On Self-Similarly Expanding Eshelby Inclusions: Spherical Inclusion with Dilatational Eigenstrain

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Abstract

For a subsonically self-similarly expanding spherical inclusion with dilatational transformation strain in a linear elastic solid, the governing system of partial differential equations is shown to be elliptic under scaling of uniform stretching of the variables, and the resulting elliptic equation is solved by satisfying the Hadamard jump conditions on the moving boundary. The solution has the Eshelby constant stress property for the interior domain, and can thus be used for the expanding inhomogeneity with transformation strain according to Eshelby (1957). The driving force on the moving boundary is also obtained.

Key words: Eshelby inclusions, transformation strain, self-similarity, elastodynamics, expanding inclusions, driving force

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1 Introduction

The subsonically self-similarly expanding Eshelby inclusion models for early times (Barenblatt, 1996) the motion of expanding regions of inhomogeneities with transformation strain. Expanding inclusions may be good models for deep earthquakes (Backus and Mulcahy, 1976; Rice, 1980), and also for stress waves induced martensitic transformations (Yang, Escobar and Clifton, 2009).

We consider the equation of linear elastodynamics which are invariant under scaling of uniform stretch of the variables $\hat{\mathbf{x}} = \alpha \mathbf{x}, \hat{t} = \alpha t, \hat{\mathbf{u}} = \alpha \mathbf{u}$, for $\alpha > 0$, with the invariance allowing for a self-similar solution in terms of the variable $\mathbf{u}/t$. We show that for a self-similarly expanding spherical inclusion with general eigenstrain the system of governing partial differential equations is elliptic in the domain $(|\mathbf{x}|/t) < \min\{a, b\}$, (with a and b being the pressure wave and shear speeds, respectively), while for a self-similarly expanding spherical inclusion with dilatational eigenstrain the system reduces to an elliptic ordinary differential equation, which we then solve by matching the interior Taylor expansion to the exterior Laurent expansion by the Hadamard jump conditions at the inclusion boundary. The solution exhibits the Eshelby constant stress property for the interior domain, and this can be used for solving the self-similarly expanding inhomogeneity with transformation strain by the Eshelby equivalent inclusion method.

We also obtain the driving force (per unit area) (Atkinson and Eshelby, 1968; Freund, 1972; Eshelby, 1970) on the boundary of the expanding inclusion, which is independent of time. To initiate a self-similar motion, an initial particle velocity at time $t = 0$ needs to be given on the outside of the boundary of the inclusion with limiting zero radius, which is found from the solution of the problem for the jump in particle velocity across the moving boundary, which depends only on the eigenstrain and the expansion velocity V , independent of time.

2 Governing equation for a self-similarly expanding spherical inclusion

2.1 Governing equation for self-similarly expanding spherical inclusion with general uniform eigenstrain

A dynamic spherical inclusion with uniform transformation strain ϵ_{lm}^* is expanding in a self-similar manner in an infinite linear isotropic elastic medium, which is centered at the origin of a Cartesian coordinate system, and at time $t > 0$, occupies the sphere

$$\{\mathbf{x} = (x_1, x_2, x_3) | |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \leq Vt\}, \quad (2.1)$$

with V being the expanding speed of the spherical inclusion. The uniform eigenstrain (transformation strain) in the interior of the sphere is expressed as

$$\epsilon_{lm}^*(\mathbf{x}, t) = \epsilon_{ml}^* H(Vt - |\mathbf{x}|), \quad (2.2)$$

where H is the Heaviside step function, and ϵ_{ml}^* is the uniform eigenstrain. We will consider the case of dilatational eigenstrain when

$$\epsilon_{lm}^* = \delta_{ml} \epsilon^*, \quad (2.3)$$

where δ is the Kronecker symbol, and ϵ^* is the uniform dilatational eigenstrain.

The governing equations of linear elastodynamics for the displacement field $\mathbf{u}(\mathbf{x}, t)$ (Willis, 1965; Markenscoff and Ni, 2010) are, for $t > 0$,

$$\rho \frac{\partial^2 u_j}{\partial t^2} - C_{jklm} \frac{\partial^2 u_l}{\partial x_k \partial x_m} = -C_{jklm} \epsilon_{lm}^* \frac{\partial}{\partial x_k} H(Vt - |\mathbf{x}|), \quad (2.4)$$

and they can be rewritten as

$$L_{jl}\left\{\frac{\partial}{\partial t}, \nabla\right\}u_l = -K_j(\nabla)H(Vt - |\mathbf{x}|), \quad (2.5)$$

with the initial and boundary conditions

$$\mathbf{u}(\mathbf{x}, 0^+) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0^+) = 0, \quad (2.6)$$

and

$$\mathbf{u}(\infty, t) = 0, \quad (2.7)$$

where we define

$$L_{jl}\left\{\frac{\partial}{\partial t}, \nabla\right\} = \rho \frac{\partial^2}{\partial t^2} \delta_{jl} - C_{jklm} \frac{\partial^2}{\partial x_k \partial x_m}, \quad (2.8)$$

$$K_j(\nabla) = C_{jklm} \epsilon_{lm}^* \frac{\partial}{\partial x_k}, \quad (2.9)$$

with ρ being the mass density of the elastic medium, C_{jklm} the elastic constants, and where and throughout, the summation convention is used.

The governing equations of linear elastodynamics (2.4) are invariant under scaling of variables $\mathbf{x}$, t , and the displacement $\mathbf{u}$. In fact, if $\mathbf{u}(\mathbf{x}, t)$ solves (2.4) and we define a scaling of uniform stretch of the variables, for $\alpha > 0$,

$$\hat{x}_i = \alpha x_i, \quad \hat{t} = \alpha t, \quad \hat{u}_i = \alpha u_i, \quad (2.10)$$

then $\hat{\mathbf{u}}(\hat{\mathbf{x}}, \hat{t})$ solves the equation

$$\rho \frac{\partial^2 \hat{u}_j}{\partial \hat{t}^2} - C_{jklm} \frac{\partial^2 \hat{u}_l}{\partial \hat{x}_k \partial \hat{x}_m} = -C_{jklm} \epsilon_{lm}^* \frac{\partial}{\partial \hat{x}_k} H(V\hat{t} - |\hat{\mathbf{x}}|). \quad (2.11)$$

The initial and boundary conditions (2.6) and (2.7) are clearly invariant under the stretching of the variables.

Markenscoff and Ni (2010) considered the expanding spherical inclusion with dilatational eigenstrain under general motion $R(t) = R_0 + l(t)$. It can be easily shown that the motion is self-similar if and only if $R_0 = 0$ and $l(t) = Vt$.

According to the general theory of dimensional analysis (Barenblatt, 1979, Bluman and Kumei, 1989), since the equation of motion (2.4) is scaling invariant, the number of its independent variables can be reduced by one, and it allows for a self-similar solution. To see this explicitly, since after the stretching of the variables, the displacement

$$\hat{\mathbf{u}}(\hat{x}, \hat{t}) = \alpha \mathbf{u}(\alpha \mathbf{x}, \alpha t) \quad (2.12)$$

also solves the governing equation (2.4), from the uniqueness (Wheeler and Sternberg, 1968) of the solution of the governing equation (2.4) with the initial and boundary conditions (2.6) and (2.7), it follows that

$$\alpha \mathbf{u}(\alpha \mathbf{x}, \alpha t) = \mathbf{u}(\mathbf{x}, t). \quad (2.13)$$

If we define, for $t > 0$,

$$\mathbf{z} = \frac{\mathbf{x}}{t} \quad \text{and} \quad \phi(\mathbf{z}, t) = \frac{\mathbf{u}(\mathbf{x}, t)}{t}, \quad (2.14)$$

then, we have

$$\alpha \mathbf{u}(\alpha \mathbf{x}, \alpha t) = t \phi\left(\frac{\alpha \mathbf{x}}{\alpha t}, \alpha t\right) = t \phi(\mathbf{z}, \alpha t). \quad (2.15)$$

From (2.13), (2.14) and (2.15), it follows that

$$\phi(\mathbf{z}, \alpha t) = \phi(\mathbf{z}, t). \quad (2.16)$$

Since $\alpha > 0$ is arbitrary, it follows that ϕ is actually independent of t , and

$$\mathbf{u}(\mathbf{x}, t) = t \phi(\mathbf{z}). \quad (2.17)$$

Hence, u_l/t , $\partial u_l/\partial x_i = \partial \phi(\mathbf{x}/t)/\partial(x_i/t)$, and components of the stress are homogeneous functions of $(\mathbf{x}, t)$ of order zero, and are hence self-similar.

Since Equation (2.4) is scaling invariant under scaling (2.10) of stretching coordinates, the self-similar solution of the displacement field is written in the new dependent variable ϕ and new independent variable $\mathbf{z} = \mathbf{x}/t$ as

$$u_l(\mathbf{x}, t) = t\phi_l\left(\frac{\mathbf{x}}{t}\right) = t\phi_l(\mathbf{z}). \quad (2.18)$$

Thus, we have

$$\frac{\partial \mathbf{u}}{\partial x_m} = \frac{\partial}{\partial x_m}(t\phi(\mathbf{z})) = \frac{\partial \phi}{\partial z_m}, \quad (2.19)$$

and

$$\frac{\partial^2 \mathbf{u}}{\partial x_k \partial x_m} = \frac{1}{t} \frac{\partial^2 \phi}{\partial z_k \partial z_m}. \quad (2.20)$$

Also

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial t}(t\phi(\mathbf{z})) = \phi(\mathbf{z}) - z_m \frac{\partial \phi}{\partial z_m}, \quad (2.21)$$

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\partial}{\partial t}(\phi(\mathbf{z}) - z_m \frac{\partial \phi}{\partial z_m}) = \frac{z_m z_k}{t} \frac{\partial^2 \phi}{\partial z_k \partial z_m}. \quad (2.22)$$

By using (2.18) - (2.22), in terms of the new variables $\mathbf{z}$ and ϕ , the left hand side of Equation (2.4) is expressed as

$$\rho \frac{\partial^2 u_j}{\partial t^2} - C_{jklm} \frac{\partial^2 u_l}{\partial x_k \partial x_m} = \frac{1}{t} (\rho z_k z_m \frac{\partial^2 \phi_j}{\partial z_k \partial z_m} - C_{jklm} \frac{\partial^2 \phi_l}{\partial z_k \partial z_m}); \quad (2.23)$$

the right hand side of Equation (2.4) is expressed as

$$-C_{jklm} \epsilon_{lm}^* \frac{\partial}{\partial x_k} [H(V^2 - |\mathbf{z}|^2)] = -\frac{1}{t} C_{jklm} \epsilon_{lm}^* \frac{\partial}{\partial z_k} [H(V - |\mathbf{z}|)]. \quad (2.24)$$

Combining (2.23) and (2.24), the governing equations of elastodynamics (2.4) for a self-similarly expanding spherical inclusion reduce to a system of partial differential equations with three independent variables z_i , $i = 1, 2, 3$,

$$(\rho\delta_{jl}(z_k\frac{\partial}{\partial z_k})(z_m\frac{\partial}{\partial z_m}) - C_{jklm}\frac{\partial^2}{\partial z_k\partial z_m})\phi_l = -C_{jklm}\epsilon_{lm}^*\frac{\partial}{\partial z_k}[H(V - |\mathbf{z}|)]. \quad (2.25)$$

By using similar definitions as (2.8) and (2.9), we have

$$L_{jl}(z_m\frac{\partial}{\partial z_m}, \nabla_z) = \rho(z_m\frac{\partial}{\partial z_m})^2\delta_{jl} - C_{jklm}\frac{\partial^2}{\partial z_k\partial z_m}, \quad (2.26)$$

$$K_j(\nabla_z) = C_{jklm}\epsilon_{lm}^*\frac{\partial}{\partial z_k}, \quad (2.27)$$

we rewrite Equation (2.25) as

$$L_{jl}((z_k\frac{\partial}{\partial z_k}), \nabla_z)\phi_l(\mathbf{z}) = -K_j(\nabla_z)H(V - |\mathbf{z}|), \quad (2.28)$$

where $\nabla_z = (\partial/\partial z_1, \partial/\partial z_2, \partial/\partial z_3)$.

Thus, by scaling of uniform stretching of the variables, we derived the governing system of partial differential equations for self-similarly expanding spherical inclusion with general eigenstrain.

2.2 Ellipticity of the governing system of equations for self-similar expanding spherical inclusion

We will show that the governing equation (2.25), or equivalently, (2.28), is a linear elliptic system of second order partial differential equations in the region $|\mathbf{z}| < \min\{a, b\}$, with $a = \sqrt{(\lambda + 2\mu)/\rho}$ and $b = \sqrt{\mu/\rho}$ being the pressure wave and shear wave speeds, where λ and μ are Lamé's constants. It may be noted that for most elastic materials $a > b$, that is $a^2 - b^2 = (\lambda + \mu)/\rho = (K + \mu/3)/\rho > 0$, here K is the bulk modulus of the material; while there are materials having negative bulk modulus and such that $K + \mu/3 < 0$, which leads to $a < b$, see e.g. Wang and Lakes (2005).

The condition for the ellipticity of a linear system of partial differential equations is discussed in, e.g., John (1955), namely, that if the determinant

of the characteristic form of the system is nonzero, then the system is elliptic. Hence for the system (2.28), if the determinant

$$\det[L_{jl}((z_k \xi_k), \boldsymbol{\xi})] \neq 0, \quad (2.29)$$

for all $\boldsymbol{\xi} \neq 0$, then the system (2.28) is elliptic.

It is well-known in elastodynamics that the determinant of the operator L_{jl} in an isotropic medium is expressed as

$$\begin{aligned} \det[L_{jl}(\gamma, \boldsymbol{\xi})] &= (\rho\gamma^2 - \mu|\boldsymbol{\xi}|^2)^2(\rho\gamma^2 - (\lambda + 2\mu)|\boldsymbol{\xi}|^2) \\ &= \rho^3(\gamma^2 - b^2|\boldsymbol{\xi}|^2)^2(\gamma^2 - a^2|\boldsymbol{\xi}|^2). \end{aligned} \quad (2.30)$$

Thus, for (2.26) we have

$$\det[L_{jl}((z_k \xi_k), \boldsymbol{\xi})] = \rho^3((z_k \xi_k)^2 - b^2|\boldsymbol{\xi}|^2)^2((z_k \xi_k)^2 - a^2|\boldsymbol{\xi}|^2). \quad (2.31)$$

If we assume $|\mathbf{z}| < \min\{a, b\}$, then by the Cauchy-Schwarz inequality, we have

$$(z_k \xi_k)^2 \leq |\mathbf{z}|^2 |\boldsymbol{\xi}|^2 < a^2 |\boldsymbol{\xi}|^2, b^2 |\boldsymbol{\xi}|^2. \quad (2.32)$$

Therefore, from (2.31) and (2.32), the condition (2.29) is satisfied, and we have proved the ellipticity of the system of partial differential equations of motion (2.28) in the region $|\mathbf{z}| < \min\{a, b\}$. And when $|\mathbf{z}|$ is outside this region, *i.e.*, when $|\mathbf{z}| > \min\{a, b\}$, the system of partial differential equations (2.28) is hyperbolic.

2.3 Governing equation for self-similarly expanding spherical inclusion with dilatational eigenstrain

We use the spherical coordinates r, θ, ψ , and the displacement vector is (u_r, u_θ, u_ψ) . For an isotropic medium, the expanding spherical inclusion with

dilatational eigenstrain corresponds to a spherical symmetry, which implies that the displacement components

$$u_\theta = u_\psi = 0, \quad (2.33)$$

and u_r only depends on r .

Note that for an isotropic medium, we have the elastic constants

$$C_{jklm} = \lambda \delta_{jk} \delta_{lm} + \mu (\delta_{jl} \delta_{mk} + \delta_{jm} \delta_{kl}). \quad (2.34)$$

Without loss of generality, we choose the field point being at $(r, 0, 0)$. Then we define $z = r/t$ and $u_r/t = \phi(r/t) = \phi(z)$. On the right hand side of Equation (2.28), $|\mathbf{z}| = z$ and only the derivative with respect to z is nonzero, and by using (2.3), (2.33), and (2.34), the term on the right hand side of (2.28) is reduced to

$$(3\lambda + 2\mu)\epsilon^* \frac{\partial}{\partial z} H(V - z). \quad (2.35)$$

Moreover, by using the expression of the operator L_{jl} in spherical coordinates (see, *e.g.*, Malvern, 1969, p. 672), similarly, the left hand side of (2.28) is reduced to

$$(\lambda + 2\mu) \frac{\partial}{\partial z} [z^{-2} \frac{\partial}{\partial z} (z^2 \phi)] - \rho z^2 \frac{\partial^2}{\partial z^2} \phi. \quad (2.36)$$

Therefore, the system of partial differential equations (2.28) reduces to the ordinary differential equation

$$(1 - \frac{z^2}{a^2}) \frac{d^2 \phi}{dz^2} + \frac{2}{z} \frac{d\phi}{dz} - \frac{2\phi}{z^2} = -\frac{(3\lambda + 2\mu)\epsilon^*}{(\lambda + 2\mu)} \delta(z - V), \quad (2.37)$$

where $\delta(\cdot)$ is the Dirac delta function. Equation (2.37) is elliptic when $(1 - z^2/a^2) > 0$, or $z < a$. For $z > a$, the equation of elastodynamics is hyperbolic and the uniqueness theorem requires the radiation condition at infinity, so

that, from (2.7), with the boundary condition

$$\phi(\infty) = 0. \quad (2.38)$$

3 Solution of the self-similarly expanding spherical inclusion with dilatational eigenstrain

We seek the solution of Equation (2.37) with the boundary condition (2.38) for $V < a$ (subsonic motion) in the form of power series. In the interior domain of the inclusion, the solution is denoted by ϕ_{in} and has a Taylor expansion, for $z < V$,

$$\phi_{in}(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (3.1)$$

where a_n are undetermined constants. In the exterior domain of the inclusion within $z < a$, the solution is denoted by ϕ_{out} and has a Laurent expansion plus an extra term to fit the delta function term on the right hand side of (2.37),

$$\phi_{out}(z) = B(z - V)H(z - V) + \sum_{n=-\infty}^{\infty} b_n z^n, \quad (3.2)$$

where B and b_n are undetermined constants. For $z > a$, the solution is denoted by ϕ_{far} , and has a Laurent expansion,

$$\phi_{far}(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad (3.3)$$

with c_n being undetermined constants, which satisfies (2.37) with homogeneous right hand side since from $z > a$ and $V < a$ (subsonic motion), it follows that $\delta(z - V) = 0$.

By substituting the expansions (3.1) for ϕ_{in} into the equation (2.37) with homogeneous right hand side

$$\left(1 - \frac{z^2}{a^2}\right) \frac{d^2 \phi}{dz^2} + \frac{2}{z} \frac{d\phi}{dz} - \frac{2}{z^2} \phi = 0, \quad (3.4)$$

and equating the like terms, we have

$$a_0 = 0, \quad a_1 \neq 0, \quad a_n = 0 \quad \text{for } n \geq 2, \quad (3.5)$$

so that

$$\phi_{in}(z) = a_1 z. \quad (3.6)$$

For the case $z < a$, by substituting (3.2) for ϕ_{out} into the equation (2.37), noting that, since $(z - V)\delta(z - V) = 0$,

$$\frac{d}{dz}[(z - V)H(z - V)] = H(z - V), \quad (3.7)$$

and

$$\frac{d^2}{dz^2}[(z - V)H(z - V)] = \delta(z - V), \quad (3.8)$$

and by equating the like terms, we obtain

$$B = -\left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right)\frac{a^2\epsilon^*}{(a^2 - V^2)}, \quad (3.9)$$

$$b_0 \neq 0, \quad b_1 \neq 0, \quad b_{-2} = -\frac{a^2}{3}(b_0 - BV), \quad (3.10)$$

while all other coefficients $b_{\pm n}$ are zero. Thus, we have

$$\phi_{out}(z) = B(x - V) + b_0 + b_1 z + \frac{b_{-2}}{z^2} \equiv B_0 + B_1 z + \frac{b_{-2}}{z^2}, \quad (3.11)$$

where $B_0 = b_0 - BV$ and $B_1 = B + b_1$.

For the case when $z > a$, similarly, by substituting the expansion (3.3) into the equation (2.37) with homogeneous right hand side, and by equating the like terms, we have

$$c_0 + \frac{3c_{-2}}{a^2} = 0, \quad (3.12)$$

and all other coefficients $c_{\pm n}$ are zero. Hence, for $z > a$, we have

$$\phi_{far}(z) = c_0 + c_{-2}z^{-2}. \quad (3.13)$$

By using the boundary condition (2.38) and Equation (3.12), from (3.13), we obtain, for $z > a$,

$$\phi_{far}(z) = 0. \quad (3.14)$$

In the solutions (3.6) and (3.11), the three undetermined constants a_1, b_0, b_1 (or equivalently, a_1, B_0, B_1) will be determined by three conditions: first the continuity of the displacement u_r on the moving boundary $z = V$; second the Hadamard jump condition for the normal stress on the moving boundary $z = V$; and third the continuity of the displacement u_r at the wave front $z = a$, as follows:

- (1) On the moving boundary $r = Vt$, or $z = V$, the displacement $u_r = t\phi(z)$ is continuous, *i.e.*,

$$\phi_{in}(V) = \phi_{out}(V), \quad (3.15)$$

which, by equating (3.6) and (3.11), gives

$$a_1 = (1 - \frac{a^2}{3V^2})\frac{B_0}{V} + B_1. \quad (3.16)$$

- (2) on the moving boundary $r = Vt$, or $z = V$, the Hadamard jump condition

$$[[\boldsymbol{\sigma} \cdot \mathbf{n}]] = [[\sigma_{rr}]] = -\rho V [[\frac{\partial u_r}{\partial t}]] \quad (3.17)$$

is satisfied where the symbol $[[\cdot]]$ denotes the jump of a variable across the interface, and is defined by

$$[[A]] = A^+ - A^-, \quad (3.18)$$

with A^+ and A^- being the limiting values of A from either side of the interface, respectively, provided that the limits exist.

Notice that at the field point $(r, 0, 0)$,

$$\sigma_{rr} = (\lambda + 2\mu) \frac{\partial u_r}{\partial r} + 2\lambda \frac{u_r}{r} - C_{11jk} \epsilon_{jk}^*. \quad (3.19)$$

So that on the moving boundary $r = Vt$, we have

$$[[\sigma_{rr}]] = (\lambda + 2\mu) \left[\left[\frac{\partial u_r}{\partial r} \right] \right] + (3\lambda + 2\mu) \epsilon^*, \quad (3.20)$$

since $[[u_r]] = 0$. We compute the jumps $[[\partial u_r / \partial t]]$ and $[[\partial u_r / \partial r]]$ in (3.17) and (3.20). From (3.6) and (3.11), we have

$$(u_r)_{in} = t\phi_{in} = a_1 r, \quad (3.21)$$

and

$$(u_r)_{out} = t\phi_{out} = t(1 - \frac{a^2 t^2}{3r^2})B_0 + B_1 r. \quad (3.22)$$

From (3.21) and (3.22), we have

$$\frac{\partial (u_r)_{in}}{\partial t} = 0, \quad \frac{\partial (u_r)_{out}}{\partial t} = (1 - \frac{a^2 t^2}{r^2})B_0, \quad (3.23)$$

and

$$\frac{\partial (u_r)_{in}}{\partial r} = a_1, \quad \frac{\partial (u_r)_{out}}{\partial r} = \frac{2a^2 t^3}{3r^3} B_0 + B_1. \quad (3.24)$$

From (3.23) and (3.24), we have the jumps at the moving boundary $r = Vt$:

$$\left[\left[\frac{\partial (u_r)}{\partial t} \right] \right] = - \frac{(a^2 - V^2)}{V^2} B_0, \quad (3.25)$$

$$\left[\left[\frac{\partial (u_r)}{\partial r} \right] \right] = \frac{(a^2 - V^2)}{V^3} B_0. \quad (3.26)$$

From (3.20) and (3.26), it follows

$$[[\sigma_{rr}]] = (\lambda + 2\mu) \frac{(a^2 - V^2)}{V^3} B_0 + (3\lambda + 2\mu) \epsilon^*. \quad (3.27)$$

Then substituting (3.16), (3.25) and (3.27) into the Hadamard jump condition (3.17), we obtain the value B_0 :

$$B_0 = - \frac{(3\lambda + 2\mu) a^2 V^3 \epsilon^*}{(\lambda + 2\mu)(a^2 - V^2)^2}. \quad (3.28)$$

- (3) By the continuity of the displacement u_r at the wave front $r = at$, or $z = a$, from (3.14), $\phi_{far} = 0$, hence, we have that the limit of the displacement from $r < at$ to $r = at$ is zero,

$$\lim_{r \rightarrow (at)^-} u_r(r) = 0. \quad (3.29)$$

Thus, from the expression (3.22), we have

$$(u_r)_{out}(at) = t\left(\frac{2}{3}B_0 + B_1a\right) = 0, \quad (3.30)$$

which leads to

$$B_1 = -\frac{2}{3a}B_0. \quad (3.31)$$

Thus, from (3.16), (3.28), and (3.31), we have

$$a_1 = \frac{(3\lambda + 2\mu)}{(\lambda + 2\mu)} \frac{a(2V + a)\epsilon^*}{2(a + V)^2}, \quad (3.32)$$

$$B_0 = -\frac{(3\lambda + 2\mu)}{(\lambda + 2\mu)} \frac{a^2V^3\epsilon^*}{(a^2 - V^2)^2}, \quad (3.33)$$

$$B_1 = \frac{2(3\lambda + 2\mu)}{3(\lambda + 2\mu)} \frac{aV^3\epsilon^*}{(a^2 - V^2)^2}. \quad (3.34)$$

Therefore, we have the following solution for the field quantities:

(1) The displacement

For a point in the interior domain of the inclusion $r < Vt$, or, $z < V$

$$u_r(\text{interior}) = \frac{(3\lambda + 2\mu)\epsilon^* r(2V + a)}{3\rho a(V + a)^2}, \quad (3.35)$$

for a point in the exterior domain of the inclusion $Vt < r < at$, or, $V < z < a$,

$$u_r(\text{exterior}) = \frac{(3\lambda + 2\mu)\epsilon^* V^3(at - r)^2(2r + at)}{3\rho a(a^2 - V^2)^2 r^2}. \quad (3.36)$$

(2) The displacement gradient

For $r < Vt$,

$$\frac{\partial u_r}{\partial r}(\text{interior}) = \frac{(3\lambda + 2\mu)\epsilon^* (a + 2V)}{3\rho a(a + V)^2}, \quad (3.37)$$

and, for $at > r > Vt$,

$$\frac{\partial u_r}{\partial r}(\text{exterior}) = \frac{2(3\lambda + 2\mu)\epsilon^* V^3(r^3 - a^3 t^3)}{3\rho a(a^2 - V^2)^2 r^3}. \quad (3.38)$$

(3) The particle velocity

For $r < Vt$,

$$\frac{\partial u_r}{\partial t}(\text{interior}) = 0, \quad (3.39)$$

and, for $at > r > Vt$,

$$\frac{\partial u_r}{\partial t}(\text{exterior}) = \frac{(3\lambda + 2\mu)\epsilon^* V^3(a^2 t^2 - r^2)}{\rho(a^2 - V^2)^2 r^2}. \quad (3.40)$$

(4) The particle acceleration

For $r < Vt$,

$$\frac{\partial^2 u_r}{\partial t^2}(\text{interior}) = 0, \quad (3.41)$$

and, for $at > r > Vt$,

$$\frac{\partial^2 u_r}{\partial t^2}(\text{exterior}) = \frac{2(3\lambda + 2\mu)\epsilon^* V^3 a^2 t}{\rho(a^2 - V^2)^2 r^2}. \quad (3.42)$$

For $r > at$, all field quantities are zero. The above solutions are consistent with the results in Xiao and Luo (2003), and Markenscoff and Ni (2010).

Equation (3.37) shows that the displacement gradient in the interior domain of the inclusion is constant and independent of the position, thus in the interior domain of the inclusion the stress, which is given by

$$\sigma_{ml} = C_{mljk} \left(\frac{\partial u_j}{\partial x_k} - \epsilon_{jk}^* \right), \quad (3.43)$$

is also constant, that is the constant stress Eshelby property in the interior domain.

Equations (3.25) and (3.28) give the jump of the particle velocity across the moving inclusion boundary $r = Vt$:

$$\left[\left[\frac{\partial u_r}{\partial t} \right] \right] = \frac{(3\lambda + 2\mu)\epsilon^* V}{\rho(a^2 - V^2)}. \quad (3.44)$$

We note that this is independent of time and is also the value of the particle velocity on the outside of the inclusion in the limiting time $t \rightarrow 0$ when the radius of the inclusion $r = Vt$ also tends to zero. Thus Equation (3.44) gives the initial condition of the particle velocity (on the outside surface of the inclusion) needed to initiate the self-similar expanding motion with velocity V for a spherical inclusion with dilatational eigenstrain ϵ^* . All other points have initial particle velocity zero and the initial displacement is also zero.

4 The driving force on the moving boundary of a self-similarly expanding spherical inclusion with dilatational eigenstrain

The driving force (configurational force, or self force) on the boundary $|\mathbf{x}| = Vt$ of the self-similarly expanding spherical inclusion with dilatational eigenstrain is defined in terms of the energy-release rate for a moving singularity. The energy-release rate for a moving defect with velocity $\mathbf{v}$ was defined by Eshelby (1970), Freund (1972, 1990), and Markenscoff and Singh (2015), as the net flux through a contour S surrounding the defect, moving with it and shrinking onto it:

$$\dot{\mathcal{E}} = - \lim_{S \rightarrow 0} \int_S [n_j \sigma_{ij} \dot{u}_j + v_n (W + T)] dS, \quad (4.1)$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the outward normal of the moving boundary, $v_n = (\mathbf{v} \cdot \mathbf{n})$, W and T are the potential and kinetic energy densities, respectively. It was proved by (Freund and Clifton, 1974) that, for any defect less singular than the crack ($r^{-1/2}$ singular), the energy-release rate (4.1) is path-independent. Thus it is also path-independent when the defect is a jump in the field quantities as for the moving boundary of the self-similar expanding spherical inclusion.

We choose the integral surface S_ϵ as two spheres $|\mathbf{x}| = Vt \pm \epsilon$ enclosing the moving boundary S_d , the sphere $|\mathbf{x}| = Vt$, of the expanding spherical inclusion for an infinitesimal number ϵ . The energy-release rate is

$$\begin{aligned}
\dot{\mathcal{E}} &= - \lim_{S_\epsilon \rightarrow S_d} \int_{S_\epsilon} [n_j \sigma_{ij} \dot{u}_j + v_n (W + T)] dS \\
&= \int_{S_d} (n_j [[\sigma_{ij} \dot{u}_j]] + V [[W + T]]) dS.
\end{aligned} \tag{4.2}$$

Note the identity

$$[[AB]] = \langle A \rangle [[B]] + [[A]] \langle B \rangle, \tag{4.3}$$

where we define the symbol $\langle \rangle$ by

$$\langle A \rangle = \frac{1}{2} (A^+ + A^-). \tag{4.4}$$

Using (4.3), we have

$$\begin{aligned}
n_j [[\sigma_{ij} \dot{u}_j]] &= n_j \langle \sigma_{ij} \rangle [[\dot{u}_i]] + n_j [[\sigma_{ij}]] \langle \dot{u}_i \rangle \\
&= -V n_j \langle \sigma_{ij} \rangle \frac{\partial u_i}{\partial \mathbf{n}} - \frac{V}{\rho} [[\dot{u}_i]] \langle \dot{u}_i \rangle \\
&= -V \langle \sigma_{ij} \rangle \frac{\partial u_i}{\partial x_j} - \frac{V}{\rho} [[\dot{u}_i]] \langle \dot{u}_i \rangle.
\end{aligned} \tag{4.5}$$

where the following well-known Hadamard jump conditions across the interface are used

$$n_j [[\sigma_{ij}]] = -V \rho [[\dot{u}_i]], \tag{4.6}$$

$$[[\dot{u}_i]] = -V [[\frac{\partial u_i}{\partial \mathbf{n}}]], \tag{4.7}$$

and the following relation (Eshelby,1970) is used

$$\frac{\partial u_i}{\partial \mathbf{n}} n_j = \frac{\partial u_i}{\partial x_j}. \tag{4.8}$$

And notice that

$$[[T]] = \frac{1}{\rho} \langle \dot{u}_i \rangle [[\dot{u}_i]]. \tag{4.9}$$

Substituting (4.5) and (4.9) into (4.10), we have

$$\dot{\mathcal{E}} = \int_{\dot{S}_d} -V([W] - \langle \sigma_{ij} \rangle \left[\left[\frac{\partial u_i}{\partial x_j} \right] \right]) dS. \quad (4.10)$$

The driving force per unit area in the direction of the normal to the boundary is written as

$$f = [W] - \langle \sigma_{ij} \rangle \left[\left[\frac{\partial u_i}{\partial x_j} \right] \right]. \quad (4.11)$$

For inclusions, the strain and stress are given as

$$\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} - \epsilon_{ij}^*, \quad \sigma_{ij} = C_{ijkl} \left(\frac{\partial u_k}{\partial x_l} - \epsilon_{kl}^* \right), \quad (4.12)$$

we have

$$[W] = \frac{1}{2} C_{ijkl} \left[\left(\frac{\partial u_k}{\partial x_l} - \epsilon_{kl}^* \right) \left(\frac{\partial u_i}{\partial x_j} - \epsilon_{ij}^* \right) \right] = \langle \sigma_{ij} \rangle \left[\left[\frac{\partial u_i}{\partial x_j} - \epsilon_{ij}^* \right] \right]. \quad (4.13)$$

From (4.11) and (4.13), the driving force on the moving boundary of expanding inclusion is given by (Markenscoff and Ni, 2010)

$$f = - \langle \sigma_{ml} \rangle \left[\left[\epsilon_{ml}^* \right] \right] = \langle \sigma_{ml} \rangle \epsilon_{ml}^*. \quad (4.14)$$

Using (4.12) and the dilatational eigenstrain (2.3) in (4.14), we have

$$\begin{aligned} f &= \langle \tau_{11} + \tau_{22} + \tau_{33} \rangle \epsilon^* - \frac{3}{2} (3\lambda + 2\mu) \epsilon^{*2} \\ &= \langle \tau_{rr} + \tau_{\theta\theta} + \tau_{\phi\phi} \rangle \epsilon^* - \frac{3}{2} (3\lambda + 2\mu) \epsilon^{*2}, \end{aligned} \quad (4.15)$$

where

$$\tau_{ml} = C_{mljk} \frac{\partial u_j}{\partial x_k} \quad (4.16)$$

are the components of the elastic stress tensor in Cartesian coordinates, and $\tau_{rr}, \tau_{\theta\theta}, \tau_{\phi\phi}$ are the components in spherical coordinates, which are expressed

as

$$\tau_{rr} = (\lambda + 2\mu) \frac{\partial u_r}{\partial r} + 2\lambda \frac{u_r}{r}, \quad (4.17)$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = \lambda \frac{\partial u_r}{\partial r} + 2(\lambda + \mu) \frac{u_r}{r}. \quad (4.18)$$

Hence by substituting (4.17) and (4.18) into (4.15) and using the solution for u_r inside and outside the inclusion given in (3.35) and (3.36), we obtain the driving force on the moving boundary $r = Vt$, as

$$\begin{aligned} f &= (3\lambda + 2\mu) \left\langle \frac{\partial u_r}{\partial r} + 2 \frac{u_r}{r} \right\rangle \epsilon^* - \frac{3}{2} (3\lambda + 2\mu) \epsilon^{*2} \\ &= - \frac{2\mu(3\lambda + 2\mu) \epsilon^{*2}}{(\lambda + 2\mu)} - \frac{(3\lambda + 2\mu)^2 \epsilon^{*2} V^2 (3a - V)}{2(\lambda + 2\mu)(a + V)(a^2 - V^2)}. \end{aligned} \quad (4.19)$$

We notice that the driving force is independent of time. The first term in (4.19) coincides with the driving force for the static inclusion as given by Eshelby (1970) and Gavazza (1977).

5 Conclusions

The early motion of a subsonically expanding Eshelby spherical inclusion with transformation strain is captured by the self-similar motion which is obtained here within the framework of the self-similar equations of elastodynamics which become elliptic. The solution was obtained and it exhibits the constant stress Eshelby property in the interior domain, which allows for the application of the equivalent inclusion method for expanding inhomogeneities. The jump condition for the particle velocity obtained by the solution determines the initial condition for a particular expansion velocity. The driving force on the inclusion boundary is also computed.

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