Improved Bounds on the Probability of the Union of Events Some of Whose Intersections are Empty

Kunikazu Yoda^{a,*}, András Prékopa^a

^a*RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, U.S.A.*

Abstract

We formulate a linear program whose optimal objective function value can be used in other formulations to yield improved upper and lower bounds on the probability of the union of events if we know some empty intersections of small numbers of events. The LP relaxation of an extension of the maximum independent set problem provides an upper bound on the largest number of events that have a nonempty intersection. We present numerical experiments demonstrating the effectiveness of our formulation.

Keywords: probability bounds, union of events, empty intersections, maximum independent set problem
2010 MSC: Primary: 60C05; secondary: 90C05

1. Introduction

Computing the probability of the union of events is important in reliability theory, stochastic programming, and other sciences concerned with stochastic systems. In network reliability, consider a communication network with nodes and links, each with a probability of failure. The two-terminal reliability of a pair of nodes is the probability of the union of events, each of which occurs when a path between the two nodes consists of links without failure. The all-terminal reliability is the probability of the union of events, each of which occurs when a spanning tree of the network consists of links without failure. In probabilistic constrained stochastic programming, a joint probabilistic constraint for random variables $\xi_1, \dots, \xi_n$ specifies a lower bound on $\mathbb{P}(\xi_1 \leq x_1 \cap \dots \cap \xi_n \leq x_n) = 1 - \mathbb{P}(\xi_1 > x_1 \cup \dots \cup \xi_n > x_n)$, which involves the probability of the union of events. Although it is very hard to compute the exact probability of the union of a large number of events, we can compute its approximation by using the probabilities of individual events and intersections of a small number of events.

Let $\{A_1, \dots, A_n\}$ be any set of n ($\in \mathbb{Z}_{>0}$) events in an arbitrary probability space and

$$N := \{1, \dots, n\}$$

be the set of positive integers at most n . For any finite set I , we designate its cardinality (i.e., the number of elements in I) by $|I|$. For any subset $I \subseteq N$, we introduce a notation for the probability of the intersection of events A_i for all

$i \in I$ as follows:

$$p_I := \mathbb{P} \left(\bigcap_{j \in I} A_j \right).$$

We introduce binomial moments S_i for all $i \in N$ as follows:

$$S_i := \sum_{I \subseteq N: |I|=i} p_I. \quad (1)$$

Let ν denote the random number of events among $A_1, \dots, A_n$ that occur. Then we have the following relation for all $i \in N$:

$$\mathbb{E} \left[\binom{\nu}{i} \right] = \sum_{j=0}^n \binom{j}{i} \mathbb{P}(\nu = j) = S_i,$$

where we employ the extended definition of binomial coefficients, in which $\binom{j}{i} = 0$ for all $i, j \in \mathbb{Z}_{\geq 0}$ such that $i > j$. The value S_i is called the i -th binomial moment of ν .

The classical inclusion-exclusion principle [1] (see also Prékopa [2]) gives the probability of the union of events by using binomial moments S_i for all $i \in N$ as follows:

$$\mathbb{P}(A_1 \cup \dots \cup A_n) = S_1 - S_2 + \dots + (-1)^{n-1} S_n. \quad (2)$$

However, this formula is impractical if the number of events n is large, in which case the calculation of S_i is intractable unless i is small enough (close to 1) or large enough (close to n) since we need $\binom{n}{i}$ sums in (1). We can still calculate S_i for a few small i and compute lower and upper bounds on the probability. Let $m \in N$ be the largest number of events the probability of whose intersection is used in approximating the probability of the union and

$$M := \{1, \dots, m\}.$$

*Corresponding author

Email addresses: kunikazu.yoda@rutgers.edu (Kunikazu Yoda), prekopa@rutcor.rutgers.edu (András Prékopa)

be the set of positive integers at most m ; we use only p_I for all $I \subseteq N$ such that $|I| \in M$, from which $S_1, \dots, S_m$ can be calculated by (1). In practice, we usually consider a small $m \ll n$, mostly such as $m = 2, 3, 4$. The well-known Bonferroni inequalities (or bounds) [3] state that for any $m \in N$,

$$\mathbb{P}(A_1 \cup \dots \cup A_n) \begin{cases} \geq \\ \leq \end{cases} S_1 - S_2 + \dots + (-1)^{m-1} S_m \quad \begin{cases} \text{if } m \text{ is even} \\ \text{if } m \text{ is odd} \end{cases}.$$

These bounds are usually very weak, often out of $[0, 1]$. Then the best possible (also called sharp) bounds using $S_1, \dots, S_m$ for a small m have been found. In the case $m = 2$, the sharp lower and upper bounds expressed as closed forms in terms of n, S_1, S_2 are obtained by Dawson & Sankoff [4] and Kwerel [5], respectively (see also Prékopa [6] and Boros & Prékopa [7]). In the case $m = 3$, the sharp bounds expressed as closed forms in terms of n, S_1, S_2, S_3 are obtained by Boros & Prékopa [7]. In the case $m = 4$, the sharp upper bound expressed as a closed form in terms of n, S_1, S_2, S_3, S_4 is obtained by Boros & Prékopa [7]. For a general m , Prékopa [6] observed that all these sharp bounds are optimal objective function values of certain linear programs, known as binomial moment problems. Furthermore, sharp bounds on the probabilities that exactly/at least r events occur are given in Prékopa [8], and the bounds on the probabilities and expectations of convex functions of discrete random variables are given in Prékopa [9]. A binomial moment S_i carries an aggregated information over event probabilities p_I , using every one of which without aggregation we can obtain better bounds. Hailperin [10] formulated linear programs with an exponential number of variables, known as Boolean probability bounding problems, which give sharp bounds using disaggregated event probabilities.

A pair of binomial moment problems (also called the aggregated LP problems, for probability bounds of the union of events) is formulated as follows [6]:

$$\begin{aligned} \min/\max \quad & \sum_{j \in N} x_j \\ \text{subject to} \quad & \sum_{j \in N} \binom{j}{i} x_j = S_i \quad \text{for } i \in M \\ & x_j \geq 0 \quad \text{for } j \in N. \end{aligned} \quad (3)$$

The optimal objective function values of these minimization and maximization problems are the sharp lower and upper bounds, respectively, on the probability of the union of n events that can be computed using only the aggregated information S_i for all $i \in M$.

A pair of Boolean probability bounding problems (also called the disaggregated LP problems, for probability bounds

of the union of events) is formulated as follows [10]:

$$\begin{aligned} \min/\max \quad & \sum_{J \subseteq N: |J| \in N} x_J \\ \text{subject to} \quad & \sum_{J \subseteq N: |J| \in N} a_{IJ} x_J = p_I \quad \text{for } I \subseteq N: |I| \in M \\ & x_J \geq 0 \quad \text{for } J \subseteq N: |J| \in N, \end{aligned} \quad (4a)$$

where we define

$$a_{IJ} := \begin{cases} 1 & \text{if } I \subseteq J \\ 0 & \text{otherwise} \end{cases}. \quad (4b)$$

The optimal objective function values of these minimization and maximization problems are the sharp lower and upper bounds, respectively, on the probability of the union of n events that can be computed using only the disaggregated information p_I for all $I \subseteq N$ such that $|I| \in M$. Since such a set of event probabilities is almost always the most detailed information we can use, the bounds are the best possible we can expect in general. These problems are, however, impractical owing to an exponential number $(2^n - 1)$ of the decision variables x_J .

Although we cannot solve the disaggregated LP problems for a large number of events n in practice, we may extract useful information from disaggregated event probabilities without dealing with an exponential size and use it with their aggregations to obtain improved bounds.

2. Improved bounds

We assume that all probabilities of intersections of small numbers of events (formally, p_I for all $I \subseteq N$ such that $|I| \in M \subset N$) are known and some of them are 0 (i.e., such intersections are empty). The smallest number of events that have an empty intersection is 2 since we may use only nonempty individual events when computing their union; hence, we assume that $m \geq 2$. For any nonempty subsets $I, J \subseteq N$ such that $I \subseteq J$ and any $\varepsilon \in [0, 1]$, we have the implication $p_I \leq \varepsilon \implies p_J \leq \varepsilon$. In particular when $\varepsilon = 0$, we have the following implication:

$$p_I = 0 \implies p_J = 0,$$

which suggests that we are likely to have more empty intersections of a larger number of events even if we have a few empty intersections of small numbers of events; if at least one intersection is empty, then the intersection of the largest number of events is empty (i.e., $p_N = 0$, where $|N| = n$). Therefore, there exists the largest number $r \in \{1, \dots, n-1\}$ of events that have a nonempty intersection; all intersections of a larger number of events are empty (i.e., $p_I = 0$ for all I such that $|I| \in \{r+1, \dots, n\}$). We want to find r , but since it is impossible to find the exact number with known limited information about event probabilities, we find an upper bound on it instead.

Fist, we consider the case $m = 2$ for ease of understanding. We can find the sharp upper bound by solving the maximum independent set problem as follows. Consider a simple graph $G = (V, E)$ of $n = |V|$ vertices corresponding to n events $A_1, \dots, A_n$, respectively. The graph G has an edge between two distinct vertices $i, j \in V$ if and only if the intersection of the corresponding two events is empty: $(i, j) \in E \iff \mathbb{P}(A_i \cap A_j) = 0$. An independent set (also known as a stable set) is a subset of V that has no edge between its vertices. The size of an independent set is the number of vertices in it. The maximum independent set problem is to find a largest independent set for a given undirected graph. Thus, the intersection of the events corresponding to the vertices in any larger subset than a maximum independent set is empty since the subset must contain a pair of connected vertices. The intersection of the events corresponding to the vertices in a maximum independent set could be empty or nonempty although all pairwise intersections of events are nonempty. Therefore, the size of a maximum independent set is the sharp upper bound knowing only all probabilities of individual and pairwise intersections of events. The problem is $\mathcal{NP}$ -hard, so it is unlikely that we can find the exact solution efficiently. Instead, we can solve the following LP relaxation efficiently:

$$\begin{aligned} & \text{maximize} && \sum_{j \in V} x_j \\ & \text{subject to} && x_i + x_j \leq 1 \text{ for } (i, j) \in E, \\ & && 0 \leq x_j \leq 1 \text{ for } j \in V. \end{aligned}$$

Now, we consider the general case $m \geq 2$, forgetting about the graph representation but thinking of extending the mathematical representation. We formulate a generalization as the following linear program:

$$\begin{aligned} & \text{maximize} && z = \sum_{j \in N} x_j \\ & \text{subject to} && \sum_{j \in I} x_j \leq k - 1 \\ & && \text{for } I \subseteq N : |I| = k, p_I = 0 \\ & && \text{for } k = 2, \dots, m \\ & && 0 \leq x_j \leq 1 \text{ for } j \in N. \end{aligned} \quad (5)$$

Although the optimal objective function value of this relaxed problem is an upper bound on that of the underlying 0-1 integer program, which is itself the sharp upper bound knowing only all probabilities of intersections of at most m events, such an upper bound is still good, if not the best, for our purpose.

Theorem 2.1. *The optimal objective function value z^* of the linear program (5) gives an upper bound on the largest number of events that have a nonempty intersection:*

$$\begin{aligned} & p_I = 0 \text{ for all } I \subseteq N : |I| = i \quad (\iff S_i = 0) \\ & \text{for all } i = \lfloor z^* \rfloor + 1, \dots, n. \end{aligned}$$

Proof. The equivalence to $S_i = 0$ is derived by (1). Any subset $I \subseteq N$ corresponds to a subset of events $\{A_i : i \in I\}$. Consider the underlying 0-1 integer program, where the constraint $0 \leq x_j \leq 1$ is replaced with $x_j \in \{0, 1\}$ in (5). A solution to the problem represents a subset of N , where x_j is 1 if j is in the subset and is 0 otherwise for each $j \in N$; the objective function value is the cardinality of the subset; a feasible solution is a subset that contains no subset with a cardinality at most m having an empty intersection of its events. The problem is to find a feasible solution with the largest cardinality. Consequently, any subset with a larger cardinality than the optimal objective function value, which must contain a subset with a cardinality at most m having an empty intersection, has an empty intersection of its events. Therefore, the optimal objective function value is the sharp upper bound on the largest number of events that have a nonempty intersection knowing only all probabilities of intersections of at most m events. Since a relaxation of a maximization problem gives an upper bound on the objective function value of the original problem, the claim follows. $\square$

Let us introduce the following notations:

$$\begin{aligned} & z^* : \text{the optimal objective function value of (5)} \\ & u := \lfloor z^* \rfloor, \quad U := \{1, \dots, u\}. \end{aligned} \quad (6)$$

If $u \leq m$, then since $S_1, \dots, S_u$ are known and $S_{u+1} = \dots = S_n = 0$, we can use the inclusion-exclusion principle (2) to compute the exact probability. Hence we assume that $u > m$. To obtain improved lower and upper bounds, we use u in binomial moment problems (3) as follows:

$$\begin{aligned} & \text{min/max} && \sum_{j \in N} x_j \\ & \text{subject to} && \sum_{j \in N} \binom{j}{i} x_j = S_i \text{ for } i = 1, \dots, m \\ & && \sum_{j \in N} \binom{j}{i} x_j = 0 \text{ for } i = u + 1, \dots, n \\ & && x_j \geq 0 \text{ for } j \in N. \end{aligned}$$

Since solutions to these problems give $x_j = 0$ for $j = u + 1, \dots, n$, we can simplify the problems as follows:

$$\begin{aligned} & \text{min/max} && \sum_{j \in U} x_j \\ & \text{subject to} && \sum_{j \in U} \binom{j}{i} x_j = S_i \text{ for } i \in M \\ & && x_j \geq 0 \text{ for } j \in U. \end{aligned} \quad (7)$$

The problems (7) are obtained by including more constraints to the problems (3). Consequently, the optimal objective function values of the minimization and maximization problems in the former are no less and no more, respectively, than those in the latter; that is, both lower and upper bounds from the former are at least as good as those from the latter.

Corollary 2.1. *The optimal objective function values of the minimization and maximization problems (7) give equal or improved lower and upper bounds, respectively, on the probability of the union of events against those from binomial moment problems (3).*

The problems (7) are very similar to (3). The only difference is that N (equivalently n) is replaced with U (equivalently u). The optimal objective function values of (3) have closed forms for $m = 2, 3, 4$ [4, 5, 7]. Accordingly, those of (7) also have closed forms.

Corollary 2.2. *Let u be the positive integer computed by (5) and (6). The lower and upper bounds on the probability of the union of events given by the optimal objective function values of (7) have closed forms for $m = 2, 3, 4$ as follows:*

- The upper bound for $m = 2$:

$$S_1 - \frac{2}{u} S_2.$$

- The lower bound for $m = 3$:

$$\frac{i+2u-1}{(i+1)u} S_1 - \frac{2(2i+u-2)}{i(i+1)u} S_2 + \frac{6}{i(i+1)u} S_3,$$

where $i := 1 + \left\lfloor \frac{2((u-2)S_2 - 3S_3)}{(u-1)S_1 - 2S_2} \right\rfloor$.

- The upper bound for $m = 4$:

$$S_1 - \frac{2((i-1)(i-2) + (2i-1)u)}{i(i+1)u} S_2 + \frac{6(2i+u-4)}{i(i+1)u} S_3 - \frac{24}{i(i+1)u} S_4,$$

where

$$i := \left\lfloor \frac{2(u-2)S_2 + 3(u-5)S_3 - 12S_4}{(u-2)S_2 - 3S_3} \right\rfloor.$$

The closed form lower bound for $m = 2$ and upper bound for $m = 3$ for (7) are the same as the original closed form bounds for those m for (3), respectively, since they do not depend on n . As far as we know, the closed form lower bound for $m = 4$ for (3) has not been found yet, but it can be shown that it does not depend on n , so any lower bound for $m = 4$ by our formulations (7) is also the same as that from binomial moment problems (3).

Applying the same type of reformulation as that of obtaining (7) from binomial moment problems (3), we can obtain a reformulation of Boolean probability bounding problems (4) using U as follows:

$$\begin{aligned} & \min/\max \sum_{J \subseteq N: |J| \in U} x_J \\ & \text{subject to} \sum_{J \subseteq N: |J| \in U} a_{IJ} x_J = p_I \quad \text{for } I \subseteq N: |I| \in M \\ & \quad x_J \geq 0 \quad \text{for } J \subseteq N: |J| \in U. \end{aligned}$$

Although the number of the decision variables x_J is still exponentially large, the reduction from the original problem is exponentially large as well.

3. Numerical experiments

We carried out numerical experiments to compare probability bounds by our formulations to those by binomial moment problems with data sets that are expected to contain empty intersections of events. We used the GurobiTM Optimizer (version 6.0) [11] for solving linear programs; when we solve binomial moment problems, for which numerical inaccuracy of the feasibility of solutions is sometimes encountered, we set the feasibility tolerance parameter ('FeasibilityTol') to its minimum value (1e-9). As mentioned following Corollary 2.2, improvements on bounds can be seen only in upper bounds for $m = 2$, lower bounds for $m = 3$, and upper bounds for $m = 4$; bounds in the other cases for $m = 2, 3, 4$ are always unchanged.

3.1. Subsets of a set

Given a set and its subsets, each associated with an individual event probability, we want to approximate the probability of the union of the events. We consider a case in which the given set is a unit square $\{(x, y) \mid x \in [0, 1], y \in [0, 1]\}$ on a plane, its subsets are rectangles whose sides are parallel to those of the unit square containing them, and the event probability associated with a rectangle is its area. We generate a test data set as follows. For each $i \in N$, we choose independently and uniformly randomly two distinct x -values x_{i1}, x_{i2} such that $0 \leq x_{i1} < x_{i2} \leq 1$ and two distinct y -values y_{i1}, y_{i2} such that $0 \leq y_{i1} < y_{i2} \leq 1$ to construct the rectangle formed by the x, y -values, which we associate with an event A_i whose probability is the area of the rectangle as follows:

$$\mathbb{P}(A_i) := (x_{i2} - x_{i1})(y_{i2} - y_{i1}) \in (0, 1].$$

For any subset $I \subseteq N$, we define the probability of the intersection of events A_i for all $i \in I$ as the area of the intersection of the rectangles, which is also a rectangle or empty, as follows:

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) := \begin{cases} (x_{I2} - x_{I1})(y_{I2} - y_{I1}) & \in (0, 1] \\ \quad \text{if } x_{I2} > x_{I1} \text{ and } y_{I2} > y_{I1} \\ 0 & \text{otherwise} \end{cases},$$

where we define

$$\begin{aligned} x_{I1} &:= \max_{i \in I} \{x_{i1}\}, & x_{I2} &:= \min_{i \in I} \{x_{i2}\} \\ y_{I1} &:= \max_{i \in I} \{y_{i1}\}, & y_{I2} &:= \min_{i \in I} \{y_{i2}\}. \end{aligned}$$

The result of the experiment is shown in Table 1. The first column n shows the number of events (rectangles), with each of which we generated test data sets. The second column 'Exact' shows the exact probability calculated by the inclusion-exclusion principle (2). For each m in the third column, we computed lower and upper bounds ('LB' and 'UB' columns, respectively) on the probability by our formulations (5)–(7) and by binomial moment problems (3) in the fourth through seventh columns. Each number in 'Exact', 'LB', and 'UB' columns is the arithmetic average over five test data sets with the same n .

Table 1: Lower and upper bounds on probabilities concerning subsets of a set (rectangles in a unit square)

(2)		Ours (5)–(7)			Orig (3)	
n	Exact	m	LB	UB	LB	UB
10	0.58781	2	0.49450	0.83593	0.49450	0.98191
		3	0.53772	0.62360	0.51397	0.62360
		4	0.58173	0.59674	0.58173	0.61159
15	0.68110	2	0.53035	1.24706	0.53035	1.52289
		3	0.58024	0.77992	0.55488	0.77992
		4	0.64619	0.73165	0.64619	0.75471
20	0.73385	2	0.57037	1.46132	0.57037	1.71017
		3	0.61238	0.85053	0.59090	0.85053
		4	0.69259	0.80407	0.69259	0.82730
25	0.78572	2	0.61959	1.90242	0.61959	2.30043
		3	0.65690	0.94528	0.63601	0.94528
		4	0.74032	0.88939	0.74032	0.91942
30	0.82538	2	0.60623	2.33700	0.60623	2.86952
		3	0.66040	1.14469	0.63200	1.14469
		4	0.75078	1.02568	0.75078	1.08940

3.2. Probabilistic Boolean satisfiability problem (PSAT)

Given a logical function of Boolean variables and probabilities that the variables take 0 or 1, we want to approximate the probability that the function value takes 1 (i.e., the logical function is satisfied by realizations of the random Boolean variables). The satisfiability of a propositional logic formula (also known as Boolean expression) in conjunctive normal form (CNF), e.g., $(x_1 \vee x_2) \wedge (x_2 \vee \bar{x}_3)$, is equivalent to the unsatisfiability of its logical complement (also called negation) in disjunctive normal form (DNF), e.g., $(\bar{x}_1 \wedge \bar{x}_2) \vee (\bar{x}_2 \wedge x_3)$. We consider the latter form.

We generate a test data set for PSAT according to an example in Boros & Prékopa [12]. Let $v, s \in \mathbb{Z}_{>0}$ such that $v > s$ be the number of Boolean variables and the average number of literals (each of which is either a single variable x_j or its negation $\bar{x}_j$) in a conjunctive clause (which is a conjunction of literals), respectively. Let $V := \{1, \dots, v\}$ denote the set of positive integers at most v . First, we create a constant Boolean vector $\mathbf{b} \in \{0, 1\}^v$ each of whose entries is 1 with probability $1/2$ independent of any other. Second, we generate a random Boolean expression in DNF that is unsatisfied by $\mathbf{b}$. For each $i \in N$, we consider independently a random polynomial of v variables $(x_1, \dots, x_v) = \mathbf{x}$ of the form

$$Q_i(\mathbf{x}) := \prod_{j \in P_i} x_j \prod_{j \in N_i} (1 - x_j),$$

where disjoint subsets $P_i, N_i \subseteq V$ are randomly generated in such a way that for each $j \in V$, we determine $j \in P_i$ with probability $s/2v$, $j \in N_i$ with probability $s/2v$, or $j \in V \setminus (P_i \cup N_i)$ with probability $1 - s/v$, mutually exclusively. If $Q_i(\mathbf{b}) \neq 0$ then we repeat the same step until we get $Q_i(\mathbf{b}) = 0$ and then proceed to the next i . To avoid trivialities, we skip those polynomials that have degrees

less than 2. Thus the polynomial

$$Q(\mathbf{x}) := Q_1(\mathbf{x}) + \dots + Q_n(\mathbf{x})$$

can be regarded as a Boolean expression in DNF that is unsatisfiable, because $Q(\mathbf{b}) = 0$. Third, let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_v)$ be a random Boolean vector each of whose entry ξ_i takes 1 with probability q_i and 0 with $1 - q_i$ independent of any other, where we used a constant probability vector $\mathbf{q} = (q_1, \dots, q_v) := (1/2, \dots, 1/2)$ in this experiment. Finally, we define a set of events $\{A_1, \dots, A_n\}$ in such a way that each A_i occurs if and only if $Q_i(\boldsymbol{\xi}) = 1$. The probability of the event A_i is expressed as follows:

$$\mathbb{P}(A_i) = \mathbb{P}(Q_i(\boldsymbol{\xi}) = 1) = Q_i(\mathbf{q}).$$

For any subset $I \subseteq N$, we define the following polynomial:

$$Q_I(\mathbf{x}) := \begin{cases} \prod_{j \in P_I} x_j \prod_{j \in N_I} (1 - x_j) & \text{if } P_I \cap N_I = \emptyset \\ 0 & \text{otherwise} \end{cases},$$

where $P_I := \bigcup_{i \in I} P_i$ and $N_I := \bigcup_{i \in I} N_i$. The probability of the intersection of events A_i for all $i \in I$ is expressed as follows:

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \mathbb{P}(Q_I(\boldsymbol{\xi}) = 1) = Q_I(\mathbf{q}).$$

The probability that $Q(\boldsymbol{\xi})$ is unsatisfied, which must be positive since $Q(\mathbf{x})$ is unsatisfiable, has the following equalities:

$$\begin{aligned} \mathbb{P}(Q(\boldsymbol{\xi}) = 0) &= \mathbb{P}(Q_i(\boldsymbol{\xi}) = 0 \text{ for all } i \in N) \\ &= 1 - \mathbb{P}(Q_i(\boldsymbol{\xi}) = 1 \text{ for at least one } i \in N) \\ &= 1 - \mathbb{P}(A_1 \cup \dots \cup A_n). \end{aligned}$$

Therefore, upper and lower bounds (denoted by UB and LB, respectively) on the probability of the union of A_i for all $i \in N$ can be used to calculate lower and upper bounds (by $1 - \text{UB}$ and $1 - \text{LB}$, respectively) on the probability that $Q(\boldsymbol{\xi})$ is unsatisfied (i.e., its logical complement in CNF is satisfied).

The result of the experiment is shown in Table 2. The notations are similar to those of Table 1 explained in Section 3.1. We used the same $v = 20$ and varying s and n to generate a test data set. Each number in ‘LB’ and ‘UB’ columns is the arithmetic average over five test data sets with the same parameters n , v , and s .

4. Applications

Approximating the probability of the union of events has applications in network reliability (Boros et al. [13], Prékopa & Boros [14], and Gao & Prékopa [15]; see also Ball et al. [16]) and system reliability (Habib & Szántai [17] and Unuvar et al. [18]; see also a survey by Chao et al. [19]).

Table 2: Lower and upper bounds on probabilities concerning PSAT

$v = 20$			Ours (5)–(7)		Orig (3)	
n	s	m	LB	UB	LB	UB
40	7	2	0.51320	0.81551	0.51320	0.83334
		3	0.52137	0.59186	0.51728	0.59186
		4	0.57541	0.58965	0.57541	0.59075
40	10	2	0.09957	0.10546	0.09957	0.10579
		3	0.09964	0.10005	0.09961	0.10005
		4	0.10002	0.10005	0.10002	0.10005
70	7	2	0.57081	1.19882	0.57081	1.22141
		3	0.57846	0.74975	0.57464	0.74975
		4	0.69255	0.74573	0.69255	0.74774
70	10	2	0.18083	0.19670	0.18083	0.19718
		3	0.18090	0.18169	0.18086	0.18169
		4	0.18166	0.18169	0.18166	0.18169
100	7	2	0.67961	1.83928	0.67961	1.87594
		3	0.68723	0.95915	0.68342	0.95915
		4	0.81281	0.95183	0.81281	0.95549
100	10	2	0.25763	0.29790	0.25763	0.29873
		3	0.25785	0.26123	0.25774	0.26123
		4	0.26103	0.26122	0.26103	0.26123

In most real-world examples, not a small number of intersections of events are often empty. Many intersections of half-spaces used in expressing the complement of a Euclidean polyhedron are empty (Bukszár et al. [20]); empty intersections of convex sets would also appear in a similar problem. In the probabilistic Boolean satisfiability problem, as inferred by our example in Section 3.2, the conjunction of many conjunctive clauses is likely to be a contradiction. Example 2 in Prékopa [21] concerns calculating the transparency of a film darkened by a finite number of emulsion spheres, in which many intersections of circles with random locations and radii in the two-dimensional Euclidean plane are empty, similarly to our example in Section 3.1.

The methodology presented in this paper is also useful in theories and applications of random sets (Takács [22], Prékopa [21], Debreu [23], and Molchanov [24]). In this case, we are led to linear programs with random right-hand side vectors, which help us to develop statistical properties of random sets.

5. Acknowledgments

We are grateful to Prof. Endre Boros for his valuable comments. This research was supported by NSF Grant CMMI-0856663.

- [1] A. de Moivre, The doctrine of chances: or, a method of calculating the probability of events in play, W. Pearson, 1718.
- [2] A. Prékopa, Inclusion-exclusion formula, in: M. Hazewinkel (Ed.), Encyclopedia of Mathematics, Supplement III, Kluwer Academic Publishers, 2001, pp. 199–201.
- [3] C. E. Bonferroni, Teoria statistica delle classi e calcolo delle probabilità, Pubblicazioni del R Istituto Superiore di Scienze Economiche e Commerciali di Firenze 8 (1936) 3–62.
- [4] D. A. Dawson, D. Sankoff, An inequality for probabilities, Proceedings of the American Mathematical Society 18 (3) (1967) 504–507.

- [5] S. M. Kwerel, Most stringent bounds on aggregated probabilities of partially specified dependent probability systems, Journal of the American Statistical Association 70 (350) (1975) 472–479.
- [6] A. Prékopa, Boole-Bonferroni inequalities and linear programming, Operations Research 36 (1) (1988) 145–162.
- [7] E. Boros, A. Prékopa, Closed form two-sided bounds for probabilities that at least r and exactly r out of n events occur, Mathematics of Operations Research 14 (2) (1989) 317–342.
- [8] A. Prékopa, Sharp bounds on probabilities using linear programming, Operations Research 38 (2) (1990) 227–239.
- [9] A. Prékopa, The discrete moment problem and linear programming, Discrete Applied Mathematics 27 (3) (1990) 235–254.
- [10] T. Hailperin, Best possible inequalities for the probability of a logical function of events, The American Mathematical Monthly 72 (4) (1965) 343–359.
- [11] Gurobi Optimization, Inc., Gurobi optimizer reference manual, <http://www.gurobi.com> (2015 (accessed August 27, 2015)).
- [12] E. Boros, A. Prékopa, Probabilistic bounds and algorithms for the maximum satisfiability problem, Annals of Operations Research 21 (1) (1989) 109–126.
- [13] E. Boros, A. Prékopa, K.-W. Lih, The use of binomial moments for bounding network reliability, in: F. Roberts, F. Hwang, C. Monma (Eds.), Reliability of Computer and Communication Networks, Vol. 5 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 1991, pp. 197–212.
- [14] A. Prékopa, E. Boros, On the existence of a feasible flow in a stochastic transportation network, Operations Research 39 (1) (1991) 119–129.
- [15] L. Gao, A. Prékopa, On performance prediction of cellular telephone networks, RUTCOR Research Report 44-2001, Rutgers Center for Operations Research (2001).
- [16] M. O. Ball, C. J. Colbourn, J. S. Provan, Network reliability, in: M. O. Ball, T. L. Magnanti, C. L. Monma, G. L. Nemhauser (Eds.), Network Models, Vol. 7 of Handbooks in Operations Research and Management Science, Elsevier Science B.V., 1995, Ch. 11, pp. 673–762.
- [17] A. Habib, T. Szántai, New bounds on the reliability of the consecutive k -out-of- r -from- n :F system, Reliability Engineering & System Safety 68 (2) (2000) 97–104.
- [18] M. Unuvar, E. E. Ozguven, A. Prékopa, Optimal capacity design under k -out-of- n and consecutive k -out-of- n type probabilistic constraints, Annals of Operations Research 226 (1) (2015) 643–657.
- [19] M. T. Chao, J. C. Fu, M. V. Koutras, Survey of reliability studies of consecutive- k -out-of- n :F & related systems, IEEE Transactions on Reliability 44 (1) (1995) 120–127.
- [20] J. Bukszár, R. Henrion, M. Hujter, T. Szántai, Polyhedral inclusion-exclusion, Stochastic Programming E-Print Series 2004 (17) (2004 (accessed August 27, 2015)), <http://www.speps.org>.
- [21] A. Prékopa, On secondary processes generated by a random point distribution of Poisson type, Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica 1 (1958) 153–170.
- [22] L. Takács, On the method of inclusion and exclusion, Journal of the American Statistical Association 62 (317) (1967) 102–113.
- [23] G. Debreu, Mathematical Economics: Twenty Papers of Gerard Debreu, no. 4 in Econometric Society Monographs, Cambridge University Press, 1986.
- [24] I. Molchanov, Theory of Random Sets, Probability and Its Applications, Springer Science+Business Media, 2006.