Frequency-Domain Subsystem Identification with Application to Modeling Human Control Behavior

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Abstract

We present a frequency-domain subsystem identification algorithm that identifies unknown feedback and feedforward subsystems that are interconnected with a known subsystem. This method requires only accessible input and output measurements, applies to linear time-invariant subsystems, and uses a candidate-pool approach to ensure asymptotic stability of the identified closed-loop transfer function. We analyze the algorithm in the cases of noiseless and noisy data. The main analytic result of this paper shows that the coefficients of the identified feedback and feedforward transfer functions are arbitrarily close to the true coefficients if the data noise is sufficiently small and the candidate pool is sufficiently dense. This subsystem identification approach has application to modeling the control behavior of humans interacting with and receiving feedback from a dynamic system. We apply the algorithm to data from a human-in-the-loop experiment to model a human's control behavior.

Keywords: subsystem identification; human-in-the-loop; human motor control

1. Introduction

Humans learn to interact with many complex dynamic systems. For example, humans learn to drive cars, fly planes, and ride unicycles. Moreover, humans learn to control all of these systems with virtually no *a priori* information. The control strategies that humans learn and the approaches used to learn them are currently unknown. The internal model hypothesis proposes that humans construct models of their body and the physical world, and these models are used for control [1, 2]. Although studies (e.g., [3–10]) provide evidence in support of the internal model hypothesis, this existing evidence is not conclusive [11].

Consider a scenario where a human interacts with a dynamic system by using feedback y_t and external information r_t (e.g., a command) to generate a control u_t as shown in Figure 1. In this scenario, the human is an unknown subsystem, which can include both feedback and feedforward. Modeling the human's control strategy can be viewed as a subsystem identification (SSID) problem, where r_t and y_t are measured and the dynamic system with which the human interacts is assumed to be known. The internal signals that the human uses to construct u_t are inaccessible (i.e., unmeasurable). For example, if u_t is the sum of feedback and feedforward terms, then these individual terms are inaccessible.

Existing methods for SSID are given in [12–24]. Specifically, [12–14] present methods for static subsystems, while [15–24] present methods for dynamic subsystems. In the dynamic SSID literature, the approaches in [15–19] are restricted to open-loop SSID, that is, identification of subsystems interconnected without feedback. We note that [16, 19] use open-loop SSID to model the dynamics of human subsystems. Specifically, [16] identifies a transfer

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Figure 1: Modeling a human’s control strategy can be viewed as an SSID problem, where r_t and y_t are measured and the dynamic system with which the human interacts is assumed to be known.

function that models a human’s precision grip force dynamics, whereas [19] identifies two transfer functions that together model a human’s oculomotor subsystem.

In contrast to [12–19], we focus on dynamic closed-loop SSID, that is, identification of dynamic subsystems with feedback. Existing dynamic closed-loop SSID methods include [20–24]. In particular, [20] identifies a transfer function that models the behavior of a human subject interacting in feedback with a mechanical system. However, the method in [20] applies to systems with feedback only, that is, systems without feedforward. We note that the methods in [20–24] are time-domain techniques and yield identified models that may not result in an asymptotically stable closed-loop system.

This paper presents a new closed-loop SSID technique that: i) identifies feedback and feedforward subsystems, and ii) ensures asymptotic stability of the identified closed-loop transfer function. A closed-loop SSID method that addresses both i) and ii) is a new contribution of this paper. The method relies on a candidate-pool approach to accomplish ii). Another contribution of this paper is an analysis of the properties of the SSID algorithm in the cases of noiseless and noisy data. Our main analytic result shows that the coefficients of the identified feedback and feedforward transfer functions are arbitrarily close to the true coefficients if the data noise is sufficiently small and the candidate pool is sufficiently dense.

Characteristics i) and ii) of the SSID algorithm are motivated by the application to modeling human control behavior. First, humans generally use both anticipatory (feedforward) and reactive (feedback) control [1, 2], which motivates i). Second, if a human-in-the-loop system has a bounded output, then it is desirable to identify subsystems that result in an asymptotically stable closed-loop transfer function, thus motivating ii). In addition, human control behavior is band limited; specifically, humans cannot produce motion with arbitrarily high frequency. Thus, we are interested in identifying models over a limited frequency range, which motivates the development of a new SSID technique in the frequency domain.

The frequency-domain SSID technique in this paper identifies unknown feedback and feedforward subsystems that are interconnected with a known subsystem, where all internal signals are assumed to be inaccessible, as shown in Figure 2. We present numerical examples to demonstrate the properties of the SSID algorithm. We also apply the SSID algorithm to data from an experiment designed to model the control behavior of a human subject interacting with a dynamic system. System identification approaches are also used in [25–31] to model human control behavior. However, [25–31] do not model both feedback and feedforward subsystems and do not ensure stability of the identified closed-loop model. A preliminary version of the SSID algorithm in this paper appeared in the conference proceedings [32]. However, this paper goes beyond the work of [32] by analyzing the properties of the SSID algorithm.

Figure 2: The input r and output y of this linear time-invariant system are measured, but all internal signals are inaccessible.

2. Problem Formulation

Consider the linear time-invariant system shown in Figure 2, where r , y , σ_r , and σ_y are the Laplace transforms of the input, output, input noise, and output noise, respectively, and for $i = 0, 1, 2$, $G_i: \mathbb{C} \rightarrow \mathbb{C}$ is a real rational transfer function. If $\sigma_r = 0$ and $\sigma_y = 0$, then the closed-loop transfer function from r to y is given by

$$\tilde{G}(s) \triangleq \frac{G_0(s)G_1(s) + G_0(s)G_2(s)}{1 + G_0(s)G_2(s)}. \quad (1)$$

Next, let N be a positive integer, and define $\mathcal{N} \triangleq \{1, 2, \dots, N\}$. For all $k \in \mathcal{N}$, let $\omega_k \in (0, \infty)$, where $\omega_1 < \dots < \omega_N$. Furthermore, for all $k \in \mathcal{N}$, define the *closed-loop frequency response data*

$$H(\omega_k) \triangleq \frac{y(j\omega_k)}{r(j\omega_k)} = \tilde{G}(j\omega_k) + \sigma(j\omega_k), \quad (2)$$

where $\sigma(s) \triangleq [\tilde{G}(s)\sigma_r(s) + \sigma_y(s)]/r(s)$. If $\sigma(j\omega_k) \equiv 0$, then $\{H(\omega_k)\}_{k=1}^N$ is noiseless. In contrast, if $\sigma(j\omega_k) \not\equiv 0$, then $\{H(\omega_k)\}_{k=1}^N$ is noisy.

We present a method to identify G_1 and G_2 provided that G_0 and $\{H(\omega_k)\}_{k=1}^N$ are known and $G_0 \neq 0$. In this case, the closed-loop frequency response data $\{H(\omega_k)\}_{k=1}^N$ can be obtained from the accessible signals r and y and does not depend on the internal signals, which are not assumed to be measured.

For $i = 0, 1, 2$, G_i can be expressed as $G_i(s) = N_i(s)/D_i(s)$, where N_i and D_i are coprime, and D_i is monic. The degrees of N_i and D_i are denoted by $n_i \triangleq \deg N_i$ and $d_i \triangleq \deg D_i$. Thus, (1) can be expressed as

$$\tilde{G}(s) = \frac{N_0(s) [D_2(s)N_1(s) + D_1(s)N_2(s)]}{D_1(s) [D_0(s)D_2(s) + N_0(s)N_2(s)]}.$$

We make the following assumptions:

- (A1) d_1 , d_2 , n_1 , and n_2 are known.
- (A2) $d_0 + d_2 > n_0 + n_2$.
- (A3) $N > d_0 + d_1 + d_2 + n_0 + \max\{n_1 + d_2, n_2 + d_1\}$.
- (A4) If $\lambda \in \mathbb{C}$ and $D_1(\lambda) [D_0(\lambda)D_2(\lambda) + N_0(\lambda)N_2(\lambda)] = 0$, then $\text{Re } \lambda < 0$.

Assumption (A1) can be replaced by the assumption that upper bounds on d_1 , d_2 , n_1 , and n_2 are known. However, we invoke (A1) for clarity of the presentation. Assumption (A2) states that the loop transfer function G_0G_2 is strictly proper. Assumption (A3) implies that the number N of frequency response data points is sufficiently large. This assumption ensures that the minimization problem solved in the SSID has a unique solution. Assumption

(A4) implies that $\tilde{G}$ is asymptotically stable, that is, the poles of $\tilde{G}$ are in the open-left-half complex plane.

Define $d \triangleq d_1 + d_2 + n_2 + 1$, and for all nonnegative integers j , let $\Gamma_j : \mathbb{C} \rightarrow \mathbb{C}^{j+1}$ be given by $\Gamma_j(s) \triangleq [s^j \ s^{j-1} \ \dots \ s \ 1]^T$. Consider the functions $\mathcal{N}_1 : \mathbb{C} \times \mathbb{R}^{n_1+1} \rightarrow \mathbb{C}$ and $\mathcal{D}_1, \mathcal{N}_2, \mathcal{D}_2 : \mathbb{C} \times \mathbb{R}^d \rightarrow \mathbb{C}$ given by

$$\begin{aligned}\mathcal{N}_1(s, \beta) &\triangleq \Gamma_{n_1}^T(s) \beta, & \mathcal{D}_1(s, \phi) &\triangleq s^{d_1} + \Gamma_{d_1-1}^T(s) E_1 \phi, \\ \mathcal{N}_2(s, \phi) &\triangleq \Gamma_{n_2}^T(s) E_2 \phi, & \mathcal{D}_2(s, \phi) &\triangleq s^{d_2} + \Gamma_{d_2-1}^T(s) E_3 \phi,\end{aligned}$$

where $E_1 \triangleq [I_{d_1} \ 0_{d_1 \times (d_2+n_2+1)}] \in \mathbb{R}^{d_1 \times d}$, $E_2 \triangleq [0_{(n_2+1) \times d_1} \ I_{n_2+1} \ 0_{(n_2+1) \times d_2}] \in \mathbb{R}^{(n_2+1) \times d}$, $E_3 \triangleq [0_{d_2 \times (d_1+n_2+1)} \ I_{d_2}] \in \mathbb{R}^{d_2 \times d}$, $\beta \in \mathbb{R}^{n_1+1}$, and $\phi \in \mathbb{R}^d$. Next, consider the functions $\mathcal{G}_1 : \mathbb{C} \times \mathbb{R}^{n_1+1} \times \mathbb{R}^d \rightarrow \mathbb{C}$ and $\mathcal{G}_2 : \mathbb{C} \times \mathbb{R}^d \rightarrow \mathbb{C}$ given by

$$\mathcal{G}_1(s, \beta, \phi) \triangleq \frac{\mathcal{N}_1(s, \beta)}{\mathcal{D}_1(s, \phi)}, \quad \mathcal{G}_2(s, \phi) \triangleq \frac{\mathcal{N}_2(s, \phi)}{\mathcal{D}_2(s, \phi)},$$

which, for each $\beta \in \mathbb{R}^{n_1+1}$ and $\phi \in \mathbb{R}^d$, are real rational transfer functions.

Our objective is to determine β and ϕ such that $\mathcal{G}_1$ and $\mathcal{G}_2$ approximate G_1 and G_2 , respectively. To achieve this objective, consider the cost function

$$J(\beta, \phi) \triangleq \sum_{k=1}^N \left| \frac{N_0(j\omega_k) [\mathcal{D}_2(j\omega_k, \phi) \mathcal{N}_1(j\omega_k, \beta) + \mathcal{D}_1(j\omega_k, \phi) \mathcal{N}_2(j\omega_k, \phi)]}{\mathcal{D}_1(j\omega_k, \phi) [D_0(j\omega_k) \mathcal{D}_2(j\omega_k, \phi) + N_0(j\omega_k) \mathcal{N}_2(j\omega_k, \phi)]} - H(\omega_k) \right|^2, \quad (3)$$

which is a measure of the difference between the frequency response data $\{H(\omega_k)\}_{k=1}^N$ and the closed-loop transfer function obtained from the estimates $\mathcal{G}_1$ and $\mathcal{G}_2$. Note that J is nonlinear and nonconvex in (β, ϕ) .

Let $\beta_* \in \mathbb{R}^{n_1+1}$ and $\phi_* \in \mathbb{R}^d$ be such that, for all $s \in \mathbb{C}$, $N_1(s) = \mathcal{N}_1(s, \beta_*)$, $D_1(s) = \mathcal{D}_1(s, \phi_*)$, $N_2(s) = \mathcal{N}_2(s, \phi_*)$, and $D_2(s) = \mathcal{D}_2(s, \phi_*)$. If $\sigma(j\omega_k) \equiv 0$, then $J(\beta_*, \phi_*) = 0$, which implies that the true parameters β_* and ϕ_* minimize the cost function (3) if the closed-loop frequency response data $\{H(\omega_k)\}_{k=1}^N$ is noiseless.

3. Subsystem Identification

We now develop an SSID algorithm to obtain estimates of β_* and ϕ_* , and thus estimates of G_1 and G_2 . Consider the function $\tilde{\mathcal{G}} : \mathbb{C} \times \mathbb{R}^{n_1+1} \times \mathbb{R}^d \rightarrow \mathbb{C}$ defined by

$$\tilde{\mathcal{G}}(s, \beta, \phi) \triangleq \frac{\tilde{\mathcal{N}}_1(s, \phi) \beta + \tilde{\mathcal{N}}_2(s, \phi)}{\tilde{\mathcal{D}}(s, \phi)}, \quad (4)$$

where

$$\tilde{\mathcal{D}}(s, \phi) \triangleq \mathcal{D}_1(s, \phi) [D_0(s) \mathcal{D}_2(s, \phi) + N_0(s) \mathcal{N}_2(s, \phi)], \quad (5)$$

$$\tilde{\mathcal{N}}_1(s, \phi) \triangleq N_0(s) \mathcal{D}_2(s, \phi) \Gamma_{n_1}^T(s), \quad (6)$$

$$\tilde{\mathcal{N}}_2(s, \phi) \triangleq N_0(s) \mathcal{D}_1(s, \phi) \mathcal{N}_2(s, \phi). \quad (7)$$

Note that $\tilde{\mathcal{G}}(s, \beta, \phi)$ is the closed-loop transfer function obtained using β and ϕ . It follows from (3)–(7) that

$$J(\beta, \phi) = \sum_{k=1}^N |\tilde{\mathcal{G}}(j\omega_k, \beta, \phi) - H(\omega_k)|^2. \quad (8)$$

Next, we define

$$\Omega_0(\phi) \triangleq \sum_{k=1}^N \left| \frac{\tilde{\mathcal{N}}_2(j\omega_k, \phi)}{\tilde{\mathcal{D}}(j\omega_k, \phi)} - H(\omega_k) \right|^2 \in \mathbb{R}, \quad (9)$$

$$\Omega_1(\phi) \triangleq 2\text{Re} \sum_{k=1}^N \left[\frac{\tilde{N}_1(j\omega_k, \phi)}{\tilde{D}(j\omega_k, \phi)} \right]^* \left[\frac{\tilde{N}_2(j\omega_k, \phi)}{\tilde{D}(j\omega_k, \phi)} - H(\omega_k) \right] \in \mathbb{R}^{n_1+1}, \quad (10)$$

$$\Omega_2(\phi) \triangleq \text{Re} \sum_{k=1}^N \left[\frac{\tilde{N}_1(j\omega_k, \phi)}{\tilde{D}(j\omega_k, \phi)} \right]^* \left[\frac{\tilde{N}_1(j\omega_k, \phi)}{\tilde{D}(j\omega_k, \phi)} \right] \in \mathbb{R}^{(n_1+1) \times (n_1+1)}, \quad (11)$$

where $[\cdot]^*$ denotes the complex conjugate transpose. Thus, (8) can be expressed as

$$J(\beta, \phi) = \beta^T \Omega_2(\phi) \beta + \Omega_1^T(\phi) \beta + \Omega_0(\phi), \quad (12)$$

which is convex in β , because, for each $\phi \in \mathbb{R}^d$, $\Omega_2(\phi)$ is positive semidefinite.

Define the set of $\phi \in \mathbb{R}^d$ such that $\mathcal{G}(s, \beta, \phi)$ is asymptotically stable, which is given by $\mathcal{S} \triangleq \{\phi \in \mathbb{R}^d : \tilde{D}(s, \phi) \text{ is Hurwitz}\}$.

The following result provides sufficient conditions such that $\Omega_2(\phi)$ is positive definite. The proof is in Appendix A.

Proposition 1. Consider Ω_2 given by (11), and assume (A1)–(A3) are satisfied. Let $\phi \in \mathcal{S}$. Then, $\Omega_2(\phi)$ is positive definite.

For each $\phi \in \mathcal{S}$, the following result provides the global minimizer of $J(\beta, \phi)$. The proof is in Appendix A.

Proposition 2. Consider J given by (12), and assume (A1)–(A3) are satisfied. Let $\phi \in \mathcal{S}$, and let $\beta \in \mathbb{R}^{n_1+1} \setminus \{-\frac{1}{2}\Omega_2^{-1}(\phi)\Omega_1(\phi)\}$. Then,

$$J\left(-\frac{1}{2}\Omega_2^{-1}(\phi)\Omega_1(\phi), \phi\right) = \Omega_0(\phi) - \frac{1}{4}\Omega_1^T(\phi)\Omega_2^{-1}(\phi)\Omega_1(\phi) < J(\beta, \phi).$$

Next, let M be a positive integer, and let $\Phi \subseteq \mathcal{S}$ be a set with M elements. We call Φ the *candidate pool*. Define $\mathcal{M} \triangleq \{1, 2, \dots, M\}$. We now create a candidate sequence using the M elements in the candidate pool Φ . For $i, j \in \mathcal{M}$, let $\phi_i, \phi_j \in \Phi$ be such that if $i \neq j$, then $\phi_i \neq \phi_j$. The sequence $\{\phi_i\}_{i=1}^M$ is not unique; however, the order of the sequence is selected arbitrarily.

Now, for all $i \in \mathcal{M}$, define the quadratic cost function

$$\mathcal{J}_i(\beta) \triangleq J(\beta, \phi) \Big|_{\phi=\phi_i} = \beta^T \Omega_2(\phi_i) \beta + \Omega_1^T(\phi_i) \beta + \Omega_0(\phi_i). \quad (13)$$

Since $\phi_1, \dots, \phi_M \in \Phi \subseteq \mathcal{S}$, Proposition 1 implies that $\Omega_2(\phi_1), \dots, \Omega_2(\phi_M)$ are positive definite. The following result is a consequence of Proposition 2 and provides the unique global minimizer of $\mathcal{J}_i$ for each $i \in \mathcal{M}$.

Proposition 3. Consider $\mathcal{J}_i$ given by (13), assume (A1)–(A3) are satisfied, and assume $\Phi \subseteq \mathcal{S}$. Let $i \in \mathcal{M}$, and let $\beta \in \mathbb{R}^{n_1+1} \setminus \{-\frac{1}{2}\Omega_2^{-1}(\phi_i)\Omega_1(\phi_i)\}$. Then,

$$\mathcal{J}_i\left(-\frac{1}{2}\Omega_2^{-1}(\phi_i)\Omega_1(\phi_i)\right) = \Omega_0(\phi_i) - \frac{1}{4}\Omega_1^T(\phi_i)\Omega_2^{-1}(\phi_i)\Omega_1(\phi_i) < \mathcal{J}_i(\beta).$$

Proposition 3 implies that for each $i \in \mathcal{M}$, $\beta_i \triangleq -\frac{1}{2}\Omega_2^{-1}(\phi_i)\Omega_1(\phi_i)$ is the global minimizer of $\mathcal{J}_i$. Next, let $\ell \in \mathcal{M}$ be the smallest integer such that $\mathcal{J}_\ell(\beta_\ell) = \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$. Then, the identified parameters are $\beta^+ \triangleq \beta_\ell$ and $\phi^+ \triangleq \phi_\ell$, and the identified transfer functions are $G_1^+(s) \triangleq \mathcal{G}_1(s, \beta^+, \phi^+)$ and $G_2^+(s) \triangleq \mathcal{G}_2(s, \phi^+)$. Note that $\arg \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$ is not necessarily unique. In this case, $\ell \in \mathcal{M}$ is the smallest integer such that $\mathcal{J}_\ell(\beta_\ell) = \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$. However, in practice, $\arg \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$ is generally unique. We now summarize this SSID method.

Algorithm 1. Consider the closed-loop transfer function (1), where G_0 is known. Assume $\{H(\omega_k)\}_{k=1}^N$ is known, and (A1)–(A4) are satisfied. Then, the subsystem identification algorithm is as follows:

Step 1. Generate the candidate pool $\Phi \subseteq \mathcal{S}$ and candidate sequence $\{\phi_i\}_{i=1}^M$.

Step 2. For each $i \in \mathcal{M}$, find $\beta_i \triangleq -\frac{1}{2}\Omega_2^{-1}(\phi_i)\Omega_1(\phi_i)$, which is the global minimizer of $\mathcal{J}_i$.

Step 3. Find the smallest integer $\ell \in \mathcal{M}$ such that $\mathcal{J}_\ell(\beta_\ell) = \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$.

Step 4. The identified parameters are $\beta^+ = \beta_\ell$ and $\phi^+ = \phi_\ell$.

Step 5. The identified transfer functions are $G_1^+(s) = \mathcal{G}_1(s, \beta^+, \phi^+)$ and $G_2^+(s) = \mathcal{G}_2(s, \phi^+)$.

In the next two sections, we analyze Algorithm 1. Specifically, we analyze how the identified parameters β^+ and ϕ^+ relate to the true parameters β_* and ϕ_* .

4. Analysis with Noiseless Frequency Response Data

In this section, we analyze Algorithm 1 under the assumption of noiseless frequency response data. We assume that for all $k \in \mathcal{N}$, $\sigma(j\omega_k) = 0$, which implies that for all $k \in \mathcal{N}$, $H(\omega_k) = \tilde{G}(j\omega_k)$. The following result relates β_* and ϕ_* if $\sigma(j\omega_k) \equiv 0$. The proof is in Appendix B.

Proposition 4. Assume (A1)–(A4) are satisfied, and assume for all $k \in \mathcal{N}$, $\sigma(j\omega_k) = 0$. Then, $\beta_* = -\frac{1}{2}\Omega_2^{-1}(\phi_*)\Omega_1(\phi_*)$ and $J(\beta_*, \phi_*) = 0$.

The following result provides sufficient conditions on $\beta \in \mathbb{R}^{n_1+1}$ and $\phi \in \mathcal{S}$ such that $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{G}(s)$. The proof is in Appendix B.

Proposition 5. Let $\beta \in \mathbb{R}^{n_1+1}$ and $\phi \in \mathbb{R}^d$. Assume (A3) is satisfied, and assume $\sum_{k=1}^N |\tilde{\mathcal{G}}(j\omega_k, \beta, \phi) - \tilde{G}(j\omega_k)| = 0$. Then, for all $s \in \mathbb{C}$, $\tilde{\mathcal{G}}(s, \beta, \phi) = \tilde{G}(s)$.

Proposition 5 provides sufficient conditions such that $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{G}(s) \equiv \tilde{\mathcal{G}}(s, \beta_*, \phi_*)$; however, these conditions are not sufficient to conclude that $[\beta^T \ \phi^T]^T$ equals $[\beta_*^T \ \phi_*^T]^T$. The following example demonstrates this case.

Example 1. Consider the closed-loop transfer function (1), where $G_0(s) = 1/(s+2)$, $G_1(s) = (s+3)/(s+1)$, and $G_2(s) = 1$. Note that $\beta_* = [1 \ 3]^T$, $\phi_* = [1 \ 1]^T$, and $\tilde{G}(s) = (2s+4)/(s^2+4s+3)$. Let $\beta = [3 \ 7]^T \neq \beta_*$ and $\phi = [3 \ -1]^T \neq \phi_*$, and it follows that $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{G}(s) \equiv \tilde{\mathcal{G}}(s, \beta_*, \phi_*)$. $\triangle$

Example 1 shows that there can exist $\beta \in \mathbb{R}^{n_1+1}$ and $\phi \in \mathcal{S}$ such that $[\beta^T \ \phi^T]^T \neq [\beta_*^T \ \phi_*^T]^T$ and $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{\mathcal{G}}(s, \beta_*, \phi_*)$. In this case, the SSID problem is not well posed, because J does not have a unique minimizer, and (β_*, ϕ_*) cannot be determined uniquely from the noiseless closed-loop frequency response data $\{H(\omega_k)\}_{k=1}^N$.

We now impose additional assumption to ensure that if $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{G}(s)$, then $\beta = \beta_*$ and $\phi = \phi_*$. First, let $\Psi \subseteq \mathbb{R}^d$ be a compact and perfect (i.e., closed with no isolated point) set containing ϕ_* . Note that Ψ can be selected sufficiently large to ensure $\phi_* \in \Psi$. We assume Ψ is known. In practice, Ψ is used to generate the candidate pool. For the remainder of this paper, we impose the assumption:

(A5) If $\beta \in \mathbb{R}^{n_1+1}$, $\phi \in \Psi \cap \mathcal{S}$, and $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{G}(s)$, then $\beta = \beta_*$ and $\phi = \phi_*$.

Assumption (A5) implies that $[\beta_*^T \ \phi_*^T]^T$ is the only element in $\mathbb{R}^{n_1+1} \times (\Psi \cap \mathcal{S})$ that yields the closed-loop transfer function $\tilde{G}$.

4.1. ϕ_* in the candidate pool Φ

The following result provides sufficient conditions such that the identified transfer functions G_1^+ and G_2^+ are equal to G_1 and G_2 , respectively.

Theorem 1. Consider the closed-loop transfer function (1), where G_0 is known. Assume $\{H(\omega_k)\}_{k=1}^N$ is known, (A1)–(A5) are satisfied, and for all $k \in \mathcal{N}$, $\sigma(j\omega_k) = 0$. Furthermore, consider Algorithm 1 with the candidate pool $\Phi \subseteq (\Psi \cap \mathcal{S})$, and assume $\phi_* \in \Phi$. Let β^+ and ϕ^+ denote the identified parameters obtained from Algorithm 1. Then, $\beta^+ = \beta_*$ and $\phi^+ = \phi_*$.

Proof. Since $\phi_* \in \Phi$, it follows that there exists $m \in \mathcal{M}$ such that $\phi_m = \phi_*$. Next, since for all $k \in \mathcal{N}$, $\sigma(j\omega_k) = 0$, it follows from Proposition 3 and Proposition 4 that $\beta_m = -\frac{1}{2}\Omega_2^{-1}(\phi_m)\Omega_1(\phi_m) = -\frac{1}{2}\Omega_2^{-1}(\phi_*)\Omega_1(\phi_*) = \beta_*$. Thus, (13) and Proposition 4 imply that $\mathcal{J}_m(\beta_m) = J(\beta_m, \phi_m) = J(\beta_*, \phi_*) = 0$.

Since for all $i \in \mathcal{M}$, $\mathcal{J}_i(\beta_i) \geq 0$ and $\mathcal{J}_m(\beta_m) = 0$, it follows that $J(\beta^+, \phi^+) = \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i) = \mathcal{J}_m(\beta_m) = 0$. Since for all $k \in \mathcal{N}$, $H(\omega_k) = \tilde{G}(j\omega_k)$, it follows from (8) that $0 = J(\beta^+, \phi^+) = \sum_{k=1}^N |\tilde{\mathcal{G}}(j\omega_k, \beta^+, \phi^+) - \tilde{G}(j\omega_k)|^2$. Thus, Proposition 5 implies that $\tilde{\mathcal{G}}(s, \beta^+, \phi^+) \equiv \tilde{G}(s)$. Since, in addition, $\phi^+ \in \Phi \subseteq (\Psi \cap \mathcal{S})$, (A5) implies that $\beta^+ = \beta_*$ and $\phi^+ = \phi_*$. $\square$

4.2. ϕ_* not necessarily in the candidate pool Φ

We now extend the analysis in the previous section to address the case where ϕ_* is not necessarily in the candidate pool. It follows from Proposition 1 that for all $\phi \in \mathcal{S}$, $\Omega_2(\phi)$ is positive definite, which implies that $\theta: \mathcal{S} \rightarrow \mathbb{R}^{n_1+1}$ given by

$$\theta(\phi) \triangleq -\frac{1}{2}\Omega_2^{-1}(\phi)\Omega_1(\phi) \quad (14)$$

is well defined on $\mathcal{S}$. Define $Q: \mathcal{S} \rightarrow [0, \infty)$ by

$$Q(\phi) \triangleq J(\theta(\phi), \phi). \quad (15)$$

The following result addresses continuity of θ and Q . The proof is in Appendix C.

Proposition 6. Assume (A1)–(A3) are satisfied. Then, θ and Q are continuous on $\mathcal{S}$.

It follows from (A4) that $\tilde{\mathcal{D}}(s, \phi_*)$ is Hurwitz. Thus, there exists $\rho < 0$ such that if $\lambda \in \mathbb{C}$ and $\tilde{\mathcal{D}}(\lambda, \phi_*) = 0$, then $\text{Re } \lambda < \rho$. We assume ρ is known. Note that $\rho < 0$ can be selected such that $|\rho|$ is arbitrarily small, which ensures that $\mathcal{S}_\rho \triangleq \{\phi \in \mathbb{R}^d: \tilde{\mathcal{D}}(s + \rho, \phi) \text{ is Hurwitz}\}$ contains ϕ_* . In practice, $\mathcal{S}_\rho$ is used to generate the candidate pool.

The following propositions are needed for the main result of this section. The proofs are in Appendix D. Note that $\overline{\Psi \cap \mathcal{S}_\rho}$ denotes the closure of $\Psi \cap \mathcal{S}_\rho$.

Proposition 7. $\Psi \cap \mathcal{S}_\rho$ is bounded and contains no isolated points.

Proposition 8. $\overline{\Psi \cap \mathcal{S}_\rho} \subseteq \mathcal{S}$ is compact.

Let $\mathbb{Z}^+$ denote the set of positive integers, let $n \in \mathbb{Z}^+$, and define the *open ball of radius $\epsilon > 0$ centered at $c \in \mathbb{R}^n$* by $\mathbb{B}_\epsilon(c) \triangleq \{x \in \mathbb{R}^n: \|x - c\| < \epsilon\}$. We now define a convergent sequence of finite sets.

Definition 1. Let $n \in \mathbb{Z}^+$, and let $\Delta \subseteq \mathbb{R}^n$ be bounded and contain no isolated points. For all $j \in \mathbb{Z}^+$, let $\Delta_j \subseteq \Delta$ be a finite set. Then, $\{\Delta_j\}_{j=1}^\infty$ *converges to Δ* if for each $x \in \Delta$, there exists a sequence $\{x_j: x_j \in \Delta_j\}_{j=1}^\infty$ such that for all $\epsilon > 0$, there exists $L \in \mathbb{Z}^+$ such that for $j > L$, $x_j \in \mathbb{B}_\epsilon(x)$.

In the following result, we consider Algorithm 1 with a sequence of candidate pools that converges to $\Psi \cap \mathcal{S}_\rho$. This result demonstrates that a sufficiently dense candidate pool yields identified parameters β^+ and ϕ^+ that are arbitrarily close to β_* and ϕ_* .

Theorem 2. Consider the closed-loop transfer function (1), where G_0 is known. Assume $\{H(\omega_k)\}_{k=1}^N$ is known, (A1)–(A5) are satisfied, and for all $k \in \mathcal{N}$, $\sigma(j\omega_k) = 0$. For all $j \in \mathbb{Z}^+$, let $\Lambda_j \subseteq (\Psi \cap \mathcal{S}_\rho)$ be a finite set such that $\{\Lambda_j\}_{j=1}^\infty$ converges to $\Psi \cap \mathcal{S}_\rho$. For each $j \in \mathbb{Z}^+$, let β_j^+ and ϕ_j^+ denote the identified parameters obtained from Algorithm 1 with the candidate pool $\Phi = \Lambda_j$. Then, for all $\epsilon > 0$, there exists $L \in \mathbb{Z}^+$ such that if $j > L$, then $\beta_j^+ \in \mathbb{B}_\epsilon(\beta_*)$ and $\phi_j^+ \in \mathbb{B}_\epsilon(\phi_*)$.

Proof. Let $\epsilon > 0$. Since θ is continuous on $\mathcal{S}$ and $\phi_* \in \mathcal{S}$, it follows that there exists $\delta > 0$ such that for all $x \in \mathbb{B}_\delta(\phi_*)$, $\theta(x) \in \mathbb{B}_\epsilon(\theta(\phi_*))$.

Define $\epsilon_1 \triangleq \min\{\epsilon, \delta\}$, $\Lambda_c \triangleq \overline{\Psi \cap \mathcal{S}_\rho}$, and

$$\Lambda_{\epsilon_1} \triangleq \Lambda_c \setminus \mathbb{B}_{\epsilon_1}(\phi_*) = \Lambda_c \cap \{x \in \mathbb{R}^d : \|x - \phi_*\| \geq \epsilon_1\}. \quad (16)$$

It follows from Proposition 8 that $\Lambda_c \subseteq \mathcal{S}$ is compact. Since Λ_c is compact, and $\{x \in \mathbb{R}^d : \|x - \phi_*\| \geq \epsilon_1\}$ is closed, it follows from (16) that Λ_{ϵ_1} is compact.

Proposition 6 implies that Q is continuous on $\Lambda_{\epsilon_1} \subseteq \Lambda_c \subseteq \mathcal{S}$. Next, define $Q_{\epsilon_1} \triangleq \min_{x \in \Lambda_{\epsilon_1}} Q(x)$, which exists because Q is continuous on the compact set Λ_{ϵ_1} [33, Theorem 7.7]. Assume for contradiction that $Q_{\epsilon_1} = 0$. Thus, there exists $\phi \in \Lambda_{\epsilon_1}$ such that $Q(\phi) = 0$. Since for all $k \in \mathcal{N}$, $H(\omega_k) = \tilde{G}(j\omega_k)$, it follows from (8) and (15) that $0 = Q(\phi) = J(\theta(\phi), \phi) = \sum_{k=1}^N |\tilde{\mathcal{G}}(j\omega_k, \theta(\phi), \phi) - \tilde{G}(j\omega_k)|^2$. Thus, Proposition 5 implies that $\tilde{\mathcal{G}}(s, \theta(\phi), \phi) \equiv \tilde{G}(s)$, and it follows from (A5) that $\phi = \phi_* \notin \Lambda_{\epsilon_1}$, which is a contradiction. Thus, $Q_{\epsilon_1} > 0$.

Next, since for all $k \in \mathcal{N}$, $\sigma(j\omega_k) = 0$, it follows from Proposition 4, (14), and (15) that $\beta_* = \theta(\phi_*)$ and $Q(\phi_*) = J(\beta_*, \phi_*) = 0$. Furthermore, since Q is continuous on Λ_c , it follows that there exists $\delta_1 > 0$ such that for all $x \in \Lambda_c \cap \mathbb{B}_{\delta_1}(\phi_*)$, $Q(x) < Q_{\epsilon_1}$. Since $\{\Lambda_j\}_{j=1}^\infty$ converges to $(\Psi \cap \mathcal{S}_\rho) \subseteq \Lambda_c$, it follows from Definition 1 that there exists a sequence $\{\phi_j : \phi_j \in \Lambda_j\}_{j=1}^\infty$ and $L \in \mathbb{Z}^+$ such that for all $j > L$, $\phi_j \in \mathbb{B}_{\min\{\epsilon_1, \delta_1\}}(\phi_*) \subseteq \mathbb{B}_{\delta_1}(\phi_*)$. Thus, for all $j > L$, $Q(\phi_j) < Q_{\epsilon_1}$.

Let $j \in \mathbb{Z}^+$ be such that $j > L$. It follows from Algorithm 1, (14), and (15) that $Q(\phi_j^+) \leq Q(\phi_j) < Q_{\epsilon_1}$. Assume for contradiction that $\phi_j^+ \notin \mathbb{B}_{\epsilon_1}(\phi_*)$. Therefore, $\phi_j^+ \in \Lambda_{\epsilon_1}$, which implies that $Q_{\epsilon_1} \leq Q(\phi_j^+)$, which is a contradiction. Thus, $\phi_j^+ \in \mathbb{B}_{\epsilon_1}(\phi_*) \subseteq \mathbb{B}_\epsilon(\phi_*)$. Since $\phi_j^+ \in \mathbb{B}_{\epsilon_1}(\phi_*) \subseteq \mathbb{B}_\delta(\phi_*)$, it follows that $\beta_j^+ = \theta(\phi_j^+) \in \mathbb{B}_\epsilon(\theta(\phi_*)) = \mathbb{B}_\epsilon(\beta_*)$. $\square$

5. Analysis with Noisy Frequency Response Data

In this section, we analyze Algorithm 1 under the assumption of noisy frequency response data. Define $\hat{\Omega}_0 : \mathcal{S} \times \mathbb{C}^N \rightarrow \mathbb{R}$ and $\hat{\Omega}_1 : \mathcal{S} \times \mathbb{C}^N \rightarrow \mathbb{R}^{n_1+1}$ by

$$\hat{\Omega}_0(\phi, \eta) \triangleq \sum_{k=1}^N \left| \frac{\tilde{\mathcal{N}}_2(j\omega_k, \phi)}{\tilde{\mathcal{D}}(j\omega_k, \phi)} - \tilde{G}(j\omega_k) - \eta_k \right|^2, \quad (17)$$

$$\hat{\Omega}_1(\phi, \eta) \triangleq 2\text{Re} \sum_{k=1}^N \left[\frac{\tilde{\mathcal{N}}_1(j\omega_k, \phi)}{\tilde{\mathcal{D}}(j\omega_k, \phi)} \right]^* \left[\frac{\tilde{\mathcal{N}}_2(j\omega_k, \phi)}{\tilde{\mathcal{D}}(j\omega_k, \phi)} - \tilde{G}(j\omega_k) - \eta_k \right], \quad (18)$$

where $\eta_1, \dots, \eta_N \in \mathbb{C}$ and $\eta \triangleq [\eta_1 \ \dots \ \eta_N]^\top$. Define $\sigma_* \triangleq [\sigma(j\omega_1) \ \dots \ \sigma(j\omega_N)]^\top \in \mathbb{C}^N$, and note that $\hat{\Omega}_0(\phi, \sigma_*) = \Omega_0(\phi)$ and $\hat{\Omega}_1(\phi, \sigma_*) = \Omega_1(\phi)$. Thus, $\hat{\Omega}_0$ and $\hat{\Omega}_1$ are extensions of Ω_0 and Ω_1 . Specifically, $\hat{\Omega}_0$ and $\hat{\Omega}_1$ are functions not only of the parameter ϕ but also the noise η .

Define $\hat{J}: \mathbb{R}^{n_1+1} \times \mathcal{S} \times \mathbb{C}^N \rightarrow [0, \infty)$, $\hat{\theta}: \mathcal{S} \times \mathbb{C}^N \rightarrow \mathbb{R}^{n_1+1}$, and $\hat{Q}: \mathcal{S} \times \mathbb{C}^N \rightarrow [0, \infty)$ by

$$\hat{J}(\beta, \phi, \eta) \triangleq \sum_{k=1}^N \left| \tilde{\mathcal{G}}(j\omega_k, \beta, \phi) - \tilde{G}(j\omega_k) - \eta_k \right|^2, \quad (19)$$

$$\hat{\theta}(\phi, \eta) \triangleq -\frac{1}{2} \Omega_2^{-1}(\phi) \hat{\Omega}_1(\phi, \eta), \quad (20)$$

$$\hat{Q}(\phi, \eta) \triangleq \hat{J}(\hat{\theta}(\phi, \eta), \phi, \eta). \quad (21)$$

Note that $\hat{J}(\beta, \phi, \sigma_*) = J(\beta, \phi)$, $\hat{\theta}(\phi, \sigma_*) = \theta(\phi)$, and $\hat{Q}(\phi, \sigma_*) = Q(\phi)$. Thus, $\hat{J}$, $\hat{\theta}$, and $\hat{Q}$ are extensions of J , θ , and Q . Specifically, $\hat{J}$, $\hat{\theta}$, and $\hat{Q}$ are functions not only of the parameters β and ϕ but also of the noise η .

It follows from (4), (11), and (17)–(19) that

$$\hat{J}(\beta, \phi, \eta) = \beta^T \Omega_2(\phi) \beta + \hat{\Omega}_1^T(\phi, \eta) \beta + \hat{\Omega}_0(\phi, \eta). \quad (22)$$

Furthermore, it follows from (14), (15), (19)–(21), and Proposition 4 that

$$\hat{\theta}(\phi_*, 0) = -\frac{1}{2} \Omega_2^{-1}(\phi_*) \hat{\Omega}_1(\phi_*, 0) = \beta_*, \quad (23)$$

$$\hat{Q}(\phi_*, 0) = \hat{J}(\beta_*, \phi_*, 0) = 0. \quad (24)$$

The following result is an extension of Proposition 6. The proof is in Appendix C.

Proposition 9. Assume (A1)–(A3) are satisfied. Then, $\hat{\theta}$ and $\hat{Q}$ are continuous on $\mathcal{S} \times \mathbb{C}^N$.

5.1. ϕ_* in the candidate pool Φ

The following result provides sufficient conditions such that the identified parameter ϕ^+ equals ϕ_* . This result also shows that if the norm of the noise σ_* is sufficiently small, then the identified parameter β^+ is arbitrarily close to β_* .

Theorem 3. Consider the closed-loop transfer function (1), where G_0 is known. Assume $\{H(\omega_k)\}_{k=1}^N$ is known and (A1)–(A5) are satisfied. Furthermore, consider Algorithm 1 with the candidate pool $\Phi \subseteq (\Psi \cap \mathcal{S})$, and assume $\phi_* \in \Phi$. Let β^+ and ϕ^+ denote the identified parameters obtained from Algorithm 1. Then, there exists $\delta_0 > 0$ such that if $\|\sigma_*\| < \delta_0$, then $\phi^+ = \phi_*$. Furthermore, for all $\epsilon > 0$, there exists $\delta \in (0, \delta_0)$ such that if $\|\sigma_*\| < \delta$, then $\beta^+ \in \mathbb{B}_\epsilon(\beta_*)$.

Proof. Let $\phi \in \Phi \setminus \{\phi_*\}$, and assume for contradiction that $\hat{Q}(\phi, 0) = 0$. It follows from (19) and (21) that $0 = \hat{Q}(\phi, 0) = \hat{J}(\hat{\theta}(\phi, 0), \phi, 0) = \sum_{k=1}^N |\tilde{\mathcal{G}}(j\omega_k, \hat{\theta}(\phi, 0), \phi) - \tilde{G}(j\omega_k)|^2$. Thus, Proposition 5 implies that $\tilde{\mathcal{G}}(s, \hat{\theta}(\phi, 0), \phi) \equiv \tilde{G}(s)$, and it follows from (A5) that $\phi = \phi_*$, which is a contradiction. Therefore, $\hat{Q}(\phi, 0) > 0$.

Define $U \triangleq \min_{x \in \Phi \setminus \{\phi_*\}} \hat{Q}(x, 0) > 0$. Since $\hat{Q}$ is continuous on $\mathcal{S} \times \mathbb{C}^N$, it follows that for each $i \in \mathcal{M}$, $\hat{Q}(\phi_i, \cdot)$ is continuous on $\mathbb{C}^N$. Thus, for each $i \in \mathcal{M}$, there exists $\delta_i > 0$ such that for all $\eta \in \{x \in \mathbb{C}^N: \|x\| < \delta_i\}$,

$$|\hat{Q}(\phi_i, \eta) - \hat{Q}(\phi_i, 0)| < U/2. \quad (25)$$

Define $\delta_0 \triangleq \min_{i \in \mathcal{M}} \delta_i > 0$, and assume that $\|\sigma_*\| < \delta_0$. Since $\phi_* \in \Phi$, it follows that there exists $m \in \mathcal{M}$ such that $\phi_m = \phi_*$. Since $\hat{Q}(\phi_m, 0) = \hat{Q}(\phi_*, 0) = 0$, it follows from (25) that $\hat{Q}(\phi_m, \sigma_*) = |\hat{Q}(\phi_m, \sigma_*) - \hat{Q}(\phi_m, 0)| < U/2$. Let $j \in \mathcal{M} \setminus \{m\}$. It follows from (25) that $-U/2 < \hat{Q}(\phi_j, \sigma_*) - \hat{Q}(\phi_j, 0)$, which implies that $\hat{Q}(\phi_j, \sigma_*) > \hat{Q}(\phi_j, 0) - U/2$. Since, in addition, $\hat{Q}(\phi_j, 0) \geq U$, it follows that $\hat{Q}(\phi_j, \sigma_*) > U/2$. Therefore, $\hat{Q}(\phi_m, \sigma_*) < \hat{Q}(\phi_j, \sigma_*)$,

which implies that $Q(\phi_m) < Q(\phi_j)$. Thus, (13), (14), (15), and Proposition 3 imply that $\mathcal{J}_m(\beta_m) < \mathcal{J}_j(\beta_j)$. Therefore, it follows from Algorithm 1 that $\phi^+ = \phi_m = \phi_*$ and

$$\beta^+ = \beta_m = \theta(\phi_*) = \hat{\theta}(\phi_*, \sigma_*). \quad (26)$$

Let $\epsilon > 0$. Since $\hat{\theta}$ is continuous on $\mathcal{S} \times \mathbb{C}^N$, it follows that $\hat{\theta}(\phi_*, \cdot)$ is continuous on $\mathbb{C}^N$. Therefore, there exists $\delta \in (0, \delta_0)$ such that for all $\eta \in \{x \in \mathbb{C}^N : \|x\| < \delta\}$, $\hat{\theta}(\phi_*, \eta) \in \mathbb{B}_\epsilon(\hat{\theta}(\phi_*, 0))$. Finally, assume $\|\sigma_*\| < \delta$, and (23) and (26) imply that $\beta^+ \in \mathbb{B}_\epsilon(\beta_*)$. $\square$

5.2. ϕ_* not necessarily in the candidate pool Φ

We now extend the analysis in the previous section to address the case where ϕ_* is not necessarily in the candidate pool. In the following result, we consider Algorithm 1 with a sequence of candidate pools that converges to $\Psi \cap \mathcal{S}_\rho$. This result demonstrates that a sufficiently dense candidate pool and sufficiently small noise σ_* yield identified parameters β^+ and ϕ^+ that are arbitrarily close to β_* and ϕ_* .

Theorem 4. Consider the closed-loop transfer function (1), where G_0 is known. Assume $\{H(\omega_k)\}_{k=1}^N$ is known and (A1)–(A5) are satisfied. For all $j \in \mathbb{Z}^+$, let $\Lambda_j \subseteq (\Psi \cap \mathcal{S}_\rho)$ be a finite set such that $\{\Lambda_j\}_{j=1}^\infty$ converges to $\Psi \cap \mathcal{S}_\rho$. For each $j \in \mathbb{Z}^+$, let β_j^+ and ϕ_j^+ denote the identified parameters obtained from Algorithm 1 with the candidate pool $\Phi = \Lambda_j$. Then, for all $\epsilon > 0$, there exist $\delta > 0$ and $L \in \mathbb{Z}^+$ such that if $\|\sigma_*\| < \delta$ and $j > L$, then $\beta_j^+ \in \mathbb{B}_\epsilon(\beta_*)$ and $\phi_j^+ \in \mathbb{B}_\epsilon(\phi_*)$.

Proof. Let $\epsilon > 0$. Since $\hat{\theta}$ is continuous on $\mathcal{S} \times \mathbb{C}^N$ and $\phi_* \in \mathcal{S}$, it follows that there exists $\delta_0 > 0$ such that for all $\phi \in \mathbb{B}_{\delta_0}(\phi_*)$ and all $\eta \in \{x \in \mathbb{C}^N : \|x\| < \delta_0\}$,

$$\hat{\theta}(\phi, \eta) \in \mathbb{B}_\epsilon(\hat{\theta}(\phi_*, 0)). \quad (27)$$

Define $\epsilon_1 \triangleq \min\{\epsilon, \delta_0\}$, $\Lambda_c \triangleq \overline{\Psi \cap \mathcal{S}_\rho}$, and

$$\Lambda_{\epsilon_1} \triangleq \Lambda_c \setminus \mathbb{B}_{\epsilon_1}(\phi_*) = \Lambda_c \cap \{x \in \mathbb{R}^d : \|x - \phi_*\| \geq \epsilon_1\}. \quad (28)$$

It follows from Proposition 8 that $\Lambda_c \subseteq \mathcal{S}$ is compact. Since Λ_c is compact, and $\{x \in \mathbb{R}^d : \|x - \phi_*\| \geq \epsilon_1\}$ is closed, it follows from (28) that Λ_{ϵ_1} is compact.

Let $v > \delta_0$, and define $V \triangleq \{x \in \mathbb{C}^N : \|x\| \leq v\}$. Since $\Lambda_{\epsilon_1} \subseteq \Lambda_c \subseteq \mathcal{S}$ and $V \subseteq \mathbb{C}^N$, it follows from Proposition 9 that $\hat{Q}$ is continuous on $\Lambda_{\epsilon_1} \times V$. Next, define $\Theta : V \rightarrow [0, \infty)$ by $\Theta(\eta) \triangleq \min_{\phi \in \Lambda_{\epsilon_1}} \hat{Q}(\phi, \eta)$, which exist because Λ_{ϵ_1} is compact and $\hat{Q}$ is continuous on $\Lambda_{\epsilon_1} \times V$ [33, Theorem 7.7].

Assume for contradiction that $\Theta(0) = 0$. Thus, there exists $z \in \Lambda_{\epsilon_1}$ such that $\hat{Q}(z, 0) = 0$, and it follows from (19) and (21) that $0 = \hat{J}(\hat{\theta}(z, 0), z, 0) = \sum_{k=1}^N |\hat{\mathcal{G}}(\mathcal{J}\omega_k, \hat{\theta}(z, 0), z) - \tilde{G}(\mathcal{J}\omega_k)|^2$. Thus, Proposition 5 implies that $\tilde{\mathcal{G}}(s, \hat{\theta}(z, 0), z) \equiv \tilde{G}(s)$, and it follows from (A5) that $z = \phi_* \notin \Lambda_{\epsilon_1}$, which is a contradiction. Thus, $\Theta(0) > 0$.

Since $\hat{Q}$ is continuous on $\Lambda_{\epsilon_1} \times V$, and Λ_{ϵ_1} and V are compact, it follows from [34, Theorem 9.14] that Θ is continuous on V . Furthermore, since $\hat{Q}$ is continuous on $\mathcal{S} \times \mathbb{C}^N$, it follows that $\hat{Q}(\phi_*, \cdot)$ is continuous on V . Thus, $W : V \rightarrow \mathbb{R}$ defined by $W(\eta) \triangleq \Theta(\eta) - \hat{Q}(\phi_*, \eta)$ is continuous on V . It follows from (24) that $W(0) = \Theta(0) - \hat{Q}(\phi_*, 0) = \Theta(0) > 0$. Since, in addition, W is continuous on V , it follows that there exists $\delta_1 \in (0, v)$ such that for all $\eta \in \{x \in \mathbb{C}^N : \|x\| < \delta_1\}$, $W(\eta) > 0$. Define $\delta \triangleq \min\{\delta_0, \delta_1\} > 0$ and assume $\|\sigma_*\| < \delta$.

Since $W(\sigma_*) > 0$ and Proposition 9 implies that $\hat{Q}(\cdot, \sigma_*)$ is continuous on Λ_c , it follows from the continuity of $\hat{Q}(\cdot, \sigma_*)$ that there exists $\delta_2 > 0$ such that for all $\phi \in (\Lambda_c \cap \mathbb{B}_{\delta_2}(\phi_*))$, $|\hat{Q}(\phi, \sigma_*) - \hat{Q}(\phi_*, \sigma_*)| < W(\sigma_*)$. Thus, for all $\phi \in (\Lambda_c \cap \mathbb{B}_{\delta_2}(\phi_*))$, $\hat{Q}(\phi, \sigma_*) - \hat{Q}(\phi_*, \sigma_*) \leq |\hat{Q}(\phi, \sigma_*) - \hat{Q}(\phi_*, \sigma_*)| < W(\sigma_*) = \Theta(\sigma_*) - \hat{Q}(\phi_*, \sigma_*)$, which implies that

$$\hat{Q}(\phi, \sigma_*) < \Theta(\sigma_*). \quad (29)$$

Since $\{\Lambda_j\}_{j=1}^\infty$ converges to $(\Psi \cap \mathcal{S}_\rho) \subseteq \Lambda_c$, it follows from Definition 1 that there exists a sequence $\{\phi_j: \phi_j \in \Lambda_j\}_{j=1}^\infty$ and $L \in \mathbb{Z}^+$ such that for all $j > L$, $\phi_j \in \mathbb{B}_{\min\{\epsilon_1, \delta_2\}}(\phi_*)$. Thus, it follows from (29) that for all $j > L$, $\hat{Q}(\phi_j, \sigma_*) < \Theta(\sigma_*)$.

Let $j \in \mathbb{Z}^+$ be such that $j > L$. It follows from Algorithm 1, (14), and (15) that $Q(\phi_j^+) \leq Q(\phi_j)$. Therefore, (29) implies that $\hat{Q}(\phi_j^+, \sigma_*) \leq \hat{Q}(\phi_j, \sigma_*) < \Theta(\sigma_*)$. Assume for contradiction that $\phi_j^+ \notin \mathbb{B}_{\epsilon_1}(\phi_*)$. Therefore, $\phi_j^+ \in \Lambda_{\epsilon_1}$, which implies that $\Theta(\sigma_*) = \min_{\phi \in \Lambda_{\epsilon_1}} \hat{Q}(\phi, \sigma_*) \leq \hat{Q}(\phi_j^+, \sigma_*)$, which is a contradiction. Thus, $\phi_j^+ \in \mathbb{B}_{\epsilon_1}(\phi_*) \subseteq \mathbb{B}_\epsilon(\phi_*)$. Since $\phi_j^+ \in \mathbb{B}_{\epsilon_1}(\phi_*) \subseteq \mathbb{B}_{\delta_0}(\phi_*)$, it follows from (23) and (27) that $\beta_j^+ = \hat{\theta}(\phi_j^+, \sigma_*) \in \mathbb{B}_\epsilon(\hat{\theta}(\phi_*, 0)) = \mathbb{B}_\epsilon(\beta_*)$. $\square$

6. Numerical Examples

For all examples in this section, let $G_0(s) = 4/(s+2)$, $G_1(s) = (2.1s+3)/(s+6.5)$, and $G_2(s) = 5.4/(s+7.1)$, which implies that $\beta_* = [2.1 \ 3]^\top$ and $\phi_* = [6.5 \ 5.4 \ 7.1]^\top$. Let $\Psi = [-8, 8] \times [-8, 8] \times [-8, 8]$, which is a compact and perfect set containing ϕ_* . We now show that this example satisfies (A5). Let $\beta = [u_1 \ u_2]^\top \in \mathbb{R}^2$ and $\phi = [v_1 \ v_2 \ v_3]^\top \in \mathbb{R}^3$, and assume $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{G}(s)$, where $\tilde{G}$ is given by (1). It follows that

$$\tilde{\mathcal{G}}(s, \beta, \phi) = \frac{4[(u_1s + u_2)(s + v_3) + v_2(s + v_1)]}{(s + v_1)[(s + 2)(s + v_3) + 4v_2]} = \frac{4(2.1s^2 + 23.31s + 56.4)}{(s + 6.5)(s^2 + 9.1s + 35.8)}. \quad (30)$$

Since the roots of $s^2 + 9.1s + 35.8$ are complex, (30) implies that $v_1 = 6.5$ and $(s + 2)(s + v_3) + 4v_2 = s^2 + 9.1s + 35.8$, or equivalently, $v_3 = 7.1$ and $v_2 = 5.4$. Moreover, the numerator of (30) implies that $(u_1s + u_2)(s + v_3) + v_2(s + v_1) = 2.1s^2 + 23.31s + 56.4$, or equivalently, $u_1 = 2.1$ and $u_2 = 3$. Thus, $\beta = \beta_*$ and $\phi = \phi_*$.

For all examples in this section, let $N = 20$ and $\omega_k = 0.2\pi k$, where $k \in \mathbb{N}$.

Example 2. *Noiseless data and $\phi_* \in \Phi$.* Assume $\sigma(j\omega_k) \equiv 0$. Consider $\Pi_0 \triangleq \{\phi \in \mathbb{R}^3: E_1\phi, E_2\phi, E_3\phi \in \{-8 + 0.1k\}_{k=0}^{160}\} \subseteq \Psi$, define $\Lambda_0 \triangleq \Pi_0 \cap \mathcal{S}$, and note that $\phi_* \in \Lambda_0$. Algorithm 1 is used with the candidate pool $\Phi = \Lambda_0$ to obtain β^+ and ϕ^+ . The identified parameters are $\beta^+ = \beta_* = [2.1 \ 3]^\top$ and $\phi^+ = \phi_* = [6.5 \ 5.4 \ 7.1]^\top$, which agrees with Theorem 1. $\triangle$

Example 3. *Noiseless data and $\phi_* \notin \Phi$.* Assume $\sigma(j\omega_k) \equiv 0$. For $j = 1, \dots, 25$, consider

$$\Pi_j \triangleq \left\{ \phi \in \mathbb{R}^3: E_1\phi, E_2\phi, E_3\phi \in \left\{ -8 + \frac{16}{5 + 10(j-1)}k \right\}_{k=0}^{5+10(j-1)} \right\} \subseteq \Psi,$$

define $\Lambda_j \triangleq \Pi_j \cap \mathcal{S}_\rho$, where $\rho = -0.001$, and note that for $j = 1, \dots, 25$, $\phi_* \notin \Lambda_j$.

For $j = 1, \dots, 25$, Algorithm 1 is used with the candidate pool $\Phi = \Lambda_j$ to obtain the identified parameters β_j^+ and ϕ_j^+ . Figure 3 demonstrates that $\|\beta_j^+ - \beta_*\|_2$ and $\|\phi_j^+ - \phi_*\|_2$ are arbitrarily small for sufficient large j , which agrees with Theorem 2. Note that $\|\beta_j^+ - \beta_*\|_2$ and $\|\phi_j^+ - \phi_*\|_2$ do not decrease monotonically. The Bode plots of G_1 and G_2 , and the identified transfer functions G_1^+ and G_2^+ with $\Phi = \Lambda_1$, $\Phi = \Lambda_2$, and $\Phi = \Lambda_{25}$, are shown in Figure 4. The identified transfer functions G_1^+ and G_2^+ with $\Phi = \Lambda_{25}$ approximate G_1 and G_2 better than those with $\Phi = \Lambda_1$ and $\Phi = \Lambda_2$. $\triangle$

Example 4. *Noisy data and $\phi_* \in \Phi$.* For $i = 1, \dots, 20$, let $n_{r,i}(t)$ and $n_{y,i}(t)$ be zero-mean Gaussian white-noise realizations with variance of 4^{-i} . Moreover, for $i = 1, \dots, 20$, let $\sigma_{r,i}(j\omega)$ and $\sigma_{y,i}(j\omega)$ be the Fourier transforms of $n_{r,i}$ and $n_{y,i}$, respectively. For $i = 1, \dots, 20$, the noisy closed-loop frequency response data is $H_i(\omega_k) \triangleq \tilde{G}(j\omega_k) + \sigma_i(j\omega_k)$, where

Figure 3: *Noiseless data and $\phi_* \notin \Phi$.* For $j = 1, \dots, 25$, Algorithm 1 is used with the candidate pool Λ_j to obtain the identified parameters β_j^+ and ϕ_j^+ . Note that $\|\beta_j^+ - \beta_*\|_2$ and $\|\phi_j^+ - \phi_*\|_2$ can be made arbitrarily small if the candidate pool is sufficiently dense.

Figure 4: *Noiseless data and $\phi_* \notin \Phi$.* For $j = 1, 2, 25$, Algorithm 1 is used with the candidate pool Λ_j to obtain the identified transfer functions G_1^+ and G_2^+ . Note that G_1^+ and G_2^+ with candidate pool Λ_{25} approximate G_1 and G_2 better than those with candidate pools Λ_1 and Λ_2 .

$\sigma_i(j\omega_k) \triangleq [\tilde{G}(j\omega_k)\sigma_{r,i}(j\omega_k) + \sigma_{y,i}(j\omega_k)]/r(j\omega_k)$. For $i = 1, \dots, 20$, define $R_i \triangleq \frac{1}{N} \sum_{k=1}^N \left| \frac{\sigma_i(j\omega_k)}{\tilde{G}(j\omega_k)} \right|$, which is the frequency-averaged noise-to-signal ratio. In this example, for $i = 1, \dots, 20$, $R_i \in (0, 24)$. Specifically, $R_1 = 23.8$, $R_5 = 1.39$, $R_{10} = 4.49 \times 10^{-2}$, $R_{12} = 9.53 \times 10^{-3}$, and $R_{20} = 4.36 \times 10^{-5}$.

For $i = 1, \dots, 20$, Algorithm 1 is used with the candidate pool $\Phi = \Lambda_0$ and data $\{H_i(\omega_k)\}_{k=1}^N$ to obtain the identified parameters β_i^+ and ϕ_i^+ . Figure 5 demonstrates that for $i \geq 14$, $\phi_i^+ = \phi_*$ and for sufficiently large i , $\|\beta_i^+ - \beta_*\|_2$ is arbitrarily small, which agrees with Theorem 3. The Bode plots of G_1 and G_2 , and G_1^+ and G_2^+ with data $\{H_5(\omega_k)\}_{k=1}^N$, $\{H_{10}(\omega_k)\}_{k=1}^N$, and $\{H_{20}(\omega_k)\}_{k=1}^N$ are shown in Figure 6. The identified transfer functions G_1^+ and G_2^+ with data $\{H_{20}(\omega_k)\}_{k=1}^N$ approximate G_1 and G_2 better than those with data $\{H_5(\omega_k)\}_{k=1}^N$ and $\{H_{10}(\omega_k)\}_{k=1}^N$. The identified transfer function G_2^+ with data $\{H_{20}(\omega_k)\}_{k=1}^N$ is $G_2^+ = G_2$. $\triangle$

Figure 5: *Noisy data and $\phi_* \in \Phi$.* For $i = 1, \dots, 20$, Algorithm 1 is used with the candidate pool Λ_0 and data $\{H_i(\omega_k)\}_{k=1}^N$ to obtain the identified parameters β_i^+ and ϕ_i^+ . For $i \geq 14$, $\phi_i^+ = \phi_*$. Note that $\|\beta_i^+ - \beta_*\|_2$ can be made arbitrarily small if the norm of the noise σ_* is sufficiently small.

Figure 6: *Noisy data and $\phi_* \in \Phi$.* For $i = 5, 10, 20$, Algorithm 1 is used with the candidate pool Λ_0 and data $\{H_i(\omega_k)\}_{k=1}^N$ to obtain the identified transfer functions G_1^+ and G_2^+ . Note that G_1^+ and G_2^+ with $\{H_{20}(\omega_k)\}_{k=1}^N$ approximate G_1 and G_2 better than those with $\{H_5(\omega_k)\}_{k=1}^N$ and $\{H_{10}(\omega_k)\}_{k=1}^N$.

Example 5. *Noisy data and $\phi_* \notin \Phi$.* Consider the noisy closed-loop frequency response data $\{H_5(\omega_k)\}_{k=1}^N$, $\{H_{10}(\omega_k)\}_{k=1}^N$, $\{H_{12}(\omega_k)\}_{k=1}^N$, and $\{H_{20}(\omega_k)\}_{k=1}^N$ given in Example 4. Moreover, consider $\Lambda_1, \dots, \Lambda_{25}$ given in Example 3. For $i = 5, 10, 12, 20$, and $j = 1, \dots, 25$, Algorithm 1 is used with the candidate pool $\Phi = \Lambda_j$ and data $\{H_i(\omega_k)\}_{k=1}^N$ to obtain the identified parameters $\beta_{j,i}^+$ and $\phi_{j,i}^+$. Figure 7 demonstrates that $\|\beta_{j,i}^+ - \beta_*\|_2$ and $\|\phi_{j,i}^+ - \phi_*\|_2$ are arbitrarily small for sufficient large j and i , which agrees with Theorem 4. $\triangle$

Figure 7: *Noisy data and $\phi_* \notin \Phi$.* For $i = 5, 10, 12, 20$, and $j = 1, \dots, 25$, Algorithm 1 is used with the candidate pool Λ_j and data $\{H_i(\omega_k)\}_{k=1}^N$ to obtain the identified parameters $\beta_{j,i}^+$ and $\phi_{j,i}^+$. Note that $\|\beta_{j,i}^+ - \beta_*\|_2$ and $\|\phi_{j,i}^+ - \phi_*\|_2$ are small for large j and i .

7. Application to Modeling Human Control Behavior

In this section, we apply Algorithm 1 to data from an experiment in which human subjects learn to interact with a dynamic system. To demonstrate the SSID method, we use data for a single subject and model that subject's feedback and feedforward control behavior. In the experiment, a total of ten subjects, who are 18–35 years of age and have no known neurological or motor-control disorders, interact with a dynamic system 40 times over a 2-week period. During each interaction, a subject uses a single-degree-of-freedom joystick to control the motion of an object that is displayed on a computer screen. The object's position y_t is the output of a dynamic system G_0 , and the joystick position u_t is the input to the dynamic system G_0 . The subject has no initial knowledge of the dynamics G_0 between u_t and y_t . A reference object, whose position r_t is independent of u_t , is also displayed on the computer screen. The feedback y_t and feedforward r_t are provided visually on the computer screen, using sliding needle displays as shown in Figure 8. The subject's objective is to move the joystick and thus generate a control u_t that minimizes the magnitude of the command-following error $e_t = r_t - y_t$.

This experiment satisfies the U.S. Department of Health and Human Services Code of Federal Regulation for human subject research (45 CFR 46) and was approved by the University of Kentucky Institutional Review Board (IRB number 12-0816-P4S). The experimental methods are related to those of reaching experiments (e.g., [6, 7, 10]), which are used extensively to study human motor learning. We also note that computer-based experiments with visual feedback have been used to study motor learning [20, 30, 31]. For example, in [20], a subject also uses a joystick to control the position of an object displayed on a computer

screen. However, [20] examines feedback behavior only and is restricted to plants that are either static or have pure integrator dynamics.

Figure 8: Subjects receive visual feedback y_t and feedforward r_t .

In this experiment, the plant G_0 is the two-degree-of-freedom mass-spring-dashpot system shown in Figure 9, where the output y_t represents the position of the second mass, and the control u_t represents the force applied to the second mass. The transfer function is $G_0(s) = (m_1 s^2 + c_1 s + c_2 s + k_1 + k_2) / [(m_1 s^2 + c_1 s + k_1)(m_2 s^2 + c_2 s + k_2) + m_2 c_2 s^3 + m_2 k_2 s^2]$, where $m_1 = 1$, $m_2 = 0.5$, $c_1 = 0.56\pi$, $c_2 = 0.5\pi$, $k_1 = 0.16\pi^2$, and $k_2 = 0.5\pi^2$.

The command r_t is an 80-second chirp with frequency content between 0.1 and 0.4 Hz as shown in Figure 10. The frequency range of r_t is selected to account for the bandwidth limitations of human motion. Humans cannot replicate arbitrarily high-frequency motion, and the 0.4 Hz upper limit is selected to be within the range of motion that a human can replicate. In addition, the frequency range is selected to include the natural frequencies associated with the zeros and dominant poles of G_0 .

Figure 9: The two-degree-of-freedom mass-spring-dashpot system.

Figure 10: The reference r_t and output y_t for a single trial.

To demonstrate the SSID algorithm, we consider a single trial out of the 400 total trials (10 subjects and 40 trials per subject). In particular, we select the trial in which the time-averaged magnitude of the command-following error $\frac{1}{80} \int_0^{80} |e_t(t)| dt$ is minimized. The output y_t for this trial is shown in Figure 10. The subject's control strategy is modeled by $u(s) = G_2(s)e(s) + G_1(s)r(s)$, where e, r, u , and y are the Laplace transforms of e_t, r_t, u_t , and y_t , respectively. The closed-loop transfer function from r to y is given by (1). The time domain signals r_t and y_t are divided into two segments of 40 seconds. For each pair of

signal segments, we use the discrete Fourier transform to calculate frequency response data at the frequencies $\omega_k = 2\pi(0.1 + 0.025(k-1))$, where $k = 1, 2, \dots, N = 13$. The frequency response data from the two segments is averaged to obtain $\{H(\omega_k)\}_{k=1}^N$. We use Algorithm 1 and the data $\{H(\omega_k)\}_{k=1}^N$ to model the subject's the feedforward G_1^+ and feedback G_2^+ controllers. We identify the best-fit second-order strictly proper feedback controller G_2^+ and second-order improper feedforward controller G_1^+ , where $n_1 = 4$, $n_2 = 1$, $d_1 = 2$, and $d_2 = 2$. The selected orders are sufficiently large to admit different control strategies that make $|e_t|$ small. For example, the selected orders allow for dynamic inversion in feedforward or high gain in feedback.

We choose a candidate pool such that the magnitude of poles of the feedforward controller, and the poles and zeros of the feedback controller are less than 25 rad/s, because poles and zeros with magnitude greater than 25 rad/s have negligible effect on the Bode plot over the 0.1-to-0.4 Hz frequency range of the chirp. The candidate pool is also selected such that the peak magnitude of the feedback controller is less than 30 dB, which is an estimate of the peak gain that a human can use in feedback. Finally, we let $\rho = -0.1$. The candidate pool Φ contains approximately 160 billion elements and captures a wide range of second-order behavior over the frequency range of the chirp.

Algorithm 1 is implemented on the Lipscomb High Performance Computing Cluster at University of Kentucky. It takes approximately 3.5 hours to run Algorithm 1 using 32 Intel E5-2670@2.6Ghz cores.

Figure 11 shows the Bode plots of the identified transfer functions G_1^+ and G_2^+ , and the corresponding closed-loop transfer function $\tilde{G}^+(s) \triangleq [G_0(s)G_1^+(s) + G_0(s)G_2^+(s)]/[1 + G_0(s)G_2^+(s)]$. The identified feedforward controller G_1^+ approximates G_0^{-1} over the frequency range of r_t , which suggests that this subject learned to control the system by using the inverse dynamics G_0^{-1} in feedforward.

Figure 11: Bode plots of G_1^+ , G_2^+ , and $\tilde{G}^+$ for one trial with the minimized time-averaged magnitude of the command-following error. Note that G_1^+ approximates G_0^{-1} .

8. Conclusion

In this paper, we presented a frequency-domain SSID algorithm for identifying unknown feedback and feedforward subsystems interconnected with a known subsystem. This SSID method ensures asymptotic stability of the identified closed-loop transfer function. The method has application to modeling human control behavior (both feedback and feedforward). We demonstrated the algorithm on data from one trial of an experiment in which human subjects interacted with a dynamic system. For this trial, the identified feedforward controller approximated the inverse plant dynamics. In this case, the identified model of the control behavior agrees with the internal model hypothesis.

The main analytic results of the paper are Theorems 1–4, which describe the properties of the SSID algorithm. In particular, Theorem 4 shows that the coefficients of the identified feedback and feedforward transfer functions are arbitrarily close to the true coefficients if the candidate pool is sufficiently dense and the data noise is sufficiently small.

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Appendix A. Proofs of Propositions 1 and 2

Proof of Proposition 1. Let $\phi \in \mathcal{S}$. It follows from (11) that $\Omega_2(\phi)$ is well defined and positive semidefinite. Next, assume for contradiction that $\Omega_2(\phi)$ is not positive definite. Thus, there exists a nonzero $x \in \mathbb{R}^{n_1+1}$ such that $x^T \Omega_2(\phi) x = 0$, and it follows from (11) that $0 = x^T \Omega_2(\phi) x = \sum_{k=1}^N \left| \frac{\tilde{N}_1(j\omega_k, \phi)x}{\tilde{D}(j\omega_k, \phi)} \right|^2$. Thus, for all $k \in \mathcal{N}$, $\tilde{N}_1(j\omega_k, \phi)x = 0$. Next, it follows from (A3) that $N > d_0 + d_1 + d_2 + n_0 + \max\{n_1 + d_2, n_2 + d_1\} \geq d_2 + n_0 + n_1$. Since, $\tilde{N}_1(j\omega_1, \phi)x = \dots = \tilde{N}_1(j\omega_N, \phi)x = 0$, $\tilde{N}_1(s, \phi)x$ is a degree $d_2 + n_0 + n_1$ polynomial, and $N > d_2 + n_0 + n_1$, it follows that $\tilde{N}_1(s, \phi)x \equiv 0$. Moreover, since $\tilde{N}_1(s, \phi)x \equiv 0$, $N_0(s) \not\equiv 0$ and $s^{d_2} + \Gamma_{d_2-1}^T(s)E_3\phi \not\equiv 0$, it follows from (6) that $\Gamma_{n_1}^T(s)x \equiv 0$. Thus, $x = 0$, which is a contradiction. Therefore, $\Omega_2(\phi)$ is positive definite. $\square$

Proof of Proposition 2. Let $\phi \in \mathcal{S}$, and Proposition 1 implies that $\Omega_2(\phi)$ is positive definite. Define $x \triangleq -\frac{1}{2}\Omega_2^{-1}(\phi)\Omega_1(\phi) \in \mathbb{R}^{n_1+1}$, and let $\beta \in \mathbb{R}^{n_1+1}$ be such that $\beta \neq x$. Thus, (12) implies that

$$J(\beta, \phi) = \beta^T \Omega_2(\phi) \beta + \Omega_1^T(\phi) \beta + \Omega_0(\phi) = [\beta - x]^T \Omega_2(\phi) [\beta - x] + J(x, \phi). \quad (31)$$

Since $\Omega_2(\phi)$ is positive definite, it follows that $[\beta - x]^T \Omega_2(\phi) [\beta - x] > 0$, and (31) confirms the result. $\square$

Appendix B. Proofs of Propositions 4 and 5

Proof of Proposition 4. Since for all $k \in \mathcal{N}$, $\sigma(j\omega_k) = 0$, it follows that $H(\omega_k) = \tilde{G}(j\omega_k) = \tilde{\mathcal{G}}(j\omega_k, \beta_*, \phi_*)$. Thus, it follows from (4) that for all $k \in \mathcal{N}$,

$$\frac{\tilde{N}_2(j\omega_k, \phi_*)}{\tilde{D}(j\omega_k, \phi_*)} - H(\omega_k) = \frac{\tilde{N}_2(j\omega_k, \phi_*)}{\tilde{D}(j\omega_k, \phi_*)} - \tilde{\mathcal{G}}(j\omega_k, \beta_*, \phi_*) = -\frac{\tilde{N}_1(j\omega_k, \phi_*)\beta_*}{\tilde{D}(j\omega_k, \phi_*)}. \quad (32)$$

Next, substituting (32) into (10), and using (11) yields $\Omega_1(\phi_*) = -2\Omega_2(\phi_*)\beta_*$. Since $\phi_* \in \mathcal{S}$, Proposition 1 implies that $\Omega_2(\phi_*)$ is positive definite, and thus, $\beta_* = -\frac{1}{2}\Omega_2^{-1}(\phi_*)\Omega_1(\phi_*)$. Moreover, (8) implies that $J(\beta_*, \phi_*) = \sum_{k=1}^N |\tilde{\mathcal{G}}(j\omega_k, \beta_*, \phi_*) - H(\omega_k)|^2 = 0$. $\square$

Proof of Proposition 5. Let $\beta \in \mathbb{R}^{n_1+1}$ and $\phi \in \mathcal{S}$. Since $\sum_{k=1}^N |\tilde{\mathcal{G}}(j\omega_k, \beta, \phi) - \tilde{G}(j\omega_k)| = 0$, it follows that for all $k \in \mathcal{N}$, $\tilde{\mathcal{G}}(j\omega_k, \beta, \phi) = \tilde{G}(j\omega_k)$. Define $\tilde{\mathcal{N}}(s) \triangleq \tilde{\mathcal{N}}_1(s, \phi)\beta + \tilde{\mathcal{N}}_2(s, \phi)$, $\tilde{\mathcal{N}}_*(s) \triangleq \tilde{\mathcal{N}}_1(s, \phi_*)\beta_* + \tilde{\mathcal{N}}_2(s, \phi_*)$, and $\psi(s) \triangleq \tilde{\mathcal{D}}(s, \phi_*)\tilde{\mathcal{N}}(s) - \tilde{\mathcal{D}}(s, \phi)\tilde{\mathcal{N}}_*(s)$. Since $\deg \tilde{\mathcal{N}} \leq n_0 + \max\{n_1 + d_2, n_2 + d_1\}$, $\deg \tilde{\mathcal{N}}_* \leq n_0 + \max\{n_1 + d_2, n_2 + d_1\}$, and $\deg \tilde{\mathcal{D}}(s, \phi) = \deg \tilde{\mathcal{D}}(s, \phi_*) = d_0 + d_1 + d_2$, it follows that $\deg \psi \leq d_0 + d_1 + d_2 + n_0 + \max\{n_1 + d_2, n_2 + d_1\}$. Since for all $k \in \mathcal{N}$, $\tilde{\mathcal{G}}(j\omega_k, \beta, \phi) = \tilde{G}(j\omega_k)$, it follows that for all $k \in \mathcal{N}$, $\psi(j\omega_k) = 0$. Next, it follows from (A3) that $\deg \psi < N$. Since for all $k \in \mathcal{N}$, $\psi(j\omega_k) = 0$, and $\deg \psi < N$, it follows that $\psi(s) \equiv 0$, which implies that $\tilde{\mathcal{D}}(s, \phi_*)\tilde{\mathcal{N}}(s) \equiv \tilde{\mathcal{D}}(s, \phi)\tilde{\mathcal{N}}_*(s)$. Thus, $\tilde{\mathcal{G}}(s, \beta, \phi) \equiv \tilde{\mathcal{G}}(s, \beta_*, \phi_*) \equiv \tilde{G}(s)$. $\square$

Appendix C. Proofs of Propositions 6 and 9

Proof of Proposition 6. Let $\phi \in \mathcal{S}$, and it follows that $\tilde{\mathcal{D}}(s, \phi)$ is Hurwitz. Thus, for all $k \in \mathcal{N}$, $\tilde{\mathcal{D}}(j\omega_k, \phi) \neq 0$, and it follows from (9)–(11) that Ω_0 , Ω_1 , and Ω_2 are well-defined and continuous on $\mathcal{S}$.

Next, it follows from Proposition 1 that $\Omega_2(\phi)$ is positive definite. Since Ω_2 is continuous on $\mathcal{S}$, it follows that each element of Ω_2 is continuous on $\mathcal{S}$, which implies that the adjugate of Ω_2 , denoted by $\text{adj } \Omega_2$, and the determinant of Ω_2 , denoted by $\det \Omega_2$, are continuous on $\mathcal{S}$. Since Ω_2 is invertible on $\mathcal{S}$ and $\Omega_2^{-1} = \frac{1}{\det \Omega_2} \text{adj } \Omega_2$, it follows that Ω_2^{-1} is continuous on $\mathcal{S}$.

Since Ω_2^{-1} and Ω_1 are continuous on $\mathcal{S}$, it follows from (14) that θ is continuous on $\mathcal{S}$. Moreover, it follows from (12) that J is continuous on $\mathbb{R}^{n_1+1} \times \mathcal{S}$. Since, in addition, θ is continuous on $\mathcal{S}$, it follows from (15) that Q is continuous on $\mathcal{S}$. $\square$

Proof of Proposition 9. It follows (17) and (18) that $\hat{\Omega}_0$ and $\hat{\Omega}_1$ are continuous on $\mathcal{S} \times \mathbb{C}^N$. Also, it follows from (11) that Ω_2 is continuous on $\mathcal{S}$.

Next, it follows from Proposition 1 that $\Omega_2(\phi)$ is positive definite. Since Ω_2 is continuous on $\mathcal{S}$, it follows that each element of Ω_2 is continuous on $\mathcal{S}$, which implies that the adjugate of Ω_2 , denoted by $\text{adj } \Omega_2$, and the determinant of Ω_2 , denoted by $\det \Omega_2$, are continuous on $\mathcal{S}$. Since Ω_2 is invertible on $\mathcal{S}$ and $\Omega_2^{-1} = \frac{1}{\det \Omega_2} \text{adj } \Omega_2$, it follows that Ω_2^{-1} is continuous on $\mathcal{S}$.

Therefore, it follows from (20) and (22) that $\hat{J}$ is continuous on $\mathbb{R}^{n_1+1} \times \mathcal{S} \times \mathbb{C}^N$ and $\hat{\theta}$ is continuous on $\mathcal{S} \times \mathbb{C}^N$. Thus, it follows from (21) that $\hat{Q}$ is continuous on $\mathcal{S} \times \mathbb{C}^N$. $\square$

Appendix D. Proofs of Propositions 7 and 8

The following notation is needed for the proofs of Propositions 7 and 8. Let $x_1, \dots, x_n \in \mathbb{C}$, and let $x = \langle x_1 \dots x_n \rangle$ be an unordered n -tuple of numbers in $\mathbb{C}$. See [35, Appendix V] for more details. Let $\mathbb{C}_{\text{sym}}^n$ denote the symmetric n^{th} power of $\mathbb{C}$ as defined in [35, Appendix V]. Let $x = \langle x_1 \dots x_n \rangle \in \mathbb{C}_{\text{sym}}^n$ and $y = \langle y_1 \dots y_n \rangle \in \mathbb{C}_{\text{sym}}^n$, and define $\xi_n(x, y) \triangleq \min_o \max_{j=1, \dots, n} |x_j - y_{o(j)}|$, where o is any permutation of $1, \dots, n$ and $o(j)$ denotes the image of j through the permutation o . It follows from [35, Lemma 3D in Appendix V] that ξ_n is a metric on $\mathbb{C}_{\text{sym}}^n$.

The following lemma is needed for the proofs of Propositions 7 and 8.

Lemma 1. Consider $h_n: \mathbb{C}_{\text{sym}}^n \rightarrow \mathbb{C}^n$ given by

$$h_n(x) \triangleq \begin{bmatrix} -\sum_{j=1}^n x_j \\ (x_1 x_2 + \dots + x_1 x_n) + (x_2 x_3 + \dots + x_2 x_n) + \dots + x_{n-1} x_n \\ \vdots \\ (-1)^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \prod_{j=i_1}^{i_k} x_j \\ \vdots \\ (-1)^n \prod_{j=1}^n x_j \end{bmatrix}, \quad (33)$$

where $x = \langle x_1 \ \cdots \ x_n \rangle \in \mathbb{C}_{\text{sym}}^n$. Consider the polynomial $P(s) = s^n + \tau_1 s^{n-1} + \cdots + \tau_{n-1} s + \tau_n$, where $\tau_1, \dots, \tau_n \in \mathbb{C}$. Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ denote the n roots of P , and define $\tau \triangleq [\ \tau_1 \ \cdots \ \tau_n \]^T \in \mathbb{C}^n$ and $\lambda \triangleq \langle \lambda_1 \ \cdots \ \lambda_n \rangle \in \mathbb{C}_{\text{sym}}^n$. Then, $h_n(\lambda) = \tau$, and h_n and h_n^{-1} are continuous.

Proof. It follows from [36, Fact 4.8.2] that $h_n(\lambda) = \tau$. Moreover, it follows from [35, Theorem 4A in Appendix V] that h_n and h_n^{-1} are continuous. $\square$

The following notation is needed for the proofs of Propositions 7 and 8. Let $\kappa \triangleq d_0 + d_2$. Let $h_{d_1} : \mathbb{C}_{\text{sym}}^{d_1} \rightarrow \mathbb{C}^{d_1}$ and $h_\kappa : \mathbb{C}_{\text{sym}}^\kappa \rightarrow \mathbb{C}^\kappa$ be functions given by (33), where n is replaced by d_1 and κ , respectively. Let $a \triangleq [\ a_1 \ a_2 \ \cdots \ a_{d_0} \]^T \in \mathbb{R}^{d_0}$ and $b \triangleq [\ b_0 \ b_1 \ \cdots \ b_{n_0} \]^T \in \mathbb{R}^{n_0+1}$ be such that $N_0(s) = \Gamma_{n_0}^T(s)b$ and $D_0(s) = s^{d_0} + \Gamma_{d_0-1}^T(s)a$. Consider the continuous function $g : \mathbb{R}^{d-d_1} \rightarrow \mathbb{R}^\kappa$ given by

$$g(z) \triangleq Az + B, \quad (34)$$

where $B \triangleq [\ a^T \ 0_{1 \times d_2} \]^T \in \mathbb{R}^\kappa$ and

$$A \triangleq \begin{bmatrix} \overbrace{0_{(\kappa-n_0-n_2-1) \times (n_2+1)}} & 1 & 0 & \cdots & 0 \\ b_0 & 0 & \cdots & 0 & a_1 & 1 & \ddots & \vdots \\ b_1 & b_0 & \ddots & \vdots & a_2 & a_1 & \ddots & 0 \\ \vdots & b_1 & \ddots & 0 & \vdots & a_2 & \ddots & 1 \\ b_{n_0} & \vdots & \ddots & b_0 & a_{d_0} & \vdots & \ddots & a_1 \\ 0 & b_{n_0} & & b_1 & 0 & a_{d_0} & & a_2 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & b_{n_0} & 0 & 0 & & a_{d_0} \end{bmatrix} \in \mathbb{R}^{\kappa \times (d-d_1)}.$$

Proof of Proposition 7. Since Ψ is compact, it follows that Ψ is bounded, which implies that $(\Psi \cap \mathcal{S}_\rho) \subseteq \Psi$ is bounded.

Let $\phi \in (\Psi \cap \mathcal{S}_\rho)$, and we show that ϕ is not an isolated point. Let $\phi = [\ x^T \ z^T \]^T$, where $x \in \mathbb{R}^{d_1}$ and $z \in \mathbb{R}^{d-d_1}$. Since $\phi \in \Psi$ and Ψ is perfect, it follows that there exists a sequence $\{\phi_i\}_{i=1}^\infty \subseteq \Psi$ that converges to ϕ . For each $i \in \mathbb{Z}^+$, let $\phi_i = [\ x_i^T \ z_i^T \]^T$, where $x_i \in \mathbb{R}^{d_1}$ and $z_i \in \mathbb{R}^{d-d_1}$. Thus, $\{x_i\}_{i=1}^\infty$ converges to x and $\{z_i\}_{i=1}^\infty$ converges to z .

Let $\mu_1, \dots, \mu_{d_1} \in \mathbb{C}$ denote the d_1 roots of $\mathcal{D}_1(s, \phi)$, and define $p \triangleq \langle \mu_1 \ \cdots \ \mu_{d_1} \rangle$. For each $i \in \mathbb{Z}^+$, let $\mu_{i,1}, \dots, \mu_{i,d_1} \in \mathbb{C}$ denote the d_1 roots of $\mathcal{D}_1(s, \phi_i)$, and define $p_i \triangleq \langle \mu_{i,1} \ \cdots \ \mu_{i,d_1} \rangle$. Lemma 1 implies that $x = h_{d_1}(p)$ and $x_i = h_{d_1}(p_i)$. Since $\lim_{i \rightarrow \infty} x_i = x$ and Lemma 1 implies that $h_{d_1}^{-1}$ is continuous, it follows that $\lim_{i \rightarrow \infty} p_i = \lim_{i \rightarrow \infty} h_{d_1}^{-1}(x_i) = h_{d_1}^{-1}(x) = p$. Since $\phi \in \mathcal{S}_\rho$, it follows that for all $j = 1, \dots, d_1$, $\text{Re } \mu_j < \rho$. Define $\epsilon_1 \triangleq \min_{j=1, \dots, d_1} \rho - \text{Re } \mu_j$. Since $\lim_{i \rightarrow \infty} p_i = p$ and ξ_{d_1} is a metric on $\mathbb{C}_{\text{sym}}^{d_1}$, it follows that there exists $L_1 \in \mathbb{Z}^+$ such that for all $i \geq L_1$, $\xi_{d_1}(p_i, p) < \epsilon_1$. Thus, for each $i \geq L_1$, there exists a permutation o_i of $1, \dots, d_1$ such that for all $j = 1, \dots, d_1$, $|\mu_{i,j} - \mu_{o_i(j)}| < \epsilon_1$. Therefore, for all $i \geq L_1$ and all $j = 1, \dots, d_1$,

$$\text{Re } \mu_{i,j} - \text{Re } \mu_{o_i(j)} \leq |\text{Re } \mu_{i,j} - \text{Re } \mu_{o_i(j)}| \leq |\mu_{i,j} - \mu_{o_i(j)}| < \epsilon_1 \leq \rho - \text{Re } \mu_{o_i(j)},$$

which implies that $\text{Re } \mu_{i,j} < \rho$.

Next, since (A2) implies that $\kappa > n_0 + n_2$, it follows that $D_0(s)\mathcal{D}_2(s, \phi) + N_0(s)\mathcal{N}_2(s, \phi)$ is a monic degree κ polynomial, which can be written as $s^\kappa + \Gamma_{\kappa-1}^T(s)\gamma$, where $\gamma \in \mathbb{R}^\kappa$. Let $\nu_1, \dots, \nu_\kappa \in \mathbb{C}$ denote the κ roots of $D_0(s)\mathcal{D}_2(s, \phi) + N_0(s)\mathcal{N}_2(s, \phi)$. For each $i \in \mathbb{Z}^+$, let $\nu_{i,1}, \dots, \nu_{i,\kappa}$ denote the κ roots of $D_0(s)\mathcal{D}_2(s, \phi_i) + N_0(s)\mathcal{N}_2(s, \phi_i)$, which can be written as $s^\kappa + \Gamma_{\kappa-1}^T(s)\gamma_i$, where $\gamma_i \in \mathbb{R}^\kappa$. Using (34), it follows that $\gamma = g(z)$ and $\gamma_i = g(z_i)$. Define

$q \triangleq \langle \nu_1 \ \cdots \ \nu_\kappa \rangle$ and $q_i \triangleq \langle \nu_{i,1} \ \cdots \ \nu_{i,\kappa} \rangle$, and Lemma 1 implies that $\gamma = g(z) = h_\kappa(q)$ and $\gamma_i = g(z_i) = h_\kappa(q_i)$. Since $\lim_{i \rightarrow \infty} z_i = z$ and g is continuous on $\mathbb{R}^{d-d_1}$, it follows that $\lim_{i \rightarrow \infty} \gamma_i = \lim_{i \rightarrow \infty} g(z_i) = g(z) = \gamma$. Since $\lim_{i \rightarrow \infty} \gamma_i = \gamma$ and Lemma 1 implies that h_κ^{-1} is continuous, it follows that $\lim_{i \rightarrow \infty} q_i = \lim_{i \rightarrow \infty} h_\kappa^{-1}(\gamma_i) = h_\kappa^{-1}(\gamma) = q$. Since $\phi \in \mathcal{S}_\rho$, it follows that for all $l = 1, \dots, \kappa$, $\operatorname{Re} \nu_l < \rho$. Define $\epsilon_2 \triangleq \min_{l=1, \dots, \kappa} \rho - \operatorname{Re} \nu_l$. Since $\lim_{i \rightarrow \infty} q_i = q$ and ξ_κ is a metric on $\mathbb{C}_{\text{sym}}^\kappa$, it follows that there exists $L_2 \in \mathbb{Z}^+$ such that for all $i \geq L_2$, $\xi_\kappa(q_i, q) < \epsilon_2$. Thus, for each $i \geq L_2$, there exists a permutation o_i of $1, \dots, \kappa$ such that for all $l = 1, \dots, \kappa$, $|\nu_{i,l} - \nu_{o_i(l)}| < \epsilon_2$. Therefore, for all $i \geq L_2$ and all $l = 1, \dots, \kappa$,

$$\operatorname{Re} \nu_{i,l} - \operatorname{Re} \nu_{o_i(l)} \leq |\operatorname{Re} \nu_{i,l} - \operatorname{Re} \nu_{o_i(l)}| \leq |\nu_{i,l} - \nu_{o_i(l)}| < \epsilon_2 \leq \rho - \operatorname{Re} \nu_{o_i(l)},$$

which implies that $\operatorname{Re} \nu_{i,l} < \rho$.

Define $L \triangleq \max\{L_1, L_2\}$. Let $i \geq L$, and (5) implies that $\mu_{i,1}, \dots, \mu_{i,d_1}$ and $\nu_{i,1}, \dots, \nu_{i,\kappa}$ are the $d_0 + d_1 + d_2$ roots of $\tilde{\mathcal{D}}(s, \phi_i)$. For all $j = 1, \dots, d_1$ and $l = 1, \dots, \kappa$, $\operatorname{Re} \mu_{i,j} < \rho$ and $\operatorname{Re} \nu_{i,l} < \rho$, which implies that $\tilde{\mathcal{D}}(s + \rho, \phi_i)$ is Hurwitz. Thus, $\phi_i \in \mathcal{S}_\rho$, which implies that $\phi_i \in (\Psi \cap \mathcal{S}_\rho)$. Therefore, $\{\phi_i\}_{i=L}^\infty \subseteq (\Psi \cap \mathcal{S}_\rho)$ converges to $\phi \in (\Psi \cap \mathcal{S}_\rho)$, which implies that ϕ is not an isolated point. $\square$

Proof of Proposition 8. Since Ψ is compact, it follows that $\overline{\Psi \cap \mathcal{S}_\rho} \subseteq (\overline{\Psi} \cap \overline{\mathcal{S}_\rho}) = \Psi \cap \overline{\mathcal{S}_\rho}$, which implies that $\overline{\Psi \cap \mathcal{S}_\rho}$ is compact.

We now show that $\overline{\Psi \cap \mathcal{S}_\rho} \subseteq \mathcal{S}$. Define $E_4 \triangleq [E_2^T \ E_3^T]^T \in \mathbb{R}^{(d-d_1) \times d}$, $\mathcal{F} \triangleq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq \rho < 0\}$, and

$$S_0 \triangleq \{\phi \in \mathbb{R}^d : \text{if } \lambda \in \mathbb{C} \text{ and } \tilde{\mathcal{D}}(\lambda, \phi) = 0, \text{ then } \lambda \in \mathcal{F}\},$$

$$S_1 \triangleq \{x \in \mathbb{R}^{d_1} : x = E_1 \phi, \text{ where } \phi \in \mathbb{R}^d \text{ and if } \lambda \in \mathbb{C} \text{ and } \mathcal{D}_1(\lambda, \phi) = 0, \text{ then } \lambda \in \mathcal{F}\},$$

$$S_2 \triangleq \{z \in \mathbb{R}^{d-d_1} : z = E_4 \phi, \text{ where } \phi \in \mathbb{R}^d \text{ and if } \lambda \in \mathbb{C} \text{ and}$$

$$D_0(\lambda) \mathcal{D}_2(\lambda, \phi) + N_0(\lambda) \mathcal{N}_2(\lambda, \phi) = 0, \text{ then } \lambda \in \mathcal{F}\}.$$

Let $x \in S_1$, $z \in S_2$, and $\phi = [x^T \ z^T]^T \in S_0$.

First, we show that S_1 is closed in $\mathbb{R}^{d_1}$. Let $\mu_1, \dots, \mu_{d_1}$ denote the d_1 roots of $\mathcal{D}_1(s, \phi)$, and define $p \triangleq \langle \mu_1, \dots, \mu_{d_1} \rangle$. Since $x \in S_1$, it follows that $\mu_1, \dots, \mu_{d_1} \in \mathcal{F}$ and $\mu \in \mathcal{F}_{\text{sym}}^{d_1}$. Lemma 1 implies that $x = h_{d_1}(p) \in h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1})$, which implies that $x \in (\mathbb{R}^{d_1} \cap h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1}))$. Thus, $S_1 \subseteq (\mathbb{R}^{d_1} \cap h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1}))$. Next, let $\hat{x} \in (\mathbb{R}^{d_1} \cap h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1}))$, and let $\hat{\mu}_1, \dots, \hat{\mu}_{d_1}$ denote the d_1 roots of $\mathcal{D}_1(s, [\hat{x}^T \ z^T]^T)$. Thus, $\hat{x} \in \mathbb{R}^{d_1}$ and $h_{d_1}^{-1}(\hat{x}) \in \mathcal{F}_{\text{sym}}^{d_1}$, which implies that $\hat{\mu}_1, \dots, \hat{\mu}_{d_1} \in \mathcal{F}$. Thus, $\hat{x} \in S_1$, which implies that $(\mathbb{R}^{d_1} \cap h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1})) \subseteq S_1$. Therefore, $S_1 = \mathbb{R}^{d_1} \cap h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1})$. Since $\mathcal{F}$ is closed in $\mathbb{C}$, it follows that $\mathcal{F}_{\text{sym}}^{d_1}$ is closed in $\mathbb{C}_{\text{sym}}^{d_1}$. Since Lemma 1 implies that $h_{d_1}^{-1}$ is continuous, it follows that $h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1})$ is closed in $\mathbb{C}^{d_1}$. Thus, $S_1 = \mathbb{R}^{d_1} \cap h_{d_1}(\mathcal{F}_{\text{sym}}^{d_1})$ is closed in $\mathbb{R}^{d_1}$.

Next, we show that S_2 is closed in $\mathbb{R}^{d-d_1}$. Since (A2) implies that $\kappa > n_0 + n_2$, it follows that $D_0(s) \mathcal{D}_2(s, \phi) + N_0(s) \mathcal{N}_2(s, \phi)$ is a monic degree κ polynomial, which can be written as $s^\kappa + \Gamma_{\kappa-1}^T(s) \gamma$, where $\gamma \in \mathbb{R}^\kappa$. Using (34), it follows that $\gamma = g(z)$. Let $\nu_1, \dots, \nu_\kappa$ denote the κ roots of $s^\kappa + \Gamma_{\kappa-1}^T(s) \gamma$, and define $q \triangleq \langle \nu_1, \dots, \nu_\kappa \rangle$. Since $z \in S_2$, it follows that $\nu_1, \dots, \nu_\kappa \in \mathcal{F}$ and $q \in \mathcal{F}_{\text{sym}}^\kappa$. Lemma 1 implies that $\gamma = h_\kappa(q) \in h_\kappa(\mathcal{F}_{\text{sym}}^\kappa)$, which implies that $z = g^{-1}(\gamma) \in g^{-1}(\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa))$. Thus, $S_2 \subseteq g^{-1}(\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa))$. Next, let $\hat{z} \in g^{-1}(\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa)) \subseteq \mathbb{R}^{d-d_1}$. Thus, $g(\hat{z}) \in (\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa))$. Let $\hat{\nu}_1, \dots, \hat{\nu}_\kappa$ denote the κ roots of $s^\kappa + \Gamma_{\kappa-1}^T(s) g(\hat{z})$. It follows from Lemma 1 that $\langle \hat{\nu}_1, \dots, \hat{\nu}_\kappa \rangle = h_\kappa^{-1}(g(\hat{z})) \in \mathcal{F}_{\text{sym}}^\kappa$, which implies that $\hat{\nu}_1, \dots, \hat{\nu}_\kappa \in \mathcal{F}$. Thus, $\hat{z} \in S_2$, which implies that $g^{-1}(\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa)) \subseteq S_2$. Therefore, $S_2 = g^{-1}(\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa))$. Since $\mathcal{F}$ is closed in $\mathbb{C}$, it follows that $\mathcal{F}_{\text{sym}}^\kappa$ is closed in $\mathbb{C}_{\text{sym}}^\kappa$. Since Lemma 1 implies that h_κ^{-1} is continuous, it follows that $h_\kappa(\mathcal{F}_{\text{sym}}^\kappa)$ is closed in $\mathbb{C}^\kappa$. Thus, $\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa)$ is closed in $\mathbb{R}^\kappa$. Since, in addition, g is continuous, it follows that $S_2 = g^{-1}(\mathbb{R}^\kappa \cap h_\kappa(\mathcal{F}_{\text{sym}}^\kappa))$ is closed in $\mathbb{R}^{d-d_1}$.

Since S_1 is closed in $\mathbb{R}^{d_1}$, S_2 is closed in $\mathbb{R}^{d-d_1}$, and (5) implies that $S_0 = S_1 \times S_2$, it follows that S_0 is closed in $\mathbb{R}^d$. Since, in addition, $\mathcal{S}_\rho \subseteq S_0$, it follows that $\overline{\mathcal{S}_\rho} \subseteq S_0$. Thus, $\overline{\Psi \cap \mathcal{S}_\rho} \subseteq (\Psi \cap \overline{\mathcal{S}_\rho}) \subseteq \overline{\mathcal{S}_\rho} \subseteq S_0 \subseteq \mathcal{S}$. $\square$

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