^s dw_2 \dots + \int_s^t dw_1 \int_s^t dw_2 \dots \\ &= \Delta_s(x_1 + u) \Delta_{t-s}(x_2 - u) + \Delta_{t-s}(x_1 + u) \Delta_s(x_2 - u) + \Delta_{t-s}(x_1 + u) \Delta_{t-s}(x_2 - u). \end{aligned}$$

Now we apply Lemma 3.5 to get

$$\begin{aligned} |\Delta_t(x_1 + u) \Delta_t(x_2 - u) - \Delta_s(x_1 + u) \Delta_s(x_2 - u)| &\leq C[s^\delta(t-s)^\delta + (t-s)^\delta s^\delta + (t-s)^{2\delta}] f_\delta(x_1 + u) f_\delta(x_2 - u) \\ &\leq C(t-s)^\delta f_\delta(x_1 + u) f_\delta(x_2 - u), \end{aligned} \quad (3.23)$$

where the last inequality follows because $0 \leq s^\delta \leq 1$ and $0 \leq (t-s)^\delta \leq 1$.

Next, using formula (4.5') of [20], (3.22) and (3.23), we can write

$$\begin{aligned} \mathbb{E}[|Z_T(t) - Z_T(s)|^2] &= \|H_{s,t,T}\|_{L^2(\mathbb{R}^2)}^2 \leq C|t-s|^{2\delta} \int_{\mathbb{R}^2} dx_1 dx_2 A_{1,T}(x_1, x_2)^2 |x_1 x_2|^{-\alpha} \times \\ &\quad \int_{\mathbb{R}^2} du_1 du_2 f_\delta(x_1 + u_1) f_\delta(x_2 - u_1) f_\delta(-x_1 + u_2) f_\delta(-x_2 - u_2) |u_1|^{-\beta} |u_2|^{-\beta} A_{2,T}(u_1) A_{2,T}(u_2) \\ &\leq C|t-s|^{2\delta} \int_{\mathbb{R}^4} dy_1 dy_2 dy_3 dy_4 A_{1,T}(y_1, y_3)^2 A_{2,T}(y_2) A_{2,T}(y_4) \times \\ &\quad f_\delta(y_1 - y_2) f_\delta(y_2 - y_3) f_\delta(y_3 - y_4) f_\delta(y_4 - y_1) |y_1|^{-\alpha} |y_2|^{-\beta} |y_3|^{-\alpha} |y_4|^{-\beta}, \end{aligned} \quad (3.24)$$

where we have applied the change of variables: $y_1 = x_1$, $y_2 = -u_1$, $y_3 = -x_2$, $y_4 = u_2$.

Since by assumption $\alpha < 1$, $\beta < 1$ and $\alpha + \beta > 1/2$, and the exponent ϵ in (3.15) and (3.16) can be chosen arbitrarily small, for a fixed δ satisfying $0 < \delta < (\alpha + \beta)/2$, we can apply Lemma 1 of [25] to conclude that the integral

$$\int_{\mathbb{R}^4} d\mathbf{y} A_{1,T}(y_1, y_3)^2 A_{2,T}(y_2) A_{2,T}(y_4) f_\delta(y_1 - y_2) f_\delta(y_2 - y_3) f_\delta(y_3 - y_4) f_\delta(y_4 - y_1) |y_1|^{-\alpha} |y_2|^{-\beta} |y_3|^{-\alpha} |y_4|^{-\beta}$$

is bounded for sufficiently large T , which in view of (3.24) implies (3.21). The proof for $Z_T(t)$ is thus complete. The proof for $Z(t)$ is similar and so we omit the details. $\square$

4 Proof of Main Results

Proof of Theorem 2.1. By Lemma 3.2, for any $0 \leq t_1 < \dots < t_n$, and constants $c_1, \dots, c_n$, we have

$$\lim_{T \rightarrow \infty} \text{Var} \left[\sum_{j=1}^n c_j \left(\tilde{Q}_T(t_j) - L_T(t_j) \right) \right] = 0.$$

Therefore the convergence of finite-dimensional distributions of $\tilde{Q}_T(t)$ to that of Brownian motion $\sigma B(t)$ follows from Lemma 3.1 with $f_Y(\cdot)$ given in (3.7) and the Cramér-Wold Device. $\square$

Proof of Theorem 2.2. In view of the well-known Prokhorov's Theorem (see, e.g., [3], p. 58), to prove the theorem, we need to show convergence of finite-dimensional distributions and tightness. The former has been established in Theorem 2.1. To prove tightness, observe that by Lemma 3.4 and the hypercontractivity inequality of the multiple Wiener-Itô integrals (see [20], Corollary 5.6), for any $T > 0$ and $0 \leq s \leq t \leq 1$, there exists a constant $C > 0$ to satisfy

$$\mathbb{E} \left[|\tilde{Q}_T(t) - \tilde{Q}_T(s)|^4 \right] \leq C_2 \left(\mathbb{E} \left[|\tilde{Q}_T(t) - \tilde{Q}_T(s)|^2 \right] \right)^2 \leq C(t-s)^2. \quad (4.1)$$

Now the tightness of the family of measures generated by the processes $\{\tilde{Q}_T(t) : T > 0\}$ in $C[0, 1]$ follows from Lemma 5.1 of [17]. $\square$

Proof of Theorem 2.3. The convergence of finite-dimensional distributions follows from Theorem 2.1. In fact, the assumptions on f and g in Theorem 2.3 imply the conditions (2.2) and (2.3) in Theorem 2.1 (see the proof of Theorem 5 in [10]). The tightness can be shown similarly as in the proof of Theorem 2.2. $\square$

Proof of Theorem 2.4. As in the proof of Theorem 2.2, we need to show convergence of finite-dimensional distributions and tightness. We first prove the convergence of finite-dimensional distributions, that is, $Z_T(t) \xrightarrow{f.d.d.} Z(t)$ as $T \rightarrow \infty$, where $Z_T(t)$ and $Z(t)$ are defined by (2.11) and (2.12), respectively.

By Lemma 3.9, the process $Z_T(t)$ defined in (2.11) has the same finite-dimensional distributions as the process $Z'_T(t)$ defined in (3.18). Therefore, taking into account the linearity of multiple Wiener-Itô integral, and applying Crámer-Wold device, to prove $Z_T(t) \xrightarrow{f.d.d.} Z(t)$, it is enough to show that as $T \rightarrow \infty$,

$$H_{t,T}(x_1, x_2) \rightarrow H_t(x_1, x_2) \quad \text{in } L^2(\mathbb{R}^2), \quad (4.2)$$

where $H_t(x_1, x_2)$ and $H_{t,T}(x_1, x_2)$ are as in (2.13) and (3.19), respectively.

First, we show pointwise convergence for a.e. $(x_1, x_2) \in \mathbb{R}^2$, that is,

$$H_{t,T}(x_1, x_2) = A_{1,T}(x_1, x_2) |x_1 x_2|^{-\alpha/2} \int_{\mathbb{R}} \Delta_t(x_1 + u) \Delta_t(x_2 - u) |u|^{-\beta} A_{2,T}(u) du \quad (4.3)$$

$$\rightarrow H_t(x_1, x_2) = |x_1 x_2|^{-\alpha/2} \int_{\mathbb{R}} \Delta_t(x_1 + u) \Delta_t(x_2 - u) |u|^{-\beta} du \quad \text{as } T \rightarrow \infty. \quad (4.4)$$

Because $L_1(x)$ is a slowly varying function, we have $A_{1,T}(x_1, x_2) \rightarrow 1$ as $T \rightarrow \infty$, where $A_{1,T}$ is as in (3.14). To show that the integral in (4.3) converges to the integral in (4.4), note first that by (3.14), $A_{2,T}(u) \rightarrow 1$ as $T \rightarrow \infty$ because $L_2(x)$ is a slowly varying function. Hence one only needs to bound the integrand properly and apply the Dominated Convergence Theorem. To this end, observe that by (3.16) for T large enough, we have

$$g_T(u; x_1, x_2) := |\Delta_t(x_1 + u)| |\Delta_t(x_2 - u)| |u|^{-\beta} A_{2,T}(u) \quad (4.5)$$

$$\leq C |\Delta_t(x_1 + u)| |\Delta_t(x_2 - u)| |u|^{-\beta} (|u|^\epsilon + |u|^{-\epsilon}) := g_\epsilon(u; x_1, x_2). \quad (4.6)$$

By choosing ϵ small enough, using Fubini Theorem and Lemma 3.7, we conclude that $g_\epsilon(\cdot; x_1, x_2) \in L^1(\mathbb{R})$ for a.e. $(x_1, x_2) \in \mathbb{R}^2$. Now (4.2) follows from (3.17) and the Dominated Convergence Theorem.

To prove tightness, first observe that by the hypercontractivity inequality of the multiple Wiener-Itô integrals (see [20], Corollary 5.6) and Lemma 3.10, for T large enough and for any $0 \leq s \leq t \leq 1$, there exists a constant $C > 0$ to satisfy

$$\mathbb{E} \left[|Z_T(t) - Z_T(s)|^4 \right] \leq C_2 \left(\mathbb{E} \left[|Z_T(t) - Z_T(s)|^2 \right] \right)^2 \leq C|t-s|^{4\delta}, \quad (4.7)$$

where δ is a fixed number within the range $0 < 4\delta < 2(\alpha + \beta)$. Since by assumption $\alpha + \beta > 1/2$, we can choose δ to satisfy $4\delta > 1$. Inequalities similar to (4.7) hold also for the limit process $Z(t)$.

In view of (4.7) and a similar inequality for $Z(t)$, it follows from Kolmogorov's criterion (see, e.g., [2] Theorem 8.1(1)) that the processes $Z_T(t)$ and $Z(t)$ admit continuous versions when T is large enough.

Now the tightness of the family of measures generated by the processes $\{Z_T(t) : T > 0\}$ in $C[0, 1]$ follows from Lemma 5.1 of [17]. Theorem 2.4 is proved. $\square$

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