Sine-Gordon modulation solutions: application to macroscopic non-lubricant friction

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Abstract

The Frenkel-Kontorova (FK) model and its continuum approximation, the sine-Gordon (SG) equation, are widely used to model a variety of important nonlinear physical systems. Many practical applications require the wave-train solution, which includes many solitons. In such cases, an important and relevant extension of these models applies Whitham's averaging procedure to the SG equation. The resulting SG modulation equations describe the behavior of important measurable system parameters that are the average of the small-scale solutions given by the SG equation.

A fundamental problem of modern physics that is the topic of this paper is the description of the transitional process from a static to a dynamic frictional regime. We have shown that the SG modulation equations are a suitable apparatus for describing this transition. The model provides relations between kinematic (rupture and slip velocities) and dynamic (shear and normal stresses) parameters of the transition process. A particular advantage of the model is its ability to describe frictional processes over a wide range of rupture and slip velocities covering seismic events ranging from regular earthquakes, with rupture velocities on the order of a few km/s, to slow slip events, with rupture velocities on the order of a few km/day.

Keywords: macroscopic friction, Frenkel-Kontorova model, sine-Gordon equation, Whitham's modulation equations, earthquake

1. Introduction

An understanding of tribology and its complicated nonlinear aspects requires a combination of experimental, theoretical and computational efforts [1, 2]. While experiments and molecular-dynamic simulations provides invaluable information about the atomic origins of static and dynamic friction, the complexities of realistic 3D systems makes it difficult to understand the general mechanisms underlying friction. In this regard, simple low-dimensional phenomenological models, such as the Tomlinson [3] and Frenkel–Kontorova (FK) [4] models are useful tools for determining the essential features of nonlinear sliding phenomena [5, 6]. These features can then be tested by experiment and molecular-dynamic simulation. Thus, development of such models is an essential part of studying friction.

The relative movement of two solids in contact is accompanied by friction occurring due to interactions between surface asperities. Under static conditions or sliding at uniform velocity, friction is usually described by the frictional coefficient, *i.e.*, the proportionality coefficient between tangential and normal stress (classical Amontons-Coulomb law). However, this description is not sufficient for sliding at non-uniform velocity. Multiple experiments have shown that in transitional regimes, friction depends on slip, sliding rate, contact time and normal stress history (see the extensive reviews [7-9]). Over the past 50 years, various approaches for the modeling of non-uniform frictional processes have been developed. Two types of models are the most common, *i.e.*, mass-spring models [10–17], and rate-and-state (Dietrich-Ruina) models [18–28].

The FK model (one of the mass-spring models) has been widely used to study micro/nanoscale friction [29-39]. We demonstrated that this model may also describe the dynamics of macroscopic non-lubricant friction [40-42]. In the model we proposed, sliding is analogous to plasticity. It occurs due to movement of a certain type of defect (a “macroscopic dislocation”) nucleated on the frictional surfaces by shear stress in the presence of asperities. The movement of dislocations (areas on the frictional surfaces with accumulated stress) requires much less external shear force than uniform displacement of frictional surfaces. The advantages of this model are: (1) it is an intrinsically dynamical model, rooted in the Newtonian equations of motions; (2) parameters used in the model have explicit and unambiguous physical correlates; (3) it describes frictional processes over a wide range of conditions, from very fast processes such as regular earthquakes down to very slow processes such as creep, silent, and slow earthquakes [40-45].

In the continuum limit the FK model is described by the sine-Gordon (SG) equation, one of the fundamental universal nonlinear equations of mathematical physics. Among other applications, this equation has been used in the theory of dislocations, Josephson junctions, self-induced transparency, commensurate-incommensurate phase transitions, charge-density waves, magnetic domain walls, etc (see [46-49] and references therein). Due to its universal character, the SG equation has been extensively investigated [46, 50-52]. The mathematical apparatus which has been developed is fully applicable to the problems considered here. What

distinguishes our approach compared to other mass-spring frictional models, in general, and FK models, in particular, is the use of the SG modulation equations rather than the standard SG equation itself. Solutions to the latter provide detailed descriptions of system parameters at small scales that are typically unmeasurable. The advantage of our approach is that it describes the average behavior of parameters, such as slip and rupture velocities and shear stress distribution, which are measurable parameters of interest.

The organization of the paper is as follows. We first describe the basics of the model, followed by the solution of the modulation equations. Applications are then considered. The paper concludes with a summary of our results.

2. Model Description

2.1. Model Derivation

It is commonly accepted that macroscopic friction results from the interaction between asperities. During relative movement of the frictional surfaces, an asperity on one frictional surface detaches from an asperity on the opposite frictional surface and attaches to the next opposing asperity, and this process continues as long as the frictional surfaces are sliding relative to each other (see illustration in Fig. 1). Neighboring asperities on the same surface interact with each other as part of an elastic solid. We will consider the asperities (Fig. 1b) on one of the frictional surfaces as forming a linear chain of balls of mass M , each ball interacting with the nearest neighbors on either side via spring forces of constant K_b (Fig. 1c). The asperities on the opposite frictional surface will be regarded as forming a rigid substrate which interacts with the masses M via a periodic potential. Then we can apply the one-dimensional FK model to describe the slip dynamics:

$$M \frac{\partial^2 u_i}{\partial t^2} - K_b (u_{i+1} - 2u_i + u_{i-1}) + F_d \sin \frac{2\pi}{b} u_i = F(x, t) - f_i(x, t, \frac{\partial u_i}{\partial t}), \quad (1)$$

where u_i is the shift of ball i relative to its equilibrium position, b is a typical distance between asperities, t is time, F_d is the amplitude of the periodic force on M associated with the periodic substrate potential, f_i is the frictional (dissipative) force on asperity i , and F is the external force. This equation has been used to study plasticity in crystalline materials, which involves the dynamics of atomic-scale edge dislocations [46, 53, 54]. To express the coefficients of equation (1) in terms of the volume and surface mechanical parameters of the frictional blocks and external conditions such as normal stress, we first consider these coefficients for the case of plasticity, i.e., at the atomic scale. We assume a sliding surface parallel to the actual frictional surface but inside the block. Supposing that it is a crystal material with volume density ρ and interatomic distances a , b and d in the directions shown on Fig. 1a, we can find the coefficients

for equation (1) (see [46, 53, 54] for details): $M = \rho abd$, $K_b = \frac{2\mu ad}{(1-\nu)b}$, $F_d = \frac{\mu b^2 a}{2\pi d}$, where μ is

the shear modulus and ν is the Poisson ratio. Now equation (1) can be written in the form (the second term on the left hand side is obtained in the continuum limit approximation):

$$\frac{c^2 \rho a d}{2\pi} \frac{\partial^2 (2\pi u / b)}{\partial (tc / b)^2} - \frac{2\mu a d}{2\pi(1-\nu)} \frac{\partial^2 (2\pi u / b)}{\partial (x / b)^2} + \frac{\mu b^2 a}{2\pi d} \sin\left(\frac{2\pi u}{b}\right) = F - f, \quad (2a)$$

where $c^2 = 2\mu / (\rho(1-\nu)) \equiv c_l^2(1-2\nu)/(1-\nu)^2$ and c_l is the longitudinal acoustic velocity. Note that $c_s < c < c_l$, where c_s is the shear wave velocity. An equivalent form is (supposing for simplicity that $a=b=d$):

$$\frac{\partial^2 (2\pi u / b)}{\partial (tc / b)^2} - \frac{\partial^2 (2\pi u / b)}{\partial (x / b)^2} + A^2 \sin\left(\frac{2\pi u}{b}\right) = (F - f) \frac{2\pi A^2}{\mu b^2} \quad (2b)$$

The dimensionless parameter A is as $A = ((1-\nu)/2)^{1/2}$. In the derivation of equation (2b), A^2 is essentially the ratio of the amplitude of two forces: one is the force amplitude between an atom and the substrate layer and the other is the force amplitude between neighboring atoms at the top layer (both forces appear when the atom shifts horizontally). This ratio is of order unity. In the derivation of this equation, the system was treated macroscopically as an elastic continuum to relate the force constants in the FK model to the elastic constants of the crystal. To describe the respective coefficients for the situation where slip occurs between two external surfaces in contact, we will use equation (2) with one significant change: we shall treat the parameter A phenomenologically, using the result for a crystal as a guide. So we assume that A likewise depends on the ratio of two relevant forces. The force amplitude (per unit area) experienced by an asperity due to neighboring asperities along the slip direction is exactly the same as it was for the case of plasticity. But the force amplitude between asperity and substrate is different and depends on the normal stress Σ_N . Indeed, when $\Sigma_N = 0$ the force is zero, since there is no interaction between asperities and a substrate. On the other hand, when Σ_N reaches the penetration hardness σ_p the interface between the two blocks disappears and the corresponding force amplitude is essentially the same as in the case of plasticity. So we regard A as a function of the ratio of Σ_N to σ_p : $A = f(\Sigma_N / \sigma_p)$. We cannot determine the exact functional form of A from our model alone, but A must reduce to zero when $\Sigma_N = 0$ and reaches its maximum of $((1-\nu)/2)^{1/2}$ when $\Sigma_N = \sigma_p$. The simplest choice is $A = ((1-\nu)/2)^{1/2} \Sigma_N / \sigma_p \approx \Sigma_N / \sigma_p$. Thus, the coefficient A reflects how deeply the asperities from two opposing surfaces interpenetrate. Its value can be considered as the ratio between actual and nominal contact areas (to within a factor $((1-\nu)/2)^{1/2}$), since this area ratio is approximately Σ_N / σ_p [55, 56].

Fig. 1

It is convenient to re-write equation (2b) in dimensionless form:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = \Sigma_s^0 - f, \quad (3)$$

where u , x and t are now in units of $b/(2\pi)$, b/A and $b/(cA)$, respectively, and Σ_s^0 and f are external shear stress and friction force per unit area, both in units of $\mu A/(2\pi)$. The derivatives $\varepsilon = \partial u / \partial x$ and $w = \partial u / \partial t$ are interpreted as the dimensionless strain and the dimensionless slip in units of $A/(2\pi)$ and $cA/(2\pi)$, respectively. It is also useful to introduce the xz component of the stress tensor $\sigma_s = 2\mu\varepsilon$. Thus, the dimensionless stress is measured in units $\mu A / \pi$.

2.2. Uniform Sliding Motion

Equation (3) in the absence of external and frictional forces is the well-known SG equation. Let us consider the existence of a classical wave solution traveling to the right with wave velocity U (in units of c) and wave number k (in units of A/b), in which u is a function of $x - Ut$. Define $\xi = x - Ut$ and $\theta = k\xi$. The basic solutions are phonons, breathers, and kinks (a particular class of solitons) [46, 57]. Only the soliton solutions represent the wave propagation necessary to model frictional dynamics. These solutions are characterized by $|U| < 1$. Integrating equation (3) for the case $\Sigma_s^0 - f = 0$ with these constraints gives a solution for u and its derivatives in terms of the elliptic Jacobi functions, cn and dn [57]:

$$\begin{aligned} u &= \arcsin[\pm cn(\beta\xi)], \\ \sigma_s &= \varepsilon = \partial u / \partial x = 2\beta dn(\beta\xi), \\ w &= \partial u / \partial t = 2\beta U dn(\beta\xi), \\ k &= 2\pi N = \frac{\pi\beta}{K(m)}, \end{aligned} \tag{4}$$

where $K(m)$ is the complete elliptic integral of the first kind of modulus m ($0 \leq m \leq 1$), $\beta = [m(1-U^2)]^{-1/2}$, and N is the density of kinks in units of A/b . Solution (4) describes an infinite sequence of interacting kinks (solitons) of one sign, which are periodic in space and time. In the context of our model, a soliton is a dislocation.

It is also useful to introduce three dimensionless variables averaged over an oscillation period.

$$W = \int \frac{wd\theta}{2\pi} = \frac{UN}{2\pi}, \Sigma_s = \int \frac{\sigma_s d\theta}{2\pi} = k. \tag{5}$$

These variables correspond to the measurable parameters of slip velocity and stress. Note that the averaged stress is proportional to the kink/dislocation density. If the density is nonzero, the stress is nonzero as well. This is a consequence of the non-homoclinic (i.e., kink) nature of the periodic solution for u . The parameters of a dislocation (amplitude of stress σ_s^0 and strain ε^0 associated with the presence of the dislocation) are: $\sigma_s^0 = \mu A / \pi$, $\varepsilon^0 = A/(2\pi)$.

Let us describe a scenario of frictional processes in terms of solutions (4) and (5). The macroscopic dislocations are nucleated on the surface by an applied shear stress in the presence of asperities. As in crystals, the mobility of a macroscopic dislocation over the frictional surface is larger than the mobility of the whole surface, since the displacement of a dislocation (a pre-stressed area) requires less external stress. So the relative sliding of two bodies occurs due to movement of dislocations. The passage of a dislocation through a particular point on the sliding

surface shifts the contacted bodies locally by a typical distance b . Such a dislocation may propagate with any velocity U ranging from 0 to c , and the average velocity of sliding, *i.e.*, the observable slip rate W , is proportional to the dislocation velocity U and dislocation density N . The parameters of a dislocation (stress amplitude and pulse width) are entirely defined by the material parameters and the normal stress and do not depend on process parameters such as dislocation density and slip rate. The characteristic width of a dislocation is $D \approx 2\pi b / A$. Usually, the dislocation width is much larger than the typical distance between asperities, ranging from 10^2 to 10^4 times the asperity size. Figure 2 schematically illustrates the frictional processes via movement of macroscopic dislocations.

Fig. 2

2.3. Non-Uniform Sliding Motion

Solutions (4) and (5) can be used to describe the frictional process in the macroscopically uniform case. However, in the non-uniform case, the macroscopic parameters such as slip velocity and stress are functions of time and space. In typical cases, the frictional surface includes many dislocations. A numerical solution for the exact positions of a multitude of microscopic dislocations is too computationally demanding to be practicable. Moreover, they cannot be measured and are therefore of little value except as they contribute on the average to the measured macroscopic dislocation.

We therefore apply a technique which simplifies equation (3) at the expense of losing non-essential details of the process such as the exact position of each dislocation. Whitham [50] developed a procedure for constructing a system of equations describing the dynamics of averaged variables (*i.e.*, the variables in equation (5)) that derive from the original dynamical equations (*i.e.*, equation (3)). For strict applicability, the average values should vary slowly in time and space, *i.e.*, the spatial scale should be larger than the size of one dislocation D and the time scale should be larger than D/c . For the present application, the variables of interest can all be expressed in terms of the two independent variables U and m . Applying this procedure to the homogeneous ($F - f = 0$) SG equation yields the system of coupled equations [54, 58]:

$$\begin{aligned} U_t \frac{\eta}{U^2 - 1} + m_t \frac{U}{2m} + U_x \frac{U\eta}{U^2 - 1} + m_x \frac{1}{2m} &= 0 \\ U_t \frac{U}{U^2 - 1} + m_t \frac{\eta}{2mm_1} + U_x \frac{1}{U^2 - 1} + m_x \frac{U\eta}{2mm_1} &= 0 \end{aligned} \quad (6)$$

where subscript labels denote the derivative with respect to the associated variable, $\eta = \frac{E}{K}$,

$m_1 = 1 - m$, and $E(m)$ is the complete elliptic integral of the second kind. The system (6) is fully integrable, *i.e.*, the solutions may be solved analytically. General solutions are obtained in the references [54, 58]. However, system (6) does not include friction. Whitham [50] also described a formal procedure for including a dissipative term. To derive modulation equations we include a

dissipative term to the equation describing conservation of wave action (equation 14.89 from [50])). Applying this procedure we obtain the Whitham equations for equation (3):

$$-\frac{4}{\pi} \frac{K}{[m(1-U^2)]^{1/2}} [U_t \frac{\eta}{U^2-1} + m_t \frac{U}{2m} + U_x \frac{U\eta}{U^2-1} + m_x \frac{1}{2m}] = D(m, U) \quad (7)$$

$$U_t \frac{U}{U^2-1} + m_t \frac{\eta}{2mm_1} + U_x \frac{1}{U^2-1} + m_x \frac{U\eta}{2mm_1} = 0$$

where $D(m, U) = \frac{1}{2\pi} \int [\Sigma_s^0 - f(u, u_t)] d\theta$. Note that equation (7) is valid only in case when D is sufficiently small. Analytical solutions for this system (if they exist) have not been found, requiring this system to be solved numerically.

3. Solution of the Modulation Equations

To solve the modulations equations (6) or (7), the initial and boundary conditions for variables U and m are required. We will use the initial value $U(t = 0, x) = 0$ throughout all the calculations. Initial values for m are obtained through the definition of Σ_s (initial distribution of shear stress) from equations (4) and (5).

3.1. Analytical Solution (No Dissipation)

The general solution of system (6) may be applied to describe the dynamics of a group of kinks with spatially monotonic initial distributions [54, 58]. Here, we will use the so-called “self-similar simple wave” analytical solution [40, 54], i.e., the solution that is defined by only one independent parameter in contrast to the general solution which depends on two independent parameters. The former is convenient for the description of specific initial distributions of kinks.

To model the transition from the static to the dynamic regime of the frictional process, we consider the following idealized problem. Suppose the point $x = 0$ divides the areas of stressed ($x < 0$) and unstressed ($x > 0$) material, modeled as a step-function $\Sigma_s(t = 0, x < 0) = \Sigma^-$, $\Sigma_s(t = 0, x > 0) = \Sigma^+ = 0$, $U(t = 0, x) = 0$ (solid line in Fig. 3). With these initial conditions, we have $\Sigma^\pm = \frac{\pi}{\sqrt{m^\pm K(m^\pm)}}$. This setting models a stress accumulation in front of an obstacle placed at position $x = 0$. Assume that at time $t = 0$ the external shear stress reaches the value necessary to overcome this obstacle. Then the dynamics of the transition (solution of system (6)) is described by the following formulae of self-similar simple wave solution [54, 58]:

$$\frac{x}{t} = V(m), U = \frac{\zeta - \zeta^-}{\zeta + \zeta^-}, \Sigma_s = k = \frac{\pi(\zeta + \zeta^-)}{K\sqrt{m\zeta\zeta^-}}, W = kU \quad (8)$$

$$V = \frac{G - \zeta^-}{G + \zeta^-}, V^+ = \frac{1 - \zeta^-}{1 + \zeta^-}, V^- = \frac{G^- - \zeta^-}{G^- + \zeta^-} \quad (9)$$

where $G = \zeta(E - K\sqrt{m_1}) / (E + K\sqrt{m_1})$, $\zeta = (1 - \sqrt{m_1})^2 / m$, $G^- \equiv G(m^-)$ and $\zeta^- \equiv \zeta(m^-)$, and m^- is defined by the transcendental relation $\Sigma^- = \frac{\pi}{K(m^-)\sqrt{m^-}}$. The variable V is the

nonlinear group velocity of the wave solution in units of c . Along a line $x/t = V = \text{constant}$ in the x - t plane, all variables are constant. The solution is represented by a region expanding in time and bounded between the lines $x/t = V(m=0) = V^- = -1$ and $x/t = V(m=1) = V^+$. Note that inside the expanding region all variables are functions of time and position. The indices $+$ and $-$ designate the leading and trailing edges of the slip pulse, respectively. Thus the solutions (8 and 9) describe the dynamics of a slip pulse with two rupture fronts (or detachment fronts, in terms of equation (3)) propagating in opposite directions with different velocities V^- and V^+ . Figure 4 qualitatively illustrates this solution.

Fig. 3

Fig. 4

The velocity of dislocation movement $U(x,t)$ (Formulae (8)) ranges from zero at the pulse trailing edge to the value V^+ at the pulse leading edge. The movement of dislocations is accompanied by slip with velocity $W(x,t)$ (Formulae (8)). The slip velocity equals zero at the trailing and leading edges of a pulse and has a maximum value at $x = 0$. Since $V(x=0) = 0$ we can find the maximum slip velocity,

$$W(x=0) = \frac{\pi(\zeta^0 - \zeta^-)}{2K\sqrt{m^0\zeta^0\zeta^-}}, \quad (10)$$

where $\zeta^0 \equiv \zeta(m^0)$ and m^0 is a solution of the equation $G(m^0) = \zeta^-$.

3.2. Solution with Dissipation

Let us consider the same problem as above, *i.e.*, the transition from the static to the dynamic regime, described now by equations (7). Note the small difference in initial conditions (dotted line in Fig. 3) compared to the case in Section 3.1 (solid line in Fig. 3): (1) $\Sigma^+ \ll \Sigma^-$ but $\Sigma^+ > 0$ and (2) the stress distribution is a smooth function of x (rather than a step-function) with approximate width π . These changes were necessary in order to obtain a numerically stable solution. For a simple velocity-dependent frictional term, $f = Cu_t$ and supposing that $F = 0$, we find $D(m,U) = CW(m,U)$, where C is the dissipation coefficient and W is the slip velocity as defined by the formulae (8). Mathematica, which employs finite difference methods for spatial discretization and applies the method of lines for the time dependence, was used to solve system (7). Figure 5 depicts the spatial and temporal distribution of a slip pulse velocity for several values of the coefficient C . One can see that the amplitude of the slip velocity decreases with an increase in C .

Fig. 5

Figure 6 shows the slip velocities at time 2π for various values of C . To examine how friction (the dissipative term) affects the rupture velocity, we normalize W to make the maximum slip velocity the same for all cases. Then we see that both velocities V^- and V^+ (left and right sides of slip pulse) are practically unchanged with increasing coefficient C for all cases considered, although the slip velocity amplitude changes by a factor greater than five. Note that this is true for any value of the initial stress Σ^- provided $\Sigma^+ \ll \Sigma^-$. Thus we conclude that while slip velocity depends on friction, the rupture velocity does not. This effect can be understood by considering the interactions among kinks. Kinks of the same sign repel each other. In the scenario considered above, the kink at position $x=0$ at time $t=0$ begins to move to the right where the density of kinks is smaller. This movement initiates two waves in two opposite directions. The velocities of these waves (rupture velocities in our context) depend on kink ‘elasticity’, i.e., the density of kinks or stress, and do not depend on the velocity of the kink itself. The latter velocity depends on friction as well as on stress. This is analogous to sound wave propagation in water or air. If dissipation is sufficiently small, the speed of sound is practically independent of dissipative processes inside a propagation medium. Yet the velocity of water or air in the medium does depend sensitively on dissipation.

Fig. 6

4. Applications

We now apply our solutions to describe 1) laboratory frictional experiments and 2) regular and slow earthquakes.

4.1 Laboratory experiments

In a series of experiments with poly(methyl-methacrylate) (PMMA) material it has been shown [59-62] that:

1. A series of slip pulses (precursors), occurring under shear stress far below the value necessary to overcome static friction, precedes sliding of the PMMA blocks;
2. The transition from static to dynamic friction occurs due to the appearance of a so-called detachment front (a sudden change of the actual contact area between frictional surfaces) propagating with speeds ranging from a few percent of the Rayleigh wave speed to in excess of shear wave (super-shear) speeds; the detachment front precedes slip;
3. There is a functional dependence of detachment front speed on the ratio of shear to normal stress (see Fig. 7).

Fig. 7.

Since we have shown that rupture (or detachment) front velocity does not depend on friction, the non-stationary solutions of the SG equation obtained above (formulae (9)) can be used to describe the dynamics of elastic energy redistribution during the transition process, i.e. the dynamics of a slip pulse. Indeed, the quantity V^+ is the velocity of the leading edge and the velocity of the first dislocation. This is the velocity of the detachment front propagating from the stressed to the unstressed area. The velocity V^- also may be interpreted as a detachment front velocity, but in contrast to V^+ , it is the velocity of the disturbance propagating through the

stressed area. These velocities are uniquely defined by the ratio of shear to normal stress (see the curves in Fig. 7); but their values are very different, and V is always larger than V^+ . One can see that the experimental values are centered about the V^- curve.

4.1 Seismic events

Fault dynamics in the earth's crust is a complicated multidisciplinary problem which has been investigated by diverse approaches [63, 64]. One such approach involves the study of frictional processes between sliding plates in contact [7, 9]. The genesis, development, and cessation of regular earthquakes continue to be one of the major problems of modern geophysics. Slow slip events have also attracted much attention as an additional source for monitoring fault dynamics. Recently discovered phenomena such as deep non-volcanic tremor (NVT) [65], low frequency earthquakes (LFE) [66, 67], very low frequency earthquakes (VLF) [68], and episodic tremor and slip (ETS) [69, 70], have enhanced and complemented our knowledge of fault dynamics. At the same time, these phenomena introduce new questions about their genesis, properties, and relation to regular earthquakes.

In previous work we have shown that our model describes the basic properties of regular earthquakes, including stress relaxation after earthquakes [40], occurrence of ambient and triggered NVT [44], and ETS events [43, 45]. The time scale of seismic events ranges widely from seconds to years. Our model accommodates this and wide range of other system parameters. For example, figure 8 depicts the calculation of rupture velocity V^- as a function of dimensionless initial shear stress Σ^- . Note a strong dependence of velocity on stress. Indeed, changes of shear stress by an order of magnitude lead to velocity changes by seven orders of magnitude, which in turn includes a multitude of seismic events, ranging from regular earthquakes to slow slip events.

Fig. 8.

5. Discussion

The FK model and its continuum approximation, the SG equation, can be used to describe dry (i.e., non-lubricant) macroscopic friction. The basic solutions of the SG equation are kinks (or solitons), phonons, and breathers. In the context of frictional sliding, the former are interpreted as areas of stressed material (macroscopic dislocations) on the frictional surfaces occurring due to interactions between asperities. The dislocation movement along a surface provides the relative sliding of two solids in contact. If the shape of asperities is ideally sinusoidal and if the material of the solids in contact is ideally elastic, the movement of dislocations is dissipationless, i.e., there is no friction. Under real conditions, the shape of asperities is not sinusoidal, the distances between asperities are not equal, and the interaction between asperities is not always elastic. As a result, the movement of macroscopic dislocations is accompanied by generation of phonons due to the structural heterogeneity of the surfaces. The

phonon mode may be interpreted as a dissipative process. Thus, in the framework of this model, friction is a dissipation of energy due to phonon generation induced by the movement of dislocations.

The width of a dislocation is usually much larger than the size of an asperity but much less than the size of a slip pulse. Indeed, for the laboratory experiments considered above [59-62], the size of asperities is about 10^{-5} - 10^{-6} m, the width of dislocations is 10^{-4} - 10^{-3} m, and the size of the slip pulse is about 10^{-2} m. In this case, a detailed dynamical analyses of each particular dislocation, including the shape of each dislocation, is unnecessary and impractical. The description of the spatial and temporal distribution of the averaged parameters is much easier and more relevant. The behavior of these averaged parameters can be compared with experimentally measured parameters. Thus, in most cases, application of the SG modulation (Whitham) equations is preferable to the SG equation itself.

However, there are some specific problems where the SG equation is more appropriate. One example is the description of so-called episodic tremor and slip events [65, 69, 70]. Propagation of a slip pulse along a subduction fault is accompanied by massive bursts of non-volcanic tremors. This periodic phenomenon has been observed in virtually all major subduction zones (65, 69-72). In the framework of our model, a moving slip pulse (one large soliton or a dislocation) generates a tremor (phonon) due to the interaction of a pulse with structural heterogeneities. The parameters of the moving pulse and the frequency content of the tremor can be described by our model [43-45].

An important limitation is that the model is based on an idealized periodicity of asperities in the substrate which will not be found most generally in realistic physical systems. However, such idealized models are a useful starting point in studying complex behavior. When compared with experiment, they indicate the degree to which deviations from ideal behavior are either important or can be neglected. Moreover, in some cases, such as experiments with PMMA material [59-62], the typical distance between asperities does conform to the model. The core of the model is the prediction of distributions of localized stress (dislocations) along the frictional surface. The mobility of these dislocation zones is greater than the mobility of the neighboring areas. Thus, the model predicts that movement of dislocations (resulting in sliding of the bulk frictional surface) requires much less external shear force than is required to uniformly displace the entire frictional surface. The assumption of periodic structures in the substrate is merely a starting point for quantitative calculation which can (at least, potentially) accommodate the effects of random, aperiodic structures. Even in the case of highly randomized asperity sizes, such areas can be expected to exist. As emphasized in this work, the measureable parameters that can be most easily compared with experiment are averages of parameters obtained by applying Witham's procedure to the SG equation. Average values can, indeed, be functionally independent of whether the distribution of asperities is periodic or aperiodic.

6. Conclusion

The significant and novel elements of our model can be summarized as follows:

- (1) The FK model has been modified and adapted to the case of non-lubricant macroscopic friction by introducing a parameter A which is the ratio between the real and nominal area of the frictional surfaces;
- (2) In the model proposed, sliding occurs due to movement of a certain type of defect, a “macroscopic dislocation” (or area of localized stress), which requires much less shear stress than uniform displacement of frictional surfaces.
- (3) To describe measureable macroscopic parameters, we use the Whitham modulation equations applied to the SG equation rather than the SG equation (the latter applies to details at the scale of individual dislocations).

The source of the transition from static to dynamic friction is the gradient of the shear stress (Fig. 3). We show that in transition processes, the rupture velocity of the trailing edge V^- , *i.e.*, the velocity of a detachment front propagating through the stressed material, is defined only by the ratio of initial shear to normal stress Σ^- and does not depend on friction. The value of this velocity is defined by the ratio between shear and normal stress. This conclusion is consistent with the experimental result that the velocity of the detachment front is defined by this ratio [59].

Note that although the model formally does not include a rupture threshold, the value of Σ^- may be considered as a threshold that is implicit in the model. As the threshold increases, the accumulated stress must increase in order to overcome it. That is why regular earthquakes “require” larger stress Σ^- than slow slip events.

The model connects the kinetic and dynamic parameters of the transition process from static to dynamic friction. It characterizes the dynamics of the process for a very wide range of rupture and slip velocities, from velocities in regular earthquakes ($V^-/c \leq 1$, $W \approx 1$ m/s) down to the velocities in slow slip events ($V^-/c \approx 10^{-7}$, $W \approx 10^{-7}$ m/s). Both velocities depend critically on the accumulated stress, *e.g.*, a change of stress by one order of magnitude can cause a velocity change by seven orders of magnitude.

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Figure Captions

Fig. 1. Schematic of a) experimental arrangement, b) asperity contact, and c) chain of masses interacting via elastic springs and placed in a periodic potential (substrate). The balls represent asperities. The sine-shaped surface is the opposite plate.

Fig. 2. Schematics of frictional sliding of two bodies via movement of a macroscopic dislocation. The slip velocity field, W , of the upper sample relative to the lower sample, illustrated by the arrows, is spatially uniform at large scales (panel (a)). W is the slip velocity w averaged over an oscillation period in time. By contrast, the w velocity field is spatially non-uniform at smaller scales, i.e., on a scale comparable with a dislocation width (panel (b)). The spatial and temporal averages of w in (b) give the uniform field in (a). The dislocation size is usually much larger than the typical size of an asperity. The relative movement on the frictional surfaces at even smaller scales (asperity size) could be larger than average if it is at the center or peak amplitude of the dislocation (panel (c)) or smaller than average if it is in between two dislocations (panel (d)).

Fig. 3. Schematics illustrating the spatial distribution of initial stress utilized in obtaining the analytical solution of Equations (6) (solid line) and the numerical solution of Equations (7) (dotted line).

Fig. 4. Schematic of spatial distribution of shear stresses σ_s and Σ_s as well as slip velocity W (a) just before a slip pulse initiation ($t=0$) and (b) during slip pulse propagation ($t>0$). The dark arrows show the directions and relative magnitudes of the “fast” (left) and “slow” (right) fronts. The slip area (shown in light grey colour) is bounded by the fronts, while the slip direction, shown by the white arrow, is directed from the stressed to unstressed area. The maximum slip velocity, hence maximum total slip, occurs at $x = 0$.

Fig. 5. Spatial and temporal distribution of slip pulse velocity for various values of coefficient C , such that $D(m, U) = CW(m, U)$ is sufficiently small, as required for the validity of equation (7) solutions. The initial stresses, chosen arbitrarily to illustrate the effect of the dissipative term, are $\Sigma^- = 0.855$ and $\Sigma^+ = 0.179$.

Fig. 6. Spatial distribution of slip pulse velocity, W , at $t = 2\pi$ for various values of the dissipation coefficient C defining a velocity-dependent friction $f = Cu_t$: $C = 0.1$ (solid line), $C = 0.5$ (dashed line) and $C = 2.0$ (dotted line). The initial stresses are $\Sigma^- = 0.855$ and $\Sigma^+ = 0.179$. The amplitude of W has been multiplied by 1.8 for the dashed line and 5.2 for the dotted line to normalize to the same maximum slip velocity in all cases. The leading and trailing edges of the slip pulse (where $W \approx 0$) determine V^- and V^+ (blue arrows), which are shown to vary little as a function of C , i.e., they are independent of friction.

Fig. 7. Detachment front velocities as a function of the ratio of shear to normal stress. Curves show theoretical results and dots represent experimental results (from [62]).

Fig. 8. The dependence of rupture velocity on the ratio of shear to normal stress. Changes of shear stress by an order of magnitude lead to changes in V by more than six orders of magnitude. The rupture velocity for the Parkfield earthquake (calculated based on observed data) and corresponding predicted shear stress are indicated by the dashed red lines on the top panel. The rupture velocity for an ETS event (calculated based on observed data) and predicted shear stress are indicated by the dashed green lines on the bottom panel.

figure 1

figure 2

figure 3

figure 4

(a) $t=0$

(b) $t>0$

figure 5

figure 6

figure 7

figure 8