A Note on Computing the Center of Uncertain Data on the Real Line

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Abstract. We consider the one-dimensional one-center problem on uncertain data. We are given a set $\mathcal{P}$ of n (weighted) uncertain points on a real line L and each uncertain point is specified by a probability density function that is a piecewise-uniform function (i.e., a histogram). The goal is to find a point c (the *center*) on L such that the maximum expected distance from c to all uncertain points of $\mathcal{P}$ is minimized. We present a linear-time algorithm for this problem.

Keywords: facility locations, one-center, uncertain data, histogram, lowest point, upper envelope, algorithms, computational geometry

1 Introduction

In the real world, data is often associated with uncertainty due to their essence, the measurement inaccuracy, sampling discrepancy, resource limitation, etc [6, 9]. In this paper, we consider the one-dimensional one-center problem on uncertain data, defined as follows.

Let L be a real line. Without loss of generality, we assume L is the x -axis. Let $\mathcal{P}$ be a set of n uncertain points $\{P_1, P_2, \dots, P_n\}$ on L , where each uncertain point $P_i \in \mathcal{P}$ is specified by its probability density function (pdf) $f_i: \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}$, which is a piecewise-uniform function (i.e., a histogram), consisting of at most $m+1$ pieces (e.g., see Fig. 1). More specifically, for each P_i , there are m sorted x -coordinates $x_{i1} < x_{i2} < \dots < x_{im}$ and $m-1$ nonnegative values $y_{i1}, y_{i2}, \dots, y_{i,m-1}$ such that $f_i(x) = y_{ij}$ for $x_{ij} \leq x < x_{i,j+1}$ with $1 \leq j \leq m-1$. For convenience of discussion, we assume $x_{i0} = -\infty$, $x_{i,m+1} = +\infty$, $y_{i0} = y_{im} = 0$, and $f_i(x) = 0$ for $x \in (-\infty, x_{i1}) \cup [x_{im}, +\infty)$.

As discussed in [1], such a histogram function f_i can be used to approximate any pdf with arbitrary precision. In particular, for the *discrete* case where each uncertain point has a finite number of discrete locations, each with a probability, it can also be incorporated by our histogram model using infinitesimal pieces at these locations. Thus, the discrete case is a special case of our histogram model.

With a little abuse of notation, for any (deterministic) point q on L , we also use q to denote the x -coordinate of q . For any uncertain point $P_i \in \mathcal{P}$, the *expected distance* from q to P_i is

$$\text{Ed}(q, P_i) = \int_{-\infty}^{+\infty} f_i(x)|x - q|dx.$$

The goal of our *one-center* problem on $\mathcal{P}$ is to find a (deterministic) point c^* on L such that the maximum expected distance from c^* to all uncertain points of $\mathcal{P}$ is minimized, and c^* is called a *center* of $\mathcal{P}$.

The algorithm proposed in our previous work [18] can solve the problem in $O(mn \log mn + n \log n \log mn)$ time. In this paper, we present an $O(mn)$ time algorithm. Since the input size of the problem is $\Theta(mn)$, our algorithm runs in linear time, which is optimal.

We should point out that our algorithm is applicable to the *weighted case* of this problem where each uncertain point $P_i \in \mathcal{P}$ has a nonnegative multiplicative weight w_i and the weighted expected distance is

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Fig. 1. Illustrating the pdf f_i of an uncertain point P_i with $m = 8$.

considered (i.e., $\text{Ed}(q, P_i) = w_i \cdot \int_{-\infty}^{+\infty} f_i(x) |x - q| dx$). To solve the weighted case, we can first reduce it to the above unweighted case by changing each value y_{ij} to $w_i \cdot y_{ij}$ for every $1 \leq i \leq n$ and $1 \leq j \leq m - 1$, and then apply our algorithm for the unweighted case. The running time is still linear. Hence, we will focus our discussion on the unweighted case.

1.1 Related Work

In our previous work [18], the problem of finding k centers for a set $\mathcal{P}$ of uncertain points on L was studied, and an algorithm of $O(mn \log mn + n \log k \log n \log mn)$ time was proposed. Therefore, when $k = 1$, the algorithm runs in $O(mn \log mn + n \log n \log mn)$ time as mentioned above. In addition, the discrete case of the above k -center problem (where each uncertain point P_i has m discrete locations, each with a probability) was solved in a faster way in $O(mn \log mn + n \log k \log mn)$ time [18], and the discrete one-center problem was solved in $O(mn)$ time [18]. Therefore, our new result in this paper for the one-center problem under the more general histogram model matches the previous result for the discrete case. We also studied the discrete one-center problem for uncertain points on tree networks and proposed a linear-time algorithm in [17].

The deterministic k -center problems are classical facility location problems have been extensively studied. The problem is NP-hard in the plane [13]. Efficient algorithms were known for special cases, e.g., finding the smallest enclosing circle (i.e., the case $k = 1$) [12], k -center on trees [4, 11, 14]. The deterministic k -center problem in the one-dimensional space is solvable in $O(n \log n)$ time [5, 7, 14]. As shown in [18], the deterministic one-center problem in the one-dimensional space can be solved in linear time.

The k -center problems on uncertain data in high-dimensional spaces have also been considered. For example, approximation algorithms were given in [8] for different problem models, e.g., the assigned model that is somewhat similar to our problem model and the unassigned model which was relatively easy because it can be reduced to the corresponding deterministic problem [8]. Foul [10] studied the problem of finding the center in the plane to minimize the maximum expected distance from the center to all uncertain points, where each uncertain point has a uniform distribution in a given rectangle. Other facility location problems on uncertain data under various models, e.g., the minmax regret [2, 3, 16], have also been studied (see [15] for a survey).

1.2 Our Techniques

To solve our one-center problem, based on observations, we reduce it to the following geometric problem. Let $\mathcal{H}$ be a set of n unimodal functions in the plane (i.e., when x changes from $-\infty$ to $+\infty$, it first monotonically decreases and then increases) and each function consists of m pieces with each piece being a parabolic arc. We wish to find the lowest point v^* in the upper envelope of the functions of $\mathcal{H}$. In the discrete version of the one-center problem, as shown in [18], each parabolic arc of every function of $\mathcal{H}$ is simply a line segment, and thus, v^* can be found by applying Megiddo's linear time linear programming algorithm [12]. In our problem, however, since the parabolic arcs of the functions of $\mathcal{H}$ may not be line segments, Megiddo's algorithm in [12] does not work any more. We present a new prune-and-search technique that can compute v^* in $O(mn)$ time. This result immediately leads to a linear time algorithm for our one-center problem on $\mathcal{P}$.

Comparing with the linear programming problem, the above geometric problem is more general (i.e., the linear programming problem is a special case of our problem). Our linear time algorithm for the problem may

Fig. 2. Illustrating the expected distance function $\text{Ed}(x, P_i)$ for an uncertain point P_i with $m = 8$; $\text{Ed}(x, P_i)$ is monotonically decreasing for $x \in (-\infty, p_i]$, and increasing for $x \in [p_i, +\infty)$.

be interesting in its own right and may find other applications as well. In fact, our result can be extended to more general unimodal functions (see the discussions in Section 4).

2 Preliminaries

A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is a *unimodal* if there exists a value x' such that for any $x_1 < x_2$, $g(x_1) \geq g(x_2)$ holds if $x_2 \leq x'$ and $g(x_1) \leq g(x_2)$ holds if $x' \leq x_1$, i.e., $g(x)$ is monotonically decreasing on $x \in (-\infty, x']$ and increasing on $x \in [x', +\infty)$.

Recall that L is the x -axis, and for any point q on L , we also use q to denote the x -coordinate of q . Therefore, the values of $\mathbb{R}$ correspond to the points of L . In the following, we will use “the values of $\mathbb{R}$ ” and “the points of L ” interchangeably.

Consider any uncertain point P_i of $\mathcal{P}$. If we consider the expected distance $\text{Ed}(x, P_i)$ as a function of the points x on L (or $x \in \mathbb{R}$), the following observation has been proved in [18]. Note that the m coordinates $x_{i1}, \dots, x_{im}$ of P_i are already given sorted.

Lemma 1. [18] *The function $\text{Ed}(x, P_i)$ for $x \in \mathbb{R}$ is unimodal. More specifically, there exists a point $p_i \in L$ such that $\text{Ed}(x, P_i)$ is monotonically decreasing on $x \in (-\infty, p_i]$ and increasing on $x \in [p_i, +\infty)$ (e.g., see Fig. 2). In addition, $\text{Ed}(x, P_i)$ is a parabolic arc (of constant complexity) on the interval $[x_{ik}, x_{i,k+1})$ for each $0 \leq k \leq m$, and can be explicitly computed in $O(m)$ time.*

The point p_i in Lemma 1 is referred to as a *centroid* of P_i and can be easily computed in $O(m)$ time after $\text{Ed}(x, P_i)$ is explicitly computed in $O(m)$ time [18]. In fact, a point $q \in L$ is a centroid of P_i if and only if $\int_{-\infty}^q f_i(x) = 0.5$ and $\int_q^{+\infty} f_i(x) = 0.5$. Note that the centroid of P_i may not be unique. This case happens when there exists an interval on the x -axis such that for any point q in this interval both $\int_{-\infty}^q f_i(x) = 0.5$ and $\int_q^{+\infty} f_i(x) = 0.5$ hold, and thus $\text{Ed}(x, P_i)$ is a constant when x is in this interval and any point q in this interval is a centroid. If the centroid of P_i is not unique, then we use p_i to refer to an arbitrary such centroid. For a similar reason, the center of $\mathcal{P}$ may also not be unique, in which case our algorithm will find one such center.

The following corollary can be easily obtained based on Lemma 1 and binary search on the sorted list of $x_{i1}, x_{i2}, \dots, x_{im}$.

Corollary 1. [18] *For each uncertain point P_i , with $O(m)$ time preprocessing, the value $\text{Ed}(x', P_i)$ for any query value x' can be computed in $O(\log m)$ time.*

Let $\mathcal{H}$ denote the set of all functions $\text{Ed}(x, P_i)$ for $i = 1, 2, \dots, n$. Since the center c^* of $\mathcal{P}$ is a point on L that minimizes the value $\max_{i=1}^n \text{Ed}(x, P_i)$, i.e., $c^* = \operatorname{argmin}_{x \in L} \max_{i=1}^n \text{Ed}(x, P_i)$, c^* is equal to the x -coordinate of the lowest point v^* in the upper envelope of $\mathcal{H}$. Therefore, to compute c^* , it is sufficient to find v^* .

As shown in [18] and mentioned in Section 1.2, in the discrete case where each uncertain point P_i has m discrete locations (each with a probability), each function $\text{Ed}(x, P_i)$ consists of $m + 1$ pieces with each piece

being a special parabolic arc: a line segment. Consequently, to compute v^* is equivalent to computing the lowest point in the upper envelope of the $n(m+1)$ extending lines of all line segments of all functions of $\mathcal{H}$, which can be done in $O(mn)$ time by Megiddo's linear time linear programming algorithm [12].

In our problem, since each piece of every function of $\mathcal{H}$ may be a general parabolic arc, Megiddo's algorithm [12], which heavily relies on the properties of lines, does not work any more. In the next section, we present a new prune-and-search algorithm that can compute v^* in $O(mn)$ time, and it can be considered as an extension of Megiddo's algorithm [12].

3 Compute the Lowest Point v^* in the Upper Envelope of $\mathcal{H}$

In this section, we present an algorithm that can compute v^* in $O(mn)$ time. Note that the lowest point of the upper envelope of $\mathcal{H}$ may not be unique, in which case we use v^* to represent an arbitrary such lowest point. We assume the preprocessing in Corollary 1 has been done for every uncertain point $P_i \in \mathcal{P}$, which takes $O(mn)$ time in total. Denote by x^* the x -coordinate of v^* . We first present an $O(n \log m)$ time algorithm for solving the following *decision problem*: given any value x' , determine whether $x' < x^*$, $x' = x^*$, or $x' > x^*$. Let $\mathcal{U}$ denote the upper envelope of $\mathcal{H}$.

3.1 A Decision Algorithm

Consider any value x' . To solve the decision problem, the main idea is to compute the intersection of $\mathcal{U}$ and the vertical line $x = x'$, and then determine whether $x' < x^*$, $x' = x^*$, or $x' > x^*$ based on the local information of $\mathcal{U}$ at the above intersection. The details are given below. Denote by $l(x')$ the vertical line $x = x'$.

For any uncertain point $P_i \in \mathcal{P}$, by Corollary 1, we can compute the intersection of the function $\text{Ed}(x, P_i)$ and $l(x')$ in $O(\log m)$ time, and let q_i denote the above intersection. In fact, the parabolic arc of $\text{Ed}(x, P_i)$ that contains q_i can also be determined. We let S_i denote the above parabolic arc. In the case that q_i is the common endpoint of two parabolic arcs, we let S_i denote the set of both parabolic arcs. Hence $|S_i| \leq 2$ holds in either case. In summary, q_i and S_i can be determined in $O(\log m)$ time.

We compute q_i and S_i for every uncertain point $P_i \in \mathcal{P}$, which takes $O(n \log m)$ time in total with the preprocessing in Corollary 1. Let $Q = \{q_i \mid 1 \leq i \leq n\}$. Next, we find the highest point $q(x')$ of Q , and clearly, $q(x')$ is the intersection of $l(x')$ and the upper envelope $\mathcal{U}$. In case $q(x')$ is equal to multiple points of Q , we let $I(x')$ denote the index set such that for each $i \in I(x')$, $q_i = q(x')$. Let $S(x') = \cup_{i \in I(x')} S_i$. Note that after q_i and S_i for every $P_i \in \mathcal{P}$ are computed, we can obtain $q(x')$, $I(x')$ and $S(x')$ in $O(n)$ time.

Based on $q(x')$ and the set $S(x')$ of parabolic arcs, it is not difficult to see that we can solve the decision problem in the following way. If all parabolic arcs of $S(x')$ are strictly decreasing at $q(x')$, then $x' < x^*$. If all parabolic arcs of $S(x')$ are strictly increasing at $q(x')$, then $x' > x^*$. Otherwise, $x' = x^*$. Clearly, with $q(x')$ and $S(x')$, the above can be determined in $O(n)$ time.

Therefore, we obtain the following result.

Lemma 2. *With $O(mn)$ time preprocessing, given any value x' , we can determine whether $x' < x^*$, $x' = x^*$, or $x' > x^*$, in $O(n \log m)$ time.*

3.2 Observations

We first give the following lemma, which will help us to prune uncertain points of $\mathcal{P}$ in our prune-and-search algorithm.

Lemma 3. *For any two uncertain points P_i and P_j , the upper envelope of $\text{Ed}(x, P_i)$ and $\text{Ed}(x, P_j)$ has $O(m)$ complexity and can be computed in $O(m)$ time.*

Proof. Consider two uncertain points P_i and P_j . We first analyze the intersections of the functions $\text{Ed}(x, P_i)$ and $\text{Ed}(x, P_j)$. Recall that each of these two functions has $m + 1$ parabolic arcs.

We use $\text{Ed}_L(x, P_i)$ to denote the part of $\text{Ed}(x, P_i)$ to the left of the centroid p_i (i.e., $x \in (-\infty, p_i]$), and use $\text{Ed}_R(x, P_i)$ to denote the part of $\text{Ed}(x, P_i)$ to the right of p_i (i.e., $x \in [p_i, +\infty)$). Hence, $\text{Ed}_L(x, P_i)$ is monotonically decreasing and $\text{Ed}_R(x, P_i)$ is monotonically increasing. We define $\text{Ed}_L(x, P_j)$ and $\text{Ed}_R(x, P_j)$ for P_j similarly.

Since $\text{Ed}_L(x, P_i)$ is monotonically increasing and $\text{Ed}_R(x, P_j)$ is monotonically decreasing, $\text{Ed}_L(x, P_i)$ can intersect $\text{Ed}_R(x, P_j)$ transversally at most once. Similarly, $\text{Ed}_R(x, P_i)$ can intersect $\text{Ed}_L(x, P_j)$ transversally at most once.

Consider $\text{Ed}_L(x, P_i)$ and $\text{Ed}_L(x, P_j)$. Since both of them are monotonically decreasing, a parabolic arc of $\text{Ed}_L(x, P_i)$ can intersect any parabolic arc of $\text{Ed}_L(x, P_j)$ transversally at most twice (this is because both parabolic arcs are decreasing). Therefore, $\text{Ed}_L(x, P_i)$ can intersect $\text{Ed}_L(x, P_j)$ transversally $O(m)$ times. Similarly, $\text{Ed}_R(x, P_i)$ can intersect $\text{Ed}_R(x, P_j)$ transversally $O(m)$ times.

The above discussion leads to the conclusion that the two functions $\text{Ed}(x, P_i)$ and $\text{Ed}(x, P_j)$ can intersect each other transversally at most $O(m)$ times. This implies that the upper envelope of $\text{Ed}(x, P_i)$ and $\text{Ed}(x, P_j)$ is of complexity $O(m)$. Since both $\text{Ed}(x, P_i)$ and $\text{Ed}(x, P_j)$ can be computed in $O(m)$ time by Lemma 1, their upper envelope can be computed by a simple sweeping algorithm in $O(m)$ time. The lemma thus follows. $\square$

Lemma 3 implies the following corollary.

Corollary 2. *For any two uncertain points P_i and P_j , in $O(m)$ time we can compute a sorted list of values $-\infty = x_0 < x_1 < x_2 < \dots < x_t < x_{t+1} = +\infty$ with $t = O(m)$ such that either $\text{Ed}(x, P_i) \leq \text{Ed}(x, P_j)$ or $\text{Ed}(x, P_i) > \text{Ed}(x, P_j)$ holds when $x \in [x_k, x_{k+1}]$ for any $0 \leq k \leq t$.*

Recall that x^* is the x -coordinate of v^* . Corollary 2 implies that if we know x^* is contained in some interval $[x_k, x_{k+1}]$ as defined in Corollary 2, then at least one of P_i and P_j can be pruned, and more specifically, if $\text{Ed}(x, P_i) \leq \text{Ed}(x, P_j)$ when $x \in [x_k, x_{k+1}]$, then for computing v^* , P_i can be discarded. In the following, let $X(i, j)$ denote the set of sorted x -coordinates as defined in Corollary 2. Thus, it holds that $|X(i, j)| \leq Cm$ for some constant C for any two uncertain points P_i and P_j of $\mathcal{P}$.

3.3 Computing the Lowest Point v^*

In the sequel, we present our algorithm for computing v^* based on the prune-and-search techniques. Below, we show that our algorithm can prune $n/4$ uncertain points of $\mathcal{P}$ in $O(mn)$ time.

We arbitrarily assign the uncertain points of $\mathcal{P}$ into $n/2$ pairs. Let Σ denote all such pairs. For each pair $(P_i, P_j) \in \Sigma$, we compute $X(i, j)$ and the upper envelope of $\text{Ed}(x, P_i)$ and $\text{Ed}(x, P_j)$ in $O(m)$ time by Lemma 3. Let $\mathcal{X} = \bigcup_{(P_i, P_j) \in \Sigma} X(i, j)$. Since Σ has $n/2$ pairs, $\mathcal{X}$ can be computed in $O(mn)$ time by Lemma 3 and $|\mathcal{X}| \leq Cmn/2$.

We find the median x_m of $\mathcal{X}$ in $O(mn)$ time. By using our decision algorithm in Lemma 2, we can determine whether $x_m < x^*$, $x_m = x^*$, or $x_m > x^*$ in $O(n \log m)$ time. If $x_m = x^*$, then x_m is the center of $\mathcal{P}$ and we are done with the algorithm. Otherwise, if $x_m < x^*$, then we eliminate all values in $\mathcal{X}$ that are smaller than or equal to x_m ; if $x_m > x^*$, then we eliminate all values in $\mathcal{X}$ that are larger than or equal to x_m . In either case, no more than $|\mathcal{X}|/2$ values of $\mathcal{X}$ will remain. Next, we continue the above procedure recursively on the remaining values of $\mathcal{X}$ until the number of remaining values is no more than $n/4$. Since initially $|\mathcal{X}| \leq Cmn/2$, the number of recursive steps is bounded by $O(\log(2Cm)) = O(\log m + \log C + 1)$.

We claim that the total running time of the above recursive procedure is $O(mn)$. Indeed, the total time for finding the medians in all recursive steps is bounded by $O(|\mathcal{X}| + |\mathcal{X}|/2 + |\mathcal{X}|/4 + \dots) = O(|\mathcal{X}|) = O(mn)$. In each recursive step, we need to call the decision algorithm once, and therefore, the total time of the decision algorithm in all recursive steps is $O(n \log^2 m)$. Hence, the above claim is proved.

We should point out that one may want to keep doing the above recursive procedure until X has at most one or two remaining values, which would need $\Theta(\log mn)$ recursive steps; but according to our above time complexity analysis, it would take $O(mn + n \log m \log mn)$ time, which may not necessarily be bounded by

$O(mn)$ (e.g., when $m = o(\log n)$). In fact, performing the above recursive steps until the remaining values of $\mathcal{X}$ is at most $n/4$ is an interesting and crucial part of our techniques.

Let $\mathcal{X}'$ be the set of remaining values of $\mathcal{X}$. Hence, $|\mathcal{X}'| \leq n/4$. Let x' and x'' denote the smallest and largest values of $\mathcal{X}'$, respectively. According to our above way of computing $\mathcal{X}'$, $\mathcal{X}'$ consists of all values of $\mathcal{X}$ in $[x', x'']$, and further, $x^* \in [x', x'']$. Since $|\mathcal{X}'| \leq n/4$, there are at most $n/4$ pairs (P_i, P_j) of Σ such that $X(i, j)$ contains a value in $\mathcal{X}'$. In other words, we have the following observation.

Observation 1 *There are at least $n/4$ pairs (P_i, P_j) of Σ such that $X(i, j)$ does not contain any value of $\mathcal{X}'$ (i.e., no value of $X(i, j)$ is in the interval $[x', x'']$).*

Let Σ' denote the set of pairs of Σ that satisfy the condition in Observation 1 (i.e., for each pair (P_i, P_j) of Σ' , no value of $X(i, j)$ is in the interval $[x', x'']$). Hence, $|\Sigma'| \geq n/4$. Consider any pair $(P_i, P_j) \in \Sigma'$. Let $X(i, j) = x_1, x_2, \dots, x_t$ with $x_0 = -\infty$ and $x_{t+1} = +\infty$. By Observation 1, since no value of $X(i, j)$ is in the interval $[x', x'']$, $[x', x'']$ is contained in $[x_k, x_{k+1}]$ for some k with $0 \leq k \leq t$. Thus, either $\text{Ed}(x, P_i) \leq \text{Ed}(x, P_j)$ or $\text{Ed}(x, P_i) > \text{Ed}(x, P_j)$ holds for $x \in [x', x'']$. Without loss of generality, we assume $\text{Ed}(x, P_i) \leq \text{Ed}(x, P_j)$ holds for $x \in [x', x'']$. Since $x^* \in [x', x'']$, we obtain that $\text{Ed}(x^*, P_i) \leq \text{Ed}(x^*, P_j)$. Therefore, P_i can be discarded. Since $|\Sigma'| \geq n/4$, we can discard at least $n/4$ uncertain points of $\mathcal{P}$.

Recall that $\mathcal{X}'$ has been computed and $|\mathcal{X}'| \leq n/4$. The above pruning procedure can be done in $O(mn)$ time. Indeed, we can find x' and x'' in $\mathcal{X}'$ in $O(n)$ time. To determine the set Σ' , suppose for each element $x \in \mathcal{X}$, we have associated the pair (P_i, P_j) with x such that $x \in X(i, j)$; then Σ' can be determined in $O(n)$ time by scanning $\mathcal{X}'$. Next, for each pair (P_i, P_j) of Σ' , we determine the interval $[x_k, x_{k+1}]$ of X_{ij} (as defined above) that contains $[x', x'']$, which can be easily done in $O(m)$ since the values of $X(i, j)$ have already been sorted and $|X(i, j)| = O(m)$ by Corollary 2. Since the upper envelope of P_i and P_j has been computed, we can also determine whether $\text{Ed}(x, P_i) \leq \text{Ed}(x, P_j)$ or $\text{Ed}(x, P_i) > \text{Ed}(x, P_j)$ holds for $x \in [x', x'']$ in $O(m)$ time. Consequently, we can prune one of P_i and P_j . Hence, the pruning procedure can be done in $O(mn)$ time.

The above discussion shows that within $O(mn)$ time, we can prune at least $n/4$ uncertain points of $\mathcal{P}$ such that the center c^* is determined only by the remaining at most $3n/4$ uncertain points of $\mathcal{P}$.

We continue the above procedure recursively on the remaining uncertain points of $\mathcal{P}$ until a constant number of uncertain points of $\mathcal{P}$ remain. The total running time satisfies the recurrence $T(m, n) = T(m, 3n/4) + O(mn)$. Solving the recurrence yields $T(m, n) = O(mn)$. Let $\mathcal{P}'$ denote the set of remaining uncertain points of $\mathcal{P}$. We have $|\mathcal{P}'| = O(1)$. Let $\mathcal{H}'$ denote the set of functions $\text{Ed}(x, P_i)$ for all $P_i \in \mathcal{P}'$, and let $\mathcal{U}'$ be the upper envelope of the functions of $\mathcal{H}'$. According to our above pruning algorithm, v^* is also the lowest point of $\mathcal{U}'$. Note that in the case that the lowest point of $\mathcal{U}'$ is not unique, our pruning algorithm above makes sure that every lowest point of $\mathcal{U}'$ is a lowest point of $\mathcal{U}$ (we omit the detailed discussions).

Finally, we find the lowest point of $\mathcal{U}'$ by constructing it explicitly, which can be done in $O(m)$ time by a sweeping algorithm, as follows. We sweep a vertical line l from left to right. During the sweeping, we maintain the intersections of l with all functions of $\mathcal{H}'$ in the order sorted by their y -coordinates (note that the number of such intersections is $O(1)$ since $|\mathcal{H}'| = O(1)$). An event happens when l hits a vertex of some function or an intersection of two functions. Clearly, all functions of $\mathcal{H}'$ have $O(m)$ vertices since $|\mathcal{H}'| = O(1)$. As discussed in the proof of Lemma 3, every two functions of $\mathcal{H}'$ can intersect (transversally) $O(m)$ times. Therefore, the number of (transversal) intersections among all functions of $\mathcal{H}'$ is $O(m)$ since $|\mathcal{H}'| = O(1)$. Hence, the total number of events is $O(m)$. Processing each event can be done in constant time since $|\mathcal{H}'| = O(1)$. Therefore, we can explicitly construct $\mathcal{U}'$ and thus find its lowest point in $O(m)$ time.

Combining all above efforts, the lowest vertex v^* of $\mathcal{H}$ can be computed in $O(mn)$ time. Thus, the center of $\mathcal{P}$ can be computed in $O(mn)$ time.

Theorem 1. *The center of $\mathcal{P}$ can be computed in $O(mn)$ time.*

4 Concluding Remarks

In fact, we can easily extend our algorithm for the above geometric problem to more general functions. Specifically, let $\mathcal{H}$ be a set of n unimodal functions in the plane such that each function consists of m pieces

with each piece being a constant-sized algebraic curve segment with the following property: any two such curve segments of all functions of $\mathcal{H}$ intersect (transversally) at most a constant number of times. Then, our techniques can be used to find the lowest point in the upper envelope of all functions of $\mathcal{H}$ in $O(mn)$ time.

It would be interesting to see whether the techniques proposed in this paper can be used to solve other related problems.

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