Lyapunov-Krasovskii characterization of the input-to-state stability for neutral systems in Hale's form [☆]

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Abstract

The primary contribution of the paper is to show that the Lyapunov-Krasovskii conditions available in the literature, for checking the input-to-state stability property of systems described by neutral functional differential equations in Hale's form, are also necessary. It is shown that the Lyapunov-Krasovskii conditions available in the literature are equivalent to the input-to-state stability property, for an enlarged class of neutral systems. Finally, a novel necessary and sufficient condition for the input-to-state stability property is provided.

Keywords: Neutral Functional Differential Equations, Converse Lyapunov-Krasovskii Theorems, Input-to-State Stability, Time Delays.

1. Introduction

Neutral functional differential equations in Hale's form (see [12], [23], [30]) describe many systems in electrical engineering (lossless transmission lines, flip-flop circuits, partial element equivalent circuits, see [1], [2], [4], [12], [30], [39] and references therein), mechanical engineering (hydraulic systems under matching conditions, see [30], [39] and references therein), bioengineering (see [41] and references therein), and often provide an alternative description of systems described by hyperbolic partial differential equations under matching conditions (see [19] and references therein). The literature on stability and control of systems described by neutral functional differential equations in Hale's form is huge. Many monographs devote entire chapters and sections to the topic (see, for instance, [4], [7], [22], [30]). It is pointed out in [23],

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Section 3.5, p. 119, that *many neutral equations (including those important for applications), and even whole classes of them can be brought to the form of a neutral functional differential equation in Hale's form*. Chapters 10 and 11, in [4], are devoted to the optimal control problem of linear and nonlinear systems described by neutral functional differential equations in Hale's form. The stabilization problem for neutral systems in Hale's form, by control Lyapunov-Krasovskii functionals, is studied in [34]. We stress here that neutral functional differential equations in Hale's form include, as special case, retarded functional differential equations. Thus, any result, shown for neutral functional differential equations in Hale's form, is inherited by retarded functional differential equations. We stress here too that Hale's form, as formulated in [12], is the classical form for a neutral equation (see [22], [30] and references therein).

In [10], [13], [14], [17], [18], [45], many converse Lyapunov theorems have been presented, for various global stability notions, for systems described by retarded functional differential equations, in very general settings. Time varying delays, disturbances, uncertainties, time-varying equations are considered. Besides the global uniform asymptotic stability, also the input-to-state stability and the weighted input-to-output stability are investigated. The reader can refer to the monograph [15] for an extensive presentation of this topic. As far as systems described by neutral equations are concerned, converse Lyapunov-Krasovskii theorems are available for the (local, uniform) asymptotic stability (see [3], [5]). Methods for constructing the quadratic Lyapunov-Krasovskii functional are also provided, for the linear case, in [20], [21], [40], [44]. The reader can refer to the monograph [22] for an extensive presentation of methods for constructing quadratic Lyapunov-Krasovskii functionals for the stability of linear systems described by both retarded and neutral type equations. Converse Lyapunov-Krasovskii methods are also used for establishing instability criteria for linear neutral systems in [29]. As far as global asymptotic stability notions are concerned, converse Lyapunov-Krasovskii theorems for nonlinear neutral systems, with strongly stable linear operator, are established in [37]. In [16], converse Lyapunov-Krasovskii results are provided for the global asymptotic stability of systems described by neutral functional differential equations with vanishing perturbations. In particular, it is shown that the lower bound for the involved functional can be given in terms of an input-to-state stable difference operator not necessarily equal to the difference operator of the system at hand. As pointed out in the introduction in [16], the establishment of converse Lyapunov results for the global asymptotic stability of systems with vanishing perturbations has opened to further developments of the input-to-state stability theory (see [26] for systems described by ordinary differential equations, see [17], [18] for systems described by retarded functional differential equations), and it is expected that the same development will appear in the area of neutral functional differential equations. We show here how those results lead actually to establishing converse Lyapunov-Krasovskii theorems for the input-to-state stability of neutral systems.

In this paper, we consider time-invariant systems described by neutral equations in Hale's form (see [12], [23]). The difference operator is nonlinear, Lipschitz on bounded sets, and can involve an arbitrary number of arbitrary constant discrete time delays. Distributed time-delays are partially covered, as the map involved in the difference operator is assumed to be independent of the argument at 0 (see Definition 5.1, p. 281, in [12], see Assumption H1, p. 245, in [16]). The standard assumption that the difference operator is input-to-state stable (see [8], [9], [24], [25], [31], [36], [38]) is not introduced here, at least as far as the main result is concerned (see Theorem 7). The map in the right-hand side of the equation is assumed to be Lipschitz on bounded sets. An arbitrary number of arbitrary discrete and distributed time-delays can appear in the right-hand side of the equation. We prove here that the sufficient Lyapunov-Krasovskii conditions for the input-to-state stability, established in [31], [38], are also necessary. Thus we

prove here that those conditions are a full characterization of the input-to-state stability property for neutral systems in Hale's form. The class of systems here considered is more general than the ones dealt with in [31] and in [38] (as far as neutral systems in Hale's form are concerned). Therefore, the results here provided extend also the results available in the literature, concerning sufficient Lyapunov-Krasovskii conditions for the input-to-state stability property of neutral systems. Finally, it is shown that also for the input-to-state stability property, the lower bound for the involved functional can be given in terms of an arbitrary input-to-state stable difference operator not necessarily equal to the difference operator of the system at hand, and with the involved map not even independent of the argument at 0. Therefore, here further necessary and sufficient Lyapunov-Krasovskii conditions, which include, for the sufficiency part, the ones provided in the literature for less general neutral systems as special case, are provided. The utility of this kind of further necessary and sufficient Lyapunov-Krasovskii conditions has been illustrated in an example in [16], concerning the global asymptotic stability in presence of vanishing perturbations, where a difference operator, with involved map not independent of the argument at 0, is used. An example is here provided, again involving a difference operator with involved map not independent of the argument at 0, concerning the input-to-state stability property. The paper is organized as follows. In Section 2, Neutral Functional Differential Equations are introduced. In Section 3, the converse Lyapunov-Krasovskii theorem for the input-to-state stability is stated. In Section 4 an illustrative example is shown. In Section 5 conclusions are drawn. The proof of the main theorem is reported in the Appendix.

Definitions and Notation

R denotes the set of real numbers, R^* denotes the extended real line $[-\infty, +\infty]$, R^+ denotes the set of non negative reals $[0, +\infty)$. For a positive integer n , the symbol $|\cdot|$ stands for the Euclidean norm of a vector in R^n . For a given positive integer m , a Lebesgue measurable function $u : R^+ \rightarrow R^m$ is said to be essentially bounded if $\text{ess sup}_{t \in R^+} |u(t)| < +\infty$, where $\text{ess sup}_{t \in R^+} |u(t)| = \inf\{a \in R^* : \lambda(\{t \in R^+ : |u(t)| > a\}) = 0\}$, λ denoting the Lebesgue measure. The symbol $\|\cdot\|_\infty$ denotes the essential supremum norm, that is, for a Lebesgue measurable and essentially bounded function $u : R^+ \rightarrow R^m$, $\|u\|_\infty = \text{ess sup}_{t \in R^+} |u(t)|$. For a real $T > 0$, with $u_{[0,T]} : R^+ \rightarrow R^m$ is indicated the function given by $u_{[0,T]}(t) = u(t)$ for all $t \in [0, T]$ and $u(t) = 0$ elsewhere. For a positive integer n , for a positive real Δ (maximum involved time-delay): C denotes the space of the continuous functions mapping $[-\Delta, 0]$ into R^n ; $W^{1,\infty}$ denotes the space of the absolutely continuous functions mapping $[-\Delta, 0]$ into R^n , with essentially bounded derivative. The supremum norm of a function in C is indicated, again, with the symbol $\|\cdot\|_\infty$. For a function $x : [-\Delta, c) \rightarrow R^n$, with $0 < c \leq +\infty$, for any real $t \in [0, c)$, x_t is the function in C defined as $x_t(\tau) = x(t + \tau)$, $\tau \in [-\Delta, 0]$. For a positive real δ , $\phi \in C$, $C_\delta(\phi) = \{\psi \in C : \|\psi - \phi\|_\infty \leq \delta\}$. For a positive real δ , $x \in R^n$, $B_\delta^n(x) = \{y \in R^n : |y - x| \leq \delta\}$. Let us here recall that a function $\gamma : R^+ \rightarrow R^+$ is: of class $\mathcal{P}$ if it is continuous, zero at zero, and positive at any positive real; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded; of class $\mathcal{L}$ if it is continuous and is monotonically decreasing to zero as its argument tends to $+\infty$. A function $\beta : R^+ \times R^+ \rightarrow R^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \geq 0$. The symbol I_d denotes the identity function defined in R^+ . A function $f : C \rightarrow R^n$ is said to be independent of the argument at 0, if there exists a positive real $\delta \in (0, \Delta]$ such that for any $\phi_i \in C$, $i = 1, 2$, satisfying $\phi_1(\tau) = \phi_2(\tau)$, $\tau \in [-\Delta, -\delta]$, the equality holds $f(\phi_1) = f(\phi_2)$ (see Definition 5.1, p. 281, in [12], see Assumption H1 in [16], see also [32]). The symbol $\circ$ denotes composition of functions. In next sections, ODE stands for ordinary differential equation, RFDE stands for retarded functional differential equation, FDE stands for functional difference equation, NFDE

stands for neutral functional differential equation, GAS stands for global asymptotic stability or globally asymptotically stable, ISS stands for input-to-state stability or input-to-state stable, LK stands for Lyapunov-Krasovskii.

2. NFDEs in Hale's form and stability definitions

Before introducing NFDEs in Hale's form, we give the following definition (see [16], [33], [36]), which will be needed later.

Definition 1. For a positive integer n and a positive real Δ , let $K : C \rightarrow R^n$ be a continuous map for which there exists a function l_K of class $\mathcal{K}_\infty$ such that

$$|K(\phi)| \leq l_K(\|\phi\|_\infty), \quad \forall \phi \in C \quad (1)$$

Let $\mathcal{D}_K : C \rightarrow R^n$ be the map defined, for $\phi \in C$, as

$$\mathcal{D}_K(\phi) = \phi(0) - K(\phi)$$

The map $\mathcal{D}_K$ is said to be an ISS-map if there exist a function Ω_K of class $\mathcal{KL}$ and a function ω_K of class $\mathcal{K}$ such that, for any $\xi_0 \in C$ and for any continuous function $u : R^+ \rightarrow R^n$, satisfying $\mathcal{D}_K \xi_0 = u(0)$, there exists, uniquely, a continuous solution $\xi_t \in C$, for all $t \in R^+$, of the FDE

$$\begin{aligned} \mathcal{D}_K \xi_t &= u(t), & t &\geq 0, \\ \xi(\tau) &= \xi_0(\tau), & \tau &\in [-\Delta, 0], \end{aligned} \quad (2)$$

and, furthermore, the solution satisfies the inequality

$$\|\xi_t\|_\infty \leq \Omega_K(\|\xi_0\|_\infty, t) + \omega_K(\|u_{[0,t]}\|_\infty) \quad (3)$$

Let us consider a system described by the following NFDE in Hale's form (see [12], [16], [23], [32] and the references therein)

$$\begin{aligned} \frac{d}{dt} \mathcal{D}x_t &= f(x_t, u(t)), & t &\geq 0, & a.e., \\ x(\tau) &= x_0(\tau), & \tau &\in [-\Delta, 0], & x_0 \in C, \end{aligned} \quad (4)$$

where $x(t) \in R^n$, n is a positive integer, $\Delta > 0$ is the maximum involved time-delay, the map $f : C \times R^m \rightarrow R^n$ is Lipschitz on bounded subsets of $C \times R^m$, m is a positive integer; the input signal $u(t) \in R^m$ is Lebesgue measurable and locally essentially bounded, $\mathcal{D} : C \rightarrow R^n$ is a map defined, for $\phi \in C$, as

$$\mathcal{D}\phi = \phi(0) - q(\phi), \quad (5)$$

where the map $q : C \rightarrow R^n$ is Lipschitz on bounded subsets of C and independent of the argument at 0. We assume $f(0, 0) = 0$ and the existence of a function l of class $\mathcal{K}_\infty$ such that

$$|q(\phi)| \leq l(\|\phi\|_\infty), \quad \forall \phi \in C. \quad (6)$$

Remark 2. Notice that the map q involved in the definition of the map $\mathcal{D}$ is assumed to be independent of the argument at 0 and that this assumption is not introduced for the map K in Definition 1. This fact plays an interesting role in forthcoming Theorem 7.

Remark 3. If, in (5), $q(\phi) \equiv 0$, $\phi \in C$ (i.e., $\mathcal{D}\phi = \phi(0)$), then (4) reduces to a RFDE.

The following Theorem 4 is an extension of results in [11] to the neutral systems here considered with Lebesgue measurable, locally essentially bounded input signals. It is provided in [16] (see Theorem 1, pp. 246-248, which concerns more general neutral systems with vanishing perturbations, see its proof as far as point (2) in Theorem 4 is concerned).

Theorem 4. ([11], [16]) The following results hold:

- 1) there exist, uniquely, a continuous solution $x(t)$ of (4), in a maximal time interval $[0, b)$, $0 < b \leq +\infty$;
- 2) the function $t \rightarrow \mathcal{D}x_t$ is locally absolutely continuous in $[0, b)$;
- 3) if $b < +\infty$, then the function $t \rightarrow x(t)$, $t \in [0, b)$, is unbounded in $[0, b)$.

Definition 5. (see [31], [38], [35], [42]) The system described by (4) is said to be ISS if there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any initial state $x_0 \in C$ and any Lebesgue measurable, locally essentially bounded input signal $u : R^+ \rightarrow R^m$, the corresponding solution exists in R^+ and, furthermore, satisfies the inequality

$$\|x_t(x_0, u)\|_\infty \leq \beta(\|x_0\|_\infty, t) + \gamma(\|u_{[0,t]}\|_\infty), \quad t \in R^+ \quad (7)$$

For a locally Lipschitz functional $V : C \rightarrow R^+$, let $D^+V(\phi, u) : C \times R^m \rightarrow R^*$ be the derivative in Driver's form (see [6] as far as RFDEs are concerned) of the functional V , along the vectors generated by the map f , defined, for $\phi \in C$, $u \in R^m$, as follows

$$D^+V(\phi, u) = \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,u}) - V(\phi)}{h}, \quad (8)$$

where, for $h \in [0, \Delta)$, $\phi_{h,u} \in C$ is defined as

$$\phi_{h,u}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h), \\ \mathcal{D}\phi + q(\psi_s) + (s+h)f(\phi, u), & s \in [-h, 0], \end{cases} \quad (9)$$

and, for $s \in [-h, 0]$, $\psi_s \in C$ is defined as

$$\psi_s(\theta) = \begin{cases} \phi(\theta + s + h), & \theta \in [-\Delta, -s-h), \\ \phi(0), & \theta \in [-s-h, 0] \end{cases} \quad (10)$$

Let us consider now the system described by the following NFDE in Hale's form, suitably obtained by (4),

$$\frac{d}{dt} \mathcal{D}x_t = f(x_t, \rho(\|x_t\|_\infty) d(t)), \quad x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad (11)$$

where ρ is a locally Lipschitz function of class $\mathcal{K}_\infty$, $x_0 \in C$, and $d \in \mathcal{M}_B$, with

$$\begin{aligned} \mathcal{M}_B = \\ \{\text{Lebesgue measurable, essentially bounded functions } p : R^+ \rightarrow R^m : \|p\|_\infty \leq 1\} \end{aligned} \quad (12)$$

Let us introduce the following definition.

Definition 6. *The system described by (4) is said to be robustly 0-GAS if there exist a locally Lipschitz function ρ of class $\mathcal{K}_\infty$ and a function β of class $\mathcal{KL}$ such that, for any initial state $x_0 \in C$ and any signal $d \in \mathcal{M}_B$, the solution of (11), with this given function ρ , denoted with $x_t^R(x_0, d)$, exists in R^+ and, furthermore, satisfies the inequality*

$$\|x_t^R(x_0, d)\|_\infty \leq \beta(\|x_0\|_\infty, t), \quad t \in R^+ \quad (13)$$

3. LK characterization of the ISS property for NFDEs in Hale's form

The main result of the paper is given by the following theorem.

Theorem 7. *The following statements are equivalent:*

- a) *the system described by (4) is ISS;*
- b) *the system described by (4) is robustly 0-GAS;*
- c) *for any continuous map $K : C \rightarrow R^n$ (independent of the argument at 0 or not) satisfying (1), with related map $\mathcal{D}_K : C \rightarrow R^n$ being an ISS-map (in the sense of Definition 1), there exist a locally Lipschitz functional $V : C \rightarrow R^+$, functions α_i , $i = 1, 2, 3$, and γ_a of class $\mathcal{K}_\infty$, a functional $N_a : C \rightarrow R^+$, and a function σ of class $\mathcal{K}$, such that the following conditions hold:*

- i) $\alpha_1(|\mathcal{D}_K \phi|) \leq V(\phi) \leq \alpha_2(N_a(\phi)), \quad \forall \phi \in C;$
- ii) $N_a(\phi) \leq \gamma_a(\|\phi\|_\infty), \quad \forall \phi \in C;$
- iii) $D^+V(\phi, u) \leq -\alpha_3(N_a(\phi)) + \sigma(|u|), \quad \forall \phi \in C, \forall u \in R^m;$

- d) *there exists a continuous map $K : C \rightarrow R^n$ (independent of the argument at 0 or not) satisfying (1), with related difference map $\mathcal{D}_K : C \rightarrow R^n$ being an ISS-map (in the sense of Definition 1), and a locally Lipschitz functional $V : C \rightarrow R^+$, functions α_i , $i = 1, 2, 3$, and γ_a of class $\mathcal{K}_\infty$, a functional $N_a : C \rightarrow R^+$, and a function σ of class $\mathcal{K}$, such that the following conditions hold:*

- i) $\alpha_1(|\mathcal{D}_K \phi|) \leq V(\phi) \leq \alpha_2(N_a(\phi)), \quad \forall \phi \in C;$
- ii) $N_a(\phi) \leq \gamma_a(\|\phi\|_\infty), \quad \forall \phi \in C;$
- iii) $D^+V(\phi, u) \leq -\alpha_3(N_a(\phi)) + \sigma(|u|), \quad \forall \phi \in C, \forall u \in R^m.$

The proof of Theorem 7 requires some lemmas and is reported in details in the Appendix, where necessary lemmas from the literature are also reported for the reader's convenience.

Corollary 8. *Let the map $\mathcal{D} : C \rightarrow R^n$ be an ISS-map (in the sense of Definition 1). Then, the following statements are equivalent:*

- 1) *the system described by (4) is ISS;*
- 2) *there exist a locally Lipschitz functional $V : C \rightarrow R^+$, functions α_i , $i = 1, 2, 3$, and γ_a of class $\mathcal{K}_\infty$, a functional $N_a : C \rightarrow R^+$, and a function σ of class $\mathcal{K}$, such that the following conditions hold:*
 - i) $\alpha_1(|\mathcal{D}\phi|) \leq V(\phi) \leq \alpha_2(N_a(\phi)), \quad \forall \phi \in C;$
 - ii) $N_a(\phi) \leq \gamma_a(\|\phi\|_\infty), \quad \forall \phi \in C;$
 - iii) $D^+V(\phi, u) \leq -\alpha_3(N_a(\phi)) + \sigma(|u|), \quad \forall \phi \in C, \forall u \in R^m.$

Proof. As far as the implication (1) $\Rightarrow$ (2) is concerned, choose $K = q$, and thus $\mathcal{D}_K = \mathcal{D}$, in item (c) of Theorem 7. Then, the implication (1) $\Rightarrow$ (2) follows from the implication (a) $\Rightarrow$ (c) in Theorem 7. As far as the implication (2) $\Rightarrow$ (1) is concerned, notice that, if item (2) holds, then item (d) of Theorem 7 holds with the same above choice of K and $\mathcal{D}_K$. Then, the implication (2) $\Rightarrow$ (1) follows from the implication (d) $\Rightarrow$ (a) in Theorem 7. $\square$

Remark 9. *The implication (2) $\Rightarrow$ (1) in Corollary 8 is proved in [31], for the case of linear strongly stable difference operator, and in [38], for more general systems, partially including the ones here considered, described by coupled RFDEs and FDEs, with input-to-state stable nonlinear difference map as hypothesized in Corollary 8. The following assumption is introduced in [31], [38].*

Assumption 10. *There exists a function $g : R^n \times C \times R^m \rightarrow R^n$, independent of the second argument at 0, such that, for any $\phi \in C$, $u \in R^m$, the equality holds $f(\phi, u) = g(\phi(0), \phi, u)$.*

Assumption 10 is a limitation, for instance, to distributed delays appearing in the right-hand side of (4). The implication (2) $\Rightarrow$ (1) in Corollary 8 thus provides an extension, as far as just sufficient LK conditions for the ISS of neutral systems are concerned, with respect to [31], [38], since, by using Theorem 4, Assumption 10 is no more required here. The implication (d) $\Rightarrow$ (a) in Theorem 7 is a further extension, as far as just sufficient LK conditions are concerned, since it can be used with any ISS difference map $\mathcal{D}_K$ (i.e., also not equal to $\mathcal{D}$), and even with related map K not independent of the argument at 0.

Remark 11. *As announced in Remark 2, the map K in item (d) of Theorem 7 is not assumed to be independent of the argument at 0, and as well, the related map $\mathcal{D}_K$ is not required to be equal to the map $\mathcal{D}$ appearing in the left-hand side of (4). Thus, the item (d) of Theorem 7 is a novel necessary and sufficient condition for the ISS property of systems described by NFDEs in Hale's form.*

Remark 12. *The reader can refer to [16], [31], [38], for examples of neutral systems whose robust 0-GAS and ISS properties are proved by means of the (necessary and) sufficient conditions provided in Theorem 7. In the following section, a further example is given, mainly in order to show the role of statement (d) in Theorem 7.*

4. Illustrative Example

The following example shows how important is statement (d) of Theorem 7. It shows how we can exploit the use of ISS-maps which are not necessarily independent of the argument at 0, and do not coincide with the map $\mathcal{D}$ that appears in the left-hand side of the NFDE (4). The example also shows that statement (d) of Theorem 7 is a novel characterization of the ISS property even for systems described by RFDEs. Let us consider a system described by the following NFDE:

$$\frac{d}{dt}\mathcal{D}x_t = \begin{bmatrix} b|x_2(t)| - b|x_2(t-\Delta)| - x_1(t) + b \int_{t-\Delta}^t |x_2(s)| ds + x_2(t) + cx_2(t-\tau) \\ -x_1(t) + b \int_{t-\Delta}^t |x_2(s)| ds - x_2(t) - cx_2(t-\tau) + u(t), \end{bmatrix} \quad (14)$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in R^2$, $u(t) \in R$, $\Delta \geq \tau > 0$ and $b \in R$, $c \in (-1, 1)$ are constants, and, for $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C$, the map $\mathcal{D} : C \rightarrow R^2$ is defined as

$$\mathcal{D}\phi = \begin{bmatrix} \phi_1(0) \\ \phi_2(0) + c\phi_2(-\tau) \end{bmatrix} \quad (15)$$

We show next that the locally Lipschitz Lyapunov functional defined, for $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C$, by

$$V(\phi) = \frac{1}{2} \left(\phi_1(0) - b \int_{-\Delta}^0 |\phi_2(s)| ds \right)^2 + \frac{1}{2} (\phi_2(0) + c\phi_2(-\tau))^2 \quad (16)$$

satisfies statement (d) of Theorem 7 and, consequently, that the system described by (14) is ISS. Notice that the equation (14) corresponds to a NFDE when $c \neq 0$. When $c = 0$, the equation (14) is a RFDE. This illustrates the applicability of statement (d) of Theorem 7 even for systems described by RFDEs, for which statement (d) of Theorem 7 provides a novel characterization of the ISS property. Let $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} : C \rightarrow R^2$ be the map defined, for all $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C$, as

$$K(\phi) = \begin{bmatrix} K_1(\phi) \\ K_2(\phi) \end{bmatrix} = \begin{bmatrix} b \int_{-\Delta}^0 |\phi_2(s)| ds \\ -c\phi_2(-\tau) \end{bmatrix} \quad (17)$$

Notice that the map K defined by (17) is not independent of the argument at 0 (see Definitions and Notation section for details) and satisfies inequality (1) with $l_K(s) = (\Delta|b| + |c|)s$, $s \in R^+$. Moreover, the corresponding map $\mathcal{D}_K$ is an ISS-map. Indeed, in this case, the equation (2) is as follows:

$$\begin{aligned} \xi(t) &= \begin{bmatrix} b \int_{-\Delta}^0 |\xi_2(s)| ds + u_1(t) \\ -c\xi_2(t-\tau) + u_2(t) \end{bmatrix}, & t \geq 0, \\ \xi(\theta) &= \xi_0(\theta), & \theta \in [-\Delta, 0], \quad \xi_0 \in C, \end{aligned} \quad (18)$$

where $\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \in R^2$, $t \geq 0$, $u_i(t) \in R$, $i = 1, 2$, $t \geq 0$, are continuous input signals, and $\mathcal{D}_K \xi_0 = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$. The existence, in R^+ , and the uniqueness of the solution of the system

described by (18) follows by standard step procedures (see [23]). The ISS property of the system described by (18) follows from the application of Corollary 6.1, p. 287, and Theorem 3.5, p. 275, in [12], to the sub-system described by the equation $\xi_2(t) = -c\xi_2(t - \tau) + u_2(t)$, and from the inequality $|\xi_1(t)| \leq |b|\Delta \sup_{\theta \in [-\Delta, 0]} |\xi_2(\theta)| + |u_1(t)|$, $t \in R^+$. Using (14) and (16), we obtain for all $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C$ and $u \in R$:

$$\begin{aligned} D^+V(\phi, u) = & \\ & - \left(\phi_1(0) - b \int_{-\Delta}^0 |\phi_2(s)| ds \right)^2 - (\phi_2(0) + c\phi_2(-\tau))^2 + (\phi_2(0) + c\phi_2(-\tau))u \leq \\ & -V(\phi) + \frac{1}{2}u^2 \end{aligned} \quad (19)$$

We choose $N_a(\phi) = V(\phi)$, $\phi \in C$. Condition (ii) in Theorem 7 holds with $\gamma_a(s) = (2 + c^2 + b^2r^2)s$, $s \in R^+$. By (19), condition (iii) in Theorem 7 holds with $\alpha_3(s) = s$ and $\sigma(s) = \frac{1}{2}s^2$, $s \in R^+$. From (16), (17), the equality follows $V(\phi) = \frac{1}{2}|\mathcal{D}_K\phi|^2$, $\phi \in C$. Therefore condition (i) of Theorem 7 holds with $\alpha_1(s) = \frac{1}{2}s^2$, $\alpha_2(s) = s$, $s \in R^+$. Thus we have shown that statement (d) of Theorem 7 holds and, consequently, that the system described by (14) is ISS.

5. Conclusions

We have provided Lyapunov-Krasovskii characterizations of the input-to-state stability property for systems described by neutral functional differential equations in Hale's form. In particular, it has been shown that the sufficient conditions available in the literature work as well for checking the input-to-state stability of the more general class of systems considered here. Moreover, it is shown that those conditions are also necessary. Finally, a novel (necessary and) sufficient condition is provided, by the use of difference operators, other than the one involved in the neutral equation, which are not independent of the argument at 0. An interesting research topic, which is left for future investigations, concerns the Lyapunov-Krasovskii characterization of stability notions for time-varying neutral systems (see [27]), as well as of neutral systems which are not in Hale's form (see [28]).

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Appendix A. Proof of Theorem 7

We will show that the implications $(d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ hold.

Let us start with the implication $(d) \Rightarrow (a)$. The following Lemma is a particular case of more general Lemma 2.14 in [15].

Lemma 13. (see Lemma 2.14, pp. 82-84, in [15]) *For any function ω of class $\mathcal{K}$ there exist a function δ of class $\mathcal{KL}$ with the following property: if $y : [0, b) \rightarrow \mathbb{R}^+$, $0 < b \leq +\infty$, is a locally absolutely continuous function such that, for some $v \in \mathbb{R}^+$, the implication holds*

$$y(t) \geq v \Rightarrow D^+y(t) \leq -\omega(y(t)), \quad t \in [0, b), \text{ a.e.}, \quad (\text{A.1})$$

where $D^+y(t)$ is the upper right-hand Dini derivative of y , then the following inequality holds

$$y(t) \leq \delta(y(0), t) + \delta(v, 0) \quad (\text{A.2})$$

The following two lemmas hold as well if Assumption 10 does not hold, as in this paper. By exploiting the results in [16], the proof is the same as the one with Assumption 10 satisfied (see [31], [38]), and therefore is omitted.

Lemma 14. ([31], [38]) *The system described by (4) is ISS if and only if there exists a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any initial state $x_0 \in W^{1,\infty}$ and any Lebesgue measurable locally essentially bounded input signal u , the solution of (4), denoted with $x_t(x_0, u)$, exists in $\mathbb{R}^+$ and, furthermore, satisfies the inequality*

$$\|x_t(x_0, u)\|_\infty \leq \beta(\|x_0\|_\infty, t) + \gamma(\|u_{[0,t]}\|_\infty), \quad t \in \mathbb{R}^+ \quad (\text{A.3})$$

Lemma 15. ([31], [38]) *Let $V : \mathbb{C} \rightarrow \mathbb{R}^+$ be a locally Lipschitz functional. Then, for any $\phi \in \mathbb{C}$, for any Lebesgue measurable, locally essentially bounded input u , the equality holds,*

$$D^+V(x_t(\phi, u), u(t)) = \limsup_{h \rightarrow 0^+} \frac{V(x_{t+h}(\phi, u)) - V(x_t(\phi, u))}{h}, \quad t \in [0, b), \text{ a.e.}, \quad (\text{A.4})$$

where $[0, b)$, $0 < b \leq +\infty$, is the maximal interval of existence of the solution of (4), corresponding to initial state ϕ and input signal u . Moreover, if the initial state $\phi \in W^{1,\infty}$, then the function $t \rightarrow V(x_t(\phi, u))$ is locally absolutely continuous in $[0, b)$.

Without any loss of generality, by Lemma 14, let us assume that the initial state $x_0 \in W^{1,\infty}$. We prove first that there exists a function $\bar{\beta}$ of class $\mathcal{KL}$ and a function $\bar{\gamma}$ of class $\mathcal{K}$ such that, for any initial state $x_0 \in W^{1,\infty}$, any $v \in R^+$, and any Lebesgue measurable input u satisfying $\|u\|_\infty \leq v$, the solution exists in R^+ and, furthermore, satisfies the inequality

$$|\mathcal{D}_K x_t| \leq \bar{\beta}(\|x_0\|_\infty, t) + \bar{\gamma}(v) \quad (\text{A.5})$$

Let $x_0 \in W^{1,\infty}$, $v \in R^+$, and $\|u\|_\infty \leq v$. Let $[0, b)$, $0 < b \leq +\infty$ be the maximal interval of existence of the solution of (4). Let $w : [0, b) \rightarrow R^+$ be the function defined, for $t \in [0, b)$, as $w(t) = V(x_t)$. By Lemma 15, this function is locally absolutely continuous in $[0, b)$. Moreover, by Lemma 15, taking into account of points (i), (iii) in item (d), the following implication holds

$$w(t) \geq \alpha_2 \circ \alpha_3^{-1} \circ 2\sigma(v) \Rightarrow D^+ w(t) \leq -\frac{1}{2} \alpha_3 \circ \alpha_2^{-1}(w(t)), \quad t \in [0, b), \text{ a.e.}, \quad (\text{A.6})$$

where $D^+ w$ denotes the upper right-hand Dini derivative of w . Therefore, from Lemma 13, there exists a function δ of class $\mathcal{KL}$ such that the inequality holds

$$w(t) \leq \delta(w(0), t) + \delta(\alpha_2 \circ \alpha_3^{-1} \circ 2\sigma(v), 0), \quad (\text{A.7})$$

from which, taking into account points (i), (ii) in item (d), the inequality follows

$$|\mathcal{D}_K x_t| \leq \alpha_1^{-1} \circ 2\delta(\alpha_2 \circ \gamma_a(\|x_0\|_\infty), t) + \alpha_1^{-1} \circ 2\delta(\alpha_2 \circ \alpha_3^{-1} \circ 2\sigma(v), 0), \quad t \in [0, b)$$

We conclude that the inequality holds

$$|\mathcal{D}_K x_t| \leq \bar{\beta}(\|x_0\|_\infty, t) + \bar{\gamma}(v), \quad t \in [0, b), \quad (\text{A.8})$$

where $\bar{\beta}$ and $\bar{\gamma}$ are the functions of class $\mathcal{KL}$ and of class $\mathcal{K}$, respectively, defined, for $s, t \in R^+$, as $\bar{\beta}(s, t) = \alpha_1^{-1} \circ 2\delta(\alpha_2 \circ \gamma_a(s), t)$, $\bar{\gamma}(s) = \alpha_1^{-1} \circ 2\delta(\alpha_2 \circ \alpha_3^{-1} \circ 2\sigma(s), 0)$. Now, let c be an arbitrary positive real in $(0, b)$ and let $v_c : R^+ \rightarrow R^n$ be defined, for $t \in [0, c]$, as $v_c(t) = \mathcal{D}_K x_t$, and, for $t \geq c$, as $v_c(t) = \mathcal{D}_K x_c$. Let us consider the FDE

$$\begin{aligned} \mathcal{D}_K \xi_t &= v_c(t), & t \in R^+, \\ \xi(\tau) &= x_0(\tau), & \tau \in [-\Delta, 0] \end{aligned} \quad (\text{A.9})$$

We have that the (unique, see Definition 1) solution of (A.9) is equal to $x_t \in C$, for $t \in [0, c]$. Moreover, from (A.8), we have, for any $c \in (0, b)$,

$$|v_c(t)| \leq \bar{\beta}(\|x_0\|_\infty, 0) + \bar{\gamma}(v), \quad t \in R^+ \quad (\text{A.10})$$

Since the map $\mathcal{D}_K$ is an ISS-map (see Definition 1), it follows that there exist a positive real M such that, for any $c \in (0, b)$, the inequality holds

$$\|x_t\|_\infty \leq M, \quad t \in [0, c] \quad (\text{A.11})$$

From the arbitrariness of $c \in (0, b)$, it follows that $\|x_t\|_\infty \leq M$, $t \in [0, b)$. And thus, from point (3) in Theorem 4, it follows that $b = +\infty$. Now, since the map $\mathcal{D}_K$ is an ISS-map (see Definition 1), it follows from (A.8) that

$$\|x_t\|_\infty \leq A_K, \quad t \in R^+, \quad (\text{A.12})$$

where $A_K = \Omega_K(\|x_0\|_\infty, 0) + \omega_k(\bar{\beta}(\|x_0\|_\infty, 0) + \bar{\gamma}(v))$, Ω_K and ω_k being the functions given in Definition 1. Again, since the map $\mathcal{D}_K$ is an ISS-map, from (A.8), taking into account the time invariant property of both the map $\mathcal{D}_K$ and the system described by (4), we have, for $t \in R^+$,

$$\begin{aligned} \|x_t\|_\infty &\leq \Omega_K\left(A_K, \frac{t}{2}\right) + \omega_K\left(\bar{\beta}\left(\|x_0\|_\infty, \frac{t}{2}\right) + \bar{\gamma}(v)\right) = \\ &\Omega_K\left(\Omega_K(\|x_0\|_\infty, 0) + \omega_K(\bar{\beta}(\|x_0\|_\infty, 0) + \bar{\gamma}(v)), \frac{t}{2}\right) + \omega_K\left(\bar{\beta}\left(\|x_0\|_\infty, \frac{t}{2}\right) + \bar{\gamma}(v)\right) \end{aligned} \quad (\text{A.13})$$

From (A.13), exploiting the inequality, for any function p of class $\mathcal{K}$,

$$p(s_1 + s_2) \leq p(2s_1) + p(2s_2), \quad s_i \in R^+, \quad i = 1, 2, \quad (\text{A.14})$$

we have

$$\begin{aligned} \|x_t\|_\infty &\leq \Omega_K\left(\Omega_K(\|x_0\|_\infty, 0) + \omega_K(2\bar{\beta}(\|x_0\|_\infty, 0)) + \omega_K \circ 2\bar{\gamma}(v), \frac{t}{2}\right) \\ &+ \omega_K\left(2\bar{\beta}\left(\|x_0\|_\infty, \frac{t}{2}\right) + \omega_K \circ 2\bar{\gamma}(v)\right) \leq \\ &\Omega_K\left(2\Omega_K(\|x_0\|_\infty, 0) + 2\omega_K(2\bar{\beta}(\|x_0\|_\infty, 0)), \frac{t}{2}\right) + \Omega_K(2\omega_K \circ 2\bar{\gamma}(v), 0) \\ &+ \omega_K\left(2\bar{\beta}\left(\|x_0\|_\infty, \frac{t}{2}\right) + \omega_K \circ 2\bar{\gamma}(v)\right) \leq \beta(\|x_0\|_\infty, t) + \gamma(v), \end{aligned} \quad (\text{A.15})$$

where β and γ are the functions of class $\mathcal{KL}$ and of class $\mathcal{K}$, respectively, defined, for $s, t \in R^+$, as

$$\begin{aligned} \beta(s, t) &= \Omega_K\left(2\Omega_K(s, 0) + 2\omega_K(2\bar{\beta}(s, 0)), \frac{t}{2}\right) + \omega_K\left(2\bar{\beta}\left(s, \frac{t}{2}\right)\right), \\ \gamma(s) &= \Omega_K(2\omega_K \circ 2\bar{\gamma}(s), 0) + \omega_K \circ 2\bar{\gamma}(s) \end{aligned} \quad (\text{A.16})$$

By causality arguments, it follows that, for any initial state $x_0 \in W^{1,\infty}$ and any Lebesgue measurable, locally essentially bounded input signal u , the solution of (4), denoted with $x_t(x_0, u)$, exists in R^+ and, furthermore, satisfies the inequality (A.3), with the functions β and γ given in (A.16). From Lemma 14, it follows that the system described by (4) is ISS. The proof of the implication (d) $\Rightarrow$ (a) is complete.

Let us prove now the implication (a) $\Rightarrow$ (b). We introduce first the following two Lemmas taken from the literature.

Lemma 16. (see [15], p. 130) *For any function α of class $\mathcal{K}_\infty$ there exists a function γ of class $\mathcal{K}_\infty$, globally Lipschitz with unit Lipschitz constant, such that, for any $s \in R^+$, the inequality holds $\gamma(s) \leq \alpha(s)$.*

Lemma 17. (Theorem 3, pp. 251, [16]) Let us consider the system described by (11) and let us denote with $x_t^R(x_0, d)$ the related solution corresponding to initial state $x_0 \in C$, $d \in \mathcal{M}_B$, in a maximal time interval $[0, b)$, $0 < b \leq +\infty$. The following statements (i) and (ii) are equivalent.

- i) There exists a function ω of class $\mathcal{KL}$ such that, for any initial state x_0 and any signal $d \in \mathcal{M}_B$, the solution $x_t^R(x_0, d)$ exists for all $t \in R^+$ and, furthermore, satisfies the inequality

$$\|x_t^R(x_0, d)\|_\infty \leq \omega(\|x_0\|_\infty, t), \quad t \in R^+. \quad (\text{A.17})$$

- ii) For any initial state $x_0 \in C$, for any signal $d \in \mathcal{M}_B$, the solution $x_t^R(x_0, d)$ exists for all $t \in R^+$ and, furthermore, the following conditions hold:

- ii₁) for every positive real M , the inequality holds

$$\sup\{\|x_t^R(x_0, d)\|_\infty, t \in R^+, x_0 \in C_M, d \in \mathcal{M}_B\} < +\infty; \quad (\text{A.18})$$

- ii₂) for every positive real ϵ there exists a positive real δ such that the inequality holds

$$\sup\{\|x_t^R(x_0, d)\|_\infty, t \in R^+, x_0 \in C_\delta, d \in \mathcal{M}_B\} < \epsilon; \quad (\text{A.19})$$

- ii₃) for every positive reals ϵ, H there exists a positive real T such that the inequality holds

$$\sup\{\|x_t^R(x_0, d)\|_\infty, t \geq T, x_0 \in C_H, d \in \mathcal{M}_B\} < \epsilon. \quad (\text{A.20})$$

Let us assume that the system described by (4) is ISS. We will show that there exists a suitable function ρ of class $\mathcal{K}_\infty$, globally Lipschitz with Lipschitz constant 1, such that the system described by (4) is *robustly* 0-GAS, with this function ρ , according to Definition 6. Let $\beta, \bar{\gamma}$ be the functions of class $\mathcal{KL}$ and $\mathcal{K}$, respectively, characterizing the ISS property as in Definition 5. Let γ be the function of class $\mathcal{K}_\infty$ defined, for $s \in R^+$, as $\gamma(s) = \max\{s, \bar{\gamma}(s)\}$. Then, for the solution of (4) with any initial state $\phi \in C$ and Lebesgue measurable, locally essentially bounded input signal u , the inequality holds

$$\|x_t(\phi, u)\|_\infty \leq \beta(\|\phi\|_\infty, t) + \gamma(\|u_{[0,t]}\|_\infty) \quad (\text{A.21})$$

Let ρ be a globally Lipschitz function of class $\mathcal{K}_\infty$, with Lipschitz constant equal to 1, such that, for $s \in R^+$ (see Lemma 16),

$$\rho(s) \leq \gamma^{-1}\left(\frac{1}{2}s\right) \quad (\text{A.22})$$

We will show that the system described by (4) is *Robustly* 0-GAS, according to Definition 6, with this function ρ . Let the input u in (4) satisfy the following equality

$$u(t) = \rho(\|x_t(\phi, u)\|_\infty)d(t), \quad t \in [0, b), \quad (\text{A.23})$$

for some $d \in \mathcal{M}_B$, where $[0, b)$, $0 < b \leq +\infty$, is the maximal interval of existence of the solution of (4), with this given input (apply Theorem 4 to equivalent system (11) with this given function ρ). First let us show that it must be $b = +\infty$. Otherwise, by contradiction, let $b < +\infty$. Let $c \in (0, b)$. Then, in $[0, c]$, the input signal u in (A.23) is Lebesgue measurable and essentially bounded. From the ISS property of system (4), taking into account of (A.22), (A.23), and recalling (12), we have, for any $t \in [0, c]$,

$$\begin{aligned}
\sup_{\tau \in [0, t]} \|x_\tau(\phi, u)\|_\infty &\leq \sup_{\tau \in [0, t]} (\beta(\|\phi\|_\infty, \tau) + \gamma(\|u_{[0, \tau]}\|_\infty)) \leq \\
&\sup_{\tau \in [0, t]} \beta(\|\phi\|_\infty, \tau) + \sup_{\tau \in [0, t]} \gamma \left(\sup_{\theta \in [0, \tau]} \rho(\|x_\theta(\phi, u)\|_\infty) \right) \leq \\
&\beta(\|\phi\|_\infty, 0) + \sup_{\tau \in [0, t]} \gamma \circ \rho \left(\sup_{\theta \in [0, \tau]} \|x_\theta(\phi, u)\|_\infty \right) \leq \\
&\beta(\|\phi\|_\infty, 0) + \sup_{\tau \in [0, t]} \gamma \circ \gamma^{-1} \left(\frac{1}{2} \sup_{\theta \in [0, \tau]} \|x_\theta(\phi, u)\|_\infty \right) \leq \\
&\beta(\|\phi\|_\infty, 0) + \frac{1}{2} \sup_{\tau \in [0, t]} \|x_\tau(\phi, u)\|_\infty
\end{aligned} \tag{A.24}$$

Then, we have, from (A.24),

$$\sup_{\tau \in [0, t]} \|x_\tau(\phi, u)\|_\infty \leq 2\beta(\|\phi\|_\infty, 0), \quad t \in [0, c], \tag{A.25}$$

and, in particular,

$$\|x_t(\phi, u)\|_\infty \leq 2\beta(\|\phi\|_\infty, 0), \quad t \in [0, c] \tag{A.26}$$

Since $c \in [0, b)$ is arbitrary, and the right-hand side of (A.26) is independent of c , it follows that

$$\|x_t(\phi, u)\|_\infty \leq 2\beta(\|\phi\|_\infty, 0), \quad t \in [0, b), \tag{A.27}$$

and the absurd boundedness of the solution in $[0, b)$, $b < +\infty$, arises (see Theorem 4). Then, $b = +\infty$, the input signal $u : R^+ \rightarrow R^m$ is Lebesgue measurable and locally essentially bounded, and the inequality (A.27) holds for any $t \in R^+$ (just repeat inequalities (A.24) for any $t \in R^+$). Again, by the ISS property of system (4), taking into account of (A.22), (A.23), and recalling (12), we have the inequalities, for $t \in R^+$,

$$\begin{aligned}
\|x_t(\phi, u)\|_\infty &\leq \beta(\|\phi\|_\infty, t) + \gamma(\|u_{[0, t]}\|_\infty) \leq \\
&\beta(\|\phi\|_\infty, t) + \gamma \left(\sup_{\tau \in [0, t]} \rho(\|x_\tau(\phi, u)\|_\infty) \right) \leq \\
&\beta(\|\phi\|_\infty, t) + \gamma \circ \rho \left(\sup_{\tau \in [0, t]} \|x_\tau(\phi, u)\|_\infty \right) \leq \\
&\beta(\|\phi\|_\infty, t) + \gamma \circ \gamma^{-1} \left(\frac{1}{2} \sup_{\tau \in [0, t]} \|x_\tau(\phi, u)\|_\infty \right) = \\
&\beta(\|\phi\|_\infty, t) + \frac{1}{2} \sup_{\tau \in [0, t]} \|x_\tau(\phi, u)\|_\infty
\end{aligned} \tag{A.28}$$

Now, for any $d \in \mathcal{M}_B$, the solution of (11), with initial state ϕ and signal $d \in \mathcal{M}_B$, is equal to the solution of (4), with same initial state and input u as in (A.23). Therefore, for any $\phi \in C$, for any $d \in \mathcal{M}_B$, the solution $x_t^R(\phi, d)$ of (11) satisfies the inequality (see (A.27))

$$\|x_t^R(\phi, d)\|_\infty \leq 2\beta(\|\phi\|_\infty, 0), \quad t \in R^+, \tag{A.29}$$

that is, for any $\phi \in C$, we have, for the solutions of (11) corresponding to $d \in \mathcal{M}_B$,

$$\sup_{d \in \mathcal{M}_B} \|x_t^R(\phi, d)\|_\infty \leq 2\beta(\|\phi\|_\infty, 0), \quad t \in R^+ \quad (\text{A.30})$$

Let us now prove that, for any given positive real H , the following limit holds for the solutions of (11) (see [17], [18], [45], as far as RFDEs are concerned),

$$\lim_{t \rightarrow +\infty} \sup_{d \in \mathcal{M}_B, \phi \in C_H} \|x_t^R(\phi, d)\|_\infty = 0 \quad (\text{A.31})$$

We have, from (A.29),

$$\limsup_{t \rightarrow +\infty} \sup_{d \in \mathcal{M}_B, \phi \in C_H} \|x_t^R(\phi, d)\|_\infty \leq 2\beta(H, 0) \quad (\text{A.32})$$

Let $p = \limsup_{t \rightarrow +\infty} \sup_{d \in \mathcal{M}_B, \phi \in C_H} \|x_t^R(\phi, d)\|_\infty$. We will show that $p = 0$. Otherwise (i.e., $p > 0$), by contradiction, let $\epsilon = \frac{p}{2}$. There exists a positive real q such that

$$\sup_{d \in \mathcal{M}_B, \phi \in C_H} \|x_t^R(\phi, d)\|_\infty \leq p + \epsilon, \quad t \geq q \quad (\text{A.33})$$

Taking into account of the time-invariant property of the system described by (11), from (A.28), (A.33) we obtain, for $t \geq q$,

$$\begin{aligned} \sup_{d \in \mathcal{M}_B, \phi \in C_H} \|x_t^R(\phi, d)\|_\infty &\leq \beta(2\beta(H, 0), t - q) + \frac{1}{2} \sup_{\theta \in [q, t]} \sup_{d \in \mathcal{M}_B, \phi \in C_H} \|x_\theta(\phi, d)\|_\infty \leq \\ &\beta(2\beta(H, 0), t - q) + \frac{1}{2}(p + \epsilon), \end{aligned} \quad (\text{A.34})$$

and, therefore,

$$\begin{aligned} p &= \limsup_{t \rightarrow +\infty} \sup_{d \in \mathcal{M}_B, \phi \in C_H} \|x_t^R(\phi, d)\|_\infty \leq \\ &\lim_{t \rightarrow +\infty} \beta(2\beta(H, 0), t - q) + \frac{1}{2}(p + \epsilon) = \frac{3}{4}p \end{aligned}$$

We thus obtain the inequality $p \leq \frac{3}{4}p$, which contradicts the assumption $p > 0$. It follows that p must be 0 (i.e., the limit (A.31) holds satisfied). From (A.30), (A.31), by Lemma 17, it follows that the system described by (4) is *robustly* 0-GAS, according to Definition 6, with ρ given by (A.22). The proof of the implication (a) $\Rightarrow$ (b) is complete.

We will show now the implication (b) $\Rightarrow$ (c). Let, for a locally Lipschitz functional $V : C \rightarrow R^+$, $D^+ V_R : C \times R^m \rightarrow R^*$ be defined (by (11)), for $\phi \in C$, $d \in B_1^m(0)$, as

$$D^+ V_R(\phi, d) = \limsup_{h \rightarrow 0^+} \frac{V(\phi_h^R) - V(\phi)}{h}, \quad (\text{A.35})$$

where, for $h \in [0, \Delta)$,

$$\phi_h^R(s) = \begin{cases} \phi(s + h), & s \in [-\Delta, -h), \\ \phi(0) + q(\psi_s) + (s + h)f(\phi, \rho(\|\phi\|_\infty)d), & s \in [-h, 0] \end{cases} \quad (\text{A.36})$$

and, for $s \in [-h, 0]$,

$$\psi_s(\theta) = \begin{cases} \phi(\theta + s + h), & \theta \in [-\Delta, -s - h), \\ \phi(0), & \theta \in [-s - h, 0] \end{cases} \quad (\text{A.37})$$

Lemma 18. ([16], Theorem 5, pp. 253-255) *If the system described by (4) is robustly 0-GAS, then there exist a functional $V : C \rightarrow R^+$, Lipschitz on bounded sets, and functions $\omega_1, \omega_2, \omega_3$ of class $\mathcal{K}_\infty$, such that, $\forall \phi \in C, d \in B_1^m(0)$, the following inequalities hold:*

$$i) \quad \omega_1(\|\phi\|_\infty) \leq V(\phi) \leq \omega_2(\|\phi\|_\infty);$$

$$ii) \quad D^+V_R(\phi, d) \leq -\omega_3(\|\phi\|_\infty),$$

where $D^+V_R : C \times B_1^m(0) \rightarrow R^*$ is defined in (A.35).

Remark 19. *The inequalities in (i), in Lemma 18, are the same as the ones invoked in point (c) of Theorem 5 in [16], namely inequalities (21) in [16]. As far as the upper-bound of the LK functional derivative is concerned, in Theorem 5 in [16] it is given in terms of the LK functional itself (see (28), point (c), in [16]). By point (i), it follows that the inequality (28) in [16] implies the inequality in point (ii) of Lemma 18.*

Lemma 20. (see Lemma 2.3, pp. 62-64, in [15]) *Let $\alpha : R^+ \rightarrow R^+$ be an unbounded, strictly increasing function such that $\alpha(0) = 0$, and*

$$\lim_{s \rightarrow 0^+} \alpha(s) = 0$$

Then, there exists a function γ of class $\mathcal{K}_\infty$ such that

$$\alpha(s) \leq \gamma(s), \quad s \in R^+ \quad (\text{A.38})$$

Lemma 21. *Let $V : C \rightarrow R^+$ be a functional, Lipschitz on bounded sets, such that there exists functions α_1, α_2 of class $\mathcal{K}_\infty$ such that the inequalities hold*

$$\alpha_1(\|\phi\|_\infty) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty), \quad \forall \phi \in C \quad (\text{A.39})$$

Then, the following conditions are equivalent:

- i) *there exist a function α of class $\mathcal{K}_\infty$ and a function ρ of class $\mathcal{K}$ such that, for any $\phi \in C, u \in R^m$ such that $\|\phi\|_\infty \geq \rho(|u|)$, the inequality holds*

$$D^+V(\phi, u) \leq -\alpha(\|\phi\|_\infty); \quad (\text{A.40})$$

- ii) *there exist a function γ of class $\mathcal{K}_\infty$ and a function σ of class $\mathcal{K}$ such that, for any $\phi \in C, u \in R^m$, the inequality holds*

$$D^+V(\phi, u) \leq -\gamma(\|\phi\|_\infty) + \sigma(|u|) \quad (\text{A.41})$$

Proof. Clearly (ii) implies (i). Let us show that (i) implies (ii). First notice that, from (A.39) and since $f(0, 0) = 0$, it follows that $D^+V(0, 0) = 0$. From the Lipschitz property, on bounded sets, of the map f and of the functional V , it follows that for any positive reals H, δ , there exists a positive real $L_{H,\delta}$ such that the following inequality holds for any $\phi \in C_H, u \in B_\delta^m(0)$,

$$D^+V(\phi, u) \leq -\alpha(\|\phi\|_\infty) + L_{H,\delta}|u| \quad (\text{A.42})$$

Indeed, for given positive reals H, δ , let L_f be a Lipschitz constant for the map f in $C_H \times B_\delta^m(0)$. Let $\bar{H} = H + 2l(H) + \sup_{\phi \in C_H, u \in B_\delta^m(0)} |f(\phi, u)|$, where l is the function given in (6). Let L_V be a Lipschitz constant for the functional V in $C_{\bar{H}}$. Then, taking into account (i), (8), and the independence, of the map q , of the argument at 0, the following equalities/inequalities hold

$$\begin{aligned}
D^+ V(\phi, u) &= \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,u}) - V(\phi)}{h} = \\
&\limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,u}) - V(\phi_{h,0}) + V(\phi_{h,0}) - V(\phi)}{h} \leq \\
&\limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,u}) - V(\phi_{h,0})}{h} + \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,0}) - V(\phi)}{h} \leq \\
&\limsup_{h \rightarrow 0^+} \frac{L_V \|\phi_{h,u} - \phi_{h,0}\|_\infty}{h} + D^+ V(\phi, 0) \leq \\
&\limsup_{h \rightarrow 0^+} \frac{L_V h |f(\phi, u) - f(\phi, 0)|}{h} - \alpha(\|\phi\|_\infty) \leq L_V L_f |u| - \alpha(\|\phi\|_\infty) \quad (\text{A.43})
\end{aligned}$$

Thus, (A.42) holds, with $L_{H,\delta} = L_V L_f$. Now, by the same reasoning used in Remark 2.4 in [43], let, for $s \in R^+$,

$$\begin{aligned}
\gamma(s) &= \alpha(s), \\
\beta(s) &= s + \sup_{\psi \in C, v \in R^m} \{ D^+ V(\psi, v) + \alpha \circ \rho(|v|) : |v| \leq s, \|\psi\|_\infty \leq \rho(|v|) \} \quad (\text{A.44})
\end{aligned}$$

The function β is well defined because of the equality $D^+ V(0, 0) = 0$ and (A.42). Moreover, it is zero at zero, strictly increasing, unbounded and $\lim_{s \rightarrow 0^+} \beta(s) = 0$. By Lemma 20, there exists a function σ of class $\mathcal{K}_\infty$ such that $\beta(s) \leq \sigma(s)$, $s \in R^+$. Now, for any given $\phi \in C$, $u \in R^m$, one of the following cases hold:

- 1) $\|\phi\|_\infty \leq \rho(|u|)$;
- 2) $\|\phi\|_\infty > \rho(|u|)$.

In the case (1), the following inequality holds

$$\sigma(|u|) \geq \beta(|u|) \geq D^+ V(\phi, u) + \alpha \circ \rho(|u|) \geq D^+ V(\phi, u) + \gamma(\|\phi\|_\infty), \quad (\text{A.45})$$

and thus the inequality (A.41) holds in the case (1). In the case (2), taking into account of (i), we have,

$$D^+ V(\phi, u) \leq -\alpha(\|\phi\|_\infty) \leq -\gamma(\|\phi\|_\infty) + \sigma(|u|), \quad (\text{A.46})$$

and thus the inequality (A.41) holds in the case (2) as well. The proof of the lemma is complete. $\square$

We have thus the following results: if the system described by (4) is *robustly* 0-GAS, there exist a functional $V : C \rightarrow R^+$, Lipschitz on bounded sets, functions $\omega_1, \omega_2, \omega_3$ of class $\mathcal{K}_\infty$,

a locally Lipschitz function ρ , of class $\mathcal{K}_\infty$, such that, for any $\phi \in C$ and $u \in R^m$, satisfying $|u| \leq \rho(\|\phi\|_\infty)$ (i.e., for some $d \in B_1^m(0)$, $u = \rho(\|\phi\|_\infty)d$, which implies $D^+V(\phi, u) = D^+V_R(\phi, d)$), the inequalities hold (see Lemma 18)

a) $\omega_1(\|\phi\|_\infty) \leq V(\phi) \leq \omega_2(\|\phi\|_\infty)$,

b) $D^+V(\phi, u) \leq -\omega_3(\|\phi\|_\infty)$;

if the point (b) holds whenever $|u| \leq \rho(\|\phi\|_\infty)$ (i.e., whenever $\|\phi\|_\infty \geq \tilde{\rho}(|u|)$, $\tilde{\rho}(s) = \rho^{-1}(s)$, $s \in R^+$), then there exist a function δ_3 of class $\mathcal{K}_\infty$ and a function δ_4 of class $\mathcal{K}$ such that the inequality holds, for any $\phi \in C$, $u \in R^m$ (see Lemma 21)

$$D^+V(\phi, u) \leq -\delta_3(\|\phi\|_\infty) + \delta_4(|u|).$$

We conclude that, if the system described by (4) is *robustly* 0-GAS, for any map K as in item (c), there exists a locally Lipschitz functional $V : C \rightarrow R^+$ satisfying points (i), (ii), (iii) of item (c), with $N_a(\phi) = \|\phi\|_\infty$, $\phi \in C$, and, for $s \in R^+$, $\gamma_a(s) = s$, $\alpha_1(s) = \omega_1 \circ (I_d + l_K)^{-1}(s)$ (where l_K is the function of class $\mathcal{K}_\infty$ in (1)), $\alpha_2(s) = \omega_2(s)$, $\alpha_3(s) = \delta_3(s)$, $\sigma(s) = \delta_4(s)$. The proof of the implication (b) $\Rightarrow$ (c) is complete.

The implication (c) $\Rightarrow$ (d) is obvious. Just choose $K(\phi) = \frac{1}{2}\phi(-\Delta)$, $\phi \in C$. By Theorem 3.5, p. 275, in [12], it follows that the related map $\mathcal{D}_K$ is an ISS-map. So, for this map K , the item (d) readily follows from item (c).

The proof of Theorem 7 is complete.