Bifurcations and strange nonchaotic attractors in a phase oscillator model of glacial-interglacial cycles

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Abstract

Glacial-interglacial cycles are large variations in continental ice mass and greenhouse gases, which have dominated climate variability over the Quaternary. The dominant periodicity of the cycles is ~ 40 kyr before the so-called middle Pleistocene transition between ~ 1.2 and ~ 0.7 Myr ago, and it is ~ 100 kyr after the transition. In this paper, the dynamics of glacial-interglacial cycles are investigated using a phase oscillator model forced by the time-varying incoming solar radiation (insolation). We analyze the bifurcations of the system and show that strange nonchaotic attractors appear through nonsmooth saddle-node bifurcations of tori. The bifurcation analysis indicates that mode-locking is likely to occur for the 41 kyr glacial cycles but not likely for the 100 kyr glacial cycles. The sequence of mode-locked 41 kyr cycles are robust to small parameter changes. However, the sequence of 100 kyr glacial cycles can be sensitive to parameter changes when the system has a strange nonchaotic attractor.

Keywords: Glacial-interglacial cycles, ice age, quasiperiodically forced dynamical systems, strange nonchaotic attractor, SNA, nonsmooth saddle-node bifurcations, middle-Pleistocene transition

1. Introduction

The glacial-interglacial cycles or simply glacial cycles are alternations between cold (glacial) and warm (interglacial) periods, which occurred over the last 3 million years (Myr). They are characterized by large fluctuations in the continental ice volume, which can be estimated, among others, from

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17 the oxygen isotope ratio $\delta^{18}O$ of benthic foraminifera sampled in deep-sea
18 cores (see Fig. 1) [1–5]. A large value of $\delta^{18}O$ indicates a large volume of ice
19 sheets. The dominant period of the glacial cycles is ~ 40 kyr before ~ 1.2 Myr
20 ago and ~ 100 kyr after ~ 0.7 Myr ago. This frequency change accompanying
21 an increase amplitude is called the middle-Pleistocene transition (MPT) (see
22 [6] and references therein).

23 There is abundant empirical evidence [1–10] that glacial cycles are con-
24 trolled by the changes in the parameters of Earth’s orbit (eccentricity e and
25 longitude of perihelion ϖ) and its obliquity ε [11]. Milankovitch’s theory
26 provides a physical mechanism for this control [12]: the orbital and obliq-
27 uity changes determine the seasonal and spatial distributions of insolation
28 (this is called the *astronomical forcing*), and, specifically, northern hemi-
29 sphere summer insolation controls the interannual accumulation of snow and
30 the subsequent growth of ice sheets. Milankovitch’s mechanism is nowadays
31 viewed as realistic, but it is considered to be only one element of a com-
32 plex nonlinear dynamical process. For example, the glacial cycles of the last
33 0.7 Myr have a dominant spectral power near 100 kyr, while the astronomical
34 forcing has negligible power near 100 kyr (so called the *100-kyr cycle prob-*
35 *lem* [3]). Thus, some nonlinear transformation mechanism must exist from
36 astronomical forcing to glacial cycles [7].

37 The glacial cycles have been investigated using various models ranging
38 from simple conceptual models to complex coupled general circulation mod-
39 els (for example, [13]), between which there are Earth Systems Models of
40 Intermediate Complexity (EMICs) [14–16]. Among others, conceptual mod-
41 els provide theoretical frameworks to identify the dynamics of glacial cycles
42 (for example, see [1, 17–29] and references therein). They are presented un-
43 der the form of low-dimensional dynamical systems forced by a measure of
44 northern hemisphere insolation or, more generally, by a linear combination of
45 $e \sin \varpi$ (the *climatic precession*) and obliquity ε . Since these quantities are
46 well-approximated as quasiperiodic functions of time over the past several
47 million years [11], the models of glacial cycles may be viewed as quasiperi-
48 odically forced dynamical systems.

49 Let $\mathbb{T}^N = \mathbb{R}^N / (2\pi\mathbb{Z})^N$ be an N -dimensional torus. Quasiperiodically
50 forced dynamical systems can be represented in a skew-product form:

$$\begin{cases} \dot{\theta} = \omega, & \theta \in \mathbb{T}^N, \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta), & \mathbf{x} \in \mathbb{R}^M, \end{cases} \quad (1)$$

51 where $\theta = (\theta_1, \theta_2, \dots, \theta_N)^T$ is the phase of the drive system, $\mathbf{x} = (x_1, x_2, \dots, x_M)^T$
52 is the state of the response system, $\mathbf{f}(\mathbf{x}, \theta)$ is a periodic function in each phase
53 θ_i , and $\omega = (\omega_1, \omega_2, \dots, \omega_N)^T$ is a vector of incommensurate frequencies such
54 that $k_1\omega_1 + k_2\omega_2 + \dots + k_N\omega_N = 0$ does not hold for any set of integers, k_1 ,
55 $k_2, \dots, k_N$, except for the trivial solution $k_1 = k_2 = \dots = k_N = 0$. In the
56 models of glacial cycles, $\theta(t)$ corresponds to the phase of the astronomical
57 forcing, and $\mathbf{x}(t)$ corresponds to the climate state.

58 Quasiperiodically forced systems can exhibit intermediate dynamics be-
59 tween quasiperiodicity and chaos, so-called *strange nonchaotic attractors*
60 (SNAs) [32, 33]. An SNA is a geometrically strange attractor for which
61 typical Lyapunov exponents are nonpositive [32] (see [30, 31] for compre-
62 hensive reviews). The system (1) is characterized by $(N + M)$ Lyapunov
63 exponents. Among them, N Lyapunov exponents are trivially zero. They
64 correspond to the phase equations of the drive system. When the system
65 has an SNA, the nontrivial largest Lyapunov exponent is negative in general.
66 Thus, under a common quasiperiodic forcing, trajectories $\mathbf{x}(t)$, which start
67 from different initial conditions $\mathbf{x}(0)$, approach (or synchronize to) a unique,
68 or one of a finite number of possible trajectories as time elapses (this is a
69 *synchronizing property*) [34, 35]. However, related to the strange geometry of
70 SNAs, trajectories $\mathbf{x}(t)$ have nonexponential sensitivity on initial phases $\theta(0)$
71 [36–39] or on parameter values of the systems [40]. SNAs have been observed
72 in many laboratory experiments [35, 41–46] but very rarely in nature so far.
73 Linder *et al.* show that the brightness changes of some RRc Lyrae stars have
74 nonchaotic and fractal properties of SNAs [47].

75 Recently, the authors of this paper showed that several models of glacial
76 cycles exhibit SNAs [27, 28]. This means that the relationship from the phase
77 of astronomical forcing θ to the climate state $\mathbf{x}$ is represented by a geomet-
78 rically strange set. When a model of glacial cycles exhibits an SNA, the
79 sequences of glacial cycles synchronize under the same astronomical forcing,
80 but simultaneously they can be sensitive to parameter changes. Ivashchenko
81 *et al.* showed that the autocorrelation function of the benthic isotopic data
82 has self-similarity characteristic of SNAs [48].

83 However, so far, the birth mechanism of SNAs has not been elucidated
84 in the models of glacial cycles. In this paper, we introduce a phase oscilla-
85 tor model of glacial cycles, whose bifurcation analysis is easier than higher-
86 dimensional models, and show the birth mechanism of SNAs in this model.
87 Based on the bifurcation analysis of the phase model, we suggest that the
88 41-kyr cycles, typical before the MPT, are little sensitive to the parameters.

89 By contrast the phase of 100-kyr cycles, which characterise climate after the
90 MPT, can be strongly sensitive to small parameter changes.

91 The remainder of this article is organized as follows. In Section 2, a phase
92 oscillator model for glacial cycles is introduced. In Section 3, the bifurca-
93 tions exhibited by the model are analyzed for two types of external forcing:
94 an ideal two-frequency quasiperiodic forcing and the astronomical forcing.
95 In Section 4, we discuss parameter sensitivity of the phase model, and the
96 MPT. Section 5 summarizes this article. In Appendix, we characterize the
97 strangeness of attractors using the phase sensitivity exponent [36].

98 2. Model

99 Consistently with the above discussion, the astronomical forcing $F_A(t)$ is
100 calculated by the following formula [26]:

$$F_A(t) = \frac{1}{a} \sum_{i=1}^{35} (s_i \sin \omega_i t + c_i \cos \omega_i t), \quad (2)$$

101 where we set $a = 23.58 \text{ W/m}^2$ so that $F_A(t)$ has unit variance. The fre-
102 quencies ω_i and coefficients, s_i and c_i , are based on [26], and they are listed
103 in [27] in descending order of power $s_i^2 + c_i^2$. The first three periods corre-
104 spond to the climatic precession: $2\pi/\omega_1 \approx 23.7 \text{ kyr}$, $2\pi/\omega_2 \approx 22.4 \text{ kyr}$, and
105 $2\pi/\omega_3 \approx 19.0 \text{ kyr}$. The fourth period $2\pi/\omega_4 \approx 41.0 \text{ kyr}$ corresponds to the
106 obliquity change [11]. Unless mentioned otherwise, we use the time unit of
107 10 kyr and the angular unit of radian for the variables and parameters of the
108 model.

109 Glacial cycles are described here by a forced oscillator, as suggested by
110 recent analyses [8–10]. Further, the global ice volume $V(\phi)$ is modeled as
111 a 2π -periodic function of the phase of the oscillator $\phi(t) \in \mathbb{T}^1 = \mathbb{R}^1/(2\pi\mathbb{Z})$,
112 i.e., $V(\phi) = V(\phi + 2\pi)$. We express the ice volume $V(\phi)$ as a Fourier sine
113 series up to the second harmonic:

$$V(\phi) = -\sin \phi - \frac{\delta}{2} \sin 2\phi, \quad (3)$$

114 where δ is the modification parameter. For $|\delta| < 1$, the ice volume $V(\phi)$ has
115 a unique minimum at $\phi_{\min} = \cos^{-1}[(\sqrt{1+8\delta^2}-1)/(4\delta)]$ and a maximum at
116 $\phi_{\max} = 2\pi - \phi_{\min}$, as shown in Fig. 2(a).

117 In general, the phase motion of a limit-cycle oscillator under a weak
 118 forcing $\mathbf{p}(t)$ is described by $\dot{\phi} = \beta + \mathbf{Z}(\phi) \cdot \mathbf{p}(t)$, where β is the natural
 119 frequency of the oscillator, and $\mathbf{Z}(\phi)$ is the sensitivity function with $\mathbf{Z}(\phi) =$
 120 $\mathbf{Z}(\phi + 2\pi)$ [49]. Based on this general phase equation, we suppose the phase
 121 motion of glacial cycles as follows:

$$\dot{\phi} = \beta - \alpha V'(\phi)[1 + \gamma F(t)], \quad (4)$$

122 where $V'(\phi) = \frac{dV(\phi)}{d\phi}$ is the gradient of the ice volume, $F(t)$ is a forcing
 123 function such as $F_A(t)$, and the parameters α , β , and γ are assumed to be
 124 positive. The specific term, $-\alpha V'(\phi)[1 + \gamma F(t)]$, provides a forcing to move
 125 toward an ice volume minimum for a large astronomical forcing with $F(t) >$
 126 $-1/\gamma$ and to move toward an ice volume maximum for a small astronomical
 127 forcing with $F(t) < -1/\gamma$. We can also write Eq. (4) as $\dot{\phi} = -\frac{\partial U(\phi, t)}{\partial \phi}$ using
 128 a time-dependent potential $U(\phi, t) = -\beta\phi + \alpha V(\phi)[1 + \gamma F(t)]$ (Fig. 2(a)).
 129 In this paper, we restrict ourselves to the simple case $|\delta| < 1/4$, where the
 130 number of local minima of $U(\phi, t)$ as a function of ϕ is at most one.

131 Let us first consider the case of constant forcing $F(t) = F_c$ is considered.
 132 The bifurcation diagram of Eq. (4) for $F(t) = F_c$ is shown in Fig 2(b). For
 133 low insolation $F_c < -\frac{1}{\gamma} \left(\frac{\beta}{\alpha(1+\delta)} + 1 \right)$, a stable equilibrium point, or a stable
 134 *node*, exists near the ice volume maximum $\phi_{\max}$, and an unstable equilibrium
 135 point, or a *saddle*, near the minimum $\phi_{\min}$. Thus, the system approaches a
 136 stable glacial state. For $-\frac{1}{\gamma} \left(\frac{\beta}{\alpha(1+\delta)} + 1 \right) < F_c < \frac{1}{\gamma} \left(\frac{\beta}{\alpha(1-\delta)} - 1 \right)$, there is
 137 no equilibrium point, and glacial-interglacial oscillations emerge. For high
 138 insolation $F_c > \frac{1}{\gamma} \left(\frac{\beta}{\alpha(1-\delta)} - 1 \right)$, a stable node exists near the ice volume
 139 minimum $\phi_{\min}$, and a saddle near the maximum $\phi_{\max}$. Thus, the system
 140 approaches a stable interglacial state.

141 Next, the case of astronomical forcing $F(t) = F_A(t)$ is studied. Unless
 142 otherwise noted, differential equations are integrated by the Runge–Kutta
 143 4th order method with a step size of 100 yr (namely $h = 0.01$). Figure 3
 144 shows a simulated ice volume trajectory $V(\phi(t))$ for $\alpha = \gamma = 1.0$, $\beta = 1.0006$,
 145 and $\delta = 0.24$ with initial condition $\phi(t_0) = 0$ at $t_0 = -20$ Myr. Here,
 146 assuming $\alpha = \gamma = 1.0$ and $\delta = 0.24$, the value of β was tuned to maximize
 147 the Pearson’s correlation coefficient r over the past 700 kyr between the ice
 148 volume reconstruction [5] and a simulated ice volume trajectory $V(\phi(t))$ with
 149 $\phi(t_0) = 0$ at $t_0 = -20$ Myr. This yields $r = 0.76$ for $\beta = 1.0006$.

150 Generally, trajectories $\phi(t)$ can depend on initial conditions $\phi(t_0)$. How-

151 ever, due to the astronomical forcing $F_A(t)$, trajectories starting from differ-
 152 ent initial phases $\phi(t_0)$ can converge to a single or some pieces of trajectories
 153 after a certain transient time, as shown in Figs. 4(a) and 4(b). The transient
 154 time can last for several million years for the case of 100-kyr cycles, but it is
 155 only several cycles for the case of 41-kyr cycles.

156 It should be noted that the asymmetric sawtooth pattern of the ice vol-
 157 ume trajectory $V(\phi(t))$ is not due to the subtle asymmetry of function $V(\phi)$
 158 caused by the second harmonic $-(\delta/2)\sin 2\phi$ but due to the specific form
 159 of Eq. (4); the phase $\phi(t)$ increases slowly in glaciation phases and quickly
 160 in deglaciation phases, on average. The qualitative features of the following
 161 bifurcation analysis do not change even if the second harmonic is omitted or
 162 changed in the range $|\delta| < 1/4$, but we keep the term because it increases
 163 the agreement with data, for example, by about 10% from $r = 0.64$ for $\delta = 0$
 164 to $r = 0.76$ for $\delta = 0.24$.

165 3. Bifurcation analysis

166 We study bifurcations of the model for the astronomical forcing $F_A(t)$
 167 and an ideal two-frequency quasiperiodic forcing described below. We set
 168 $\gamma = 1.0$ and $\delta = 0.24$, and regard α and β as control parameters.

169 3.1. The case of the ideal two-frequency quasiperiodic forcing

170 We start from the two-frequency quasiperiodic forcing

$$F_G(t) = \sin \omega_1 t + \sin \omega_G t, \quad (5)$$

171 where ω_1 is the most prominent frequency of the climatic precession ($2\pi/\omega_1 \approx$
 172 23.7 kyr), and $\omega_G = \omega_1 \times (\sqrt{5} - 1)/2$ ($2\pi/\omega_G \approx 38.4$ kyr) is a frequency near
 173 the dominant frequency of obliquity change $2\pi/\omega_4 \approx 41.0$ kyr [11], which
 174 was chosen to be most incommensurate with ω_1 . Defining phase variables,
 175 $\theta_1 = \omega_1 t \pmod{2\pi}$ and $\theta_G = \omega_G t \pmod{2\pi}$, the model can be expressed in
 176 the skew-product form

$$\begin{aligned} \dot{\phi} &= \beta + \alpha(\cos \phi + \delta \cos 2\phi)[1 + \gamma F_G(\theta_1, \theta_G)], \\ \dot{\theta}_1 &= \omega_1, \quad \dot{\theta}_G = \omega_G, \end{aligned} \quad (6)$$

177 where $F_G(t)$ is redefined as $F_G(\theta_1, \theta_G) = \sin \theta_1 + \sin \theta_G$, and the phase space is
 178 $\mathbb{T}^3$. Similar pendulum-type systems but with additive quasiperiodic forcing

179 have been studied in [50–53]. The attractor of the system is obtained by
 180 solving Eq. (6) forward for a long time. In the particular case of this model,
 181 the unstable invariant set of the system is found by solving Eq. (6) backward
 182 for a long time (this is called the *repeller*) [52]. We plot the attractor and
 183 the repeller at a Poincaré surface of section (PS) at a certain constant value
 184 of $\theta_1 (\approx 2.168184)$.¹

The system (6) has three Lyapunov exponents: one is the Lyapunov exponent λ_ϕ in the direction of ϕ , and the other two are trivially zero, corresponding to $\dot{\theta}_1 = \omega_1$ and $\dot{\theta}_G = \omega_G$. The dynamics of Eq. (6) are classified by the Lyapunov exponent λ_ϕ and the winding number W . The winding number W is defined as

$$W = \lim_{t \rightarrow \infty} \frac{\phi(t) - \phi(0)}{t}$$

185 for the same Eq. (6) but with ϕ on the line $\mathbb{R}$. The dynamics are referred to
 186 as *mode-locked* if the winding number W is constant with respect to slight
 187 changes of β [54]. When the dynamics are mode-locked, the model has a
 188 two-frequency quasiperiodic attractor and a two-frequency quasiperiodic re-
 189 peller, which are represented by a stable and an unstable invariant curve
 190 in PS, respectively (Fig. 5(a) for example), and the winding number W is
 191 rationally related to the forcing frequencies, i.e., $W = (k/m)\omega_1 + (l/m)\omega_G$
 192 ($k, l \in \mathbb{Z}, m \in \mathbb{N}$) [54]. The denominator m corresponds to the multiplicity of
 193 each invariant curve in the direction of ϕ [50]. The two-frequency quasiperi-
 194 odic attractors are characterized by the negative Lyapunov exponent $\lambda_\phi < 0$.
 195 When the dynamics are non-mode-locked, we find three-frequency quasiperi-
 196 odic motion when $\lambda_\phi = 0$ and SNAs when $\lambda_\phi < 0$, as is the case of the general
 197 quasiperiodically forced circle map [55] (see Figs. 5(c) and 5(f)).

198 The regime diagram for $F_G(t)$ is shown in Fig. 6(a), which is obtained
 199 by taking a grid size of $\Delta\alpha = 0.01$ and $\Delta\beta = 2.5 \times 10^{-3}$. The dynamics are
 200 numerically regarded as mode-locked if the change of W is less than 10^{-4}
 201 when β is changed either by $\pm\Delta\beta$ (white regions). The non-mode-locked
 202 regions are divided into regions with $|\lambda_\phi| < 5.0 \times 10^{-4}$ (green) and regions
 203 with $\lambda_\phi \leq -5.0 \times 10^{-4}$ (magenta). Under these numerical criteria, we find
 204 typically the two-frequency quasiperiodic attractors in mode-locked regions
 205 (white), three-frequency quasiperiodic motion in non-mode-locked regions

¹In numerical calculations for Fig. 5, a step size $h = 2\pi/\omega_1 \times 0.002$ was used so that θ_1 takes a constant phase in every 500 steps.

206 with $|\lambda_\phi| < 5.0 \times 10^{-4}$ (green), and SNAs in non-mode-locked regions with
 207 $\lambda_\phi < -5.0 \times 10^{-4}$ (magenta). The boundaries of each regime depend on the
 208 grid size and thresholds slightly,² but the overall structure is fairly robust.

209 The strangeness of attractor in the non-mode-locked regions with negative
 210 Lyapunov exponent (magenta) may also be assessed with the phase sensitiv-
 211 ity exponent [36], which characterizes the sensitivity of ϕ with respect to the
 212 changes in θ_1 or θ_G . The phase sensitivity exponent is computed in the Ap-
 213 pendix, and the results show that the attractor is smooth in the mode-locked
 214 regions, and strange in the non-mode-locked regions with negative Lyapunov
 215 exponent.

216 The mode-locked regions form Arnol'd tongue-like structures in the pa-
 217 rameter plane, but the width of each mode-locked region first increases and
 218 then decreases to values close to zero. Such leaf-like Arnol'd tongues are
 219 characteristic of mode-locking in quasiperiodically forced systems [37], and
 220 they have been found in several models of glacial cycles [26, 28].

221 The system exhibits two types of bifurcations on the boundaries of mode-
 222 locked regions. The *smooth saddle-node bifurcation of tori* occurs on the
 223 boundaries where the Lyapunov exponent λ_ϕ changes from negative to zero
 224 (between white and green regions). The *nonsmooth saddle-node bifurcation of*
 225 *tori* [37, 55, 56] occurs on the boundaries where λ_ϕ remains negative (between
 226 white and magenta regions). Here, “tori” means the quasiperiodic attractor
 227 and repeller in the mode-locked regions (In higher-dimensional systems, the
 228 latter need not be a repeller but an unstable quasiperiodic torus of saddle
 229 type). In this paper, we focus on the bifurcations for the mode-locked region
 230 with $W = 0$, but they occur in the other mode-locked regions as well in the
 231 same manner.

232 In the smooth saddle-node bifurcation, the distance between stable and

²For the $(N+1)$ -frequency quasiperiodic motion with the Lyapunov exponent λ_ϕ of zero, the order of the computed Lyapunov exponent converges to zero inversely proportionally to the averaging time T of the local Lyapunov exponent. Typically, we have $|\lambda_\phi| \sim O(10^{-5})$ for $T = 10^5$. Thus, the threshold at $\lambda_\phi = -5.0 \times 10^{-4}$ is effective to distinguish the regimes with and without the Lyapunov exponent λ_ϕ of zero. The changes in the threshold between $-10^{-5} < \lambda_\phi < -10^{-4}$ shift the boundaries between $(N+1)$ -frequency quasiperiodic motion (green) and SNA (magenta) in the direction parallel to nearby mode-locked regions. This uncertainty of the boundaries is typically less than 0.1 with respect to α and β . The choice of the threshold for the winding number W is also based on the numerical convergence speed. The above choice gives consistent results with the analysis of the phase sensitivity exponent shown in Appendix.

233 unstable invariant curves decreases to zero for every value of θ_G , as shown in
 234 Fig. 5(b). At the bifurcation point $\beta = \beta_c$, the stable and unstable invari-
 235 ant curves merge at every point in θ_G yielding three-frequency quasiperiodic
 236 motion, as shown in Fig. 5(c). Specifically, define the distance $d(\theta_1, \theta_G)$ be-
 237 tween a point on attractor and a point on the repeller at the same (θ_1, θ_G) ,
 238 and consider its maximum $d_{\max}$ and its minimum $d_{\min}$ on $(\theta_1, \theta_G) \in \mathbb{T}^2$ [57].
 239 As shown in Fig. 7(a), they obey a scaling law $d_{\min} \simeq C_1 |\beta_c - \beta|^{0.5}$ and
 240 $d_{\max} \simeq C_2 |\beta_c - \beta|^{0.5}$ ($C_1 \leq C_2$). This scaling clearly holds with $C_1 = C_2$ in
 241 the small forcing limit $\gamma \rightarrow 0$, where the smooth saddle-node bifurcation of
 242 tori degenerates into that of equilibrium points.

243 On the other hand, in the non-smooth saddle-node bifurcation, the colli-
 244 sion of the stable invariant curves and unstable invariant curves occurs only
 245 in a countable dense set of θ_G , as demonstrated in Fig. 5(e), and it creates an
 246 SNA and a strange repeller. An SNA and a strange repeller for $\alpha = 2.2$ and
 247 $\beta = 1.4$ are shown in Fig. 5(f). At the bifurcation point $\beta = \beta_c$, the minimal
 248 distance $d_{\min}$ decreases to zero, but the maximal distance $d_{\max}$ is strictly pos-
 249 itive, as shown in Fig. 7(b) [57]. In the nonsmooth saddle-node bifurcation,
 250 the distances do not show the scaling low $|\beta_c - \beta|^{0.5}$, which appears in the
 251 smooth saddle-node bifurcation.

252 3.2. The case of the astronomical forcing

253 We now consider the case of the astronomical forcing $F_A(t)$, where Eq. (6)
 254 is defined using $F_A(\theta_1, \dots, \theta_{35}) = \frac{1}{a} \sum_{i=1}^{35} (s_i \sin \theta_i + c_i \cos \theta_i)$, $\theta_i = \omega_i$ instead
 255 of $F_G(\theta_1, \theta_G)$. The regime diagram for $F_A(t)$ is shown in Fig. 6(b), which is
 256 produced in the same manner as Fig. 6(a). As a natural extension from two-
 257 frequency forcing, the system exhibits a 35-frequency quasiperiodic attractor
 258 in each mode-locked region (white), 36-frequency quasiperiodic motion in
 259 non-mode-locked regions with $\lambda_\phi = 0$ (green), and an SNA in non-mode-
 260 locked regions with $\lambda_\phi < 0$ (magenta). The strangeness of attractor in non-
 261 mode-locked regions with negative Lyapunov exponent is assessed by the
 262 phase sensitivity exponent [37] in Appendix, which numerically shows that
 263 the attractor is smooth in the mode-locked regions and strange in the non-
 264 mode-locked regions with negative Lyapunov exponent.

265 As is the case of two-frequency forcing, smooth and nonsmooth saddle-
 266 node bifurcations occur at the boundaries of mode-locked regions. Due to
 267 high dimensionality, the attractor and repeller are not visible but we can cal-
 268 culate the maximal distance $d_{\max}$ and the minimal distance $d_{\min}$ between the
 269 attractor and the repeller. On the boundaries with zero Lyapunov exponent,

the smooth saddle-node bifurcation occurs creating 36-frequency quasiperiodic motion, with the scaling law $d_{\min} \simeq C_1 |\beta_c - \beta|^{0.5}$ and $d_{\max} \simeq C_2 |\beta_c - \beta|^{0.5}$, as shown in Fig. 7(c). On the boundaries with negative Lyapunov exponent, the nonsmooth saddle-node bifurcation occurs creating an SNA and a strange repeller. At the bifurcation point $\beta = \beta_c$, the minimal distance $d_{\min}$ decreases to zero, but the maximal distance $d_{\max}$ is strictly positive, as shown in Fig. 7(d). In the nonsmooth saddle-node bifurcation, the distances do not show the scaling law $|\beta_c - \beta|^{0.5}$, which appears in the smooth saddle-node bifurcation.

4. Discussions

4.1. Parameter sensitivity

A number of previous studies report parameter sensitivity of the sequence of glacial cycles generated by simulation models [1, 20, 21, 24, 27–29, 58]. In some models, parameter sensitivity is attributed to thresholds with discontinuity [1], and in other models, it is attributed to a chaotic property, that is, the exponential sensitivity to initial conditions [29]. However, parameter sensitivity can appear even when the models are smooth dynamical systems and nonchaotic. An example is Saltzman-Maasch (1990) model [59] as pointed out in [28]. Our phase oscillator model also shows such a parameter sensitivity.

To indicate “the region of 100 kyr cycles” in parameter space, we again calculate the Pearson’s correlation coefficient r over the past 700 kyr between the ice volume reconstruction [5] and a simulated ice volume trajectory with $\phi(t_0) = 0$ at $t_0 = -20$ Myr. The parameter points with high correlation $r > 0.7$ are plotted by symbol “+” in Fig. 6(b).³ They are distributed in a narrow region, where the winding number W is constrained between 0.58 and 0.66. Around the region of 100 kyr cycles, the mode-locked states are unlikely to occur because they have little measure. Even if the mode-locked states occurred, they would be fragile against small parameter changes.

If the 100 kyr cycles are non-mode-locked, the dynamics of SNA or the dynamics of $(N + 1)$ -frequency quasiperiodic motion with the Lyapunov exponent of zero are candidates for the 100 kyr cycles. The dynamics of SNAs

³High values of correlation coefficient r with $r > 0.7$ can be obtained in regions with 36-frequency quasiperiodic motion (green) or in mode-locked regions with $m \geq 2$ (white) by chance because trajectories depend on initial conditions over the past 700 kyr.

show the synchronizing property under the same astronomical forcing, as shown in Fig. 4(a), which is often considered as an important feature for the models of glacial cycles. However, in the SNA regime, the sequence of glacial cycles can be highly sensitive to parameter changes. Two simulated ice volume trajectories for two slightly different parameter values $\beta = 1.0006$ (red) and $\beta = 1.0001$ (blue) are shown in Fig. 8(a). The deviation of the trajectories with nearby parameters is reminiscent of the well-known butterfly effect in chaotic systems but this system is nonchaotic and even has a negative Lyapunov exponent λ_ϕ for these parameter values of β . Since SNAs have strange geometrical structures as shown in Fig. 5(f), subtle changes of parameter values can induce large shifts of the points in the attractors.

On the other hand, the 41 kyr cycles is caused by the mode-locking to the 41 kyr component of astronomical forcing (due to obliquity motion) in this model. The mode-locking region of the 41 kyr cycles is relatively wide (see the white region labeled by $W = \omega_4$ in Fig. 6(b)). Thus, this mode-locking is likely to occur. This is consistent with the previous study [22], which proposes that the phase of 41 kyr cycles is locked by the astronomical forcing.

The sequence of the mode-locked 41 kyr cycles is relatively robust to parameter changes. Figure 8(b) shows ice volume trajectories corresponding to different values of β in the mode-locked region: $\beta = 1.61$, $\beta = 1.65$, and $\beta = 1.69$. The other parameters are $\alpha = 0.8$, $\gamma = 1$, and $\delta = 0.24$.

4.2. Middle Pleistocene transition

The middle Pleistocene transition (MPT) began ~ 1.2 Myr ago and was complete by ~ 0.7 Myr ago, through which the average period of glacial cycles changed from ~ 40 kyr to ~ 100 kyr, accompanying an increase of amplitude [6]. A full investigation of the causes and dynamics of the MPT is beyond the scope of this paper. In particular, a phase model provides no information on amplitude. Yet, we note that a frequency change similar to the one observed during the MPT is obtained when the parameter β is changed over the period of MPT, as shown in Fig. 1. The parameter β is fixed at $\beta = 1.65$ in the model-locked region with $W = \omega_4$ until 1300 kyr ago (see Fig. 6(b)). Then, β is decreased linearly until it reaches $\beta = 0.9$ at 600 kyr ago. After 600 kyr ago, β is kept at $\beta = 0.9$. The other parameters are set at $\alpha = 0.8$, $\gamma = 1.0$, and $\delta = 0.24$. The sequence of the glacial cycles in Fig. 1 is robust to changes in initial conditions because of the strong synchronizing property of the 41-kyr cycles before the MPT (cf. Fig. 4(b)).

339 5. Summary

340 We introduced a phase oscillator model of glacial cycles and analyzed the
 341 bifurcations of the model for the ideal two-frequency quasiperiodic forcing
 342 and for the astronomical forcing. It was shown that SNAs appear through
 343 nonsmooth saddle-node bifurcations of tori in the model. Based on the re-
 344 sults for the phase oscillator model, we conjecture that the bifurcations from
 345 quasiperiodic attractors to SNAs found in oscillator models of glacial cy-
 346 cles [28] are also nonsmooth saddle-node bifurcations. The regime diagram
 347 in Fig. 6(b) indicates that mode-locking is likely to occur for the 41 kyr
 348 glacial cycles but not likely for the 100 kyr glacial cycles. The sequence of
 349 mode-locked 41 kyr cycles is robust to small parameter changes. However,
 350 the sequence of 100 kyr glacial cycles can be sensitive to parameter changes
 351 when the system has an SNA.

352 Long transient dynamics of million-year scale can be observed for the 100-
 353 kyr glacial cycles though it can vanish quickly for the 41-kyr cycles (Figs. 4(a)
 354 and 4(b)). Given that the Quaternary ice age is ~ 3 Myr, and that there can
 355 be perturbations to the system, transient trajectories may be important to
 356 understand the glacial cycles, not only just on the attractor. This problem
 357 will be explored elsewhere [60].

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365 Appendix. Strangeness of attractors in the non-mode-locked re- 366 gions with the negative Lyapunov exponent

367 We present numerical support for the existence of SNAs in the non-mode-
 368 locked regions with the negative Lyapunov exponent $\lambda_\phi < 0$ (magenta regions
 369 of Figs. 6(a) and (b)). Denote formally the phase equation of an oscillator
 370 under an N -frequency forcing by

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\phi} = f(\phi, \theta), \end{cases} \quad (7)$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_N)^T$ and $\omega = (\omega_1, \omega_2, \dots, \omega_N)^T$. When the system (7) has an N -frequency quasiperiodic attractor, it is represented by a certain smooth function $\phi = H(\theta)$. On the other hand, when the system (7) has an SNA, the relationship between θ and ϕ is discontinuous.

The phase sensitivity function $\Gamma(t)$ of variable ϕ with respect to variable θ_i is defined as

$$\Gamma(t) = \min_{\{\phi(0), \theta(0)\}} \left\{ \max_{0 \leq \tau \leq t} \left| \frac{\partial \phi(\tau)}{\partial \theta_i} \right| \right\}. \quad (8)$$

This derivative $\partial \phi(\tau)/\partial \theta_i$ is obtained by integrating

$$\frac{d}{dt} \left(\frac{\partial \phi}{\partial \theta_i} \right) = \frac{\partial f(\phi, \theta)}{\partial \phi} \left(\frac{\partial \phi}{\partial \theta_i} \right) + \frac{\partial f(\phi, \theta)}{\partial \theta_i}, \quad (9)$$

along the solution of Eq. (7). When the system (7) has an N -frequency quasiperiodic attractor, the state $(\theta(t), \phi(t))$ and the quantity $\partial \phi(t)/\partial \theta_i$, which started from some initial conditions, approach the attractor and its derivative $H'(\theta(t))$, respectively, as time t increases [36]. Thus, $\Gamma(t)$ is bounded for N -frequency quasiperiodic attractors. If we write the phase sensitivity function formally as $\Gamma(t) \simeq t^\mu$ for large t , the exponent μ is zero for N -frequency quasiperiodic attractors. The exponent μ is called the *phase sensitivity exponent*. On the other hand, SNAs do not have finite derivatives by definition, but we can calculate the quantity $\partial \phi(t)/\partial \theta_i$ (and $\Gamma(t)$) along orbits approaching an SNA. It is known that the phase sensitivity function grows as $\Gamma(t) \simeq t^\mu$ with $\mu > 0$ for SNAs (see [31, 36] for detail).

For the case of two-frequency forcing $F_G(\theta_1, \theta_G)$, we calculate the phase sensitivity function $\Gamma(t)$ with respect to θ_G , changing parameter β , where the other parameters are set to $\alpha = 2.2$, $\gamma = 1$, and $\delta = 0.24$, as in Fig. 7(b). For the case of astronomical forcing $F_A(\theta_1, \dots, \theta_{35})$, the phase sensitivity function $\Gamma(t)$ is calculated with respect to θ_4 , changing parameter β , where the other parameters are set to $\alpha = 1.4$, $\gamma = 1$, and $\delta = 0.24$, as in Fig. 7(d). The phase sensitivity exponent μ is measured from the growth of the phase sensitivity function over the time interval $[10^5, 10^7]$.

Figures 9(a) and 9(b) show the winding number W , the phase sensitivity function μ , and the value of phase sensitivity function Γ at $t = 10^7$ as functions of β . Mode-locked regions and non mode-locked regions are shown by thick points and thin points, respectively, in the graph of the winding number W . The phase sensitivity exponent μ takes a positive value enough away from zero in the non-mode-locked regions. This means that the attractors in the non-mode-locked regions are typically strange. On the other

hand, the phase sensitivity exponent μ takes the value of zero in most of the mode-locked regions, which indicates N -frequency quasiperiodic attractors. In some regions ($1.4075 \lesssim \beta \lesssim 1.41$ and $1.4225 \lesssim \beta \lesssim 1.435$ in Fig. 9(a)), the numerical value of μ obtained with a simulation length of 10^7 can be positive due to slow convergence. However, the value of the phase sensitivity function $\Gamma(10^7)$ itself is small when compared to non-mode-locked regions.

References

- [1] D. Paillard, Quaternary glaciations: from observations to theories, *Quat. Sci. Rev.* **107**, 11 (2015).
- [2] J. Imbrie, E. A. Boyle, S. C. Clemens, A. Duffy, W. E. Howard, G. Kukla, J. Kutzbach, D. G. Martinson, A. McIntyre, A. C. Mix, B. Molfino, J. J. Morley, L. C. Peterson, N. G. Pisias, W. L. Prell, M. E. Raymo, N. J. Shackleton, and J. R. Toggweiler, On the structure and origin of major glaciation cycles. 1. Linear responses to Milankovitch forcing, *Paleoceanography*, **7**, 701 (1992).
- [3] J. Imbrie, A. Berger, E. A. Boyle, S. C. Clemens, A. Duffy, W. E. Howard, G. Kukla, J. Kutzbach, D. G. Martinson, A. McIntyre, A. C. Mix, B. Molfino, J. J. Morley, L. C. Peterson, N. G. Pisias, W. L. Prell, M. E. Raymo, N. J. Shackleton, and J. R. Toggweiler, *Paleoceanography*, On the structure and origin of major glaciation cycles. 2. The 100,000-year cycle, **8**, 699 (1993).
- [4] W. F. Ruddiman, Orbital insolation, ice volume, and greenhouse gases, *Quat. Sci. Rev.* **22**, 1597 (2003).
- [5] L. E. Lisiecki and M. E. Raymo, A Pliocene-Pleistocene stack of 57 globally distributed benthic $\delta^{18}\text{O}$ records, *Paleoceanography* **20**, PA1003 (2005).
- [6] P. U. Clark, D. Archer, D. Pollard, J. D. Blum, J. A. Rial, V. Brovkin, A. C. Mix, N. G. Pisias, and M. Roy, The middle Pleistocene transition: characteristics, mechanisms, and implications for long-term changes in atmospheric pCO_2 , *Quat. Sci. Rev.* **25**, 3150 (2006).

- [7] J. D. Hays, J. Imbrie, and N. J. Shackleton, Variations in the Earth's orbit: Pacemakers of the ice ages, *Science* **194**, 1121 (1976).
- [8] L. Lisiecki, Links between eccentricity forcing and the 100,000-year glacial cycle, *Nature Geosci.* **3**, 349 (2010).
- [9] J. Nie, J. King, and X. Fang, Late Pliocene-early Pleistocene 100-ka problem, *Geophys. Res. Lett.* **35**, L21606 (2008).
- [10] J. A. Rial, J. Oh, and E. Reischmann, Synchronization of the climate system to eccentricity forcing and the 100,000-year problem, *Nature Geosci.* **6**, 289 (2013).
- [11] A. L. Berger, Long-term variations of daily insolation and Quaternary climatic change, *J. Atmos. Sci.* **35**, 2362 (1978); A. Berger and M. F. Loutre, Insolation values for the climate of the last 10 million years, *Quat. Sci. Rev.* **10**, 297 (1991).
- [12] M. Milankovitch, *Kanon der Erdbestrahlung und seine Anwendung auf das Eiszeitenproblem* (Königlich Serbische Akademie, Belgrade, 1941).
- [13] A. Abe-Ouchi, F. Saito, K. Kawamura, M. E. Raymo, J. Okuno, K. Takahashi, and H. Blatter, Insolation-driven 100,000-year glacial cycles and hysteresis of ice-sheet volume, *Nature* **500**, 190 (2013).
- [14] M. Claussen, L. A. Mysak, A. J. Weaver, M. Crucifix, T. Fichefet, M.-F. Loutre, S. L. Weber, J. Alcamo, V. A. Alexeev, A. Berger, R. Calov, A. Ganopolski, H. Goosse, G. Lohmann, F. Lunkeit, I. I. Mokhov, V. Petoukhov, P. Stone, and Z. Wang, Earth system models of intermediate complexity: closing the gap in the spectrum of climate system models, *Climate Dynamics* **18**, 579 (2002).
- [15] H. Gallée, J. P. van Ypersele, T. Fichefet, I. Marsiat, C. Tricot, and A. Berger, Simulation of the last glacial cycle by a coupled, sectorially averaged climate-ice sheet model. Part II: response to insolation and CO₂ variation, *J. Geophys. Res.* **97**, 15713 (1992).
- [16] A. Ganopolski, R. Calov, and M. Claussen, Simulation of the last glacial cycle with a coupled climate ice-sheet model of intermediate complexity, *Clim. Past* **6**, 229 (2010).

- 466 [17] M. Crucifix, Traditional and novel approaches to palaeoclimate mod-
467 elling, *Quat. Sci. Rev.* **57**, 1 (2012).
- 468 [18] M. Ghil and S. Childress, *Topics in geophysical fluid dynamics: at-*
469 *mospheric dynamics, dynamo theory and climate dynamics* (Applied
470 Mathematical Sciences, vol. 60. Berlin, Germany: Springer, 1987).
- 471 [19] B. Saltzman, Dynamical paleoclimatology: generalized theory of global
472 climate change (international geophysics. International Geophysics Se-
473 ries, vol. 80. New York, NY: Academic Press, 2001).
- 474 [20] J. Weertman, Milankovitch solar radiation variations and ice age ice
475 sheet sizes, *Nature* **261**, 17 (1976).
- 476 [21] J. Oerlemans, Glacial cycles and ice-sheet modelling, *Climate Change*
477 **4**, 353 (1982).
- 478 [22] Y. Ashkenazy, and E. Tziperman, Are the 41 kyr glacial oscillations
479 a linear response to Milankovitch forcing? *Quat. Sci. Rev.* **23**, 1879
480 (2004).
- 481 [23] E. Tziperman, M. E. Raymo, P. Huybers, and C. Wunsch, Conse-
482 quences of pacing the Pleistocene 100 kyr ice ages by nonlinear phase
483 locking to Milankovitch forcing, *Paleoceanography* **21**, PA4206 (2006).
- 484 [24] A. M. Hogg, Glacial cycles and carbon dioxide: A conceptual model,
485 *Geophys. Res. Lett.* **35**, L01701 (2008).
- 486 [25] M. Crucifix, Oscillators and relaxation phenomena in Pleistocene cli-
487 mate theory, *Philos. Trans. R. Soc. A* **370**, 1140 (2012).
- 488 [26] B. De Saedeleer, M. Crucifix, and S. Wiczorek, Is the astronomical
489 forcing a reliable and unique pacemaker for climate? A conceptual
490 model study, *Clim. Dyn.* **40**, 273 (2013).
- 491 [27] T. Mitsui and K. Aihara, Dynamics between order and chaos in con-
492 ceptual models of glacial cycles, *Clim. Dyn.* **42**, 3087 (2014).
- 493 [28] M. Crucifix, Why could ice ages be unpredictable? *Clim. Past* **9**, 2253
494 (2013).

- 495 [29] I. Daruka and P. D. Ditlevsen, Changing climatic response: a concep-
496 tual model for glacial cycles and the Mid-Pleistocene Transition, *Clim.*
497 *Past Discuss.* **10**, 1101 (2014).
- 498 [30] A. Prasad, S. S. Negi, and R. Ramaswamy, Strange nonchaotic attrac-
499 tors, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **11**, 291 (2001).
- 500 [31] U. Feudel, S. Kuznetsov, and A. Pikovsky, *Strange Nonchaotic Attrac-*
501 *tors* (World Scientific, Singapore, 2006).
- 502 [32] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Strange attractors
503 that are not chaotic, *Physica D* **13**, 261 (1984).
- 504 [33] K. Kaneko, Fractalization of torus, *Prog. Theor. Phys.* **71**, 1112 (1984).
- 505 [34] R. Ramaswamy, Synchronization of strange nonchaotic attractors,
506 *Phys. Rev. E*, **56**, 7294 (1997).
- 507 [35] S. Uennohara, T. Mitsui, Y. Hirata, T. Morie, Y. Horio, and K. Ai-
508 hara, Experimental distinction between chaotic and strange nonchaotic
509 attractors on the basis of consistency, *Chaos* **23**, 023110 (2013).
- 510 [36] A. Pikovsky and U. Feudel, Characterizing strange nonchaotic attrac-
511 tors, *Chaos* **5**, 253 (1995).
- 512 [37] U. Feudel, J. Kurths, and A. Pikovsky, Strange non-chaotic attractor
513 in a quasiperiodically forced circle map, *Physica D* **88**, 176 (1995).
- 514 [38] P. Glendinning, T. H. Jäger, and G. Keller, How chaotic are strange
515 nonchaotic attractors?, *Nonlinearity* **19**, 2005 (2006).
- 516 [39] T. Mitsui, Subdiffusion due to strange nonchaotic dynamics: A numer-
517 ical study, *Phys. Rev. E*, **83**, 066212 (2011).
- 518 [40] T. Nishikawa and K. Kaneko, Fractalization of a torus as a strange
519 nonchaotic attractor, *Phys Rev E* **54**, 6114 (1996).
- 520 [41] W. Ditto, M. Spano, H. Savage, S. Rauseo, J. Heagy, and E. Ott,
521 Experimental observation of a strange nonchaotic attractor, *Phys. Rev.*
522 *Lett.* **65**, 533 (1990).

- [42] T. Zhou, F. Moss, and A. Bulsara, Observation of a strange nonchaotic attractor in a multistable potential, *Phys. Rev. A* **45**, 5394 (1992).
- [43] T. Yang and K. Bilimgut, Experimental results of strange nonchaotic phenomenon in a second-order quasi-periodically forced electronic circuit, *Phys. Lett. A* **236**, 494 (1997).
- [44] K. Thamilmaran, D. Senthilkumar, A. Venkatesan, and M. Lakshmanan, Experimental realization of strange nonchaotic attractors in a quasiperiodically forced electronic circuit, *Phys. Rev. E* **74**, 036205 (2006).
- [45] G. Ruiz and P. Parmananda, Experimental observation of strange nonchaotic attractors in a driven excitable system, *Phys. Lett. A* **367**, 478 (2007).
- [46] T. Mitsui, S. Uenohara, T. Morie, Y. Horio, and K. Aihara, Torus-doubling process via strange nonchaotic attractors, *Phys. Lett. A*, **376**, 1907 (2012).
- [47] J. F. Lindner, V. Kohar, B. Kia, M. Hippke, J. G. Learned, and W. L. Ditto, Strange nonchaotic stars, *Phys. Rev. Lett.* **114**, 054101 (2015).
- [48] N. N. Ivashchenko, V. M. Kotlyakov, D. M. Sonechkin, and N. V. Vakulenko, On bifurcations inducing glacial cycle lengthening during Pliocene/Pleistocene epoch, **24**, 1440018 (2014).
- [49] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
- [50] A. Bondeson, E. Ott, and T. M. Antonsen, Quasiperiodically forced damped pendula and Schrödinger equation with quasiperiodic potentials: Implication of their equivalence, *Phys. Rev. Lett.* **55**, 2103 (1985).
- [51] F. J. Romeiras, A. Bondeson, E. Ott, T. M. Antonsen Jr, and C. Grebogi, Quasiperiodically forced dynamical systems with strange nonchaotic attractors, *Physica D* **26**, 277 (1987).
- [52] E. Neumann and A. Pikovsky, Quasiperiodically driven Josephson junctions: strange nonchaotic attractors, symmetries and transport, *Eur. Phys. J. B* **26**, 219 (2002).

- 555 [53] S. P. Kuznetsov and E. Neumann, Torus fractalization and singularities
556 in the current-voltage characteristics for the quasiperiodically forced
557 Josephson junction, *Europhys. Lett.* **61**, 20 (2003).
- 558 [54] K. Bjerklöv and T. Jäger, Rotation numbers for quasiperiodically
559 forced circle maps – mode-locking vs strict monotonicity, *J. Amer.*
560 *Math. Soc.* **22**, 353 (2008).
- 561 [55] U. Feudel, C. Grebogi, and E. Ott, Phase-locking in quasiperiodically
562 forced systems, *Phys. Rep.* **290**, 11 (1997).
- 563 [56] T. Jäger, The creation of strange non-chaotic attractors in non-smooth
564 saddle-node bifurcations, *Mem. Am. Math. Soc.* **945**, 1 (2009).
- 565 [57] À. Haro and J. Puig, Strange non-chaotic attractors in Harper maps,
566 *Chaos* **16**, 033127 (2006).
- 567 [58] A. Ganopolski and R. Calov, The role of orbital forcing, carbon dioxide
568 and regolith in 100 kyr glacial cycles, *Clim. Past* **7**, 1415 (2011).
- 569 [59] B. Saltzman and K. A. Maasch, A first-order global model of late
570 Cenozoic climatic change, *Trans. R. Soc. Edinburgh Earth Sci.* **81**,
571 315 (1990).
- 572 [60] T. Mitsui and M. Crucifix, Effects of noise on the dynamics of glacial
573 cycles, submitted.

Figure 1: (Color) Glacial cycles over the Quaternary: The global ice volume reconstructed from the oxygen isotope ratio (black solid line) [5] and the simulated ice volume $V(\phi(t))$ (red dashed line). The former is scaled as $1.3 \times \delta^{18}O - 4$ for comparison. The parameter β is fixed at $\beta = 1.65$ until -1300 kyr and fixed at $\beta = 0.9$ after -600 kyr, between which β is decreased linearly interpolating the two values. The other parameters are $\alpha = 0.8$, $\gamma = 1.0$, and $\delta = 0.24$. The sequence is robust to changes in initial conditions because of the strong synchronizing property of the 41-kyr cycles before the middle-Pleistocene transition.

Figure 2: (Color) (a) Ice volume $V(\phi)$ (black solid line) and potential $U(\phi, t)$ as functions of ϕ for $F(t) = 3$ (red solid line), $F(t) = 0$ (magenta dashed line), and $F(t) = -3$ (blue dotted line). $\alpha = 1.0$, $\beta = 1.0006$, $\gamma = 1.0$, and $\delta = 0.24$. (b) Bifurcation diagram of Eq. (4) for constant forcing $F(t) = F_c$. The positions of stable nodes and saddles are presented by black solid lines and green dotted lines, respectively. The horizontal lines are drawn at the ice volume maximum $\phi = \phi_{\max}$ and minimum $\phi = \phi_{\min}$.

Figure 3: (Color) Global ice volume reconstructed from benthic oxygen isotopic ratio (dashed line) [5] and the simulated ice volume V (blue line) using Eqs. (2) and (4). $\alpha = 1.0$, $\beta = 1.0006$, $\gamma = 1.0$, and $\delta = 0.24$. Both curves are normalized.

Figure 4: (Color) Synchronization of ice volume trajectories $V(\phi(t))$ started from different initial conditions $\phi(t_0) = 2\pi k/5$ ($k = 0, 1, 2, 3, 4$) at time $t_0 = -10$ Myr under the astronomical forcing $F_A(t)$. (a) The case of 100-kyr cycles for $\alpha = 1.0$, $\beta = 1.0006$, $\gamma = 1.0$, and $\delta = 0.24$. (b) The case of 41-kyr cycles for $\alpha = 0.80$, $\beta = 1.65$, $\gamma = 1.0$, and $\delta = 0.24$.

Figure 5: (Color) Poincaré surface of section plots at a certain constant value of $\theta_1 (\approx 2.168184)$ obtained by integrating Eq. (6) forward (red) and backward (blue) for a long time, respectively. Each panel corresponds to different values of parameters α and β : (a) in two-frequency quasiperiodic regime, (b) same but just before the smooth saddle-node bifurcation occurring at $\alpha = 0.4$, $\beta_c \approx 0.303754$, (c) in three-frequency quasiperiodic regime, (d) in two-frequency quasiperiodic regime, (e) same but just before the nonsmooth saddle-node bifurcation occurring at $\alpha = 2.2$, $\beta_c \approx 1.3964061$, and (f) in the SNA regime.

Figure 6: (Color) (a) Regime diagram for the two-frequency forcing $F_G(t)$ ($N = 2$) and (b) regime diagram for the astronomical forcing $F_A(t)$ ($N = 35$): the mode-locked regions corresponding to N -frequency quasiperiodic attractors (invariant curves in PS) (white), the regions with $|\lambda_\phi| < 5.0 \times 10^{-4}$ typically corresponding to $(N + 1)$ -frequency quasiperiodic motion (green), and the regions with $\lambda_\phi \leq -5.0 \times 10^{-4}$ typically corresponding to SNAs (magenta). Prominent mode-locked regions are shown with their rotation numbers. The parameter points corresponding to correlation coefficient r larger than 0.7 are marked by symbol “+” (see text for the details of the calculation of r).

Figure 7: (color) The minimal distance $d_{\min}$ and the maximal distance $d_{\max}$ between the attractor and the repeller as functions of β : (a) a smooth saddle-node bifurcation along $\alpha = 0.4$ for the two-frequency forcing $F_G(t)$. Inset is a log-log plot showing the scaling $\propto |\beta_c - \beta|^{0.5}$ just before the bifurcation occurring at $\beta = \beta_c$. (b) a nonsmooth saddle-node bifurcation along $\alpha = 2.2$ for $F_G(t)$. (c) a smooth saddle-node bifurcation along $\alpha = 0.4$ for the astronomical forcing $F_A(t)$. (d) a nonsmooth saddle-node bifurcation along $\alpha = 1.4$ for $F_A(t)$. The panels (a) and (c) show the results only for the region with attractor $\beta < \beta_c$.

Figure 8: (Color) (a) Simulated ice volume trajectories for two slightly different values in SNA regime: $\beta = 1.0006$ (red solid line) and $\beta = 1.0001$ (blue dashed line) ($\alpha = 1.0$, $\gamma = 1$, and $\delta = 0.24$). (b) Simulated ice volume trajectories for three different values in 41-kyr mode-locked region: $\beta = 1.61$ (red solid line), $\beta = 1.65$ (green dashed), and $\beta = 1.69$ (blue dotted line) ($\alpha = 0.8$, $\gamma = 1$, and $\delta = 0.24$).

Figure 9: The winding number W , the phase sensitivity exponent μ , and the value of the phase sensitivity function Γ at $t = 10^7$ as functions of parameter β , respectively. (a) The case for the two-frequency forcing $F_G(t)$ with parameters $\alpha = 2.2$, $\gamma = 1$, and $\delta = 0.24$. The phase sensitivity function is calculated with respect to θ_G . (b) The case for the astronomical forcing $F_A(t)$ for with parameters $\alpha = 1.0$, $\gamma = 1$, and $\delta = 0.24$. Mode-locked regions and non mode-locked regions are shown by thick points and thin points, respectively, in the graph of the winding number W . The phase sensitivity function is calculated with respect to θ_4 . The exponent μ is calculated from the growth of the phase sensitivity function over the time interval $[10^5, 10^7]$. The phase sensitivity function is minimized for 10 initial conditions.