Simulation of wind velocities on long span structures: a novel stochastic wave based model

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Abstract

A methodology is presented for efficient and accurate modeling of correlated wind velocities along long span structures at a virtually infinite number of points. Currently, the standard approach is to model wind velocities as discrete components of a multivariate stochastic vector process, characterized by a Cross-Spectral Density Matrix. To simulate sample realizations of the vector process, the Spectral Representation Method is one of the most commonly used, which involves a Cholesky decomposition of the Cross-Spectral Density Matrix. However, it is a well known issue that as the length of the structure, and consequently the size of the vector process, increases, this Cholesky decomposition breaks down numerically. Alternatively, this paper introduces the use of the frequency-wavenumber spectrum to model the wind velocities as a stochastic “wave,” continuous in both space and time. This allows the wind velocities to be modeled at a virtually infinite number of points along the length of the structure. In this paper, the relationship

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between the Cross Spectral Density Matrix and the frequency-wavenumber spectrum is first examined. The frequency-wavenumber spectrum is then derived for wind velocities. Numerical examples are carried out demonstrating that the simulated wave samples exhibit the desired spectral and coherence characteristics. The efficiency of this method, specifically through the use of the Fast Fourier Transform, is also demonstrated.

Keywords: Monte Carlo simulation, Spectral Representation Method, Wind velocity fields, Stochastic waves, Frequency-wavenumber spectrum

1. Introduction & motivation

For tall buildings and long bridges alike, wind loads are often the governing design loads in structural design. This is especially true in the case of long-span bridges, where the dynamic behavior under wind loading is likely to be the governing design condition. Consider the well known case of the Tacoma-Narrows bridge which failed spectacularly under dynamic wind loading: the aeroelastic flutter that eventually lead to its failure [1] would never be accounted for in a static analysis. In order to account for the randomness in the wind loading, this is often considered probabilistically [2]. To that end, Monte-Carlo simulation is typically used in practice. Sample realizations of wind velocity fields are generated, and the response of the bridge is calculated. Ensemble statistics can then be calculated from the responses.

Wind velocity fields are, by nature, a three dimensional vector field. That is, at any point $\mathbf{x} = (x, y, z)$ in space, the wind velocity is a vector with three components $\mathbf{u}(\mathbf{x}, t) = [u_x(\mathbf{x}, t), u_y(\mathbf{x}, t), u_z(\mathbf{x}, t)]^T$, corresponding to the three Euclidean dimensions. It should be noted, though, that for clar-

ity and simplicity, a scalar wind velocity field is considered throughout the remainder of this paper along with its spatial variation in a single direction (i.e. horizontally along the length of the bridge deck, x). For a bridge, these simplifications are fairly reasonable. It is implied that the cross-correlation between velocity components is negligible. This is a quite common assumption in practice [3–6]. Research efforts looking at the effect of this assumption have found that although there is an observable correlation, the magnitude is negligible when compared to the overall uncertainty in the wind velocity [4, 7–10].

The most widely used method, and standard practice, for generating wind velocities along the length of a bridge is to model the wind velocities as a multi-variate process. That is, the wind field is discretized into discrete components of a multi-variate vector process, $\mathbf{u}(t) = [u_1(t), \dots, u_j(t), \dots, u_n(t)]^T$. The j^{th} component ($u_j(t)$) of the vector $\mathbf{u}(t)$ corresponds to the wind velocity field $u(x, t)$ at x_j . Sample realizations are then simulated using a wide range of methodologies. The Spectral Representation Method (SRM) is one of the most commonly used today and will be considered in this paper. Simulation of sample realizations of the multi-variate processes within the SRM centers around the Cross-Spectral Density Matrix (CSDM) which is decomposed using usually Cholesky’s decomposition [11–13]. Alternatively, a modal decomposition can be used [14–16]. It is well known, however, that for both modal and Cholesky decompositions, there is an issue when the number of components in the vector process becomes large (on the order of 20–30 points). These decompositions break down due to the fact that neighboring points are highly correlated and so, as the CSDM grows in size, it

becomes increasingly close to singular.

Recent research efforts have centered on approximate techniques to simplify this decomposition. Some examples of this approach include the following non-exhaustive list. Yang, Chang, and Chang [17] developed an explicit form for the Cholesky decomposition under certain assumptions. In [18], Cao et. al. developed algebraic expressions to explicitly express the Cholesky decomposition. Li et. al. [19] proposed a simplified formulation, which breaks the n -variate vector process, with n correlated components, into smaller sized independent processes. These smaller vector processes (or even univariate processes) avoid the issues of large dimensionality, and are more computationally efficient. In [20], Ding et. al. reduce the computational effort by limiting the number of Cholesky decompositions necessary. However, the decomposition itself is not circumvented. For seismic applications, rather than wind, Gao et. al. [21] proposed the use of the lagged coherency matrix rather than the CSDM, since it is smoothly varying in frequency, and can then be coupled with interpolation methods to reduce the need for Cholesky decompositions. All of these methods, however, involve approximations to some degree. Furthermore, the majority of these methods remain quite computationally intensive, with many decompositions necessary for each simulation.

In this work, an alternative formulation will be presented in which a virtually infinite number of spatial points can be simulated *exactly*. This is achieved by modeling the wind velocities as a continuous ‘wave’ in space-time rather than a discrete vector in space and continuous only in time. Furthermore, the sample generation is comparatively efficient due to use of the Fast

67 Fourier Transform (FFT). This formulation was initially presented in [22],
 68 but is significantly expanded upon in this paper. This paper is organized as
 69 follows. Section 2 reviews the SRM for multi-variate processes. Section 3
 70 derives the frequency wavenumber spectrum used to describe the stochastic
 71 wave and establishes its relationship to the conventional multi-variate de-
 72 scription of wind velocities. Section 4 expands upon the discussion of Section
 73 3 and provides a detailed discussion of the numerical implementation of sim-
 74 ulating stochastic waves, specifically for the wind velocity problem. Finally,
 75 Section 5 carries out two numerical examples to show the proposed formula-
 76 tion’s validity, effectiveness, and accuracy. It should be noted that all of the
 77 theory and examples in this work assume stationarity and homogeneity.

78 **2. Review of multi-variate stochastic process simulation based on** 79 **the SRM**

80 When modeling wind velocities along a long span, one must consider both
 81 the stochastic temporal and spatial variabilities. In order to model the spatial
 82 variation, the wind velocity “field” is typically modeled as a vector process,
 83 with discrete components. In this review, let us consider only stationary
 84 and Gaussian vector processes, for simplicity. For the non-stationary case, a
 85 discussion can be found in [12]. For the non-Gaussian case, a discussion can
 86 be found in [23].

87 With that in mind, let us define the notion of a *vector process*. A vector

process is made up of individual, discrete, random processes:

$$\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_j(t) \\ \vdots \\ X_n(t) \end{bmatrix} \quad (1)$$

where $X_j(t)$ is component j of the n -variate vector process. Individually, each component is a univariate stochastic process, which can be considered in the conventional sense. However, when considered as a whole, the correlation between different components is important. These correlations are modeled via the Cross Correlation Matrix (CCM), $R_{jk}(\tau)$, or the Cross Spectral Density Matrix (CSDM), $S_{jk}(\omega)$. The cross-correlations and cross-spectra form Fourier transform pairs, just as in the stationary univariate case (i.e. Wiener-Khinchin theorem/transform [24–27]):

$$S_{jk}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{jk}(\tau) e^{-i\omega\tau} d\tau \quad (2a)$$

$$R_{jk}(\tau) = \int_{-\infty}^{\infty} S_{jk}(\omega) e^{i\omega\tau} d\omega \quad (2b)$$

Considering here the homogeneous case (i.e. when the statistical properties do not vary spatially), the CSDM is typically constructed from a prescribed auto-spectrum, $S_{jj}(\omega)$ ¹ and a coherence model $\gamma(\omega, \xi)$ [28] as follows:

$$S_{jk}(\omega, \xi) = \sqrt{S_{jj}(\omega) S_{kk}(\omega)} \cdot \gamma(\omega, \xi) \quad (3)$$

¹note: repeated indices here do not imply summation

where ξ is the separation distance between nodes j and k . It should be noted that $S_{jj}(\omega) = S_{kk}(\omega)$ for every value of j and k . The auto-spectrum, S_{jj} , and the coherence function, γ , are chosen for the problem at hand. There are a multitude of options in the literature for most practical problems. For wind problems, as considered in this work, the auto-spectrum can be chosen to be the Kaimal Spectrum [29] (among various available options) and the coherence model is typically chosen to be the Davenport coherence function [30].

With the CSDM known — either from prescribed properties or estimated from data — the SRM can be used to simulate sample realizations of the vector process. First, the CSDM is decomposed as:

$$\mathbf{S}(\omega) = \mathbf{H}(\omega) \mathbf{H}^{T*}(\omega) \quad (4)$$

where the superscript T denotes the matrix transpose, and $*$ denotes complex conjugate. This can be thought of as a typical Cholesky decomposition, and thus $\mathbf{H}$ is lower triangular:

$$\mathbf{H}(\omega) = \begin{bmatrix} H_{11}(\omega) & 0 & 0 & \cdots & 0 \\ H_{21}(\omega) & H_{22}(\omega) & 0 & \cdots & 0 \\ H_{31}(\omega) & H_{32}(\omega) & H_{33}(\omega) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n1}(\omega) & H_{n2}(\omega) & H_{n3}(\omega) & \cdots & H_{nn}(\omega) \end{bmatrix} \quad (5)$$

See [11–13] for discussion of the properties and restrictions on this matrix.

From the decomposition of Eqns. (4) and (5), sample realizations can be

generated from the following formula, as $N \rightarrow \infty$ [11–13]:

$$X_j(t) = 2 \sum_{m=1}^j \sum_{l=0}^{N-1} |H_{jm}(\omega_l)| \sqrt{\Delta\omega} \cos(\omega_l t + \Theta_{jm}(\omega_l) + \phi_{ml}); \quad j = 1, 2, \dots, n \quad (6)$$

where:

$$\omega_l = l\Delta\omega, \quad l = 0, 1, \dots, N-1 \quad (7a)$$

$$\Delta\omega = \frac{\omega_u}{N} \quad (7b)$$

$$\Theta_{jm}(\omega_l) = \tan^{-1} \left(\frac{\text{Im}[H_{jm}(\omega_l)]}{\text{Re}[H_{jm}(\omega_l)]} \right) \quad (7c)$$

where $\text{Im}[\cdot]$ and $\text{Re}[\cdot]$ denote the imaginary and real parts, respectively; and ϕ_{ml} represents independent random phase angles between $[0, 2\pi]$. In this multi-variate case, ϕ_{ml} is a $n \times N$ matrix: ϕ_{ml} can be thought of as n independent series of length N . In the form shown in Eqn. (6), the generated samples are not ergodic, in contrast to the typical univariate SRM samples [31]. An alternative form in [12, 32] double-indexes the frequencies in order to overcome this limitation.

As was mentioned earlier, it is well known that the decomposition of Eqn. 4 breaks down as $\mathbf{S}(\omega)$ grows (or $n \rightarrow \infty$). This is independent of the decomposition used (i.e. modal or Cholesky), due to the fact that as $n \rightarrow \infty$, $\Delta x \rightarrow 0$, and these neighboring points become highly correlated. This correlation causes $\mathbf{S}(\omega)$ to become increasingly closer to singular, and the decompositions break down numerically. Furthermore, it should be noted that $\mathbf{S}(\omega)$ needs to be decomposed at each frequency ω_l , which quickly adds up in computational cost.

3. Modeling wind as a stochastic wave

3.1. Derivation of frequency-wavenumber spectrum

In this section, we propose the use of the frequency-wavenumber spectrum, $S(\omega, \kappa)$, to model the wind velocities as a stochastic “wave” continuous in both space and time. This allows the wind velocities to be modeled at a virtually infinite number of points along the length of the structure. The Cross-Spectral Density Matrix (CSDM), or auto-spectrum and coherence model, can be transformed to the frequency-wavenumber spectrum through the following integral expression [28]:

$$S(\omega, \kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \cdot \gamma(\xi, \omega) e^{i\kappa\xi} d\xi \quad (8)$$

where $S(\omega)$ is the auto-spectrum, $\gamma(\xi, \omega)$ is the coherence model, ξ is the spatial separation distance, ω is the angular frequency, and κ is the wavenumber. It should be pointed out here that wavenumber representations have been used previously [33, 34] in wind engineering. These works model the wind velocity field as a frozen field, and describe the time variation by advecting the frozen field with a mean wind speed. The frequency-wavenumber model proposed here does not follow this assumption.

Several choices exist for the auto-spectrum, $S(\omega)$, in Eqn. (8). Some of the more commonly used in practice include: von Karman’s spectrum [35, 36], Davenport’s spectrum [37], Kaimal’s spectrum [29], Simiu and Scanlan’s spectrum [4], and Solari and Piccardo’s spectrum [38] which models the full co-spectral dependencies across components. In this study, the Kaimal spectrum (which is of the same functional form as Simiu and Scanlan’s) will

be used, as this is one of the classical expressions used in wind engineering. A second example in Section 5.3 will also return to the von Karman spectrum. The Kaimal two-sided spectrum is defined (using parameter definitions from [4]) as [29]:

$$S_K(\omega) = \frac{1}{2} \cdot \frac{200}{2\pi} \cdot u_*^2 \cdot \frac{z}{U(z)} \cdot \frac{1}{\left[1 + 50 \frac{\omega z}{2\pi U(z)}\right]^{\frac{5}{3}}} \quad (9)$$

where z is the height above ground, $U(z)$ is the average wind velocity at height z , and u_* is the velocity friction component defined as:

$$u_* = \frac{kU(z)}{\ln\left(\frac{z}{z_0}\right)} \quad (10)$$

where k is von Karman's constant ($k \cong 0.4$), and z_0 is a parameter describing the ground roughness.

Similarly, there are several choices for the coherence model, $\gamma(\xi, \omega)$, in Eqn. (8). Davenport's coherence function [30] is among the classical expressions. Kristensen and Jenson [39] proposed a model based on Taylor's hypothesis. More recently, Krenk [40] proposed a coherence model to account for the observation that the normalized co-spectral densities do not go to unity at $\omega = 0$. For the purposes of this study, the Davenport coherence function will be used, defined as [30]:

$$\gamma(\xi, \omega) = e^{-\frac{\lambda \omega \xi}{2\pi U(z)}} \quad (11)$$

where λ is a decay parameter, often chosen to be between 7 and 10. A second example, using Krenk's coherence function, will also be carried out in Section 5.3.

Combining Eqns. (8), (9), and (11), the frequency-wavenumber spectrum, $S(\omega, \kappa)$, can be expressed as:

$$S(\omega, \kappa) = \underbrace{\frac{1}{2} \cdot \frac{200}{2\pi} \cdot u_*^2 \cdot \frac{z}{U(z)} \cdot \frac{1}{\left[1 + 50 \frac{\omega z}{2\pi U(z)}\right]^{\frac{5}{3}}}}_{\text{Kaimal spectrum}} \times \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{\lambda \omega \xi}{2\pi U(z)}} e^{i\kappa \xi} d\xi \right]}_{\text{Fourier integral of Davenport coherence}} \quad (12)$$

The Fourier integral in the second term has an analytical solution [41]. Therefore, Eqn. (12) simplifies to:

$$S(\omega, \kappa) = \frac{1}{2} \cdot \frac{200}{2\pi} \cdot u_*^2 \cdot \frac{z}{U(z)} \cdot \frac{1}{\left[1 + 50 \frac{\omega z}{2\pi U(z)}\right]^{\frac{5}{3}}} \times \left[\frac{\left(\frac{\lambda \omega}{\pi U(z)}\right)}{\kappa^2 + \left(\frac{\lambda \omega}{2\pi U(z)}\right)^2} \right] \quad (13)$$

The frequency-wavenumber spectrum in Eqn. (13) is a generalized, closed form expression for modeling wind velocity fields in space-time as a stochastic wave. Sample realizations can be generated using the SRM formulation which will be discussed later in Section 4.

3.2. Verification of frequency-wavenumber spectrum

In order for the frequency-wavenumber spectrum of Eqn. (13) to be valid, the spectral density $S(\omega)$, at any point in the spatial domain, should be the originally prescribed Kaimal spectrum, $S_K(\omega)$. From basic multi-variate probability, any dimension of a multi-dimensional process can be integrated out to give the uni-variate spectral density in the remaining dimension(s) (e.g., [42]). Thus, for the two dimensional case of the wave being considered in this chapter:

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega, \kappa) d\kappa \quad (14)$$

Here, the scaling of $\frac{1}{2\pi}$ is added to be consistent with definitions used throughout this paper. For simplicity, a quantity $\rho(\omega)$ is defined, such that the frequency-wavenumber spectrum in Eqn. (13) can be rewritten as:

$$S(\omega, \kappa) = S_K(\omega) \times \left[\frac{\rho(\omega)}{\kappa^2 + \left(\frac{\rho(\omega)}{2}\right)^2} \right] \quad (15)$$

where $\rho(\omega)$ is given by:

$$\rho(\omega) = \frac{\lambda\omega}{\pi U(z)} \quad (16)$$

Substituting Eqn. (15) into Eqn. (14), and solving:

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_K(\omega) \times \left[\frac{\rho(\omega)}{\kappa^2 + \left(\frac{\rho(\omega)}{2}\right)^2} \right] d\kappa \quad (17a)$$

$$= \frac{1}{2\pi} S_K(\omega) \rho(\omega) \int_{-\infty}^{\infty} \frac{1}{\kappa^2 + \left(\frac{\rho(\omega)}{2}\right)^2} d\kappa \quad (17b)$$

$$= \frac{1}{2\pi} S_K(\omega) \rho(\omega) \times 2\pi \sqrt{\frac{1}{\rho^2(\omega)}} \quad (17c)$$

$$= S_K(\omega) \quad (17d)$$

Therefore, it can be seen that at any point in the spatial domain, the auto-spectral characteristics are intact and still reflect the prescribed Kaimal spectrum.

It is also required that the coherence is retained throughout the spatial domain, however this is non-trivial to explore analytically. In Section 5, the auto-spectral and coherence characteristics of generated samples will be verified through numerical tests.

191 4. Simulation of wind velocities as a wave

192 The Spectral Representation Method (SRM) can again be used to simu-
 193 late sample realizations of a stochastic wave from its frequency-wavenumber
 194 spectrum [43, 44]:

$$u(x, t) = \sqrt{2} \sum_{l=1}^{N_\kappa} \sum_{m=1}^{N_\omega} \sum_{I_\omega=\pm 1} \sqrt{2 \cdot S_f(I_\omega \omega_m, \kappa_l) \cdot \Delta\omega \Delta\kappa} \cdot \cos[I_\omega \omega_m t + \kappa_l x + \phi_{ml}^{I_\omega}] \quad (18)$$

where

$$\omega_m = m \Delta\omega \quad (19a)$$

$$\Delta\omega = \frac{\omega_u}{N_\omega} \quad (19b)$$

$$\kappa_l = l \Delta\kappa \quad (19c)$$

$$\Delta\kappa = \frac{\kappa_u}{N_\kappa} \quad (19d)$$

195 However, the simulation becomes more efficient if a frequency shifting
 196 method is adopted, as proposed by Zerva [45]. In the standard SRM, the
 197 frequencies are discretized such that:

$$\omega_j = j \Delta\omega \quad (20)$$

198 Zerva [45] showed it is advantageous to shift the frequencies such that:

$$\omega_j = \left(j + \frac{1}{2}\right) \Delta\omega \quad (21)$$

199 There are several benefits to this formulation, as discussed in [45]. Worth
 200 noting are its (a) increased ensemble convergence rate, (b) simplified adap-
 201 tion to FFT, and (c) preservation of ergodicity properties. Specific to this
 202 example, note that there is a singularity at $S(\omega = 0, \kappa = 0)$ in Eqn. (13).

203 This singularity is avoided by using this discretization, as the value for
 204 $S\left(\omega = \frac{\Delta\omega}{2}, \kappa = \frac{\Delta\kappa}{2}\right)$ is finite. In the case of a wave, both frequency, ω , and
 205 wave-number, κ , are discretized in this matter. Using this discretization, the
 206 SRM formula of Eqn. (18) can be used directly.

207 Use of the FFT can dramatically increase the computational efficiency of
 208 simulation. In fact, the effects are compounded for this multi-dimensional
 209 case compared to the univariate, one-dimensional, stationary case. Efficiency
 210 gains of multiple orders of magnitude are thus possible. In order to make use
 211 of the FFT, Eqn. (18) is rearranged as:

$$u(x, t) = \text{Re} \left[\sum_{l=0}^{N_\kappa-1} \sum_{m=0}^{N_\omega-1} \left\{ B_{lm}^{(1)} \exp[i\omega_m t + i\kappa_l x] + B_{lm}^{(2)} \exp[-i\omega_m t + i\kappa_l x] \right\} \right] \quad (22)$$

212 where, $\text{Re}[\cdot]$ denotes the real part, and:

$$B_{lm}^{(n)} = 2\sqrt{S(\omega_m, \kappa_n) \cdot \Delta\omega \Delta\kappa} e^{i\phi_{lm}^{(n)}} \quad (23)$$

213 Using a pseudo-notation for the FFT, Eqn. (22) can be re-written as a series
 214 of FFTs:

$$u(x, t) = \text{Re} \left\{ \text{FFT}_\kappa [\text{FFT}_\omega (\mathbf{B}^{(1)})] + \text{FFT}_\kappa [\text{IFFT}_\omega (\mathbf{B}^{(2)})] \right\} \quad (24)$$

215 where $\text{FFT}(\cdot)$ and $\text{IFFT}(\cdot)$ denote the Fast Fourier Transform and Inverse
 216 Fast Fourier Transform, respectively, and subscripts ω and κ denote along
 217 which dimension the (I)FFT is computed.

218 Once the wave is simulated, time histories of individual points in space
 219 can be picked out, as necessary or desired. As a conceptual example, consider
 220 a bridge with a 1,000 meter main span. The designer builds a Finite Element
 221 model of the bridge with 10 meter segments representing the deck, and thus

Figure 1: Simulation of wind velocities as a wave: The top image shows the simulated wave sample, and the bottom shows the individual “slices” that make up the time histories at each point of interest along the deck of the bridge

would like wind velocity time histories for each of the 100 nodes along the span. Using the conventional multi-variate methods, this would be extremely challenging or essentially impossible. In fact, those methods break down after about 20-30 points. Thus, the wind ‘field’ is modeled as a continuous wave as described above. With the continuous wave sample, the individual time histories at the 100 locations are simply the “slices” where x is held constant, for each of the desired x_i . This is illustrated in Figure 1. The continuous wave is shown on top, with slices highlighted in blue. These slices are then shown below as individual time histories.

5. Numerical examples

In this section, two numerical examples are considered and the generated samples are tested to verify that their spectral characteristics match the prescribed auto-spectra and coherence functions. In other words, it is ensured that the samples simulated via the wave method and multi-variate method are *statistically* equivalent.

5.1. Example 1: Problem definition

The first example considered is a bridge with a 1600 meter long main span. The other parameter definitions are as follows:

- $L = 1600$ m (main span length)
- $z = 50.0$ m
- $z_0 = 0.03$ m
- $U(z) = 40.0$ m/s
- $\lambda = 10.0$

Although the wind velocities are being modeled as a continuous wave, the wave is still discretized for numerical simulation. The frequency-wavenumber discretization used is:

- $\omega_u = 8\pi$ rad/s
- $N_\omega = 1,024$
- $\Delta\omega = \frac{\omega_u}{1023} \approx 0.0246$ rad/s

$$\bullet N_\kappa = 16,368$$

$$\bullet \kappa_u = \frac{2(N_\kappa - 1)}{L} \pi = \frac{16367}{800} \pi \approx 64.273 \text{ m}^{-1}$$

$$\bullet \Delta\kappa = \frac{2\pi}{L} = \frac{\pi}{800} \approx 0.0039 \text{ m}^{-1}$$

Due to the frequency shifting paradigm of [45] as described above, the upper-cutoff frequency and wavenumber are actually shifted to:

$$\bullet \omega'_u = \omega_u + \frac{\Delta\omega}{2} \approx 25.1450 \text{ rad/s}$$

$$\bullet \kappa'_u = \kappa_u + \frac{\Delta\kappa}{2} \approx 64.2750 \text{ m}^{-1}$$

Time and space are discretized as follows:

$$\bullet N_t = 2N_\omega = 2,048$$

$$\bullet T = \frac{2\pi}{\Delta\omega} \approx 255.6251 \text{ s}$$

$$\bullet \Delta t = \frac{T}{N_t} = \frac{\left(\frac{2\pi}{\Delta\omega}\right)}{N_t} \approx 0.1249$$

$$\bullet N_x = 2N_\kappa = 32,736$$

$$\bullet \Delta x = \frac{L}{N_x} \approx 0.0489 \text{ m}$$

There are some items in the discretization worth discussing. First, note that $N_t = 2N_\omega$ and $N_x = 2N_\kappa$. This is done because when using the FFT, the frequency is actually doubled and then padded with zeros. This is in accordance with the formulation in [44]. Also note that $N_x = 32,736$. This high N_x is chosen (a) to demonstrate the ability of this method to simulate a very high number of points in space, and (b) it was found through trial and error that this combination of κ_u and $\Delta\kappa$ produced the best results - in

terms of avoiding aliasing as well as closeness of fit to target coherence. All N 's are chosen as powers of 2 to optimize the FFT efficiency. Finally, the specific relationship between frequency and time and space and wavenumber are in accordance with the MATLAB FFT routines which were used in this work.

5.2. Example 1: Results

In this numerical example, 5,000 samples are generated from the frequency-wavenumber spectrum. At the start of the simulation, an index is chosen at random. In the results shown here, this index was $j = 476$, for which $x_j = 23.22$ m. The auto-spectrum was estimated at this location in space for each of the 5,000 samples, and the ensemble average of these auto-spectra is shown in Figure 2 compared with the prescribed Kaimal spectrum. For sample function ‘slice’ $u(t, x_j)$, the auto-spectrum is estimated from:

$$S_{jj}(\omega) = \frac{\left| \int_0^T u(t, x_j) e^{-i\omega t} dt \right|^2}{2\pi T} \quad (25a)$$

$$= \frac{|\text{FFT}(\mathbf{u}^{(j)})|^2}{\Delta\omega M_\omega^2} \quad (25b)$$

where $\text{FFT}(\cdot)$ is following the notation defined earlier, $\mathbf{u}^{(j)}$ is a vector such that $u_k^{(j)} = u(t_k, x_j)$, and $M_\omega = 2N_\omega$ is the number of points used for the FFT, as described above.

The estimated auto-spectrum matches the target very nicely. Aside from very close to $\omega = 0$, the two are indistinguishable. Thus, as was shown in Eqn. (17), the auto-spectral characteristics are retained in the wave model proposed.

Figure 2: Comparison of the estimated spectra from 5000 generated samples using the wave model vs. the prescribed Kaimal auto-spectrum

Additionally, the cross-spectra are estimated for $S_{j(j+50)}(\omega)$, $S_{j(j+100)}(\omega)$, and $S_{j(j+200)}(\omega)$. These indices correspond to $x_{526} = 25.66$ m, $x_{576} = 28.10$ m, $x_{676} = 32.99$ m. Therefore, the separation distances are: $\xi_{476,526} = 2.44$ m, $\xi_{476,576} = 4.89$ m, $\xi_{476,676} = 9.76$ m. For sample function slices $X(t, x_j)$ and $X(t, x_k)$, the cross-spectra are estimated from:

$$S_{jk}(\omega) = \frac{\int_0^T u(t, x_j) e^{i\omega t} dt \int_0^T u(t, x_k) e^{-i\omega t} dt}{2\pi T} \quad (26a)$$

$$= \frac{\text{IFFT}(\mathbf{u}^{(j)}) \text{FFT}(\mathbf{u}^{(k)})}{\Delta\omega M_\omega^2} \quad (26b)$$

From these cross-spectra, the coherence functions are estimated as:

$$\gamma(\omega, \xi_{jk}) = \frac{S_{jk}(\omega)}{S_{jj}(\omega) S_{kk}(\omega)} \quad (27)$$

These three estimated coherence functions are plotted against the target Davenport coherence models for those specified separation distances in Figure 3. For all three, there is clear agreement between the target and the estimated. The first two are in fact perfectly aligned with the target. The

Figure 3: Comparison of the estimated coherence functions from 5000 generated samples using the wave model vs. the prescribed Davenport coherence function

third, for $\xi_{476,676} = 9.78$ shown in Figure 3(c), though, is quite noisy. This is unfortunately inescapable. However, it is not an issue with the simulated samples, but rather with the estimation of the coherence and cross-spectra. It is well known that the coherence is very difficult to capture and is very sensitive [28], especially as the separation distance grows. These results are therefore perfectly acceptable.

A sample realization of the simulated wave is shown in Figure 4. Figure

4(a) shows the entire wave sample in space and time. Figure 4(b) then shows sample time histories taken at $x = 0$ m, $x = 800$ m, and $x = 1600$ m. Lastly, Figure 4(c) shows zoom-in time histories at $x = 23.22$ m and $x = 25.66$ m, over-layed on top of each other to show the spatial variation between neighboring points in space.

Finally, the computational efficiency of the proposed algorithm is examined in Figure 5. This figure plots the CPU time (in seconds) to simulate a sample "wave" versus a discretization parameter, N . Note, that in this case $N_\omega = N_\kappa = N$ (i.e. there are $M^2 = 4N^2$ points in the frequency-wavenumber domain). These CPU times were computed using MATLAB(R) R2013a, 64-bit, on desktop workstation with a Intel(R) Core(TM) i7-2600 processor and 16GB RAM. For a problem of the scale of the above example, simulation of a single sample realization of the "wave" takes around 10 seconds. Note that this is independent of the number of spatial points considered, as they are just slices along the *continuous* wave. Using a multi-variate process simulation approach, this would take many hours, if even tractable.

5.3. Example 2: Problem description

In this second example, the von Karman auto-spectrum [35, 36] and Krenk coherence [40] will be used to demonstrate the flexibility of the proposed methodology. The example is similar to the second example in [46]. The von Karman auto-spectrum is defined as:

$$S_{VK}(\omega) = \frac{4L_u\sigma_u^2}{U(z) \left(1 + 70.8 \left(\frac{\omega L_u}{2\pi U(z)}\right)^2\right)^{5/6}} \quad (28)$$

(a) Generated sample wave in space and time

(b) Generated time histories. Top: $x = 0$ m. Middle: $x = 800$ m. Bottom: $x = 1600$ m.

(c) Spatial variation between neighboring points

Figure 4: Example 1: Generated sample realization of wind velocities modeled as a stochastic wave

Figure 5: Computational efficiency of proposed methodology. CPU time in seconds vs. domain discretization, where $N_\omega = N_\kappa = N$.

where L_u is the integral length scale, and σ_u is the standard deviation of the turbulence. The Krenk coherence function is defined as:

$$\gamma(\xi, \omega) = (1 - 0.35\alpha\xi) e^{-0.7\alpha\xi} \quad (29)$$

where:

$$\alpha = \sqrt{\left(\frac{\omega}{U(z)}\right)^2 + \left(\frac{1}{l_u}\right)^2} \quad (30)$$

and $l_u = 1.34L_u$.

Substituting Eqns. (28) and (29) into Eqn. (8) and taking the Fourier transform of the coherence term yields:

$$S(\omega, \kappa) = \underbrace{\left[\frac{4L_u\sigma_u^2}{U(z) \left(1 + 70.8 \left(\frac{\omega L_u}{2\pi U(z)}\right)^2\right)^{5/6}} \right]}_{\text{von Karman Spectrum}} \times \underbrace{\left[\frac{(0.7\alpha)^3 + 2.1\alpha\kappa^2}{2\pi((0.7\alpha)^2 + \kappa^2)^2} \right]}_{\text{Fourier Integral of Krenk Coherence}} \quad (31)$$

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323 The parameter definitions for this example are as follows:

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12 324 • $z = 50$ m

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15 325 • $L_u = 150$ m

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17 326 • $U(z) = 35$ m/s

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20 327 • $\sigma_u = 5$ m/s

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23 328 All other paramters were the same as in Example 1 (Section 5.1).

24 25 329 *5.4. Example 2: Results*

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27 330 Once again, 5,000 samples are generated from the frequency-wavenumber
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29 331 spectrum. At the start of the simulation, an index is chosen at random. In
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31 332 the results shown here, this index was $j = 26,221$, for which $x_j = 1,281.5$ m.
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33 333 The auto-spectrum is estimated at this location in space for each of the 5,000
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35 334 samples, and the ensemble average of these auto-spectra is shown in Figure 6
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37 335 compared with the prescribed von Karman spectrum. The auto-spectra are
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39 336 once again estimated using Eqn. (25). The estimated auto-spectrum again
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41 337 matches the target very nicely, with similar behavior to that in Example 1.

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43 338 The cross-spectra are also again estimated, using Eqn. (26), for $(j, j+50)$,
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45 339 $(j, j+100)$, and $(j, j+200)$. These indices correspond to coordinates of
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47 340 $x_{26271} = 1,284$ m, $x_{26321} = 1,286.4$ m, and $x_{26421} = 1291.3$ m and separation
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49 341 distances of $\xi_{j,j+50} = 2.44$ m, $\xi_{j,j+100} = 4.89$ m, and $\xi_{j,j+200} = 9.78$ m. From
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51 342 the cross-spectra, the coherence functions are again estimated using Eqn.
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53 343 (27). The estimated coherence functions for the three separation distances
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55 344 are plotted against the target Krenk coherence function in Figure 7. For all
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57 345 three, there is again clear agreement between the target and the estimated.

Figure 6: Comparison of the estimated spectra from 5000 generated samples using the wave model vs. the prescribed von Karman auto-spectrum

A similar behavior as in Example 1 is again witnessed, with noise increasing with separation distance. This is again an artifact of the estimation techniques. Note that the coherence at $\xi = 9.78$ m in Figure 7(c) actually becomes negative. This is the distinction of the Krenk coherence, which allows it to more appropriately characterize turbulence along long-span structures [40], by ensuring that the average turbulence integrates to zero.

Finally, a sample realization of the simulated wave is plotted in Figure 8. Figure 8(a) shows the entire wave sample in space and time. Figure 8(b) then shows sample time histories taken at $x = 0$ m, $x = 800$ m, and $x = 1600$ m. Lastly, Figure 8(c) shows zoom-in time histories at $x = 1,281.5$ m and $x = 1,284$ m, over-layed on top of each other to show the spatial variation between neighboring points in space.

Figure 7: Comparison of the estimated coherence functions from 5000 generated samples using the wave model vs. the prescribed Krenk coherence function

(a) Generated sample wave in space and time

(b) Generated time histories. Top: $x = 0$ m. Middle: $x = 800$ m. Bottom: $x = 1600$ m.

(c) Spatial variation between neighboring points

Figure 8: Example 2: Generated sample realization of wind velocities modeled as a stochastic wave

6. Conclusions

A novel approach to simulating wind velocity time histories along long-span structures was presented. In contrast to the existing methodologies of modeling these wind velocities as stochastic vector processes, it was proposed to model them as a stochastic wave which is continuous in space and time. This allows the simulation of wind velocities at a virtually infinite number of points in space, a feat which was virtually impossible up to this point. Furthermore, the method was shown to be accurate, robust, and more efficient than previous methodologies.

This methodology could potentially be extended to other cases. The formulation in this paper was for one-dimensional, homogeneous in space, and stationary in time, velocity fields. Alternatively, each of these ‘categories’ could be extended to multi-dimensionality, non-homogeneity, and non-stationarity, respectively.

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