Experiments on a non-smoothly-forced oscillator

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Abstract

This paper describes some typical behavior encountered in the response of a harmonically-excited mechanical system in which a severe nonlinearity occurs due to an impact. Although such systems have received considerable recent attention (most of it from a theoretical viewpoint), the system scrutinized in this paper also involves a discrete input of energy at the impact condition. That is, it is kicked when contact is made. One of the motivations for this work is related to a classic pinball machine in which a ball striking a bumper experiences a sudden impulse, introducing additional unpredictability to the motion of the ball. A one-dimensional analog of a pinball machine was the subject of a detailed mathematical study in [1], and the current paper details behavior obtained from a mechanical experiment and describes dynamics not observed in a conventional (passive) impact oscillator.

1. Introduction

It is well established that non-smooth dynamical systems are capable of exhibiting a wide variety of fascinating behavior [2, 3]. Some of this behavior, e.g., low-velocity grazing bifurcations, are typically *not* exhibited by smooth dynamical systems [4], and some experimental studies have been conducted [5, 6, 7]. The relative maturity of this subject (at least from a theoretical perspective) is reflected in a research monograph and the extensive list of references contained therein [8].

A recent paper [1] described some interesting behavior in a unidirectional analog of a pinball machine. That is, an otherwise linear oscillator experiences a sudden kick when it comes into contact with a barrier located at a given position. In their work, they showed some interesting bifurcation structure associated with this form of non-smooth forcing. The current paper describes an experimental study of just this type of system. Certain similarities and differences are noted. For example, in [1], they chose parameter values (for example, very high viscous damping) such that the

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system could be modeled as a first-order dynamical system, and thus facilitate mathematical modeling of the system as a discrete map. In the current study, the damping is relatively light (typical for most structural and mechanical systems) and the experimental data is compared with results from numerical simulation of the governing differential equation. Even in the absence of external periodic excitation the system is capable of exhibiting steady oscillatory behavior when the discrete addition of energy at impact balances the energy dissipation due to damping.

It is also worth mentioning that aspects of the unpredictability associated with the movement of a pinball is not unrelated to the roulette wheel for example [9, 10].

2. The Underlying Linear System

2.1. Free Vibration

Since the underlying, i.e., non-contacting, system is linear we can initially make use of some closed-form analytic solutions. It is the initiation of contact that causes the system to suddenly transition into nonlinear behavior.

Consider an old fashioned pinball machine as shown in Fig. 1(a). We choose to model this system using the one-dimensional system shown schematically in Fig. 1(b). Clearly, this is a somewhat simplistic analog but we shall see that it is still capable of exhibiting interesting dynamical behavior. This is the classic linear oscillator in which a point mass is attached to a linear (Hookean) spring and linear (viscous) dashpot. In the current study we consider the spring to provide the restoring force that replaces the effect of gravity and the pinball that rolls (downhill) due to the sloping surface.

Initially suppose we simply have $c = y(t) = 0$ and that there is no impact barrier. The equation of motion is simply $\ddot{x} = -\omega_n^2 x$, where $\omega_n = \sqrt{k/m}$ is a constant (the natural frequency), and given some non-zero initial conditions we get simple harmonic motion about the equilibrium position $x(t) = A \cos(\omega_n t + \phi)$. The velocity is $\dot{x}(t) = -A\omega_n \sin(\omega_n t + \phi)$ and it is useful to plot the phase trajectory $(x(t), \dot{x}(t))$ to show ellipses ($x^2 + (\dot{x}/\omega_n)^2 = A^2$) and the familiar exchange of energy between potential and kinetic. This behavior is shown schematically in Fig. 2 in parts (a), (d) and (g). In part (a) we make use of the conceptual aid of a small ball rolling on a parabolic surface (which was also used directly as an experimental system in [6]). Part (d) shows a typical time series (with $x(0) = 1.0, \dot{x}(t) = 0$, and thus $A = 1$ and $\phi = 0$), with the corresponding elliptical phase trajectory in part (g).

Suppose we place an impact barrier to the left of the equilibrium position (at the bottom of the curve) and the small ball that can otherwise roll along the surface without friction. Now assume the ball is started from rest towards the

right hand end, i.e., the initial conditions are $(x(0) = 1, \dot{x}(0) = 0)$ and the impact barrier prevents the ball from passing to the left of the barrier. If we further assume that the mass experiences a perfect elastic rebound then we can obtain the behavior shown in parts (b) and (e) when the barrier is located at $x_0 = -0.5$ and parts (c) and (f) when the barrier is located at $x_0 = 0.0$. Both of these are also contained within the ellipse in part (g) in which the mass experiences a sudden reversal of the sign of the velocity at impact. If we add linear viscous damping or allow some discrete loss of energy at impact (via a coefficient of restitution less than one) then the motion would ultimately decay.

2.2. Forced Vibration

Now consider the case in which the support for the system is shaken harmonically. The equation of motion is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 2\zeta\omega_n\dot{y} + \omega_n^2y, \quad (1)$$

in which linear viscous damping has been added ($\zeta = c/c_c$), where $c_c = 2m\omega_n$. In general we will consider a lightly damped mechanical system for which $\zeta \ll 1$. It is convenient to introduce a new relative coordinate defined by $z = x - y$, and assuming the base excitation is harmonic $y(t) = y_0 \cos(\omega t + \phi)$ leads to

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2z = -\ddot{y} = \omega^2y_0 \cos(\omega t + \phi). \quad (2)$$

Focusing on the steady-state solution (i.e., after transients have decayed) we have an amplitude response given by:

$$|z/y_0| = \frac{\Omega^2}{\sqrt{[1 - \Omega^2]^2 + [2\zeta\Omega]^2}}, \quad (3)$$

where we have introduced the frequency ratio $\Omega = \omega/\omega_n$. The familiar resonance phenomenon is observed when $\Omega \rightarrow 1$. A typical amplitude-frequency response diagram is shown in Fig. 3(a), in which the damping ratio is fixed ($\zeta = 0.06$, or 6% of critical). The peak response occurs at resonance and with this value of damping corresponds to a peak response of $|z/y_0| = 8.33$.

If we now include the impact barrier located at some distance z_0 from equilibrium, we observe the range of frequencies over which the steady-state amplitude exceeds the location of the impact. For example, Fig. 3(b) shows the response but now plotted in terms of the frequency ratio and the base excitation amplitude (nondimensionalized by the location of the impact barrier). Thus, suppose we have the impact barrier located at $z_0 = 60$ mm (where we note that $z_0 \equiv x_0$ from the unforced case), and a base excitation amplitude of $y_0 = 10$ mm, we have a steady-state response that at least matches the impact barrier location over the range $0.95 < \Omega < 1.07$. This range corresponds to the input ratio

$|y_0/z_0| = 0.167$. This provides a convenient nondimensionalization, since the response scales linearly with the forcing amplitude, at least in the non-contacting regime. For example, a forcing amplitude of $y_0 = 5$ mm and an impact location of $z_0 = 30$ mm should give the same qualitative behavior, at least in the non-contacting range.

2.3. The Poincaré Section

The previous section gave the standard expression for the amplitude of response of a mass subject to base excitation. Together with the amplitude response we have the corresponding phase difference between input and output with a sudden shift at resonance. While the amplitude is a key response measure relative to whether the response meets the impact, in many circumstances (especially in experimental data acquisition and issues related to stability) it is convenient to use the concept of Poincaré sampling as a measure of the response. This is a standard tool in nonlinear dynamical systems [11]. For linear systems, since we have access to an exact analytical solution, we can obtain expressions for instantaneous position and velocity whenever a chosen forcing phase is met.

Writing the steady-state response in the form

$$z(t)/y_0 = a \cos(\omega t) + b \sin(\omega t), \quad (4)$$

we can substitute back into eq. 2 to obtain

$$a = \frac{\Omega^2 [(1 - \Omega^2) \cos \phi - (2\zeta\Omega) \sin \phi]}{(1 - \Omega^2)^2 + (2\zeta\Omega)^2} \quad (5)$$

and

$$b = \frac{\Omega^2 [(1 - \Omega^2) \sin \phi + (2\zeta\Omega) \cos \phi]}{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}. \quad (6)$$

The equivalence of this response to the previous format is clear since $A = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$. We can examine this response at various instants of time (and not necessarily when the oscillation is at a maximum). The steady-state response is now characterized, for example, by $z/y_0 = a$ and $\dot{z}/y_0 = b\omega$ when $t = 2n\pi/\Omega$, but any convenient phase can be chosen. For example, the position of the mass whenever the forcing phase passes through $\phi = -\pi/4$ is shown in Fig. 4.

Given the cosine forcing function we see that taking $\phi = 0$ would recover the amplitude response. The Poincaré ‘velocity’ can alternatively be tracked to give another choice in capturing the response of the system. Suppose we again set the barrier at $|y_0/z_0| = 0.167$, and the sampling phase is again $\phi = -\pi/4$. In this case, and starting from a low forcing frequency we encounter contact when $z/y_0 = -0.45$ under a slowly *increasing* forcing frequency, and from

a high forcing frequency we encounter contact when $z/y_0 = -5.99$ under slowly *decreasing* forcing frequency. An alternative approach might be to reset the initial conditions (at zero say) for each frequency increment. This would generate significant transients and might pick-up remote solutions. This is an important consideration, as we shall see later, since there is a sensitivity to initial conditions in which there are co-existing responses. Investigating what happens after contact is initiated, that is, within the gray areas in Figs. 3 and 4 and access to simple analysis is lost, is the primary goal of this paper.

Another technique commonly utilized in nonlinear dynamics is the use of time-lag embedding, and again this proves to be especially useful in an experimental context. In many experiments only a single state variable is measured. For example, if the position is measured then it can be possible to extract velocity information through differentiation. But this process tends to amplify noise. A topologically equivalent phase space can be obtained by plotting the position versus the position delayed by a certain amount of time. We shall use a quarter-cycle lag (since for a linear system the position and velocity are separated by a phase angle of a quarter of a period). This approach can also be used when viewing the Poincaré section based on a stroboscopic sampling of the response at two phases, separated by a fixed time-lag. Of course, given the three-dimensional phase space under consideration $(z, \dot{z}, \phi)$, Poincaré sampling of the flow produces a convenient two-dimensional map in the plane. In the general case, a trajectory can be sampled using a variety of means, for example, by penetration of a certain surface of section based on a state variable, but in the current study the periodic nature of the external excitation provides a natural choice of stroboscopic sampling. More details of time-lag embedding can be found in [12].

2.4. The Impact Condition

When the mass comes into contact with the impact barrier it experiences an assumed instantaneous rebound. This effect is often modeled by a coefficient of restitution (r) that relates the velocity before ($\dot{z}_+$) and after ($\dot{z}_-$) using the simple rule:

$$\dot{z}_+ = -r\dot{z}_- + V, \quad (7)$$

applied whenever $z = z_0$. However, importantly, we also add a velocity to the mass (V) when contact is made (in analogy to the pinball machine). The parameters r and V are both measured from the experiment to be described. In subsequent sections we shall refer to the standard impact case ($V = 0$) as *passive*, otherwise we have what we shall term an *active* impact ($V \neq 0$). In this paper the equation of motion together with the collision condition are handled

numerically via time-step refinement to accurately model the non-smooth (contact) characteristic [13]. It is interesting to note that extracting a Poincaré mapping from a trajectory is numerically similar to establishing the impact condition [14].

3. The Experimental System

Various photographic images of the experimental system are shown in Fig. 5. A mass slides on a low-friction linear bearing. It is attached to an arrangement of springs that provide a linear restoring force and maintains an equilibrium under unforced conditions. If the motion is ‘large enough’ the mass triggers a laser that instantaneously activates a solenoid trigger, causing a sudden motion of a plunger that impacts the mass. The solenoid trigger is fixed in place at a given location away from equilibrium, but the equilibrium location itself can be changed via adjustment of the springs. With the trigger switched off the mass may simply impact the stopper, and this tends to extract energy from the motion. We term this situation the ‘passive’ case. With the trigger switched the mass experiences a discrete input of energy at impact and we term this situation as the ‘active’ case. The slider base is mounted on a table that is harmonically shaken by a Scotch-yoke mechanism (as shown in part (c)). This arrangement transmits a sinusoidal force through the base to the mass, thus providing a physical mimicry of the model shown in Fig. 1. The motor of the Scotch-yoke was maintained by a closed loop (PID) controller; the quality of the harmonic signal is detailed in [15].

3.1. Data Acquisition

A PASCO rotary transducer provides a measurement of the horizontal position of the mass, at a sampling rate of 1 kHz, using LabVIEW software. Since the transducer is mounted to the slider and the mass, it furnishes the relative position, $z(t)$. The impact surface is placed at a distance $-z_0$ from equilibrium. A photo (laser) sensor detects when the mass makes contact and causes a solenoid valve to impact the mass. The relative position is measured for a variety of base excitation conditions (in terms of amplitude and frequency), but in general we shall keep the amplitude of harmonic shaking fixed at 10 mm, and observe the response under varying forcing frequencies and proximity of the impact location to the equilibrium of the spring-mass system. The instantaneous location of the mass was extracted whenever the Scotch-yoke arm passed a fixed forcing phase using a reflective sensor. This provided a convenient means of extracting a Poincaré sampling of the behavior.

Additionally, a scheme was devised to assess the relative dominance of a response when it was not unique. That is, for parameter ranges in which there were co-existing attractors, the forcing was changed suddenly, and randomly,

to impart perturbations (large and small) that initiated transients. By tracking the decay of these transients and their ultimate destinations, it was then possible to build-up basins of attraction.

3.2. System Identification

The spring-mass-damper is characterized by a natural frequency and damping ratio. These parameters were estimated using a variety of standard techniques (frequency spectra, logarithmic decrement, etc), and the values $\omega_n = 0.94$ Hz and $\zeta = 0.06$ gave the best overall fit. The natural frequency value conveniently fit within the middle of the 0.2 Hz - 1.8 Hz range over which the Scotch provided a strongly harmonic input. Based on a comparison of velocities before, and after, contact for a variety of conditions led to an estimation of the coefficient of restitution of $r = 0.78$. For the active case the momentum input provided by the solenoid was estimated at $V = 0.175\text{m/s}$.

4. Results for the Passive Case

We shall initially consider the passive case, that is, with the solenoid trigger switched off, and the mass experiences a rebound when it contacts the stop ($V = 0$ in eq. 7). It is convenient to summarize behavior in terms of a bifurcation diagram. Figure 6(a) shows a response envelop as a function of the forcing frequency, with $z_0 = 60\text{mm}$. This is essentially the same as that shown in Fig. 3(a) but with impact occurring on the vicinity of resonance. The response is now nondimensionalized with respect to the impact barrier location (in contrast to that used in section 2.2), such that impact occurs at $z/z_0 = -1$, and the forcing frequency is still nondimensionalized with respect to the linear natural frequency of the system without impact, i.e., $\Omega = \omega/\omega_n$. The response envelop is based on the extraction of local maxima and minima over many cycles and hence a single data point represents a tight cluster of very periodic behavior. We shall color code the responses according to the direction of frequency sweep: light blue for a sweep-up, and red for a sweep-down¹. Under low forcing frequency conditions the system response is not sufficient to cause contact: the system is effectively linear. As the amplitude grows contact occurs at $\Omega \approx 0.9$. The motion then grows in a period-one impacting motion until it suddenly drops off back to a non-impacting linear oscillation, a little over $\Omega \approx 1.2$. Starting from a relatively high initial frequency and then under reducing forcing frequency conditions the motion again grows as resonance is approached. Another contact event occurs close to $\Omega \approx 1.1$. There is a range of frequencies over which co-existing oscillations occur (a regime of hysteresis). The equivalent numerical result is shown in Fig. 6(c).

¹there is an unfortunate yet unavoidable graphical artifact where in some instances the color plotted last tends to overlie previous colors, thus making the last color plotted artificially dominant.

The frequency range over which the large impacting solutions persists is slightly greater in numerical simulations, this is partly due to the shrinking basin of attraction, a feature that we shall return to later. Importantly, the large-amplitude impacting response occurs over a greater frequency range than the contact range suggested from the linear theory in section 2.2. The overall the qualitative agreement between simulations and experiment is good.

Parts (b) and (d) show an alternative bifurcation diagram in terms of a Poincaré displacement. Again this is effectively the numerical analog of Fig. 4 outside of the contacting region. The instantaneous position of the mass is plotted whenever the forcing phase passes through a prescribed value, $z_p \equiv z(\phi = -\pi/4)$, for example. This projection tends to highlight the periodic nature of a response (or lack thereof).

5. Results for the Active Case

Suppose we leave all the parameters values at the same levels as in the previous section but with the solenoid trigger now activated. We shall also consider an interesting regime in which the impact barrier is placed sufficiently close to equilibrium that there exists a sustained periodic oscillation due to the impact force alone, even under conditions of zero external base excitation. This underlies some interesting quasi-periodic behavior that is effectively illustrated using spectrograms [16].

5.1. Impact barrier relatively far from equilibrium

Figure 7 shows a repeat of the study from the previous section but with an active impact force. Outside of the main resonance the system is again linear. However, we now see that the initiation of the impact is associated with some random-like behavior. This has been identified as a period-doubling cascade culminating in chaos and is associated with one of the generic outcomes after a *grazing bifurcation* [4, 17, 18]. The regime of periodic large-amplitude impacting motion is also extended in this case. The correspondence between experimental data and numerical simulation is still good, but not quite as good as in the passive case. Clearly the relatively simplistic modeling of the impact kick adds an additional modeling uncertainty. We shall see later that the discrete increase in velocity immediately after impact is not quite the constant magnitude assumed in the numerical model. Some of the responses in this case reach relatively large amplitude in the experiment and may encounter further nonlinear effects.

A close-up view of grazing is shown in Fig. 8(a), and corresponds to the dashed box in Fig. 7(d). However, in this case a variety of initial conditions were generated and we observe some interesting behavior. In contrast to the system described in [1], this system shows bifurcation behavior associated with grazing from *either* direction. From

the low frequency regime we see a sudden transition from period-one to period-four behavior. On subsequent increase in the forcing frequency the motion undergoes sudden shifts to other periodic behavior. Likewise, if we start in the high frequency regime, and gradually reduce the forcing frequency, we also observe a sudden transition from period-one to period-four motion. But an important feature is that there are various ranges of hysteresis, and a typical bifurcation (under the slow change of the control parameter) is strongly influenced by which attractor is currently being tracked and how it loses stability. For example, at the specific frequency of $\Omega = 1.09$ (indicated by the vertical dashed line) there are actually *four* co-existing stable periodic attractors. Figure 8(b) shows the basins of attraction for these responses. For example, the dark blue area covers those initial conditions that ultimately converge onto the period-one non-impacting behavior also indicated by blue in part (a). It is interesting to see a relatively small area of initial conditions (black) that leads to a large-amplitude impacting period-one response. However, *most* initial conditions lead either to the period-four (red) or period-five (yellow) response.

Thus, sweeping through a control parameter to produce a bifurcation needs to be augmented by an investigation of the initial conditions: a daunting task even for a low-order dynamical system. This behavior is also too sensitive to be adequately captured in the experiments. This was partly because the force associated with the active impact was not quite constant, and together with the inevitable presence of experimental noise, tended to blur this subtle behavior.

5.2. *Impact barrier relatively close to equilibrium*

If we now move the impact location closer to equilibrium, i.e., from $z_0 = 60\text{mm}$ to $z_0 = 22.5\text{mm}$, we have two basic possibilities, even in the case where the Scotch-yoke is switched off: the mass is started from rest (or close to it) and simply stays there; but we can also observe a steady-state motion. The case with no external excitation (the Scotch-yoke is switched off) is shown in Fig. 9 as a time series in part (a) and as a time-lag embedded phase projection in part (b). The numerical simulation gave essentially the same result.

When the base excitation is turned on we get the results summarized in Fig. 10. Again the response is color-coded according to the direction of sweep: up (light blue) and down (red). Now we see the purely impact response extending all the way to resonance. It is characterized as quasi-periodicity, with the impact forcing remaining somewhat constant and the external base excitation gradually increasing. Some higher-order periodic responses appear in the numerical results (Fig. 10(c) and (d)), although this is not so clear in the experimental results due to a small variation in the impulse force and the presence of noise. The difference in some of the colors between experiment and simulation

is simply a sensitivity to initial conditions determining which of the available attractors is picked-up under changing forcing frequency. Again there is a graphical bias according to the order of color plotting. At the higher end of the frequency range the experimental large-amplitude impact response appears to drop-off in the vicinity of $\Omega = 1.7$, an effect not seen in the numerical results over the same range, but did occur for slightly higher frequencies, where the impact response gives way to the effectively linear non-impacting response.

Thus we see significant regions of hysteresis. Over the frequency range $0 < \Omega < 1$ non-impacting, essentially linear behavior, co-exists with a relatively large amplitude quasi-periodic response. Figure 11 shows randomly-generated initial conditions at a forcing frequency of $\Omega = 0.6$, using time-lag coordinates (with a quarter-cycle lag) and the aforementioned Poincaré section. This specific frequency can be seen in Fig 10(b). When the system has reached one of these two steady-states, it can be perturbed into the basin of attraction of the other response, providing the perturbation is large enough to cause the transient to cross the separatrix [19].

There are a number of ways in which the quasi-periodic motion can be characterized. In contrast to the regularity of the time series shown in Fig. 9(a), quasi-periodicity possesses a somewhat random-looking time series. However, the behavior is still governed primarily by two (incommensurate) frequencies. Figure 12 shows two phase projections using time-lag coordinates for $\Omega = 0.64$. Superimposed on the trajectories is the Poincaré mapping (in red). Part (a) is sampled using the external forcing mechanism, and part (b) using the impact-force period. The closed-curve nature of the map and the underlying toroidal phase projection are indicative of quasi-periodicity. This result co-exists with a stable equilibrium (stationary) response, which of these two behaviors results depends on the initial conditions as mentioned previously.

A final verification of this behavior is illustrated using the spectrogram, an alternative to the bifurcation diagram but emphasizing the frequency content of the motion. In Fig. 13(a) is plotted the spectrogram for the case in which $z_0 = 60\text{mm}$, and thus provides a frequency analog corresponding to the up-sweep (blue data) in the bifurcation diagram of Figs. 7(a) and (b). The frequency content of the response is given by f (which has been nondimensionalized by the natural frequency ω_n). In these figures the whiter shade indicates a frequency spike (the contours are on a log scale), with the yellow-to-orange-to-red scale corresponding to a transition to the noise floor (dark shades). The onset of chaos ($\Omega \approx 0.9$) is characterized by a sudden increase in frequency content reflecting the highly nonlinear random-like behavior, and this transitions to high-amplitude (noisier) periodic impacting motion from $\Omega < 1.1 < 1.35$. Outside the

resonant contact frequency range, the magnitude of the response frequency is dominated by the forcing frequency. The down-sweep (red) data in Fig. 7(a) and (b) has the frequency content shown in Fig. 13(b). The range of chaotic behavior is smaller and the high-amplitude impacting motion is not present.

Not surprisingly, the case of the close-proximity ($z_0 = 22.5\text{mm}$) active system under external forcing results in richer spectral content. In Fig. 13(c) we show the up-sweep case. Again this corresponds to the light blue data in Fig. 10(a) and (b). The purely impact-forced response remains fairly constant over the low frequency range. A few other frequencies can be detected and the transition into a chaotic impacting response is followed by the strongly nonlinear periodic impact response until the non-contacting (linear) response takes over when $\Omega > 1.75$. We note that although the dominant response below resonance is quasi-periodicity, since the two frequencies are changing we do observe a specific forcing frequency for which they are equal. This occurs when $\Omega \approx 0.75$ and a *periodic* response appears. This is confirmed by the very small periodic window at this value in the bifurcation diagrams and most clearly seen in Fig. 10(b). The down-sweep case in Fig. 13(d) corresponds to the red data points in Fig. 10(b). This is dominated by the forcing frequency (and to a lesser extent its multiples), and even within the broadband chaotic range it is possible to see small periodic windows, a feature not uncommon within chaotic regimes [11].

The equivalent spectrograms extracted from the numerical simulation are even cleaner (obviously without the inevitable noise of experimental data), but are not reproduced here for conciseness. Rather, we show a numerically-generated spectrogram in Figure 14, in which the distance between the equilibrium position and the location of the impact is used as the control parameter, with no external excitation. We see that if the position of the wall is sufficiently close to equilibrium then impacting behavior can be maintained until a location of about 36 mm. The vertical white dashed line at 22.5 mm corresponds approximately to the type of experimental result shown in Fig. 9. When the wall-to-equilibrium is greater than 36 mm, the energy gain at impact is insufficient to overcome the energy lost through damping and the system settles to its stationary, equilibrium position. The frequency of this purely impulsively-forced behavior does not change much with proximity to the impact wall (approximately 1.5 Hz), although of course the magnitude of the response is more difficult to determine from (coloring of) the spectrogram.

The bifurcation diagrams produced in this paper are the result of quasi-static (slow) sweeps in the forcing frequencies. Hence, any transients were minimized, and the occurrence of co-existing responses were directly the result of changing the direction of sweep. This is not say that additional, ‘remote’ responses aren’t present. In order to uncover

the full range of behavior requires the system to be subject to large perturbations at every value of the forcing parameter, and indeed, at all values of the parameters and initial conditions. One specific example of this approach was shown in Figure 11 in which only two co-existing behaviors were obtained. A more general example of this approach can be found in [20], but is not pursued further here.

6. Concluding Remarks

This paper has described the behavior of an otherwise forced linear oscillator that is also subject to a sudden contact. The impact not only causes an (assumed) instantaneous reversal of velocity but also subjects the system to a discrete input of energy at impact. In some ways this system is analogous to a pinball machine in which a rolling ball is kicked whenever it comes into contact with a stopper. Here we are making a single-degree-of-freedom simplification, replacing gravity (the sloping surface) with a set of springs, and shaking the system horizontally. This is an example of a non-smooth dynamical system with a degree of unpredictability. The range of interesting dynamical behavior is captured by bifurcation diagrams and usefully augmented by spectrograms. A close agreement between the behavior of a mechanical experiment and numerical simulations is demonstrated.

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(a)

(b)

Figura 1: (a) a pinball machine, (b) A spring-mass-damper with an immovable impact barrier and base excitation.

Figure 2: The free response of an undamped impact oscillator. (a) - (c) a schematic of a small ball rolling on a parabolic surface, (d) - (f) some typical time series, (g) elliptical phase trajectories cropped by the impact barrier.

Figura 3: The harmonic response, (a) response amplitude-frequency response, (b) forcing amplitude-frequency response.

Figura 4: The harmonic response in terms of a Poincaré section, that is, position sampled at a given forcing phase, in this case $\phi = -\pi/4$.

Figura 5: Photographic images of the experimental set-up. (a) the overall system, (b) the laser trigger, (c) the Scotch-yoke forcing mechanism, (d) the solenoid.

Figura 6: Response of the impacting system under passive conditions using bifurcation diagrams. (a) Response amplitude envelope, experimental data, (b) Poincaré displacement, experimental data, (c) Response amplitude envelope, numerical, (d) Poincaré displacement, numerical. $z_0 = 60\text{mm}$, $y_0 = 10\text{mm}$.

Figure 7: Response of the impacting system under active conditions using bifurcation diagrams. (a) Response amplitude envelope, experimental data, (b) Poincaré displacement, experimental data, (c) Response amplitude envelope, numerical, (d) Poincaré displacement, numerical. $z_0 = 60\text{mm}$, $y_0 = 10\text{mm}$.

Figura 8: Detailed numerical results in the vicinity of grazing, (a) bifurcation diagram, (b) basins of attraction at $\Omega = 1.09$. $z_0 = 60\text{mm}$, $y_0 = 10\text{mm}$.

Figura 9: The dimensional impulse response of the system, (a) time series, (b) phase projection using time-delay coordinates. $y(t) = 0$. Note, the phase projection is rotated 90 degrees from conventional to match with the time series.

Figura 10: Response of the impacting system under active conditions using bifurcation diagrams. (a) Response amplitude envelope, experimental data, (b) Poincaré displacement, experimental data, (c) Response amplitude envelope, numerical, (d) Poincaré displacement, numerical. $z_0 = 22.5\text{mm}$, $y_0 = 10\text{mm}$.

Figura 11: Randomly generated experimental initial conditions at a forcing frequency $\Omega = 0.6$ (below grazing). The blue dots are attracted to the quasi-periodic response, and the red data points are attracted to the (non-impacting) periodic response.

Figura 12: *Quasi-periodic response of the experimental impacting system under active conditions ($z_0 = 22.5\text{mm}$) in terms of reconstructed phase trajectories using time-lag embedding: (a) using the forcing phase as Poincaré trigger, (b) using the impact period as Poincaré trigger. $\Omega = 0.64$.*

Figura 13: Experimental response spectrogram based on the FFT. (a) sweep-up ($z_0 = 60$ mm). (b) sweep-down ($z_0 = 60$ mm), (c) sweep-up ($z_0 = 22.5$ mm), (d) sweep-down ($z_0 = 22.5$ mm).

Figura 14: Numerical response spectrogram based on the FFT as a function of distance between equilibrium position and the impact location.