Stability Criteria with Nonlinear Eigenvalues for Diagonalizable Nonlinear Systems

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Abstract

Stability of the linear system, especially the diagonalizable system, can be verified by computing the signs of the real parts of its eigenvalues. In this paper, we generalize this result to the diagonalizable nonlinear system in terms of nonlinear eigenvalues. According to the main results, the unique equilibrium point of the diagonalizable system is globally asymptotically stable if a specific set of nonlinear eigenvalues are not positive at each point in the state space.

Keywords: Nonlinear systems; nonlinear eigenvalues; diagonal form; stability

1. Introduction

Stability is one of the fundamental concepts in the field of systems and control, and the Lyapunov function [1, 2] and density function [3, 4] are key tools of asymptotic stability analysis. In the linear case, the eigenvalue also plays an important role. Stability of the linear system can be verified by checking the signs of real parts of eigenvalues. The proof for the diagonalizable system can be separated into two steps: diagonalizing the system and deriving the asymptotic stability conditions for the one-dimensional systems. In this paper, we generalize this result to diagonalizable nonlinear systems [5] by using the recently defined nonlinear eigenvalues, which are functions of the state variables [5, 6, 7].

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The diagonalizable nonlinear system is the system that can be transformed into n one-dimensional subsystems, called the diagonal form, which is a generalization of the diagonal linear system, and even if the original system is real, its diagonal form can be complex. In contrast to the linear case, the class of diagonalizable systems is not obvious, and even for one-dimensional complex systems, there is no universal criterion for global asymptotic stability. Thus, global asymptotic stability analysis of the diagonalizable nonlinear system can be separated into two steps: clarifying the class of diagonalizable nonlinear systems; deriving global asymptotic stability conditions for the one-dimensional complex systems. These two problems are tackled in this paper by using nonlinear eigenvalues.

First, we derive a condition for verifying diagonalizability. References [5, 6] deal with diagonalizability of the continuous-time and discrete-time nonlinear systems. Note that the eigenvalues defined in this paper are dual concepts of those in [5] but in contrast to [5], a diagonalizing coordinate transformation is directly obtained from our condition.

Second, we derive a global asymptotic stability condition for one-dimensional complex nonlinear systems in terms of nonlinear eigenvalues, which has not been presented in the literature to the authors' best knowledge although related results are found in [8, 9, 10, 11, 12] as remarked later. According to our condition, the unique equilibrium point is globally asymptotically stable if the real part of a specific nonlinear eigenvalue at each point is not positive. That is, global asymptotic stability of the one-dimensional complex system can be verified without finding the specific function such as the Lyapunov or density function. The nonlinear eigenvalue corresponds to the eigenvalue of the linearized system at an equilibrium point if it is defined at the point, and thus our condition can be interpreted as a generalization of Lyapunov's indirect method and the classical asymptotic stability criterion for linear systems with eigenvalues.

Finally, based on the conditions for diagonalizability and asymptotic stability of one-dimensional systems, we present a global asymptotic stability condition for diagonalizable nonlinear systems with nonlinear eigenvalues. It is worth mentioning that our results can also be used for estimating a region of attraction.

There are several similar results to our asymptotic stability condition for one-dimensional complex systems [8, 9, 10, 11, 12], where global asymptotic stability criteria are given for two-dimensional and more general (real) nonlinear systems in terms of the eigenvalues of the linearized systems. By

applying these criteria to one-dimensional complex systems, the system is globally asymptotically stable if the real part of the eigenvalue of the linearized system at each point is negative and if additional conditions hold. In contrast, our condition only requires that the real part of the eigenvalue is not positive at each point. Therefore, our condition is weaker than any of the conventional criteria.

Another application of nonlinear eigenvalues and eigenvectors is to observability criteria [7], and one of the criteria can be interpreted as the Popov-Belevitch-Hautus observability test for nonlinear systems. The results in [7] and ours indicate that the nonlinear eigenvalues potentially play similar roles to the eigenvalues of linear systems theory.

2. Problem Settings

2.1. Mathematical Preliminaries

First, let us briefly summarize the notions of differential geometry used in this paper. Note that we will leave out the arguments of functions when it is clear from the context. Moreover, we will focus on the analytic case. For more general definitions, see [13, 14]. Let $\mathbf{R}$ and $\mathbf{C}$ be the fields of real and complex numbers, respectively. Let $D_0 \subset \mathbf{R}^n$ be a domain (an open and connected subset) containing the origin in its interior. A mapping $\varphi : D_0 \rightarrow \mathbf{C}^n$ is said to be complex analytic if it is a restriction of a complex analytic function defined on a domain containing D_0 in $\mathbf{C}^n$. Let $\mathcal{K}_0$ be the set of complex analytic functions $\varphi : D_0 \rightarrow \mathbf{C}^n$ with variables $x_1, x_2, \dots, x_n$, and we define $x := [x_1, \dots, x_n]^T$.

For subsets $D_1 \subset \mathbf{R}^n$ and $D_2 \subset \mathbf{C}^n$, a differentiable mapping $\varphi : D_1 \rightarrow D_2$ is said to be a local diffeomorphism if for each point $x \in D_1$, there exists an open subset $U_x \subset D_1$ containing x in its interior, such that the restriction of φ to U_x , $\varphi|_{U_x} : U_x \rightarrow \varphi(U_x)$ is bijective, and its inverse mapping is differentiable. Moreover, if φ is complex analytic, and the inverse of $\varphi|_{U_x}$ is also complex analytic, then φ is called a local analytic diffeomorphism. Suppose that $D_1 \subset \mathbf{R}^n$ and $D_2 \subset \mathbf{C}^n$ contain the origin in their interiors. We denote the set of local analytic diffeomorphisms $\varphi : D_1 \rightarrow D_2$ satisfying $\varphi(\{0\}) = \varphi^{-1}(\{0\}) = \{0\}$ as $\text{Diff}_0^n(D_1)$.

For an analytic vector valued function $f(x) := [f_1(x), \dots, f_n(x)]^T : \mathbf{C}^n \rightarrow \mathbf{C}^n$, we define $\partial f(x)/\partial x := [\partial f(x)/\partial x_1, \dots, \partial f(x)/\partial x_n]$. Let U be an open

subset in $\mathbf{R}^n$ ($\mathbf{C}^n$). A differential one-form at a point $p \in U$ (in the x -coordinate) is

$$\gamma_p = \sum_{i=1}^n v_i(p)(dx_i)_p,$$

where $v_i(p)$ ($i = 1, \dots, n$) is an analytic function defined at p . If v_i ($i = 1, \dots, n$) is defined and analytic at each $p \in U$, we can represent this one-form as

$$\gamma = \sum_{i=1}^n v_i(x)dx_i.$$

By defining $dx := [dx_1, \dots, dx_n]^T$, we can make an even simpler expression,

$$\gamma = v^T(x)dx.$$

For the analytic scalar valued function and differential one-form defined on U , we define the exterior derivatives d to be operators. The exterior derivative of scalar valued function α is defined as

$$d\alpha := \sum_{j=1}^n \frac{\partial \alpha(x)}{\partial x_j} dx_j.$$

The exterior derivative of differential one-form γ is defined as

$$d\gamma := \sum_{i=1}^n \sum_{k=1}^n dv_i \wedge dx_k,$$

where $\wedge$ is the wedge product. Let us remark on an important property of the wedge product.

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \quad (i, j = 1, \dots, n).$$

A differential one-form γ is said to be (locally) exact if at each $p \in U$ there exist an open subset $V_p \subset U$ and a scalar valued function α_p defined on V_p such that $\gamma = d\alpha_p$ on V_p . Moreover, a differential one-form γ is said to be (locally) integrable if at each $p \in U$ there exist an open subset $V_p \subset U$ and scalar valued functions α_p and β_p defined on V_p such that $\gamma = \beta_p d\alpha_p$ on

V_p . It is clear that γ is exact if $\beta_p = 1$. Therefore, integrability is a weaker concept than exactness. For given differential one-forms, both exactness and integrability can be checked. According to the Poincaré lemma, a differential one-form γ is exact if and only if

$$d\gamma = 0,$$

or equivalently,

$$\frac{\partial v_i(x)}{\partial x_j} = \frac{\partial v_j(x)}{\partial x_i} \quad (i, j = 1, \dots, n).$$

According to the Frobenius theorem, a differential one-form γ is integrable if and only if

$$d\gamma \wedge \gamma = 0.$$

The Lie derivative of a scalar valued function α along a vector valued function f , denoted by $L_f\alpha$, is defined as

$$L_f\alpha(x) := \frac{\partial \alpha(x)}{\partial x} f(x).$$

The Lie derivative of a differential one-form γ along a vector valued function f , denoted by $\mathcal{L}_f\gamma$, is defined as

$$\begin{aligned} \mathcal{L}_f\gamma &:= \sum_{i=1}^n v_i(x) df_i(x) + L_f v_i(x) dx_i \\ &= v^T(x) \frac{\partial f(x)}{\partial x} dx + \left(\frac{\partial v(x)}{\partial x} f(x) \right)^T dx. \end{aligned} \quad (1)$$

The Lie bracket of vector fields w and f , denoted by $[w, f]$, is defined as

$$[w, f](x) = \frac{\partial f(x)}{\partial x} w(x) - \frac{\partial w(x)}{\partial x} f(x).$$

Finally, we remark on the Frobenius theorem in relation to Lie brackets. Consider vector valued functions w^i ($i = 1, \dots, n$). There exists a local analytic diffeomorphism φ such that $\partial\varphi(x)/\partial x = [w^1, \dots, w^n]^{-1}$ if and only if $[w^i, w^j] = 0$ for all $i, j = 1, \dots, n$.

2.2. Diagonalizable Systems

Let us consider a system described by

$$\Sigma : \frac{dx(t)}{dt} = f(x(t)),$$

where $x(t) \in \mathbf{R}^n$, $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$, and $f \in \mathcal{K}_0^n$. Suppose that the origin is an equilibrium point, i.e. $f(0) = 0$. If there exists a coordinate transformation $z = \varphi(x) \in \text{Diff}_0^n(D_0)$ such that in the z -coordinates, the system Σ becomes the following diagonal form,

$$\Sigma_D : \frac{dz_i(t)}{dt} = g_i(z_i(t)), \quad (i = 1, \dots, n),$$

where $z_i(t) \in \mathbf{C}$, and $g_i : \mathbf{C} \rightarrow \mathbf{C}$ is analytic then asymptotic stability analysis at the origin of Σ reduces to that of the individual one-dimensional subsystems of Σ_D . Note that $\varphi \in \text{Diff}_0^n(D_0)$ implies $g_i(0) = 0$ ($i = 1, \dots, n$). In the linear case, i.e., when $g_i = \lambda_i z_i$ ($\lambda_i \in \mathbf{C}; i = 1, \dots, n$), asymptotic stability of Σ can be tested by checking the sign of the real part of every λ_i . Thus, from the viewpoint of asymptotic stability analysis, it is worth researching a class of systems Σ that can be transformed into Σ_D . We call such systems diagonalizable.

Definition 1. *A system Σ is diagonalizable on a subset $D_0 \subset \mathbf{R}^n$ if there exists a coordinate transformation $z = \varphi(x) \in \text{Diff}_0^n(D_0)$ which transforms Σ into Σ_D .*

A similar definition for diagonalizability can be found [5, 6]. The difference is that we restrict the class of coordinate transformations to $\text{Diff}_0^n(D_0)$. That is, we only consider local analytic diffeomorphisms which preserve the origin.

Remark 1. *If we do not restrict the class of coordinate transformations, every system is diagonalizable. According to the straightening theorem [14], there exist local coordinates $z_1, \dots, z_n$ such that in these coordinates the vector field g takes the form $g = [0, \dots, 0, 1]^T$ in a neighborhood of a non-equilibrium point. This form is nothing but the diagonal form, but the origin is not preserved. Therefore, in the sense of [5], every system is diagonalizable. It is clear that this system has no equilibrium point and cannot directly be used for asymptotic stability analysis.*

2.3. Eigenvalues of Nonlinear Systems

In this paper, we study the diagonalizability and asymptotic stability of the system Σ in terms of nonlinear eigenvalues and eigenvectors. The concepts of eigenvalues and eigenvectors have been provided in the field of non-commutative algebra [15, 16]. We apply these definitions to the dynamical system Σ .

Definition 2. *The elements $\lambda \in \mathcal{K}_0$ and $v \in \mathcal{K}_0^n$, where $v(0) \neq 0$, are respectively called an eigenvalue and left eigenvector of a system Σ if*

$$v^T(x) \frac{\partial f(x)}{\partial x} + \left(\frac{\partial v(x)}{\partial x} f(x) \right)^T = \lambda(x) v^T(x) \quad (2)$$

holds on D_0 . They are also called a nonlinear eigenvalue and nonlinear left eigenvector, respectively, so they can be distinguished from those in linear algebra.

By multiplying (2) by dx , we get from (1),

$$\mathcal{L}_f(v^T dx) = v^T \frac{\partial f}{\partial x} dx + \left(\frac{\partial v}{\partial x} f \right)^T dx = \lambda v^T dx,$$

Thus, λ can be viewed as an eigenvalue of the Lie derivative of differential one-form $v^T dx$.

At the origin, $\lambda(0)$ and $v(0)$ are nothing but an eigenvalue and a left eigenvector of $\partial f(0)/\partial x$ if they are defined. In fact, by substituting $x = 0$ into (2), since the origin is an equilibrium point, we have

$$v^T(0) \frac{\partial f(0)}{\partial x} = \lambda(0) v^T(0).$$

Example 1. *Consider the following system:*

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + x_1(t)x_2(t), \\ \dot{x}_2(t) &= x_2(t). \end{aligned}$$

Definition (2) becomes

$$\begin{aligned} & \begin{bmatrix} v_1(x) & v_2(x) \end{bmatrix} \begin{bmatrix} 1 + x_2 & x_1 \\ 0 & 1 \end{bmatrix} \\ & + \begin{bmatrix} \frac{\partial v_1(x)}{\partial x_1} (x_1 + x_1 x_2) + \frac{\partial v_1(x)}{\partial x_2} x_2 & \frac{\partial v_2(x)}{\partial x_1} (x_1 + x_1 x_2) + \frac{\partial v_2(x)}{\partial x_2} x_2 \end{bmatrix} \\ & = \lambda(x) \begin{bmatrix} v_1(x) & v_2(x) \end{bmatrix}. \end{aligned}$$

For example, the following λ_i and v^i ($i = 1, 2, 3$) respectively satisfy the definition of a nonlinear eigenvalue and left eigenvector on $D_0 := \mathbf{R} \times (-1, \infty)$.

$$\begin{aligned}\lambda_1 &= 1 + x_2, \quad v^1 = \begin{bmatrix} 1 & -x_1 \end{bmatrix}^T, \\ \lambda_2 &= 1, \quad v^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \\ \lambda_3 &= \frac{1 + 2x_2}{1 + x_2}, \quad v^3 = \begin{bmatrix} 0 & 1 + x_2 \end{bmatrix}^T.\end{aligned}$$

Note that there are more than two eigenvalues and eigenvectors. However, v^2 and v^3 are linearly dependent, and thus $\text{span}_{\mathbf{R}}\{v^1(p), v^2(p)\} = \text{span}_{\mathbf{R}}\{v^1(p), v^3(p)\}$ for every $p \in D_0$. In particular, $\lambda_2(0) = \lambda_3(0)$ and $v^2(0) = v^3(0)$. It is obvious that the number of linearly independent eigenvectors at each $p \in D_0$ is less than or equal to 2. It is worth mentioning that not all eigenvalues are important in asymptotic stability analysis, as will be mentioned in Section 3.1.

Example 1 demonstrates that unlike the eigenvalues of linear algebra, nonlinear eigenvalues are not invariant under scalar multiplication with the nonlinear eigenvectors. In fact, multiplying equation (2) with an invertible $a \in \mathcal{K}_0$ ($a^{-1} \in \mathcal{K}_0$), we find that

$$\begin{aligned}av^T \frac{\partial f}{\partial x} + a \left(\frac{\partial v}{\partial x} f \right)^T &= a\lambda v^T, \\ av^T \frac{\partial f}{\partial x} + \left(\frac{\partial av}{\partial x} f \right)^T &= a\lambda v^T + \left(\frac{\partial a}{\partial x} f \right) v^T \\ &= \left(\lambda + \frac{1}{a} \frac{\partial a}{\partial x} f \right) av^T = \left(\lambda + \frac{1}{a} L_f a \right) av^T.\end{aligned}$$

Therefore, $\lambda + (1/a)L_f a$ is the corresponding eigenvalue to av . In general, $\lambda + (1/a)L_f a$ and λ are different functions. However, since the origin is an equilibrium point, $L_f a(0) = (\partial a(0)/\partial x)f(0) = 0$, which implies $\lambda(0) + (1/a(0))L_f a(0) = \lambda(0)$.

Example 2. We demonstrate the importance of condition $v(0) \neq 0$ in Definition 2. If this condition does not hold, λ satisfying (2) for some v , does not necessarily correspond to an eigenvalue of the linearized system at the origin. Consider the following one-dimensional linear system [5]:

$$\dot{x}(t) = x(t).$$

The linear eigenvalue of this system is 1. This is also a nonlinear eigenvalue. Next, consider the coordinate transformation $z = x^2$, which is not a diffeomorphism at the origin. In the z -coordinate, we have

$$\dot{z}(t) = 2z(t).$$

The linear eigenvalue in the z -coordinate is 2. It is possible to show that in the x -coordinate, there exists v satisfying (2) for $\lambda = 2$ while 2 is not the linear eigenvalue of the linearized system at the origin in the x -coordinate. For $\lambda = 2$, (2) is

$$v + \frac{dv}{dx}x = 2v$$

The general solution of this differential equation is $v = cx$, where c is an arbitrary constant. For any value of c , $v(0) = 0$. Therefore, $\lambda = 2$ and $v = cx$ are not a nonlinear eigenvalue and eigenvector in the x -coordinate.

3. Main Results

3.1. Diagonalizability

In the linear case, the system is diagonalizable if and only if there exist n linearly independent eigenvectors. However, in the nonlinear case, the existence of n linearly independent eigenvectors is not sufficient for diagonalizability. The following theorem shows that eigenvectors satisfying a kind of integrability conditions sufficiently ensure diagonalizability. Therefore, eigenvalues corresponding to such eigenvectors are important for diagonalizability and consequently asymptotic stability.

Theorem 1. Let $\lambda_i \in \mathcal{K}_0$ ($i = 1, \dots, n$) be eigenvalues of a system Σ , and let $v^i \in \mathcal{K}_0^n$ ($i = 1, \dots, n$) be the associated left eigenvectors of λ_i . The system Σ is diagonalizable on $D_0 \subset \mathbf{R}^n$ if and only if the following holds:

1. $\text{span}_{\mathbf{R}}[v^1(p) \cdots v^n(p)] = \mathbf{R}^n$ at each $p \in D_0$;
2. Every $v^{i\top} dx$ ($i = 1, \dots, n$) is exact on D_0 , i.e., there exists $\varphi_i \in \mathcal{K}_0$ ($i = 1, \dots, n$) such that $v^{i\top} dx = d\varphi_i$ on D_0 .

Proof. (Necessity) Consider a coordinate transformation $z = \varphi(x) \in \text{Diff}_0^n(D)$, which transforms Σ into the diagonal form Σ_D . Then, definition (2) of an

eigenvalue and left eigenvector of Σ_D holds for $\lambda = \partial g_i(z_i)/\partial z_i$ and $v = e^i$, i.e., we have

$$e^{iT} \frac{\partial g(z)}{\partial z} = e^{iT} \frac{\partial g_i(z_i)}{\partial z_i}, \quad (3)$$

where $e^i \in \mathcal{K}^n$ is a standard basis, i.e., $e^{iT} dz = dz_i$ holds. Note that $\text{span}_{\mathbf{R}}[e^1 \cdots e^n] = \mathbf{R}^n$ on $\mathbf{R}^n$.

Next, consider the original coordinates. Denote $T(x) := \partial \varphi(x)/\partial x$. Since $g(z) = (\partial \varphi(\varphi^{-1}(z))/\partial x) f(\varphi^{-1}(z))$ holds, we have

$$\frac{\partial g(\varphi(x))}{\partial z} = \left(T(x) \frac{\partial f(x)}{\partial x} + \left(\frac{\partial t_{ij}(x)}{\partial x} f(x) \right) \right) T^{-1}(x), \quad (4)$$

where $((\partial t_{ij}/\partial x)f)$ is a matrix whose (i, j) th element is $(\partial t_{ij}/\partial x)f$. From (3) and (4),

$$\begin{aligned} e^{iT} \left(T \frac{\partial f}{\partial x} + \left(\frac{\partial t_{ij}}{\partial x} f \right) \right) T^{-1} &= e^{iT} \frac{\partial g_i(\varphi_i)}{\partial z_i}, \\ e^{iT} \left(T \frac{\partial f}{\partial x} + \left(\frac{\partial t_{ij}}{\partial x} f \right) \right) &= e^{iT} T \frac{\partial g_i(\varphi_i)}{\partial z_i}, \\ e^{iT} T \frac{\partial f}{\partial x} + \left(\frac{\partial (e^{iT} T)^T}{\partial x} f \right)^T &= e^{iT} T \frac{\partial g_i(\varphi_i)}{\partial z_i} \end{aligned}$$

Thus, $\partial g_i(\varphi_i(x))/\partial z_i$ is an eigenvalue associated with left eigenvector $v^i := T^T e^i$. Since $\varphi_i(x)$ is a local analytic diffeomorphism, T^T is analytic on D_0 . Since T is nonsingular on D_0 , condition 1 holds. Moreover, $v^{iT} = e^{iT} T = \partial \varphi_i/\partial x$ satisfies condition 2.

(Sufficiency) From conditions 1 and 2, $z = \varphi(x)$ is a local analytic diffeomorphism on D_0 . After making the coordinate transformation $z = \varphi(x)$, we have

$$\frac{dz(t)}{dt} = g(z(t)). \quad (5)$$

We can show that this system is a diagonal form, i.e, its Jacobian matrix $\partial g(z)/\partial z$ is a diagonal matrix.

Let $T := \partial\varphi(x)/\partial x$. Then, from (2), we obtain

$$\begin{aligned}
v^{i\top} \frac{\partial f}{\partial x} + \left(\frac{\partial v^i}{\partial x} f \right)^\top &= \lambda_i v^{i\top}, \\
v^{i\top} \frac{\partial f}{\partial x} T^{-1} + \left(\frac{\partial v^i}{\partial x} f \right)^\top T^{-1} &= \lambda_i v^{i\top} T^{-1}, \\
v^{i\top} \frac{\partial f}{\partial x} T^{-1} + \left(\frac{\partial(v^{i\top} T^{-1})^\top}{\partial x} f \right)^\top - v^{i\top} \left(\frac{\partial t_{ij}^{-1}}{\partial x} f \right) &= \lambda_i v^{i\top} T^{-1}, \\
v^{i\top} T^{-1} \left(T \frac{\partial f}{\partial x} - T \left(\frac{\partial t_{ij}^{-1}}{\partial x} f \right) T \right) T^{-1} + \left(\frac{\partial(v^{i\top} T^{-1})^\top}{\partial x} f \right)^\top &= \lambda_i v^{i\top} T^{-1},
\end{aligned}$$

where t_{ij}^{-1} is the (i, j) th element of T^{-1} , and $((\partial t_{ij}^{-1}/\partial x)f)$ is a matrix whose (i, j) th element is $(\partial t_{ij}^{-1}/\partial x)f$. From $((\partial t_{ij}/\partial x)f) = -T((\partial t_{ij}^{-1}/\partial x)f)T$, we have

$$v^{i\top} T^{-1} \left(T \frac{\partial f}{\partial x} + \left(\frac{\partial t_{ij}}{\partial x} f \right) \right) T^{-1} + \left(\frac{\partial(v^{i\top} T^{-1})^\top}{\partial x} f \right)^\top = \lambda_i v^{i\top} T^{-1},$$

and consequently,

$$\begin{aligned}
&[v^1 \ \dots \ v^n]^\top T^{-1} \left(T \frac{\partial f}{\partial x} + \left(\frac{\partial t_{ij}}{\partial x} f \right) \right) T^{-1} \\
&\quad + \left[\frac{\partial(v^{1\top} T^{-1})^\top}{\partial x} f \ \dots \ \frac{\partial(v^{n\top} T^{-1})^\top}{\partial x} f \right]^\top \\
&= [v^1 \ \dots \ v^n]^\top T^{-1} \text{diag}\{\lambda_1, \dots, \lambda_n\}.
\end{aligned}$$

Since $[v^1 \ \dots \ v^n]^\top = T$ holds, we have

$$\left(T \frac{\partial f}{\partial x} + \left(\frac{\partial t_{ij}}{\partial x} f \right) \right) T^{-1} = \text{diag}\{\lambda_1, \dots, \lambda_n\}. \quad (6)$$

From (4) and (6)

$$\frac{\partial g(z)}{\partial z} = \left(T \frac{\partial f(\varphi^{-1}(z))}{\partial x} + \left(\frac{\partial t_{ij}}{\partial x} f \right) \right) T^{-1} = \text{diag}\{\lambda_1, \dots, \lambda_n\}.$$

Therefore, (5) is a diagonal from. $\square$

Example 3. Consider the following system:

$$\frac{dx(t)}{dt} = \begin{bmatrix} -x_1(t) \\ x_1^2(t) \end{bmatrix}.$$

Its nonlinear eigenvalues and eigenvectors on $\mathbf{R}^2$ are

$$\begin{aligned} \lambda_1 &= -1, \quad v^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \\ \lambda_2 &= 0, \quad v^2 = \begin{bmatrix} 2x_1 & 2 \end{bmatrix}^T. \end{aligned}$$

Let us check the conditions of Theorem 1. First, condition 1 holds on $\mathbf{R}^2$. Next, $v^{1T}dx = dx_1$ and $v^{2T}dx = d(x_1^2 + 2x_2)$ hold, i.e., condition 2 holds. According to Theorem 1, the system is diagonalizable on $\mathbf{R}^2$ by

$$z := \varphi(x) = \begin{bmatrix} x_1 \\ x_1^2 + 2x_2 \end{bmatrix}.$$

The diagonal form is the following linear system:

$$\frac{dz(t)}{dt} = \begin{bmatrix} -z_1(t) \\ 0 \end{bmatrix}.$$

Example 4. This example demonstrates that condition 1 does not imply integrability of each $v^{iT}dx$ ($i = 1, \dots, n$), which is a weaker property than exactness, i.e., condition 2. Consider a 3-dimensional system,

$$\frac{dx(t)}{dt} = 0.$$

Then, $v^1 := [1 \ 0 \ x_2]^T$, $v^2 = [0 \ 1 \ 0]^T$ and $v^3 = [0 \ 0 \ 1]^T$ are left eigenvectors, and condition 1 holds on $\mathbf{R}^3$. However, $v^{1T}dx = dx_1 + x_2 dx_3$ is not integrable, and thus not exact because $d(v^{1T}dx) \wedge (v^{1T}dx) = dx_2 \wedge dx_3 \wedge dx_1 \neq 0$.

3.2. Diagonalizability Condition in Dual Form

The above diagonalizability condition was presented in terms of left eigenvectors. Here, we derive another diagonalizability condition using the following right eigenvectors [5, 6].

Definition 3. The elements $\lambda \in \mathcal{K}_0$ and $w \in \mathcal{K}_0^n$, where $w(0) \neq 0$, are respectively called a right eigenvalue and right eigenvector of the system Σ if

$$\frac{\partial f}{\partial x}w - \frac{\partial w}{\partial x}f = \lambda w \tag{7}$$

holds.

Definition (7) can be expressed as $[w, f] = \lambda w$ by using the Lie bracket of vector fields. Thus, the right eigenvalue can be viewed as an eigenvalue of the Lie bracket. We have a dual of Theorem 1.

Theorem 2. *Let $\lambda_i \in \mathcal{K}_0$ ($i = 1, \dots, n$) be eigenvalues of a system Σ , and let $w^i \in \mathcal{K}_0^n$ ($i = 1, \dots, n$) be the associated right eigenvectors of λ_i . The system Σ is diagonalizable on D_0 if and only if the following holds:*

1. $\text{span}_{\mathbf{R}}\{w^1(p) \cdots w^n(p)\} = \mathbf{R}^n$ at each $p \in D_0$;
2. $[w^i, w^j] = 0$ on D_0 and for every $i, j = 1, \dots, n$.

Moreover, the coordinate transformation $z = \psi(x)$ is obtained as $(\partial\psi/\partial z) = [w^1, \dots, w^n]^{-1}$.

To construct the coordinate transformation $z = \psi(x)$ by using right eigenvectors, we need to compute the inverse of the matrix in contrast to the left case. In general, it is not guaranteed that the sets of eigenvalues defined in Definitions 2 and 3 are equivalent. However, if the system is diagonal form, it is possible to show that these two sets are equivalent.

3.3. Stability Conditions for One-dimensional Complex Systems

If the system is diagonalizable, asymptotic stability analysis of the system reduces to asymptotic stability analysis of one-dimensional subsystems. These one-dimensional subsystems can be complex systems as in the linear case. Here, we will study asymptotic stability of one-dimensional complex nonlinear systems.

Let us consider the following one-dimensional complex nonlinear system:

$$\Sigma^1 : \dot{z}(t) = g(z(t)),$$

where $z(t) \in \mathbf{C}$; $g : \mathbf{C} \rightarrow \mathbf{C}$ is analytic and $g(0) = 0$. By using $\xi(t) \in \mathbf{R}$ and $\eta(t) \in \mathbf{R}$, $z(t)$ can be expressed as $z(t) = \xi(t) + j\eta(t)$, where j is the unit imaginary number. Also, there exist functions $u(\xi, \eta), v(\xi, \eta) : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that

$$\dot{\xi}(t) + j\dot{\eta}(t) = u(\xi(t), \eta(t)) + jv(\xi(t), \eta(t)),$$

and equivalently, we have the following representation:

$$\Sigma_R^1 : \begin{bmatrix} \dot{\xi}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} u(\xi(t), \eta(t)) \\ v(\xi(t), \eta(t)) \end{bmatrix}.$$

Note that $g(0) = 0$ implies that $u(0, 0) = 0$ and $v(0, 0) = 0$. Since g is analytic, u and v satisfy the Cauchy-Riemann equations [17, 18]

$$\frac{\partial u(\xi, \eta)}{\partial \xi} = \frac{\partial v(\xi, \eta)}{\partial \eta}, \quad \frac{\partial u(\xi, \eta)}{\partial \eta} = -\frac{\partial v(\xi, \eta)}{\partial \xi}. \quad (8)$$

Thus, the Jacobian matrix denoted by $D_{(u,v)}(\xi, \eta)$, is obtained as

$$D_{(u,v)}(\xi, \eta) = \begin{bmatrix} \frac{\partial u(\xi, \eta)}{\partial \xi} & \frac{\partial u(\xi, \eta)}{\partial \eta} \\ \frac{\partial v(\xi, \eta)}{\partial \xi} & \frac{\partial v(\xi, \eta)}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u(\xi, \eta)}{\partial \xi} & -\frac{\partial v(\xi, \eta)}{\partial \xi} \\ \frac{\partial v(\xi, \eta)}{\partial \xi} & \frac{\partial u(\xi, \eta)}{\partial \xi} \end{bmatrix}.$$

A global asymptotic stability condition of the one-dimensional system Σ^1 is based on the following lemmas.

Lemma 1. *The origin of the system Σ^1 is globally (or locally) asymptotically stable if and only if the origin of the system Σ_R^1 is globally (or locally) asymptotically stable.*

Lemma 1 comes from that $\mathbf{R}^2$ and $\mathbf{C}$ are isomorphic to each other.

Lemma 2. *Suppose that the origin is an isolated equilibrium point of the system Σ^1 , or equivalently Σ_R^1 . The origin of Σ^1 , or equivalently Σ_R^1 is locally asymptotically stable if there exists a neighborhood of the origin such that $\partial u(\xi, \eta)/\partial \xi \leq 0$ and $\partial u(\xi, \eta)/\partial \xi \neq 0$.*

Proof. Let $V(\xi(t), \eta(t)) := (u^2(\xi, \eta) + v^2(\xi, \eta))/2$ be a candidate of Lyapunov function. Since the origin is an isolated equilibrium point, there exists a neighborhood $U \subset \mathbf{R}^2$ around the origin such that V is positive definite on U . Taking the time derivative of V , we have

$$\begin{aligned} \dot{V}(\xi, \eta) &= \begin{bmatrix} u(\xi, \eta) & v(\xi, \eta) \end{bmatrix} D_{(u,v)}(\xi, \eta) \begin{bmatrix} u(\xi, \eta) \\ v(\xi, \eta) \end{bmatrix} \\ &= \frac{\partial u(\xi, \eta)}{\partial \xi} (u^2(\xi, \eta) + v^2(\xi, \eta)) \leq 0 \end{aligned}$$

on U . From $\partial u(\xi, \eta)/\partial \xi \neq 0$, $\dot{V}$ is not identically zero on U . Let $S := \{(\xi, \eta) \in U : \dot{V}(\xi, \eta) = 0\}$. Then, no solution can stay in $S \setminus \{0\}$. From LaSalle's invariance principle [1, 2], the origin of the system is locally asymptotically stable. $\square$

Lemma 3. [8] *Let us consider a two-dimensional dynamical system $\dot{x} = g(x)$, where $x \in \mathbf{R}^2$ and $g : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a class C^1 function. Suppose that the origin is the unique equilibrium point and the system is locally asymptotically stable. Let $\gamma(x) = \max w(x)^T D_g(x) w(x)$ for $w(x)^T w(x) = 1$ and $w(x)^T g(x) = 0$, where $w = [w_1 \ w_2]^T$. If $\gamma(x) \leq 0$ for all $x \in \mathbf{R}^2$, then the origin is globally asymptotically stable.*

The lemma leads us to the following global asymptotic stability condition.

Theorem 3. *Suppose that the origin is the unique equilibrium point of the system Σ^1 . The origin is globally asymptotically stable if*

$$\frac{\partial u(\xi, \eta)}{\partial \xi} \leq 0 \quad (9)$$

for all $(\xi, \eta) \in \mathbf{R}^2$, and $\partial u(\xi, \eta)/\partial \xi \not\equiv 0$.

Proof. Consider Σ_R^1 . It is clear that the condition is sufficient for the conditions in Lemma 2 to hold. Thus, the origin is locally asymptotically stable. It suffices to show that γ in Lemma 3 satisfies the conditions. Compute

$$\begin{aligned} \hat{\gamma}(\xi, \eta) &:= \begin{bmatrix} w_1(\xi, \eta) & w_2(\xi, \eta) \end{bmatrix} D_{(u,v)}(\xi, \eta) \begin{bmatrix} w_1(\xi, \eta) \\ w_2(\xi, \eta) \end{bmatrix} \\ &= \frac{\partial u(\xi, \eta)}{\partial \xi} (w_1^2(\xi, \eta) + w_2^2(\xi, \eta)) \end{aligned}$$

From (9), $\hat{\gamma}(\xi, \eta) \leq 0$ on $\mathbf{R}^2$ for any functions w_1 and w_2 such that $w^T w = w_1^2 + w_2^2 = 1$. Therefore, $\gamma(\xi, \eta) = \max \hat{\gamma}(\xi, \eta) \leq 0$ on $\mathbf{R}^2$. From Lemma 3, the origin of system Σ_R^1 is globally asymptotically stable. That is, from Lemma 1, the origin of Σ^1 is globally asymptotically stable. $\square$

The uniqueness of the equilibrium point is a necessary condition for global asymptotic stability, and it is possible to show that $\partial u(\xi, \eta)/\partial \xi \not\equiv 0$ is also a necessary condition.

For the one-dimensional system Σ^1 , $dg(z)/dz$ is a nonlinear eigenvalue, and since

$$\left. \frac{dg(z)}{dz} \right|_{z=\xi+j\eta} = \frac{\partial u(\xi, \eta)}{\partial \xi} + j \frac{\partial v(\xi, \eta)}{\partial \eta}$$

holds, the real part of the eigenvalue is $\partial u(\xi, \eta)/\partial \xi$. Therefore, asymptotic stability of the system Σ^1 can be verified by checking the sign of the real

part of the nonlinear eigenvalue $dg(z)/dz$ at each point $z \in \mathbf{C}$. Note that the condition of Theorem 3 does not require that the real part of the eigenvalue at the equilibrium point be negative, i.e., Theorem 3 is weaker than Lyapunov's indirect method.

If the origin is locally asymptotically stable then based on Lemma 3, we can estimate a region of attraction of the origin, which is the set of points z_1 such that the solution $z(t)$ starting from z_1 exists for all $t \geq 0$ and satisfies $z(t) \rightarrow 0$ as $t \rightarrow \infty$. In [8], it has been shown that a set of points satisfying the condition $\gamma(u, v) \leq 0$ of Lemma 3 is a region of attraction. Suppose that there exists a neighborhood $U \subset \mathbf{C}$ of the origin such that $\partial u(\xi, \eta)/\partial \xi \leq 0$ and $\partial u(\xi, \eta)/\partial \xi \neq 0$, i.e., the condition in Lemma 3 holds for all $(\xi, \eta) \in \mathbf{R}^2$ such that $\xi + j\eta \in U$. Thus, U is a region of attraction of the origin.

Example 5. *Let us estimate a region of attraction of the complex logistic equation,*

$$\frac{dz(t)}{dt} = z(t)(1 - z(t)).$$

This system has two equilibrium points 0 and 1. Here, we can compute

$$\left. \frac{dz(1 - z)}{dz} \right|_{z=\xi+j\eta} = 1 - 2(\xi + j\eta).$$

Therefore, a region of attraction is $D := \{\xi + j\eta \in \mathbf{C} : \xi \geq 1/2\}$. Since 1 is contained in D , equilibrium point 1 is locally asymptotically stable.

Theorem 3 can be generalized to the case that both u and v are in class C^1 , i.e., the Cauchy-Riemann equations (8) do not necessarily hold. In this case, we can prove the following theorem.

Theorem 4. *Suppose that the origin is the unique equilibrium point of the system Σ . The origin is globally asymptotically stable if $w^T D_g(\xi, \eta)w \leq 0$ at each $(\xi, \eta) \in \mathbf{R}^2$ and $w \in \mathbf{R}^2$, and $w^T D_g(\xi, \eta)w \neq 0$.*

Example 6. *Consider*

$$\frac{dz(t)}{dt} = -z(t)|z(t)|^2,$$

where $|z| = \sqrt{\xi^2 + \eta^2}$. The system can be expressed as

$$\frac{d}{dt} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} -\xi(t)(\xi^2(t) + \eta^2(t)) \\ -\eta(t)(\xi^2(t) + \eta^2(t)) \end{bmatrix}.$$

The origin is a unique equilibrium point where Lyapunov's indirect theorem does not hold. We can test global asymptotic stability of the system with Theorem 4. The Jacobian matrix $D_{(u,v)}(\xi, \eta)$ is computed as

$$D_{(u,v)}(\xi, \eta) = \begin{bmatrix} -(3\xi^2 + \eta^2) & -2\xi\eta \\ -2\xi\eta & -(\xi^2 + 3\eta^2) \end{bmatrix},$$

which does not satisfy the Cauchy-Riemann equations (8). Now, let us check the condition of Theorem 4. The quadratic form $w^T D_g(\xi, \eta) w$ is obtained as follows:

$$\begin{aligned} & -(3\xi^2 + \eta^2)w_1^2 - (\xi^2 + 3\eta^2)w_2^2 - 4\xi\eta w_1 w_2 \\ &= -(\xi^2 + \eta^2)(w_1^2 + w_2^2) - 2\xi^2 w_1^2 - 2\eta^2 w_2^2 - 4\xi\eta w_1 w_2 \\ &= -(\xi^2 + \eta^2)(w_1^2 + w_2^2) - 2(\xi w_1 + \eta w_2)^2 \leq 0, \end{aligned}$$

According to Theorem 4, the origin is globally asymptotically stable.

3.4. Stability Conditions for Diagonalizable Systems

In summary, Theorems 1 and 3 yield the following Theorem on global asymptotic stability of diagonalizable systems.

Theorem 5. *Let $\lambda_i(x) \in \mathcal{K}_0$ ($i = 1, \dots, n$) be eigenvalues of the system Σ , and let $v^i \in \mathcal{K}_0^n$ ($i = 1, \dots, n$) be the associated left eigenvectors of $\lambda_i(x)$ satisfying the conditions in Theorem 1 on D_0 . Suppose that the origin of Σ is an isolated equilibrium point in D_0 . Then, the origin is locally asymptotically stable if the real part of every $\lambda_i(x)$ ($i = 1, \dots, n$) is not positive and not identically zero on D_0 . Moreover, if $D_0 = \mathbf{R}^n$ holds and the origin is the unique equilibrium point, then the origin is globally asymptotically stable.*

Proof. From Theorem 1, the system Σ is diagonalizable on D_0 . Let $z = \varphi(x)$. Then, we have the analytic system Σ_D in diagonal form, which is defined on $\varphi(D_0)$. Note that the origin is an equilibrium point of Σ_D . For each subsystem of Σ_D , the real part of every $\lambda_i(z)$ ($i = 1, \dots, n$) is not positive and not identically to zero on $\varphi(D_0)$, and thus, $\lambda_i(\xi + j\eta) = \partial u_i(\xi_i, \eta_i)/\partial \xi_i \leq 0$ and $\lambda_i(\xi + j\eta) \not\equiv 0$ ($i = 1, \dots, n$) on $\varphi(D_0)$. Therefore, the conditions of Lemmas 2 and 3 hold on $\varphi(D_0)$, and thus, a flow starting from any $z \in \varphi(D_0)$ is bounded and converges to the origin.

Consider the original system Σ . For analytic systems Σ and Σ_D , the flows $x(t; x_0)$ and $z(t; \varphi(x_0))$ starting from any initial state $x_0 \in D_0$ uniquely

exist. Moreover, the mapping $\varphi : U(x_0) \rightarrow \varphi(U(x_0))$ is homeomorphic, for a sufficiently small neighborhood $U(x_0) \subset \mathbf{R}^n$ around each $x_0 \in \mathbf{R}^n$. Therefore, $[x(t; x_0), x(t_1 + t; \varphi^{-1}(x_0))] = \varphi^{-1}([z(t; \varphi(x_0)), z(t_1 + t; \varphi(x_0))]) \subset U(x_0)$ is uniquely determined for a sufficiently small t_1 . Consequently, for any $x_0 \in D_0$, $x(t; x_0)$ exists for all $t \geq 0$ and is bounded. Furthermore, since φ preserves the origin, if $z(t; \varphi(x_0))$ converges to the origin, then $x(t; x_0)$ converges to the origin. $\square$

Remark 2. *We can derive a similar global asymptotic stability condition by using Theorem 2 instead of Theorem 1.*

The conditions of Theorem 5 are generalizations of the conventional asymptotic stability criterion for the diagonalizable linear system. Theorem 5 implies that global or local asymptotic stability can be verified if there exists a set of nonlinear eigenvalues whose corresponding eigenvectors give a coordinate transformation into the diagonalizable form.

Example 7. *Consider the following system:*

$$\begin{aligned}\dot{x}_1 &= \frac{(x_1 - x_2)(x_1^2 + 4x_1x_2 + x_2^2)}{4}, \\ \dot{x}_2 &= -\frac{(x_1 + x_2)(x_1^2 - 4x_1x_2 + x_2^2)}{4}.\end{aligned}$$

For this system, the eigenvalues and eigenvectors are

$$\begin{aligned}\lambda_1 &= \frac{3(-1+j)}{4}(-jx_1 + x_2)^2, \quad v^1 = \begin{bmatrix} -j/2 & 1/2 \end{bmatrix}^T, \\ \lambda_2 &= \frac{3(-1-j)}{4}(jx_1 + x_2)^2, \quad v^2 = \begin{bmatrix} j/2 & 1/2 \end{bmatrix}^T.\end{aligned}$$

Since $[v^1 \ v^2]$ is regular on $\mathbf{R}^2$, and since

$$\begin{aligned}v^{1T}dx &= d\left(-\frac{jx_1}{2} + \frac{x_2}{2}\right), \\ v^{2T}dx &= d\left(\frac{jx_1}{2} + \frac{x_2}{2}\right)\end{aligned}$$

hold, the conditions of Theorem 1 hold on $\mathbf{R}^2$. We can compute the region of attraction of the system on the basis of Theorem 5. The eigenvalues λ_1 and

λ_2 can be expressed as

$$\begin{aligned}\lambda_1 &= \frac{3}{4}(x_1^2 + 2x_1x_2 - x_2^2) - \frac{3j}{4}(x_1^2 - 2x_1x_2 - x_2^2), \\ \lambda_2 &= \frac{3}{4}(x_1^2 + 2x_1x_2 - x_2^2) + \frac{3j}{4}(x_1^2 - 2x_1x_2 - x_2^2).\end{aligned}$$

According to Theorem 5, a region of attraction of the system is

$$D := \{x \in \mathbf{R}^2 : x_1^2 + 2x_1x_2 - x_2^2 \leq 0\},$$

which is the domain between the two lines $x_1 = (-1 - \sqrt{2})x_2$ and $x_1 = (-1 + \sqrt{2})x_2$.

Example 8. Consider the following system:

$$\begin{aligned}\dot{x}_1 &= -x_2(-1 + 4x_1 + 4x_2 - 4x_2^2), \\ \dot{x}_2 &= -2(x_1 + x_2 - x_2^2).\end{aligned}$$

The eigenvalues and eigenvectors for this system are

$$\begin{aligned}\lambda_1 &= -1 + j, \quad v^1 = \begin{bmatrix} j & (1 + j)/2 - 2jx_2 \end{bmatrix}^T, \\ \lambda_2 &= -1 - j, \quad v^2 = \begin{bmatrix} -j & (1 - j)/2 + 2jx_2 \end{bmatrix}^T.\end{aligned}$$

Since $[v^1 \ v^2]$ is regular on $\mathbf{R}^2$, and since

$$\begin{aligned}v^{1T} dx &= d \left(jx_1 + \frac{(1 + j)x_2}{2} - jx_2^2 \right), \\ v^{2T} dx &= d \left(-jx_1 + \frac{(1 - j)x_2}{2} + jx_2^2 \right)\end{aligned}$$

hold, the conditions of Theorem 1 hold on $\mathbf{R}^2$. Moreover, the conditions of Theorem 5 hold on $\mathbf{R}^2$. Therefore, the origin is globally asymptotically stable.

4. Conclusion

In this paper, we derived asymptotic stability conditions for diagonalizable nonlinear systems. We presented conditions for diagonalizability and global asymptotic stability of one-dimensional complex nonlinear systems. Both conditions were described in terms of nonlinear eigenvalues. These two conditions yielded a global asymptotic stability condition for diagonalizable nonlinear systems. The future work includes taking care of the computational issue of nonlinear eigenvalues and finding further applications for them.

Acknowledgments

This work was partly supported by JSPS KAKENHI Grant Numbers 15K18087 and 15H02257 and JST CREST. We thank Prof. Miroslav Halás from Slovak University of Technology for fruitful discussions about nonlinear eigenvalues. We thank the reviewers for their valuable comments and suggestions that helped us to improve the paper.

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