Controllability for transmission wave/plate equations on Riemannian manifolds

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Abstract

In this paper we consider controllability for transmission system of coupling wave equations with Euler-Bernoulli equations on Riemannian manifolds. We show that such a system is exactly controllable by boundary controls only along the exterior boundary, which means there is no control on the interface of the transmission system. Our proofs rely on the Hilbert Uniqueness Method(HUM) and geometric multiplier method.

Keywords: exact controllability; transmission problem; Riemannian geometry method.

1 Introduction

Transmission systems appear in many practical control systems such as chemical reactions coupling, structural-acoustic systems and many other interactive physical processes. Among these systems are the transmission problem of hyperbolic systems, which attract much attention and have strong physical backgrounds, for example, it can describe the displacement of flexible structures consisting of two physically different types of materials. The analysis of controllability and stabilization for the transmission of hyperbolic systems has been widely carried out.

In this article we aim to discuss the controllability of transmission system of coupling wave equations with Euler-Bernoulli equations on Riemannian manifolds, of which system we considered the stabilization property in our recent work [24]. More precisely, we consider the case where we only introduce the control on the exterior boundary. That means there is no control on the interface of the transmission problem. We prove that the system is exactly controllable under some geometric assumptions on the interface. We found that such kind of geometric assumption can not be removed. The reason is that, the system may not be controllable only from exterior boundaries, due to total reflection at the interface, as already pointed out by many other authors, see [15, 17].

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1.1 Statement of the problem and the main result

Let M be a complete two dimensional Riemannian manifold of class C^3 with C^3 -metric $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. For each $x \in M$, M_x is the tangential space of M at x . Denote the set of all n order tensor fields on M by $T^n(M) = \cup_{x \in M} T_x^n(M)$, where n is a nonnegative integer. It is well known that the space $T_x^n(M)$ of n order tensor on M_x is an inner product space. Its inner product is defined by

$$\langle T_1, T_2 \rangle_{T_x^n} = \sum_{i_1, i_2, \dots, i_n=1}^2 T_1(e_{i_1}, \dots, e_{i_n}) T_2(e_{i_1}, \dots, e_{i_n}) \text{ at } x, \quad (1.1)$$

for any $T_1, T_2 \in T_x^n(M)$, where e_1, e_2 is an orthonormal basis of M_x for $x \in M$. For any $T \in T^2(M)$, the trace of T is defined by

$$tr T = \sum_{i=1}^2 T(e_i, e_i). \quad (1.2)$$

We denote by ∇ the gradient, by D the Levi-Civita connection and by $\Delta = \text{div}(\nabla)$ the Laplace-Beltrami operator in the Riemannian metric g . For any vector field H on M , DH is the covariant differential of H which is a second order tensor field in the following sense:

$$DH(X, Y) = D_Y H(X) = \langle D_Y H, X \rangle \text{ for all } X, Y \in M_x, x \in M. \quad (1.3)$$

For scalar function u we have $Du = \nabla u$.

Let Ω be an open, bounded, connected subset of M with smooth boundary Γ such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, where Ω_i , $i = 1, 2$ are two disjoint open connected bounded domains with smooth boundary. They satisfy that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = S$, $\partial\Omega_1 = S \cup \Gamma_1$ and $\partial\Omega_2 = S \cup \Gamma_2$ (See Figure 1).

Figure 1: the domain

We consider the wave equation in Ω_1 coupled with the Euler-Bernoulli plate equation in Ω_2 by the interface S with boundary controls $\phi(x, t), \psi(x, t), \zeta(x, t)$ on $\Gamma_0 \subset \Gamma$ which is a subset of the exterior boundary Γ . More precisely, we consider the following system:

$$\left\{ \begin{array}{ll} \partial_t^2 u_1(x, t) - \Delta u_1(x, t) = 0, & \text{in } \Omega_1 \times (0, +\infty), \\ \partial_t^2 u_2(x, t) + \Delta^2 u_2(x, t) - (1 - \mu) \operatorname{div}(\mathcal{K} \nabla u_2)(x, t) = 0, & \text{in } \Omega_2 \times (0, +\infty), \\ u_i(x, 0) = u_i^0(x), \partial_t u_i(x, 0) = u_i^1(x), & \text{in } \Omega_i, i = 1, 2, \\ u_1 = u_2, B_1 u_2 = 0, B_2 u_2 = \partial_{\nu_1} u_1, & \text{on } S \times (0, +\infty), \\ u_1 = 0, & \text{on } (\Gamma_1 / \Gamma_0) \times (0, +\infty), \\ u_1 = \phi, & \text{on } (\Gamma_1 \cap \Gamma_0) \times (0, +\infty), \\ u_2 = \partial_{\nu_2} u_2 = 0, & \text{on } (\Gamma_2 / \Gamma_0) \times (0, +\infty), \\ u_2 = \psi, & \text{on } (\Gamma_2 \cap \Gamma_0) \times (0, +\infty), \\ \partial_{\nu_2} u_2 = \zeta, & \text{on } (\Gamma_2 \cap \Gamma_0) \times (0, +\infty), \end{array} \right. \quad (1.4)$$

where $\mathcal{K}$ is the Gaussian curvature function on Ω_2 . Here $\nu_i = \nu_i(x)$ denotes the unit outward normal vector of Ω_i along $\partial\Omega_i = \Gamma_i \cup S$ for $i = 1, 2$. In the above system, $0 < \mu < \frac{1}{2}$ is the Poisson coefficient, and the boundary operators B_1, B_2 are defined on $\partial\Omega_2 = \Gamma_2 \cup S$ as follows:

$$\begin{aligned} B_1 y &= \Delta y - (1 - \mu) D^2 y(\tau_2, \tau_2), \\ B_2 y &= \partial_{\nu_2} \Delta y + (1 - \mu) \partial_{\tau_2} (D^2 y(\tau_2, \nu_2)) + \mathcal{K} \partial_{\nu_2} y, \end{aligned}$$

where $D^2 y$ is the Hessian of y and τ_2 is the unit tangential vector along the boundary $\partial\Omega_2 = \Gamma_2 \cup S$.

Remark 1.1. The term $(1 - \mu) \operatorname{div}(\mathcal{K} \nabla u_2)$ in the system (1.4) comes from the curvedness of the Riemannian metric g . For details, see [22, Model, pp.150]

To obtain the controllability of the problem (1.4), the following geometrical hypotheses are assumed:

Geometrical assumptions Given the triple $\{\Omega_1, S, \Omega_2\}$, there exists a vector field H on Riemannian manifold (M, g) such that the following three properties hold true:

(A.1) $DH(\cdot, \cdot)$ is strictly positive definite on $\bar{\Omega}$: there exists a constant $\rho > 0$ such that for all $x \in \bar{\Omega}$, for all $X \in M_x$ (the tangent space at x):

$$DH(X, X) \equiv \langle D_X H, X \rangle \geq \rho |X|^2. \quad (1.5)$$

(A.2) The control area $\Gamma_0 \triangleq \Gamma_0^1 \cup \Gamma_0^2$ satisfies

$$\Gamma_0^i = \{x \in \Gamma_i \mid \langle H, \nu_i \rangle > 0 \text{ on } \Gamma_i, i = 1, 2\}. \quad (1.6)$$

(A.3) The interface satisfies

$$\langle H, \nu_i \rangle = 0, \text{ on } S. \quad (1.7)$$

Remark 1.2. For any Riemannian manifold M , the existence of such a vector field H in (A.1) has been proved in [21], where some examples are given, too. See also [22]. In the framework of Euclidean metric, one can take the vector field $H = x - x_0$. This was given first in the paper [11] as well as in [12, 13]. Thus $DH(X, X) = |X|^2$ follows, which means assumption (A.1) always holds true with $\rho = 1$ for the Euclidean case.

Here we state our main result.

Theorem 1.1. *We assume the geometrical assumptions (A.1), (A.2) and (A.3) hold true. Then the transmission problem (1.4) is exactly $L^2(\Omega_1) \times L^2(\Omega_2) \times H^{-1}(\Omega_1) \times H^{-2}(\Omega_2)$ controllable by $L^2(0, T; L^2(\Gamma_0^1)) \times L^2(0, T; L^2(\Gamma_0^2)) \times L^2(0, T; L^2(\Gamma_0^2))$ controls, i.e., there exist some $T > T_0$, control functions $\phi \in L^2(0, T; L^2(\Gamma_0^1))$, $\psi \in L^2(0, T; L^2(\Gamma_0^2))$ and $\zeta \in L^2(0, T; L^2(\Gamma_0^2))$ such that the corresponding solution $(u_1, u_2, \partial_t u_1, \partial_t u_2)$ of (1.4) satisfies*

$$(u_1, u_2)(\cdot, T) = 0, \quad (\partial_t u_1, \partial_t u_2)(\cdot, T) = 0,$$

where Γ_0^i are given in (1.6) and

$$T_0 = \frac{8}{\rho} \sup_{x \in \Omega_1} |H|. \quad (1.8)$$

1.2 Literature

Controllability for transmission problems were studied by several authors, for example, for the coupled wave equations with constant coefficients, the controllability results are presented in [15] by applying the Hilbert Uniqueness Method (HUM). For the coupled wave equations with variable coefficients (describing the wave with variable propagation speed), the boundary controllability is treated in [10], which involves the research for transmission problem of anisotropic elastic materials. The exact controllability for transmission plate equations was addressed in [16]. We refer to [3, 7, 9, 18, 19] for related results on transmission problems of other hyperbolic systems.

These works mentioned above offer fruitful results regarding the transmission systems in the framework of Euclidean metric. However, the problems on the compact Riemannian manifolds have limited results compared with the problems on Euclidean spaces. The main difficulty comes from the fact that the trapped geodesics in the general Riemannian manifolds can preclude the effectiveness of the canonical multiplier which plays an important role in obtaining the global estimates in the controllability analysis. At the same time, the transmission problems on Riemannian manifolds have strong practical applications. For example, once the plates in [1, 2] have curved middle surfaces Ω , the transmission system considered by [1, 2] becomes the one on the general Riemannian manifold (Ω, g) , where g is the induced Riemannian metric.

In the present paper we consider the controllability of a coupled wave-plate system on Riemannian manifolds. The main tool we use is the geometric multiplier method, which first appeared in [21] and subsequently in [4, 6, 14, 23, 24], and many others. First, we establish multiplier equality for the coupled wave-plate system on Riemannian manifolds. Then, under the geometric assumptions on the domain, we obtain the controllability of the coupled system in Theorem 1.1.

The content of this paper is organized as follows. In Section 2, we establish the multiplier equality for the dual transmission system by the geometric multiplier method. Finally in Section 3, we present the proofs of the main results.

2 The dual transmission system.

In this section, we will discuss the dual transmission system.

$$\left\{ \begin{array}{ll} \partial_t^2 w_1(x, t) - \Delta w_1(x, t) = 0, & \text{in } \Omega_1 \times (0, +\infty), \\ \partial_t^2 w_2(x, t) + \Delta^2 w_2(x, t) - (1 - \mu) \operatorname{div}(\mathcal{K} \nabla w_2)(x, t) = 0, & \text{in } \Omega_2 \times (0, +\infty), \\ w_i(x, 0) = w_i^0(x), \partial_t w_i(x, 0) = w_i^1(x), & \text{in } \Omega_i, i = 1, 2, \\ w_1 = w_2, B_1 w_2 = 0, B_2 w_2 = \partial_{\nu_1} w_1, & \text{on } S \times (0, +\infty), \\ w_1 = 0, & \text{on } \Gamma_1 \times (0, +\infty), \\ w_2 = \partial_{\nu_2} w_2 = 0, & \text{on } \Gamma_2 \times (0, +\infty). \end{array} \right. \quad (2.1)$$

First, we write the system (2.1) as an abstract Cauchy problem to treat its well-posedness. Let

$$\mathcal{H} = \{(w_1, v_1, w_2, v_2) \in H^1(\Omega_1) \times L^2(\Omega_1) \times H^2(\Omega_2) \times L^2(\Omega_2), w_1|_S = w_2|_S\}$$

equipped with the inner product

$$\begin{aligned} & \langle (w_1, v_1, w_2, v_2), (y_1, z_1, y_2, z_2) \rangle_{\mathcal{H}} \\ &= \int_{\Omega_1} (\langle Dw_1, Dy_1 \rangle + v_1 z_1) dx + \int_{\Omega_2} (a(w_2, y_2) + v_2 z_2) dx, \end{aligned} \quad (2.2)$$

where the bilinear form $a(\cdot, \cdot)$ is defined on $H^2(\Omega_2) \times H^2(\Omega_2)$ by

$$a(y, z) = (1 - \mu) \langle D^2 y, D^2 z \rangle_{T_x^2} + \mu \Delta y \Delta z. \quad (2.3)$$

Here the inner product $\langle \cdot, \cdot \rangle_{T_x^2}$ is given in (1.1).

Then we define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\begin{pmatrix} 0 & Id & 0 & 0 \\ \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & Id \\ 0 & 0 & -\Delta^2 + (1 - \mu) \operatorname{div}(\mathcal{K} \nabla) & 0 \end{pmatrix} \quad (2.4)$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (w_1, v_1, w_2, v_2) \in \mathcal{H} \left| \begin{array}{l} (v_1, \Delta w_1, v_2, -\Delta^2 w_2) \in \mathcal{H}, \\ B_1 w_2 = 0, B_2 w_2 = \partial_{\nu_1} w_1, \text{ on } S, \\ w_1 = 0, \text{ on } \Gamma_1, \\ w_2 = \partial_{\nu_2} w_2 = 0, \text{ on } \Gamma_2. \end{array} \right. \right\} \quad (2.5)$$

By the semigroup theory, we can show that the operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ defined by (2.4) and (2.5) generates a C_0 -group on the Hilbert space $\mathcal{H}$ endowed with the inner product (2.2). Thus we can obtain the well-posedness for the problem (2.1) (See Proposition 2.1). Here we state this proposition without proof. Its proof is similar as that in [1].

Proposition 2.1. *For all given initial data $(w_1^0, w_1^1, w_2^0, w_2^1) \in \mathcal{H}$, the problem (2.1) admits a unique global weak solution*

$$(w_1, \partial_t w_1, w_2, \partial_t w_2) \in C([0, \infty); \mathcal{D}(\mathcal{A})) \cap C^1([0, \infty), \mathcal{H}). \quad (2.6)$$

We define the energy of a solution

$$w = \begin{cases} w_1, & x \in \Omega_1, \\ w_2, & x \in \Omega_2, \end{cases}$$

of the system (2.1) by

$$E(t) = E_1(t) + E_2(t), \quad (2.7)$$

where we denote

$$E_1(t) = \frac{1}{2} \int_{\Omega_1} ((\partial_t w_1(x, t))^2 + |Dw_1(x, t)|^2) dx,$$

and

$$E_2(t) = \int_{\Omega_2} ((\partial_t w_2(x, t))^2 + a(w_2, w_2)) dx.$$

To get some technical estimates for the dual system (2.1), in what follows we assume that w is a strong solution. By a classical density argument we can see that these estimates hold for a weak solution as well.

The following formula presents the relationship between the interior terms and boundary terms.

Lemma 2.2. *[8, Lemma 3.1, p.55] Let $y, v \in H^4(\Omega_2)$. Then we have*

$$\int_{\Omega_2} \left(\Delta^2 y - (1 - \mu) \operatorname{div}(\mathcal{K} \nabla y) \right) v dx = \int_{\Omega_2} a(y, v) dx - \int_{\partial\Omega_2} B_1 y \partial_{\nu_2} v d\sigma + \int_{\partial\Omega_2} B_2 y v d\sigma. \quad (2.8)$$

For the energy $E(t)$ defined in (2.7), we have

Lemma 2.3. *For any w solves the dual system (2.1), we have*

$$\partial_t E(t) = 0. \quad (2.9)$$

Lemma 2.4. *Let H be a vector field on Ω_2 . For any w_2 solves the plate equation on Ω_2 as*

$$\partial_t^2 w_2(x, t) + \Delta^2 w_2(x, t) - (1 - \mu) \operatorname{div}(\mathcal{K} \nabla w_2)(x, t) = 0, \quad (2.10)$$

we have

$$\begin{aligned} \int_0^T \int_{\Gamma_2 \cup S} \mathfrak{B}_2(w_2) dx dt &= 2 \int_{\Omega_2} \partial_t w_2 H(w_2) dx \Big|_0^T + \int_0^T \int_{\Omega_2} [(\partial_t w_2)^2 - a(w_2, w_2)] \operatorname{div} H d\sigma dt \\ &\quad + 4(1 - \mu) \int_0^T \int_{\Omega_2} DH(D^2 w_2, D^2 w_2) dx dt + 4\mu \int_0^T \int_{\Omega_2} DH(\Delta w_2, \Delta w_2) \\ &\quad + 2(1 - \mu) \int_0^T \int_{\Omega_2} \langle D^2 w_2, l(w_2) \rangle dx dt \\ &\quad + 2\mu \int_0^T \int_{\Omega_2} \Delta w_2 \cdot \operatorname{tr} l(w_2) dx dt, \end{aligned} \quad (2.11)$$

where the boundary term

$$\mathfrak{B}_2(w_2) = 2B_1(w_2)\partial_{\nu_2}(H(w_2)) - 2H(w_2)B_2(w_2) + [(\partial_t w_2)^2 - a(w_2, w_2)]\langle H, \nu_2 \rangle, \quad (2.12)$$

and

$$l(w_2) = -\mathbf{R}(Dw_2, \cdot, H, \cdot) - D^2 H(Dw_2, \cdot, \cdot). \quad (2.13)$$

Here $\mathbf{R}$ is the curvature tensor of the Levi-Civita connection D (for details see [20]).

Proof. Multiplying the plate equation (2.10) by $2H(w_2)$ yields

$$2\partial_t^2 w_2(x, t)H(w_2) + 2(\Delta^2 w_2(x, t) - (1 - \mu)\operatorname{div}(\mathcal{K}\nabla w_2)(x, t))H(w_2) = 0. \quad (2.14)$$

Using the divergence formula we have

$$2\partial_t^2 w_2(x, t)H(w_2) = 2\partial_t[\partial_t w_2 H(w_2)] - \operatorname{div}[(\partial_t w_2)^2 H] + (\partial_t w_2)^2 \operatorname{div} H. \quad (2.15)$$

From Lemma 2.2 we know

$$\begin{aligned} & \int_0^T \int_{\Omega_2} \left(\Delta^2 w_2 - (1 - \mu)\operatorname{div}(\mathcal{K}\nabla w_2) \right) 2H(w_2) dx \\ = & 2 \int_0^T \int_{\Omega_2} a(w_2, H(w_2)) dx - 2 \int_0^T \int_{\partial\Omega_2} (B_1 w_2 \partial_{\nu_2}(H(w_2)) - B_2 w_2 H(w_2)) d\sigma \end{aligned} \quad (2.16)$$

Now we deal with the term $\int_{\Omega_2} a(w_2, H(w_2)) dx$. Since $D^2 w_2$ is a symmetric second order tensor field, we have from [23, Lemma 2.7] that

$$\langle D^2 y, D^2(H(y)) \rangle_{T_x^2} = \frac{1}{2} H(|D^2 y|_{T_x^2}^2) + 2DH(D^2 y, D^2 y) + \langle D^2 y, l(y) \rangle_{T_x^2}, \quad (2.17)$$

and

$$\operatorname{tr} D^2 y \cdot \operatorname{tr} D^2(H(y)) = \frac{1}{2} H((\Delta y)^2) + 2DH(\Delta y, \Delta y) + \Delta w_2 \cdot \operatorname{tr} l(y), \quad (2.18)$$

where $l(y)$ is a lower order term respect to $a(w_2, w_2)$ given in (2.13) and " $\cdot$ " denotes the position of the variable. Combining the equalities (2.17), (2.18) and the definition of $a(y, z)$ in (2.3), we can easily have

$$\begin{aligned} a(w_2, H(w_2)) &= \frac{1}{2} H(a(w_2, w_2)) + 2(1 - \mu)DH(D^2 w_2, D^2 w_2) + 2\mu DH(\Delta w_2, \Delta w_2) \\ &\quad + (1 - \mu)\langle D^2 w_2, l(w_2) \rangle_{T_x^2} + \mu \Delta w_2 \operatorname{tr} l(w_2). \end{aligned} \quad (2.19)$$

Note that

$$\int_{\Omega_2} H(a(w_2, w_2)) dx = \int_{\partial\Omega_2} a(w_2, w_2) \langle H, \nu_2 \rangle d\sigma - \int_{\Omega_2} a(w_2, w_2) \operatorname{div} H d\sigma,$$

thus we have

$$\begin{aligned}
\int_{\Omega_2} a(w_2, H(w_2)) dx &= \frac{1}{2} \int_{\partial\Omega_2} a(w_2, w_2) \langle H, \nu_2 \rangle d\sigma - \frac{1}{2} \int_{\Omega_2} a(w_2, w_2) \operatorname{div} H dx \\
&+ 2(1-\mu) \int_{\Omega_2} DH(D^2 w_2, D^2 w_2) dx + 2\mu \int_{\Omega_2} DH(\Delta w_2, \Delta w_2) dx \\
&+ (1-\mu) \int_{\Omega_2} \langle D^2 w_2, l(w_2) \rangle_{T_x^2} dx + \mu \int_{\Omega_2} \Delta w_2 \operatorname{tr} l(w_2) dx. \quad (2.20)
\end{aligned}$$

Integrating the equality (2.15) on $(0, T) \times \Omega_2$ and combining the equalities (2.16), (2.20) to obtain the desired equality (2.11). $\square$

Lemma 2.5. *Let p be a function. For any w_2 solves the plate equation given in Lemma 2.4, we have*

$$\begin{aligned}
\int_0^T \int_{\Gamma_2 \cup S} \mathfrak{B}'_2(w_2) dx dt &= \int_{\Omega_2} p \partial_t w_2 w_2 dx \Big|_0^T - \int_0^T \int_{\Omega_2} [(\partial_t w_2)^2 - a(w_2, w_2)] p dx dt \\
&+ \int_0^T \int_{\Omega_2} [2(1-\mu) D^2 w_2 (Dp, Dw_2) + 2\mu \Delta w_2 \langle Dp, Dw_2 \rangle] dx dt \\
&+ \int_0^T \int_{\Omega_2} w_2 a(p, w_2) dx dt, \quad (2.21)
\end{aligned}$$

where the boundary term

$$\mathfrak{B}'_2(w_2) = B_1(w_2) \partial_{\nu_2} (pw_2) - pw_2 B_2(w_2). \quad (2.22)$$

Proof. We use the multiplier pw_2 . A similar computation leads to the above equality (2.21). $\square$

Lemma 2.6. *Let H be a vector field on Ω_1 . For any w_1 solves the wave equation on Ω_1 as*

$$\partial_t^2 w_1(x, t) - \Delta w_1(x, t) = 0,$$

we have

$$\begin{aligned}
\int_0^T \int_{\Gamma_1 \cup S} \mathfrak{B}_1(w_1) dx dt &= 2 \int_{\Omega_1} \partial_t w_1 H(w_1) dx \Big|_0^T + \int_0^T \int_{\Omega_1} [(\partial_t w_1)^2 - |Dw_1|^2] \operatorname{div} H dx dt \\
&+ 2 \int_0^T \int_{\Omega_1} DH(Dw_1, Dw_1) dx dt, \quad (2.23)
\end{aligned}$$

where the boundary term

$$\mathfrak{B}_1(w_1) = 2\partial_{\nu_1} w_1 H(w_1) + [(\partial_t w_1)^2 - |Dw_1|^2] \langle H, \nu_1 \rangle. \quad (2.24)$$

Proof. Using the identities in [21, Theorem 2.1] we have

$$\begin{aligned} 2 \int_{\Omega_1} \Delta w_1 H(w_1) dx &= 2 \int_{\partial\Omega_1} H(w_1) \partial_{\nu_1} w_1 d\sigma - 2 \int_{\Omega_1} DH(Dw_1, Dw_1) dx \\ &\quad - \int_{\partial\Omega_1} |Dw_1|^2 \langle H, \nu_1 \rangle d\sigma + \int_{\Omega_1} |Dw_1|^2 \operatorname{div} H(x) dx. \end{aligned} \quad (2.25)$$

Similar to the identity (2.15), we have

$$\begin{aligned} 2 \int_0^T \int_{\Omega_1} \partial_t^2 w_1(x, t) H(w_1) dx dt &= 2 \int_{\Omega_1} [\partial_t w_1 H(w_1)] dx \Big|_0^T - \int_0^T \int_{\partial\Omega_1} (\partial_t w_1)^2 \langle H, \nu_1 \rangle d\sigma dt \\ &\quad + \int_0^T \int_{\Omega_1} (\partial_t w_1)^2 \operatorname{div} H dx dt. \end{aligned} \quad (2.26)$$

Combining the identities (2.25) and (2.26) to complete the proof. $\square$

Lemma 2.7. *Let p be a function on Ω_1 . For any w_1 solves the wave equation given in Lemma 2.6, we have*

$$\begin{aligned} \int_0^T \int_{\Gamma_1 \cup S} \mathfrak{B}'_1(w_1) dx dt &= \int_{\Omega_1} \partial_t w_1 p w_1 dx \Big|_0^T - \int_0^T \int_{\Omega_1} p [(\partial_t w_1)^2 - |Dw_1|^2] dx dt \\ &\quad + \int_0^T \int_{\Omega_1} w_1 \langle Dp, Dw_1 \rangle dx dt, \end{aligned} \quad (2.27)$$

where the boundary term

$$\mathfrak{B}'_1(w_1) = \partial_{\nu_1} w_1 p w_1. \quad (2.28)$$

Proof. We use the multiplier $p w_1$. A similar computation as that in Lemma 2.6 leads to the above equality (2.27). $\square$

3 Proof of exact controllability result: Theorem 1.1.

By Hilbert uniqueness method (HUM)(see e.g. [15]), we know that the exact controllability of system (1.4) is equivalent to the observability inequality of system (2.1). Therefore, we only need to establish the following observability result.

Theorem 3.1. *We assume the geometrical assumptions (A.1), (A.2) and (A.3) hold true. Let $T_0 = 8\rho^{-1} \sup_{x \in \Omega_1} |H|$. Then for all $T > T_0$ there exists a constant $c_T > 0$ such that*

$$c_T E(0) \leq \int_0^T \int_{\Gamma_2 \cap \Gamma_0} (D^2 w_2(\nu_2, \nu_2))^2 d\sigma dt + \int_0^T \int_{\Gamma_1 \cap \Gamma_0} (\partial_{\nu_1} w_1)^2 d\sigma dt, \quad (3.1)$$

for all solutions w to problem (2.1).

Step 1. Estimates of the plate part on Ω_2 .

Using Lemma 2.4 and Lemma 2.5, we obtain

$$\begin{aligned}
& \int_0^T \int_{\Gamma_2 \cup S} (\mathfrak{B}_2(w_2) + \mathfrak{B}'_2(w_2)) dx dt \\
= & \int_{\Omega_2} \partial_t w_2 (2H(w_2) + (\operatorname{div} H - \frac{3}{2}\rho)w_2) dx \Big|_0^T + \frac{3}{2}\rho \int_0^T \int_{\Omega_2} [(\partial_t w_2)^2 - a(w_2, w_2)] d\sigma dt \\
& + 4(1 - \mu) \int_0^T \int_{\Omega_2} DH(D^2 w_2, D^2 w_2) dx dt + 4\mu \int_0^T \int_{\Omega_2} DH(\Delta w_2, \Delta w_2) dx dt \\
& + 2(1 - \mu) \int_0^T \int_{\Omega_2} \langle D^2 w_2, l(w_2) \rangle dx dt + 2\mu \int_0^T \int_{\Omega_2} \Delta w_2 \cdot \operatorname{trl}(w_2) dx dt \\
& + \int_0^T \int_{\Omega_2} [2(1 - \mu) D^2 w_2 (D(\operatorname{div} H), Dw_2) + 2\mu \Delta w_2 \langle D(\operatorname{div} H), Dw_2 \rangle] dx dt \\
& + \int_0^T \int_{\Omega_2} w_2 a(\operatorname{div} H, w_2) dx dt, \tag{3.2}
\end{aligned}$$

where we set $p = \operatorname{div} H - \frac{3}{2}\rho$ in Lemma 2.5. Then by Cauchy-Schwartz inequality we have

$$\begin{aligned}
& \int_{\Omega_2} \partial_t w_2 (2H(w_2) + (\operatorname{div} H - \frac{3}{2}\rho)w_2) dx \Big|_0^T \\
\geq & -\varepsilon (E_2(T) + E_2(0)) - C_\varepsilon (\|Dw_2(0)\|^2 + \|Dw_2(T)\|^2 + \|w_2(0)\|^2 + \|w_2(T)\|^2), \tag{3.3} \\
& 2(1 - \mu) \int_0^T \int_{\Omega_2} \langle D^2 w_2, l(w_2) \rangle dx dt + 2\mu \int_0^T \int_{\Omega_2} \Delta w_2 \cdot \operatorname{trl}(w_2) dx dt \\
\geq & -\varepsilon \int_0^T \int_{\Omega_2} a(w_2, w_2) dx dt - C_\varepsilon \int_0^T \int_{\Omega_2} (|l(w_2)|_{T_x}^2 + (\operatorname{trl}(w_2))^2) dx dt, \\
& \int_0^T \int_{\Omega_2} [2(1 - \mu) D^2 w_2 (D(\operatorname{div} H), Dw_2) + 2\mu \Delta w_2 \langle D(\operatorname{div} H), Dw_2 \rangle] dx dt \\
\geq & -\varepsilon \int_0^T \int_{\Omega_2} a(w_2, w_2) dx dt - C_\varepsilon \int_0^T \|Dw_2\|^2 dt, \tag{3.4}
\end{aligned}$$

and

$$\int_0^T \int_{\Omega_2} w_2 a(\operatorname{div} H, w_2) dx dt \geq -\varepsilon \int_0^T \int_{\Omega_2} a(w_2, w_2) dx dt - C_\varepsilon \int_0^T \|w_2\|^2 dt, \tag{3.5}$$

where $\varepsilon > 0$ can be an arbitrary small constant and $\|\cdot\|$ denotes the $L^2(\Omega_2)$ norm. Due to

the assumption (1.5) we have

$$\begin{aligned}
& 4(1-\mu) \int_0^T \int_{\Omega_2} DH(D^2 w_2, D^2 w_2) dx dt + 4\mu \int_0^T \int_{\Omega_2} DH(\Delta w_2, \Delta w_2) dx dt \\
& \geq 4(1-\mu)\rho \int_0^T \int_{\Omega_2} |D^2 w_2|_{T_x}^2 dx dt + 4\mu\rho \int_0^T \int_{\Omega_2} (\Delta w_2)^2 dx dt \\
& \geq 4\rho \int_0^T \int_{\Omega_2} a(w_2, w_2) dx dt.
\end{aligned} \tag{3.6}$$

Substituting inequalities (3.3)-(3.6) to equality (3.2) yields

$$\begin{aligned}
& \int_0^T \int_{\Gamma_2 \cup S} (\mathfrak{B}_2(w_2) + \mathfrak{B}'_2(w_2)) d\sigma dt \\
& \geq \frac{3}{2}\rho \int_0^T \int_{\Omega_2} [(\partial_t w_2)^2 + a(w_2, w_2)] dx dt - \varepsilon(E_2(T) + E_2(0)) \\
& \quad - C_\varepsilon(\|Dw_2(0)\|^2 + \|Dw_2(T)\|^2 + \|w_2(0)\|^2 + \|w_2(T)\|^2) \\
& \quad - C_\varepsilon \int_0^T (\|w_2\|^2 + \|Dw_2\|^2) dt,
\end{aligned} \tag{3.7}$$

where we notice that we shall take $\varepsilon < \frac{\rho}{2}$.

Step 2. Estimates of the wave part on Ω_1 .

Similarly, using Lemma 2.6 and 2.7 we obtain

$$\begin{aligned}
& \int_0^T \int_{\Gamma_1 \cup S} (\mathfrak{B}_1(w_1) + \mathfrak{B}'_1(w_1)) d\sigma dt \\
& = \int_{\Omega_1} \partial_t w_1 (2H(w_1) + (\operatorname{div} H - \frac{3}{2}\rho)w_1) dx|_0^T + \frac{3}{2}\rho \int_0^T \int_{\Omega_1} [(\partial_t w_1)^2 - |Dw_1|^2] dx dt \\
& \quad + 2 \int_0^T \int_{\Omega_1} DH(Dw_1, Dw_1) dx dt + \int_0^T \int_{\Omega_1} w_1 \langle Dp, Dw_1 \rangle dx dt \\
& \geq \frac{1}{4}\rho \int_0^T \int_{\Omega_1} [(\partial_t w_1)^2 + |Dw_1|^2] dx dt - \varepsilon(E_1(T) + E_1(0)) - C_\varepsilon(\|w_1(0)\|^2 + \|w_1(T)\|^2) \\
& \quad - 2 \sup_{x \in \Omega_1} |H|(E_1(0) + E_1(T)) - C_\varepsilon \int_0^T \|w_1\|^2 dt,
\end{aligned} \tag{3.8}$$

where we notice that we can take $\varepsilon < \frac{\rho}{4}$.

Step 3. We combine estimates (3.7), (3.8) and Lemma 2.3 to get

$$\begin{aligned}
& \int_0^T \int_{\Gamma_2 \cup S} (\mathfrak{B}_2(w_2) + \mathfrak{B}'_2(w_2)) d\sigma dt + \int_0^T \int_{\Gamma_1 \cup S} (\mathfrak{B}_1(w_1) + \mathfrak{B}'_1(w_1)) d\sigma dt \\
& \geq \frac{1}{2}\rho T E(0) - 2\varepsilon E(0) - 4 \sup_{x \in \Omega_1} |H| E(0) + \mathfrak{L}(w) \\
& \geq c_T E(0) + \mathfrak{L}(w),
\end{aligned} \tag{3.9}$$

for $T > T_0 \equiv 8\rho^{-1} \sup_{x \in \Omega_1} |H|$, where we have

$$\begin{aligned} \mathfrak{L}(w) &= -C_\varepsilon (\|Dw_2(0)\|^2 + \|Dw_2(T)\|^2 + \|w_2(0)\|^2 + \|w_2(T)\|^2) \\ &\quad - C_\varepsilon \int_0^T (\|w_2\|^2 + \|Dw_2\|^2) dt - C_\varepsilon (\|w_1(0)\|^2 + \|w_1(T)\|^2) \\ &\quad - C_\varepsilon \int_0^T \|w_1\|^2 dt, \end{aligned} \quad (3.10)$$

which is a lower order term with respect to the energy $E(t)$.

Next, we deal with the left hand side of the inequality (3.9). On S we have

$$\begin{aligned} H(w_1) &= \langle Dw_1, H \rangle = \partial_{\nu_1} w_1 \langle H, \nu_1 \rangle + \partial_{\tau_1} w_1 \langle H, \tau_1 \rangle = \partial_{\tau_1} w_1 \langle H, \tau_1 \rangle \\ &= \partial_{\tau_2} w_2 \langle H, \tau_2 \rangle = \langle Dw_1, H \rangle = H(w_2), \end{aligned} \quad (3.11)$$

where we notice that $\langle H, \nu \rangle = 0$ and $w_1 = w_2$ on S . Combining identity (3.11) and the boundary condition on the interface S yields

$$\mathfrak{B}_1(w_1)|_S = 2\partial_\nu w_1 H(w_1)|_S = 2B_2(w_2)H(w_2)|_S = -\mathfrak{B}_2(w_2)|_S. \quad (3.12)$$

Obviously we have

$$\mathfrak{B}'_1(w_1)|_S = (\operatorname{div} H - \frac{3}{2}\rho)w_1\partial_\nu w_1|_S = (\operatorname{div} H - \frac{3}{2}\rho)w_2B_2(w_2)|_S = -\mathfrak{B}'_2(w_2)|_S. \quad (3.13)$$

Thus we obtain

$$\int_0^T \int_S (\mathfrak{B}_2(w_2) + \mathfrak{B}'_2(w_2)) d\sigma dt + \int_0^T \int_S (\mathfrak{B}_1(w_1) + \mathfrak{B}'_1(w_1)) d\sigma dt = 0. \quad (3.14)$$

On the other hand we have on Γ_1 that

$$\begin{aligned} H(w_1) = \partial_{\nu_1} w_1 \langle H, \nu_1 \rangle + \partial_{\tau_1} w_1 \langle H, \tau_1 \rangle &= \partial_{\nu_1} w_1 \langle H, \nu_1 \rangle \\ |Dw_1|^2 &= (\partial_{\nu_1} w_1)^2, \end{aligned}$$

which yields

$$\mathfrak{B}_1(w_1)|_{\Gamma_1} = (\partial_{\nu_1} w_1)^2 \langle H, \nu_1 \rangle. \quad (3.15)$$

Due to the boundary condition $w_2 = \partial_{\nu_2} w_2 = 0$ on Γ_2 we have

$$Dw_2 = (\partial_{\nu_2} w_2)\nu_2 + (\partial_{\tau_2} w_2)\tau_2 = 0, \quad H(w_2) = 0, \quad (3.16)$$

which yield on Γ_2

$$D^2 w_2(\tau_2, \nu_2) = \langle D_{\tau_2}(Dw_2), \nu_2 \rangle = \langle 0, \nu_2 \rangle = 0, \quad (3.17)$$

and

$$D^2 w_2(\tau_2, \tau_2) = \langle D_{\tau_2}(Dw_2), \tau_2 \rangle = \langle 0, \tau_2 \rangle = 0. \quad (3.18)$$

Therefore on Γ_2 we have

$$\begin{aligned}\mathfrak{B}_2(w_2) &= 2B_1(w_2)\partial_{\nu_2}(H(w_2)) - a(w_2, w_2)\langle H, \nu_2 \rangle \\ &= (D^2w_2(\nu_2, \nu_2))^2\langle H, \nu_2 \rangle,\end{aligned}\tag{3.19}$$

where we notice on Γ_2 that

$$\begin{aligned}\partial_{\nu_2}(H(w_2)) &= D^2w_2(H, \nu_2) + Dw_2(D_{\nu_2}H) \\ &= D^2w_2((\langle H, \nu_2 \rangle \nu_2 + \langle H, \tau_2 \rangle \tau_2), \nu_2) = \langle H, \nu_2 \rangle D^2w_2(\nu_2, \nu_2).\end{aligned}\tag{3.20}$$

It's easy to know

$$\mathfrak{B}'_1(w_1)|_{\Gamma_1} = \mathfrak{B}'_2(w_2)|_{\Gamma_2} = 0.\tag{3.21}$$

Finally we get

$$\begin{aligned}& \int_0^T \int_{\Gamma_1} (\mathfrak{B}_2(w_2) + \mathfrak{B}'_2(w_2)) d\sigma dt + \int_0^T \int_{\Gamma_2} (\mathfrak{B}_1(w_1) + \mathfrak{B}'_1(w_1)) d\sigma dt \\ &= \int_0^T \int_{\Gamma_1} (\partial_{\nu_1} w_1)^2 \langle H, \nu_1 \rangle d\sigma dt + \int_0^T \int_{\Gamma_2} (D^2w_2(\nu_2, \nu_2))^2 \langle H, \nu_2 \rangle d\sigma dt.\end{aligned}\tag{3.22}$$

Combining (3.9), (3.14) and (3.22) to obtain

$$\begin{aligned}& \int_0^T \int_{\Gamma_1} (\partial_{\nu_1} w_1)^2 \langle H, \nu_1 \rangle d\sigma dt + \int_0^T \int_{\Gamma_2} (D^2w_2(\nu_2, \nu_2))^2 \langle H, \nu_2 \rangle d\sigma dt \\ &\geq c_T E(0) + \mathfrak{L}(w).\end{aligned}\tag{3.23}$$

Finally, the lower order term $\mathfrak{L}(w)$ can be absorbed using a compactness-uniqueness argument in the literature (see [13] and many others). Consequently the inequality (3.1) follows considering the definition of Γ_0^i in (1.6).

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