A Polyhedral Characterization of the Inverse-Feasible Region of a Mixed-Integer Program

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Abstract

Given a feasible solution x^0 to a mixed-integer program (MIP), the inverse MIP problem is to find an objective d such that x^0 is optimal for the MIP with objective function d , and among all such objectives, the distance from a given target objective is minimized. By using a novel expression for the MIP value function, we formulate the inverse MIP problem as a linear program (LP), albeit of exponentially large size.

Keywords: Inverse optimization, Mixed-integer programming, Superadditive duality, Value functions.

1. Introduction

Given a mixed-integer program (MIP), a feasible solution x^0 to the MIP, the inverse MIP problem is to find an *inverse-feasible* objective d , that is, an objective for which x^0 is optimal for the MIP with objective function d , and among all inverse-feasible objectives, find the one for which the (weighted) distance from a target objective d^0 is minimized.

Zhang and Liu [11] first developed a general method for solving inverse linear programs (LPs) based on LP duality. Ahuja and Orlin [1] showed that given a polynomial time solvable optimization problem, its respective inverse

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optimization problem is also polynomially solvable in most cases under the L_1 and L_∞ norms. A related stream of research is inverse combinatorial optimization; see the survey by Heuberger [7] for more details.

Less is known about inverse integer and inverse mixed-integer programs. Schaefer [8] characterized the set of inverse-feasible objectives D^* for pure integer programs as a polyhedron, implying that the inverse integer programming problem may be formulated as a linear program. Wang [9], and Wang and Duan [10] developed algorithms for solving inverse MIPs, but the structure of inverse MIPs remains an open question.

In Section 2 we provide necessary definitions and cite preliminary work. In Section 3 we give a polyhedral characterization of the set of inverse MIP objectives using a novel superadditive dual to MIP. This indicates that inverse MIPs can be solved as LPs, albeit of exponentially large size.

2. Definitions and Preliminary Results

Let $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of non-negative integers and reals, respectively. Given $A \in \mathbb{Z}_+^{m \times n}$, $G \in \mathbb{Z}_+^{m \times p}$, and $b \in \mathbb{R}_+^m$, let $D = \{(c, h) \in \mathbb{R}_+^{n+p} : Hc + Mh \leq r\}$ be a polyhedral set of objective functions of interest, where $H \in \mathbb{R}^{m_1 \times n}$, $M \in \mathbb{R}^{m_1 \times p}$, and $r \in \mathbb{R}^{m_1}$. For any $(c, h) \in D$, we define the parameterized mixed-integer program $MIP(c, h)$:

$$\begin{aligned} & \max c^T x + h^T y \\ & \text{subject to} \\ & \quad Ax + Gy \leq b, \\ & \quad x_j \in \mathbb{Z}_+, \text{ for } j \in I, \\ & \quad y_j \in \mathbb{R}_+, \text{ for } j \in C, \end{aligned} \tag{1}$$

where $|I| = n$ and $|C| = p$. For $\beta \in \mathbb{R}_+^m$, define the hyperrectangle $B(\beta) = \prod_{i=1}^m [0, \beta_i]$, and for $\beta \notin \mathbb{R}_+^m$, let $B(\beta) = \emptyset$. For $\beta \in \mathbb{R}^m$, define $\hat{B}(\beta) = B(\beta) \cap \mathbb{Z}^m$. We denote the i th column of A and G by a_i and g_i , respectively. We assume that $a_i \in \hat{B}(b)$ for each $i \in I$, as $x_i = 0$ in any feasible solution otherwise. We also assume that $a_i \neq 0$ for $i = 1, \dots, n$, and $b_i \neq 0$ for $i = 1, \dots, m$.

For any $(c, h) \in D$, the value function of $MIP(c, h)$, $z_{MIP}^{c,h} : \mathbb{R}^m \rightarrow \mathbb{R}$, is defined as

$$z_{MIP}^{c,h}(\beta) = \max \{c^T x + h^T y : (x, y) \in S(\beta)\}, \tag{2}$$

where

$$S(\beta) = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Gy \leq \beta\}. \quad (3)$$

If $S(\beta) = \emptyset$, we assume $z_{MIP}^{c,h}(\beta) = -\infty$. Note that for a fixed (c, h) , $z_{MIP}^{c,h}$ is nondecreasing and superadditive over $\mathbb{R}^m$ [3]. $SDMIP(c, h)$, the superadditive dual problem [3, 6] of $MIP(c, h)$, is given by

$$\min F^{c,h}(b) \quad (4a)$$

subject to

$$F^{c,h}(a_i) \geq c_i \text{ for all } i \in I, \quad (4b)$$

$$\bar{F}^{c,h}(g_i) \geq h_i \text{ for all } i \in C, \quad (4c)$$

$$F^{c,h} : \mathbb{R}^m \rightarrow \mathbb{R} \text{ nondecreasing and superadditive,} \quad (4d)$$

$$F^{c,h}(0) = 0, \quad (4e)$$

where $\bar{F}^{c,h}(\beta) := \lim_{\delta \rightarrow 0^+} F^{c,h}(\delta\beta)/\delta$. Blair and Jereslow [3] established that strong duality holds, and, in particular, $z_{MIP}^{c,h}$ is an optimal solution to $SDMIP(c, h)$. A straightforward extension of the approach of Schaefer [8] would introduce continuous variables for all points in the lattice $\hat{B}(b)$. However, constraints (4c) create complications, as it is not clear how to model the limit in constraints (4c). Consequently, we will consider an alternative approach based on the duals of the integer and continuous portions of the MIP.

For any $\beta \in \hat{B}(b)$ and for any $c \in \{c \geq 0 : Hc \leq r\}$, define $IP(c, \beta)$ as

$$\begin{aligned} & \max c^T x \\ & \text{subject to} \\ & Ax \leq \beta, \\ & x \in \mathbb{Z}_+^n. \end{aligned} \quad (5)$$

The value function of $IP(c, \beta)$, $z_{IP}^c : \mathbb{R}^m \rightarrow \mathbb{R}$, is defined as

$$z_{IP}^c(\beta) = \max_{x \in \mathbb{Z}_+^n} \{c^T x : Ax \leq \beta\}. \quad (6)$$

Note that z_{IP}^c is nondecreasing and superadditive over $\mathbb{R}^m$ [2]. $SDIP(c, \beta)$,

the superadditive dual [6] of $IP(c, \beta)$, is given by

$$\begin{aligned}
& \min F(\beta) \\
& \text{subject to} \\
& F(a_i) \geq c_i \text{ for all } i \in I, \\
& F : \mathbb{R}^n \rightarrow \mathbb{R} \text{ nondecreasing and superadditive,} \\
& F(0) = 0.
\end{aligned} \tag{7}$$

Strong and weak duality hold, and z_{IP}^c is an optimal solution for $SDIP(c, \beta)$ for all $\beta \in \hat{B}(b)$ [2].

For any $h \in \{h \geq 0 : Mh \leq r\}$, define $LP(h, \beta)$ as

$$\begin{aligned}
& \max h^T y \\
& \text{subject to} \\
& Gy \leq \beta, \\
& y \in \mathbb{R}_+^p.
\end{aligned} \tag{8}$$

The value function of $LP(h, \beta)$, $z_{LP}^h : \mathbb{R}^m \rightarrow \mathbb{R}$, is defined as

$$z_{LP}^h(\beta) = \max_{y \in \mathbb{R}_+^p} \{h^T y : Gy \leq \beta\}. \tag{9}$$

$DLP(h, \beta)$, the dual of $LP(h, \beta)$, is given by

$$\min \{\pi^T \beta : \pi^T G \geq h, \pi \in \mathbb{R}_+^m\}. \tag{10}$$

Strong and weak duality hold as well. We can express z_{LP}^h at each β as

$$z_{LP}^h(\beta) = \min_{\pi^T G \geq h, \pi \in \mathbb{R}_+^m} \pi^T \beta. \tag{11}$$

Proposition 1 expresses $z_{MIP}^{c,h}$ in terms of z_{IP}^c and z_{LP}^h .

Proposition 1. *For all $(c, h) \in D$ and $\beta \in \mathbb{R}^m$,*

$$z_{MIP}^{c,h}(\beta) = \max_{\beta' \in \hat{B}(\beta)} \{z_{IP}^c(\beta') + z_{LP}^h(\beta - \beta')\}. \tag{12}$$

Proof. Let $(c, h) \in D$, $\beta \in \mathbb{R}^m$, and define

$$g(\beta) = \max_{\beta' \in \hat{B}(\beta)} \{z_{IP}^c(\beta') + z_{LP}^h(\beta - \beta')\}.$$

Case 1 If $\beta \notin \mathbb{R}_+^m$, then $g(\beta) = z_{MIP}^{c,h}(\beta) = -\infty$ because $\hat{B}(\beta) = \emptyset$.

Case 2 If $\beta \in \mathbb{R}_+^m$ and $S(\beta) = \emptyset$, then $z_{MIP}^{c,h}(\beta) = -\infty$. Suppose that $g(\beta) > -\infty$. Then there exists $\tilde{\beta} \in \hat{B}(\beta)$ such that $z_{IP}^c(\tilde{\beta}) + z_{LP}^h(\beta - \tilde{\beta}) > -\infty$, which implies that there exists a feasible solution to $MIP(c, h)$, a contradiction. Thus, $g(\beta) = z_{MIP}^{c,h}(\beta) = -\infty$.

Case 3 If $z_{MIP}^{c,h}(\beta) = \infty$, then $S(\beta)$ is unbounded. Suppose that $g(\beta) < \infty$ and take

$$\beta^* \in \operatorname{argmax}_{\beta' \in \hat{B}(\beta)} \{z_{IP}^c(\beta') + z_{LP}^h(\beta - \beta')\}. \quad (13)$$

Then $z_{IP}^c(\beta^*) + z_{LP}^h(\beta - \beta^*) < \infty$, and hence there exists $(x^*, y^*) \in S(\beta)$ such that $c^T x^* + h^T y^* = z_{IP}^c(\beta^*) + z_{LP}^h(\beta - \beta^*)$. Because $S(\beta)$ is unbounded and $z_{MIP}^{c,h}(\beta) = \infty$, there exists $(\tilde{x}, \tilde{y}) \in S(\beta)$ such that $c^T \tilde{x} + h^T \tilde{y} > c^T x^* + h^T y^*$. Because $\tilde{x}$ is feasible for $IP(c, A\tilde{x})$ and $\tilde{y}$ for $LP(h, \beta - A\tilde{x})$, $z_{IP}^c(A\tilde{x}) \geq c^T \tilde{x}$ and $z_{LP}^h(\beta - A\tilde{x}) \geq h^T \tilde{y}$, respectively. Hence,

$$\begin{aligned} z_{IP}^c(A\tilde{x}) + z_{LP}^h(\beta - A\tilde{x}) &\geq c^T \tilde{x} + h^T \tilde{y} \\ &> c^T x^* + h^T y^* \\ &= z_{IP}^c(\beta^*) + z_{LP}^h(\beta - \beta^*). \end{aligned}$$

Since $A\tilde{x} \in \hat{B}(\beta)$, this contradicts (13). Hence, $g(\beta) = z_{MIP}^{c,h}(\beta) = \infty$.

Case 4 If $0 \leq z_{MIP}^{c,h}(\beta) < \infty$, then let $(x^*, y^*) \in S(\beta)$ be an optimal solution to $MIP(c, h)$. Because x^* is feasible for $IP(Ax^*)$ and y^* is feasible for $LP(\beta - Ax^*)$, $z_{IP}^c(Ax^*) \geq c^T x^*$ and $z_{LP}^h(\beta - Ax^*) \geq h^T y^*$, respectively. Since $Ax^* \in \hat{B}(\beta)$,

$$g(\beta) \geq z_{IP}^c(Ax^*) + z_{LP}^h(\beta - Ax^*) \geq c^T x^* + h^T y^*.$$

Suppose that $g(\beta) > c^T x^* + h^T y^*$. Then there exists $\tilde{\beta} \in \hat{B}(\beta)$ such that $z_{IP}^c(\tilde{\beta}) + z_{LP}^h(\beta - \tilde{\beta}) > c^T x^* + h^T y^*$, which implies there exists $(\tilde{x}, \tilde{y}) \in S(\beta)$ such that $c^T \tilde{x} + h^T \tilde{y} > c^T x^* + h^T y^*$, a contradiction. Therefore, $z_{MIP}^{c,h}(\beta) = g(\beta)$ for all $\beta \in \mathbb{R}^m$. $\square$

Corollary 2. If (x^0, y^0) is an optimal solution to $MIP(c, h)$, then

$$Ax^0 \in \operatorname{argmax}_{\beta' \in \hat{B}(b)} \{z_{IP}^c(\beta') + z_{LP}^h(b - \beta')\}. \quad (14)$$

We will use Proposition 1 to characterize the set of inverse-feasible objectives in Section 3.

3. Characterizing the Set of Inverse-Feasible Objectives

Given $(x^0, y^0) \in S(b)$, a vector $(c, h) \in D$ is *inverse-feasible* if $c^T x^0 + h^T y^0 = z_{MIP}^{c,h}(b)$; that is, (x^0, y^0) is an optimal solution to $MIP(c, h)$. In this section, we show that the set of inverse-feasible objectives,

$$D^* = \{(c, h) \in D : c^T x^0 + h^T y^0 = z_{MIP}^{c,h}(b)\}$$

is a polyhedron. Let $\beta^0 = Ax^0$. For all $\beta \in \hat{B}(b)$, we introduce variables $\phi(\beta) \in \mathbb{R}_+$ and $\pi_\beta \in \mathbb{R}_+^m$, which we will use to model the values of z_{IP}^c and z_{LP}^h on the lattice $\hat{B}(b)$, respectively. Consider the following constraints.

$$\phi(\beta_1) + \phi(\beta_2) - \phi(\beta_1 + \beta_2) \leq 0 \text{ for all } \beta_1, \beta_2, \beta_1 + \beta_2 \in \hat{B}(b), \quad (15a)$$

$$\phi(\beta_1) - \phi(\beta_2) \leq 0 \text{ for all } \beta_1 \leq \beta_2 \in \hat{B}(b), \quad (15b)$$

$$\phi(a_j) - c_j \geq 0 \text{ for } j \in I, \quad (15c)$$

$$\phi(0) = 0, \quad (15d)$$

$$\pi_{b-\beta}^T G \geq h \text{ for all } \beta \in \hat{B}(b), \quad (15e)$$

$$\phi(\beta^0) + \pi_{b-\beta^0}^T (b - \beta^0) = c^T x^0 + h^T y^0, \quad (15f)$$

$$\phi(\beta^0) + \pi_{b-\beta^0}^T (b - \beta^0) \geq \phi(\beta) + \pi_{b-\beta}^T (b - \beta) \text{ for all } \beta \in \hat{B}(b), \quad (15g)$$

$$\phi(\beta) \in \mathbb{R}_+ \text{ for all } \beta \in \hat{B}(b), \quad (15h)$$

$$\pi_{b-\beta} \in \mathbb{R}_+^m \text{ for all } \beta \in \hat{B}(b), \quad (15i)$$

$$(c, h) \in D. \quad (15j)$$

Theorem 3. *Consider the polyhedron,*

$$P = \left\{ \left(\phi, c, h, (\pi_{b-\beta})_{\beta \in \hat{B}(b)} \right) : (15a) - (15j) \right\}.$$

The set of inverse-feasible objectives is given by $D^ = \text{proj}_D(P)$, where $\text{proj}_D(P)$ is the projection of P onto D .*

Proof. $D^* \subseteq \text{proj}_D(P)$: Let $(c, h) \in D^*$ and consider the value function of $MIP(c, h)$ given by $z_{MIP}^{c,h} : \mathbb{R}^m \rightarrow \mathbb{R}_+$. From Proposition 1,

$$z_{MIP}^{c,h}(b) = \max_{\beta' \in \hat{B}(b)} \{z_{IP}^c(\beta') + z_{LP}^h(b - \beta')\}.$$

Define $\phi(\beta) = z_{IP}^c(\beta)$ for each $\beta \in \hat{B}(b)$. Because z_{IP}^c is superadditive, nondecreasing, and a feasible solution to $SDIP(c, b)$, ϕ satisfies (15a) – (15d) and (15h). For each $\beta \in \hat{B}(b)$, choose $\pi_\beta \in \operatorname{argmin}_{\pi \in \mathbb{R}_+^m, \pi^T G \geq h} \pi^T \beta$ and by (11), $z_{LP}^h(\beta) = \pi_\beta^T \beta$. Thus, (15e) and (15i) are clearly satisfied. Then by Corollary 2 and $(c, h) \in D^*$, for all $\beta \in \hat{B}(b)$,

$$c^T x^0 + h^T y^0 = \phi(\beta^0) + \pi_{b-\beta^0}^T (b - \beta^0) \geq \phi(\beta) + \pi_{b-\beta}^T (b - \beta),$$

which yields constraints (15f) and (15g). Thus, $(\phi, c, h, (\pi_{b-\beta})_{\beta \in \hat{B}(b)}) \in P$ and hence $(c, h) \in \operatorname{proj}_D(P)$.

$D^* \supseteq \operatorname{proj}_D(P)$: Let $(c, h) \in \operatorname{proj}_D(P)$, so that there exists $(\phi, c, h, (\pi_{b-\beta})_{\beta \in \hat{B}(b)}) \in P$. Let e be the m -dimensional vector of all ones. Define

$$F_{IP}(\beta) := \begin{cases} \phi(\lfloor \beta \rfloor), & \beta \in B(b), \\ \phi(b)e^T \beta, & \beta \in \mathbb{R}_+^m \setminus B(b), \\ -\infty, & \beta \notin \mathbb{R}_+^m, \end{cases}$$

which has been shown to be nondecreasing, superadditive, and feasible for $SDIP(c, b)$ [8]. Let $\mathcal{L} = \{\pi_{b-\beta} : \beta \in \hat{B}(b)\}$. We define $F_{LP} : \mathbb{R}^m \rightarrow \mathbb{R}$ as

$$F_{LP}(\beta) := \begin{cases} \min_{\pi \in \mathcal{L}} \pi^T \beta, & \beta \in \mathbb{R}_+^m \\ -\infty, & \beta \notin \mathbb{R}_+^m. \end{cases}$$

Motivated by (12), define $F_{MIP} : \mathbb{R}^m \rightarrow \mathbb{R}_+$ as

$$F_{MIP}(\beta) := \max_{\beta' \in \hat{B}(\beta)} \{F_{IP}(\beta') + F_{LP}(\beta - \beta')\}.$$

Note that for $\beta \notin \mathbb{R}_+^m$, $F_{MIP}(\beta) = -\infty$ as $\hat{B}(\beta) = \emptyset$. We show that F_{MIP} is feasible for $SDMIP(c, h)$ (4).

First, F_{MIP} satisfies (4b), as

$$F_{MIP}(a_i) = \max_{\beta' \in \hat{B}(a_i)} \{F_{IP}(\beta') + F_{LP}(a_i - \beta')\} \geq F_{IP}(a_i) \geq c_i,$$

where the first inequality was obtained by noting $a_i \in \hat{B}(a_i)$ and the second from our definition of F_{IP} and constraint (15c).

Second, F_{MIP} satisfies (4c), as

$$\begin{aligned} \bar{F}_{MIP}(g_i) &= \lim_{\delta \rightarrow 0^+} \max_{\beta' \in \hat{B}(\delta g_i)} \{F_{IP}(\beta') + F_{LP}(\delta g_i - \beta')\} \cdot \frac{1}{\delta} \\ &= \lim_{\delta \rightarrow 0^+} \max_{\beta' \in \hat{B}(\delta g_i)} \{F_{IP}(\beta') + \min_{\pi \in \mathcal{L}} \pi^T (\delta g_i - \beta')\} \cdot \frac{1}{\delta}. \end{aligned}$$

For δ sufficiently small, $\hat{B}(\delta g_i) = \{\mathbf{0}\}$. Thus we have that

$$\bar{F}_{MIP}(g_i) = \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \min_{\pi \in \mathcal{L}} \pi^T(\delta g_i) = \lim_{\delta \rightarrow 0^+} \min_{\pi \in \mathcal{L}} \pi^T g_i \geq h_i,$$

where the inequality was obtained from (15e).

Third, F_{MIP} is nondecreasing and superadditive satisfying (4d). Let $\beta_1 \leq \beta_2 \in \mathbb{R}^m$. If $\beta_1 \notin \mathbb{R}_+^m$, then monotonicity clearly holds. Suppose $\beta_1 \in \mathbb{R}_+^m$, and choose

$$\beta_1^* \in \operatorname{argmax}_{\beta' \in \hat{B}(\beta_1)} \{F_{IP}(\beta') + F_{LP}(\beta_1 - \beta')\}. \quad (16)$$

Then

$$\begin{aligned} F_{MIP}(\beta_2) &= \max_{\beta' \in \hat{B}(\beta_2)} \{F_{IP}(\beta') + F_{LP}(\beta_2 - \beta')\} \\ &\geq F_{IP}(\beta_1^*) + F_{LP}(\beta_2 - \beta_1^*) \\ &= F_{IP}(\beta_1^*) + \min_{\pi \in \mathcal{L}} \pi^T(\beta_2 - \beta_1^*) \\ &\geq F_{IP}(\beta_1^*) + \min_{\pi \in \mathcal{L}} \pi^T(\beta_1 - \beta_1^*) \\ &= F_{MIP}(\beta_1). \end{aligned}$$

where the inequalities follow from the facts that $\beta_1^* \in \hat{B}(\beta_2)$ and $\beta_1 \leq \beta_2$, respectively. Hence, $F_{MIP}(\beta_2) \geq F_{MIP}(\beta_1)$.

To show that F_{MIP} is superadditive, first note that if either β_1 or β_2 have a negative component, then superadditivity holds. Otherwise, let $\beta_1, \beta_2 \in \mathbb{R}_+^m$. Let $\beta_1^* \in \mathbb{R}_+^m$ be defined as in (16), and β_2^* defined analogously. Then

$$\begin{aligned} F_{MIP}(\beta_1 + \beta_2) &= \max_{\beta' \in \hat{B}(\beta_1 + \beta_2)} \{F_{IP}(\beta') + F_{LP}(\beta_1 + \beta_2 - \beta')\} \\ &\geq F_{IP}(\beta_1^* + \beta_2^*) + F_{LP}(\beta_1 + \beta_2 - (\beta_1^* + \beta_2^*)) \\ &= F_{IP}(\beta_1^* + \beta_2^*) + \min_{\pi \in \mathcal{L}} \pi^T(\beta_1 + \beta_2 - (\beta_1^* + \beta_2^*)). \quad (17) \end{aligned}$$

Choosing $\pi_\beta \in \operatorname{argmin}_{\pi \in \mathcal{L}} \pi^T(\beta_1 + \beta_2 - (\beta_1^* + \beta_2^*))$, and because F_{IP} is superadditive, (17) implies

$$\begin{aligned} F_{MIP}(\beta_1 + \beta_2) &\geq F_{IP}(\beta_1^*) + F_{IP}(\beta_2^*) + \pi_\beta^T(\beta_1 + \beta_2 - (\beta_1^* + \beta_2^*)) \\ &= F_{IP}(\beta_1^*) + \pi_\beta^T(\beta_1 - \beta_1^*) + F_{IP}(\beta_2^*) + \pi_\beta^T(\beta_2 - \beta_2^*) \\ &\geq F_{IP}(\beta_1^*) + \min_{\pi \in \mathcal{L}} \pi^T(\beta_1 - \beta_1^*) + F_{IP}(\beta_2^*) + \min_{\pi \in \mathcal{L}} \pi^T(\beta_2 - \beta_2^*) \\ &= F_{MIP}(\beta_1) + F_{MIP}(\beta_2), \end{aligned}$$

where the second inequality was obtained by noting $\pi_\beta \in \mathcal{L}$. Hence, F_{MIP} is superadditive.

Finally, (4e) is satisfied because $F_{MIP}(\mathbf{0}) = F_{IP}(\mathbf{0}) + F_{LP}(\mathbf{0}) = 0$, and therefore F_{MIP} is feasible for $SDMIP(c, h)$.

We show $F_{MIP}(b) = c^T x^0 + h^T y^0$. Since F_{MIP} is feasible for $SDMIP(c, h)$, $F_{MIP}(b) \geq c^T x^0 + h^T y^0$ by weak duality. Suppose that $F_{MIP}(b) > c^T x^0 + h^T y^0$. Then by definition of F_{MIP} , there exists $\tilde{\beta} \in \hat{B}(b)$ such that

$$F_{IP}(\tilde{\beta}) + F_{LP}(b - \tilde{\beta}) > c^T x^0 + h^T y^0.$$

Then by (15g),

$$\begin{aligned} \phi(\beta^0) + \pi_{b-\beta^0}^T(b - \beta^0) &\geq \phi(\tilde{\beta}) + \pi_{b-\tilde{\beta}}^T(b - \tilde{\beta}) \\ &\geq \phi(\tilde{\beta}) + \min_{\pi \in \mathcal{L}} \pi^T(b - \tilde{\beta}) \\ &= F_{IP}(\tilde{\beta}) + F_{LP}(b - \tilde{\beta}) \\ &> c^T x^0 + h^T y^0. \end{aligned}$$

This contradicts (15f), and therefore, $F_{MIP}(b) = c^T x^0 + h^T y^0$. Thus by weak duality, (x^0, y^0) is optimal for $MIP(c, h)$. Hence, $(c, h) \in D^*$. $\square$

Corollary 4. *If $C = \emptyset$, then we are left with the set of inverse-integer-feasible objectives characterized in [8].*

Proof. If $C = \emptyset$, for all $\beta \in \hat{B}(b)$, we remove $\pi_{b-\beta}$ from each of our constraints. Then, constraints (15e) and (15i) vanish, (15f) becomes $\phi(b) = c^T x^0$, and (15g) reduces to a portion of the constraints given by (15b). Thus we are left with the set of inverse-integer-feasible objectives characterized in [8]. $\square$

Corollary 5. *If $I = \emptyset$, then we are left with the set of inverse-linear-feasible objectives characterized in [1].*

Proof. If $I = \emptyset$, for all $\beta \in \hat{B}(b)$, we remove variables $\phi(\beta)$ from our constraints. Then constraints (15a) – (15d) and (15h) vanish, (15f) becomes $\pi_b^T b = h^T y^0$, and (15g) becomes $\pi_b^T b \geq \pi_{b-\beta}^T(b - \beta)$ for all $\beta \in \hat{B}(b)$. Since we are interested in $proj_D(P)$, we can also remove variables $\pi_{b-\beta}$ for $\beta \neq \mathbf{0}$ and obtain the same set of inverse-feasible objectives. We are then left with variables and constraints that describe the set of inverse-linear-feasible objectives characterized in [1]. $\square$

Using Theorem 3, we can solve inverse MIPs as LPs. Let $(c_0, h_0) \in D$ be a known target vector, and let $(v^I, v^C) \in \mathbb{R}_+^{n \times p}$ be a fixed vector of weights. The inverse mixed-integer programming problem under the L_1 norm is to find an inverse-feasible (c, h) that solves

$$\min_{(c, h) \in D^*} \left\{ \sum_{i \in I} v_i^I |c_i^0 - c_i| + \sum_{j \in C} v_j^C |h_j^0 - h_j| \right\}.$$

We introduce nonnegative variables $\alpha^I, \gamma^I \in \mathbb{R}_+^n$ and $\alpha^C, \gamma^C \in \mathbb{R}_+^p$, and formulate the inverse mixed-integer programming problem as the following linear program.

$$\begin{aligned} & \min \sum_{i \in I} v_i^I (\alpha_i^I + \gamma_i^I) + \sum_{j \in C} v_j^C (\alpha_j^C + \gamma_j^C) \\ & \text{subject to} \end{aligned}$$

$$\phi(\beta_1) + \phi(\beta_2) - \phi(\beta_1 + \beta_2) \leq 0 \text{ for all } \beta_1, \beta_2, \beta_1 + \beta_2 \in \hat{B}(b), \quad (18a)$$

$$\phi(\beta_1) - \phi(\beta_2) \leq 0 \text{ for all } \beta_1 \leq \beta_2 \in \hat{B}(b), \quad (18b)$$

$$c_i^0 + \alpha_i^I - \gamma_i^I - \phi(a_i) \leq 0 \text{ for } i \in I, \quad (18c)$$

$$\phi(0) = 0, \quad (18d)$$

$$h^0 + \alpha_j^C - \gamma_j^C - \pi_{b-\beta}^T G \leq 0 \text{ for all } \beta \in \hat{B}(b) \text{ and } j \in C, \quad (18e)$$

$$\phi(\beta^0) + \pi_{b-\beta^0}^T (b - \beta^0) = (c^0 + \alpha^I - \gamma^I)^T x^0 + (h^0 + \alpha^C - \gamma^C)^T y^0 \quad (18f)$$

$$\phi(\beta^0) + \pi_{b-\beta^0}^T (b - \beta^0) \geq \phi(\beta) + \pi_{b-\beta}^T (b - \beta) \text{ for all } \beta \in \hat{B}(b), \quad (18g)$$

$$H(c^0 + \alpha^I - \gamma^I) + M(h^0 + \alpha^C - \gamma^C) \leq r \quad (18h)$$

$$\phi(\beta) \in \mathbb{R}_+ \text{ for all } \beta \in \hat{B}(b), \quad (18i)$$

$$\pi_{b-\beta} \in \mathbb{R}_+^m \text{ for all } \beta \in \hat{B}(b), \quad (18j)$$

$$\alpha, \gamma \in \mathbb{R}_+^{n+p} \quad (18k)$$

We can also consider the problem under the L_∞ norm and give a reformulation in a similar manner to [1]. Although (18) is a linear program, its size is exponentially large.

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