1 Sensitivity analysis of a viscoelastic parameterization for gravity wave
2 dispersion in ice covered seas

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Abstracts

Viscoelastic parameterization has been proposed for modeling gravity wave propagation under various ice covers. The resulting dispersion relation predicts wave speed change and attenuation, both depend on the viscoelastic parameters. Through direct measurements of wave characteristics, these parameters may be inversely determined. To effectively use an inverse method, the present study employs ANOVA to perform a sensitivity analysis on the dispersion relation. Two dispersion models are examined. Both models have qualitatively similar sensitivity to their parameters, but different parameterization. The statistical results show that the change of wave speed is most strongly dependent on the elasticity parameter, while attenuation is most strongly dependent on the wave period. The sensitivity analysis provides additional information. For example, for high frequency waves strong interactions exist between wave period, ice thickness, and viscoelastic parameters, which result in uncertainties to the inversely determined parameters. For low frequency waves, the viscoelastic parameters may be determined from attenuation data alone. If in addition the ice cover is thick and rigid, the accuracy of ice thickness is not crucial to the inversely determined elastic parameter. These findings are tested and validated using a field data set. The method presented in this study may be used to test other viscoelastic models.

Index Terms and Keywords

Sensitivity analysis, ANOVA, wave-in-ice model, viscoelastic, inverse method

1. Introduction

One of the most apparent consequences of Arctic ice reduction is the escalation of waves (Thomson & Rogers, 2014). In response to this environmental change, operational wave models have begun improving their capabilities for the ice covered waters. For example, WAVEWATCH III (WW3) used to treat ice covered regions as islands (Tolman, 2003), now provides several options (Tolman, 2014): one assumes a constant wave attenuation coefficient, another utilizes the eddy viscosity model (Liu & Mollo-Christensen, 1998), and the third adopts the viscoelastic ice model (Wang & Shen, 2010). Significant differences in the model results have been found (Rogers, 2014). Wave scattering, extensively studied over the past three decades (e.g. Squire, 2007), is being implemented as well (Williams et al., 2013a,b). Different physical mechanisms contribute to the change of wave propagation in ice covered regions. The presence of ice introduces extra damping through ice floe interactions, bending and fracturing, boundary layer effects, in addition to the scattering process (Shen & Squire, 1998). The presence of ice, depending on its mechanical property, also changes the speed of wave propagation. Under very rigid ice, this speed is greater than in the open water (Wadhams, 1986). In soft frazil/grease ice this speed may become lower than in the open water (Wadhams & Holt, 1991).

Sea ice varies spatially and temporally in response to the external forcing. Its physical composition changes dynamically especially in the marginal ice zone (MIZ) adjacent to open water. Fig. 1 shows a LANDSAT8 image obtained on Sept. 20, 2014. In this image the ice floes formed earlier are seen to be refrozen and then broken up again. Other mechanical processes such as rafting and ridging are also known to complicate the ice cover composition.

68 To study wave propagation under ice covers, its mechanical properties are required. There
69 is no direct way to derive the mechanical properties of an ice cover due to its complexity.
70 An effective field approach has therefore been adopted to model an ice cover. Three
71 classical models exist that describe an ice cover as a collection of disjoint mass points
72 (Peters, 1950; Weitz & Keller, 1950), a thin elastic sheet (Greenhill, 1887; Wadhams,
73 1973; Fox & Squire, 1990), or a viscous layer (Keller, 1990). These separate models have
74 been combined into one viscoelastic model (Wang & Shen, 2010). This generalized
75 model, henceforth referred to as VEWS, is shown to converge to the three previous
76 models under different limiting conditions, hence is being considered a candidate for
77 describing various ice conditions. Another viscoelastic sea ice model was proposed much
78 earlier (Squire & Allan, 1980), henceforth referred to as VESA. This model is based on a
79 thin plate theory, where the ice cover is modeled as a Maxwell-Voigt material. The
80 resulting dispersion relation reduces to the thin elastic plate theory when the viscosity
81 parameter is set to zero. It further reduces to the mass loading theory with zero flexural
82 rigidity. The main difference between VEWS and VESA is how the ice layer is treated:
83 as a continuum of its own momentum equation or as a thin-plate where the internal stress
84 gradients through the thickness are constant. The dispersion relations from both models
85 predict change of wave number and attenuation of wave energy.

Fig.1.A LANDSAT8 image on Sept. 20, 2014.(LC80820052014263_Path82.jpg), centered at $77^{\circ}02'53.09''N$, $145^{\circ}34'27.52''W$. (Left) Full image; (Right) Enlarged rectangle in the full image. The 50% ice concentration contour (solid line) was about 30 km from the center location, dash line is 80% ice concentration (Saha et al., 2011).

Any viscoelastic model contains at least two material parameters: the shear modulus and the viscosity. To apply such a model to ice covers, these two parameters need to be determined. From physical arguments, the viscoelastic parameters of a sea ice cover are related to the ice cover composition and perhaps the wave frequency as well. It is extremely challenging to derive these parameters from first principles. An example is shown in De Carolis et al., 2005 where the viscosity of grease ice, an agglomeration of ice crystals, was derived. A phenomenological approach using an inverse method is an alternative. With this approach, a plausible model is adopted based on simplified physical processes. Directly measured data are then used to inversely determine the corresponding material parameters. Such inverse method is widely used in model development for a

broad range of phenomena(Tanaka&Dulikravich, 1998).However, inverse problems are usually difficult because of several reasons. First, there may be non-unique solutions. Second, the measured data may be inaccurate. Third, sensitivity of the measurable quantities may be low with respect to the model parameters. Finally, the trend of model responses to one parameter may be dependent on other parameters, i.e.,parameters' interactions. In this last case, variability of unobserved parameters may greatly influence the model response to a target parameter. Above all these difficulties, the inverse method also only works well if the model itself captures the physical processes well.

Sensitivity analysis is the study of how uncertainty in the model output is contributed by different sources of uncertainty in the model input (e.g., forcing data and process parameters) (Saltelli et al., 2010).It is an important tool used in modeling complex systems, and contributes to the understanding of structural characteristics of a model.Our goal is to further understand the two viscoelastic models. How do they behave under different plausible wave and viscoelastic conditions? When are they most responsive to small changes of any measurable quantities? Such study will help us prepare for using data to perform the inverse method more effectively.While a sensitivity analysis does not test whether a model itself is correct, when measured data show qualitative differences from model characteristics, we may then use such evidence to challenge the models themselves.

According to Campolongo et al., 2000, to examine the output response by a local method that changes one factor at a time, although widely applied among scientists, fails to account for potential interactions among different factors.Instead, variance-based methods, such

as the Sobol' method (Sobol', 1993, 2001) and ANOVA(Analysis of variance), have recently become the preferred approach for sensitivity analysis across a wide range of applications. These two methods both decompose the variance of the model output into components that result from individual parameters as well as parameter interactions. Therefore, they provide not just a rank ordering of key factors, but also a quantitative measure of the sensitivity of the outcome to each of the parameters and their interactions. In this study, we will use ANOVA, a standard statistical tool to perform our sensitivity analysis.

Ordinary applications of ANOVA begin with a large set of directly measured data. These data are collected under different values of the variables which are believed to influence the measured outcome. In the present study, we have models that describe the effect of each variable and their relationship. However, these models are complicated due to the number of inter-related parameters. The relationships between the parameters are further obscured by the implicit and nonlinear nature of their relationships. Using a statistical analysis, we may quantify and rank the ranges of variables in which their relations are most or least significant to the outcome. Due to the scarcity of field and laboratory data for this problem, doing such a study based on real measurements alone is presently unfeasible. All 'data' in this study will be generated by the models themselves. Hence this study strictly focuses on characterizing but not validating the models. Evaluations of the variable trends and their interactions from this study will be useful for the inverse analysis. Understanding the model characteristics will ultimately help to judge if the models themselves are acceptable once large amount of real data are available.

The organization of the paper is as follows. In section 2 we briefly summarize the viscoelastic models VEWS and VESA and introduce all physical parameters. In section 3 we present details of the ANOVA analysis to all parameters considered for VEWS. In section 4 we discuss the significance of each of the variables and their interactions. Section 5 summarizes the findings and discusses the implications to the inverse method. Examples using a few available data sets are given in this section as well. Section 6 discusses the influence of the viscoelastic model itself by comparing results from VEWS and VESA. Section 7 concludes our study.

2. Modeling ice as a viscoelastic material

In this section we describe two models for ice covers. We first briefly summarize the viscoelastic model VEWS proposed in Wang and Shen, 2010. Details of the derivation may be found in that reference. The simpler viscoelastic model, VESA described in Squire and Allan, 1980, will be presented next. Both viscoelastic models begin with assuming a two-layer system as shown in Fig. 2. In VEWS the top layer is described by the Voigt model

$$\tau_{mn} = -P_1 \delta_{mn} + 2GS_{mn} + 2\rho_1 \nu \dot{S}_{mn}, \quad (1)$$

where ρ_1 is the density of the ice layer; τ_{mn} , S_{mn} and $\dot{S}_{mn}$ represent the stress tensor, the strain tensor and the strain rate tensor, respectively; m and n represent x or z . G and ν are the effective shear modulus and the effective kinematic viscosity of the ice layer, respectively; P_1 is the pressure and δ_{mn} the Kronecker delta. Conceptually, the Voigt model agrees with an ice cover consisting of large solid floes interspersed in a mixture of

168 slushy ice, often found in the MIZ. The total internal stress is therefore the sum of
 169 contribution from elasticity due to the solid portion and viscosity due to the slushy
 170 portion. By assuming a simple harmonic wave with angular frequency σ , the
 171 strain S_{mn} and strain rate $\dot{S}_{mn}$ can be related to each other by

$$172 \quad \dot{S}_{mn} = -i\sigma S_{mn}. \quad (2)$$

173 The constitutive equation(1) can be simplified as

$$174 \quad \tau_{mn} = -P_1 \delta_{mn} + 2\rho_1 \nu_e \dot{S}_{mn}, \quad (3)$$

175 where ν_e is the complex equivalent kinematic viscosity

$$176 \quad \nu_e = \nu + iG/\rho_1 \sigma. \quad (4)$$

177 Introducing the potential and stream functions ϕ_1 and ψ_1 , respectively, the solution of the
 178 continuity and momentum equations for the ice layer become

$$179 \quad \nabla^2 \phi_1 = 0, \quad (5)$$

$$180 \quad \frac{\partial \psi_1}{\partial t} = \nu_e \nabla^2 \psi_1, \quad (6)$$

$$181 \quad \frac{\partial \phi_1}{\partial t} - \frac{P_1}{\rho_1} + \varphi = 0, \quad (7)$$

182 where φ is the gravitational potential. The water layer governing equations are

$$183 \quad \nabla^2 \phi_2 = 0, \quad (8)$$

$$184 \quad \frac{\partial \phi_2}{\partial t} - \frac{P_2}{\rho_2} + \varphi = 0. \quad (9)$$

185 where ϕ_2 is the velocity potential of the water layer. The solutions of the above system in
 186 terms of their harmonic components are

$$187 \quad \phi_1(x, z, t) = (A \cosh kz + B \sinh kz) e^{i(kx - \sigma t)}, \quad (10)$$

$$188 \quad \psi_1(x, z, t) = (C \cosh az + D \sinh az) e^{i(kx - \sigma t)}, \quad (11)$$

189 where $\alpha^2 = k^2 - i\sigma/\nu_e$, and

$$\phi_2(x, z, t) = E \cosh k(z + H) e^{i(kx - \sigma t)}. \quad (12)$$

The coefficients A, B, C, D , and E are related to each other through the boundary conditions.

Fig. 2. Schematic of the wave propagation in an ice covered sea, where the mechanical property of the ice cover is characterized by the equivalent shear modulus G and the equivalent viscosity ν .

The boundary conditions of the above equations include the stress-free conditions at the free surface, the compatibility of stresses and velocities at the ice-water interface, and the no flow penetration at the sea floor. These conditions form a homogeneous linear system of equations for the coefficients A, B, C and D . Setting the determinant of this system of equations to zero we obtain the dispersion relation,

$$(\sigma^2 - Q_c g k \tanh kH) \frac{\sigma^2}{v_e^2} \frac{\rho_2}{\rho_1} \frac{4k^3 \alpha v_e^2 S_k C_\alpha + N^2 S_\alpha C_k - g k S_k S_\alpha}{\tanh kH} = 0, \quad (13)$$

where

$$Q_c = 1 + \frac{\rho_1}{\rho_2} \frac{g^2 k^2 S_k C_\alpha - (N^4 + 16k^6 \alpha^2 v_e^4) S_k S_\alpha - 8k^3 \alpha v_e^2 (C_k C_\alpha - 1)}{g k (4k^3 \alpha v_e^2 S_k C_\alpha + N^2 S_\alpha C_k - g k S_k S_\alpha)}. \quad (14)$$

206 In the above, $S_k = \sinh kh$, $S_\alpha = \sinh \alpha h$, $C_k = \cosh kh$, $C_\alpha = \cosh \alpha h$, $N = \sigma + 2ik^2\nu_e$.

207 We fix $\rho_1 = 917 \text{ kg/m}^3$ and $\rho_2 = 1000 \text{ kg/m}^3$ in the rest of the analysis. For each set of
 208 input parameters, $\{\sigma, H, h, G, \nu\}$, there are two modes that are relevant (Wang & Shen,
 209 2010, 2011), which are roots of the simplified dispersion relation

$$210 \quad \sigma^2 = Q_c g k \tanh k H. \quad (15)$$

211 Both modes are complex numbers, i.e. $k = k_n + iq$. The real part is the wave number
 212 and the imaginary part is the attenuation coefficient. We denote the normalized wave
 213 number as $\kappa = k_n/k_0$, where k_0 is the open water wave number. Of the two modes, the
 214 one with κ closer to 1 contains most of the wave energy and thus is the dominant one.

215 While based on a simple linear viscoelastic assumption, VEWS possesses complex
 216 mathematical structure. As shown previously (Wang & Shen, 2010, 2011), of the two
 217 relevant modes the dominant one with κ closer to 1 changes sharply in the mid-range of
 218 G . The other mode takes over to become dominant. The physical nature of this mode
 219 switching is unknown at this point. Similar phenomenon has also been reported in wave
 220 propagating through a porous wave breaker (Dalrymple et al., 1991). In our analysis that
 221 follows, we continue as in Wang and Shen, 2010 and focus on the dominant mode only.
 222 Further scrutiny of this assumption requires a separate study.

223 Next we give a brief account of VESA. Squire and Allan, 1980 proposed a linear
 224 Maxwell-Voigt model where the ice stress τ is related to strain S via both Maxwell
 225 parameters E_1, ξ_1 and Voigt parameters E_2, ξ_2 ,

$$226 \quad \ddot{\tau} + \left(\frac{E_1}{\xi_1} + \frac{E_1}{\xi_2} + \frac{E_2}{\xi_2} \right) \dot{\tau} + \frac{E_1 E_2}{\xi_1 \xi_2} \tau = E_1 \ddot{S} + \frac{E_1 E_2}{\xi_2} \dot{S}. \quad (16)$$

227 In the limit of $E_1, \xi_1 \rightarrow \infty$, this model becomes the Voigt model described in (1). Instead
 228 of solving the full continuum equations (5-7), a thin-plate over deep water
 229 assumption was used in Squire and Allan, 1980 to derive the dispersion relation,

$$\begin{aligned} & \frac{h^3}{6} \left(E_1 \sigma + i \frac{E_1 E_2}{\xi_2} \right) k^5 + \left(\sigma - \frac{1 E_1 E_2}{\sigma \xi_1 \xi_2} + i \left(\frac{E_1}{\xi_1} + \frac{E_1}{\xi_2} + \frac{E_2}{\xi_2} \right) \right) (\rho_2 g - \rho_1 h \sigma^2) k \\ & - \rho_2 \left(\sigma - \frac{1 E_1 E_2}{\sigma \xi_1 \xi_2} + i \left(\frac{E_1}{\xi_1} + \frac{E_1}{\xi_2} + \frac{E_2}{\xi_2} \right) \right) \sigma^2 = 0. \end{aligned} \quad (17)$$

231 Letting $E_1, \xi_1 \rightarrow \infty$ and removing the deep water assumption, the above becomes

$$\left(\frac{h^3}{6} \left(\frac{E_2}{\rho_2 g} - \frac{\sigma \xi_2}{\rho_2 g} i \right) k^4 + 1 - \frac{\rho_1 h \sigma^2}{\rho_2 g} \right) k \tanh kH = \frac{\sigma^2}{g}. \quad (18)$$

233 The thin elastic plate theory for arbitrary water depth states (Fox & Squire, 1990)

$$\left(\frac{L}{\rho_2 g} k^4 + 1 - \frac{\rho_1 h \sigma^2}{\rho_2 g} \right) k \tanh kH = \frac{\sigma^2}{g}, L = \frac{E h^3}{12(1-\nu^2)} = \frac{G h^3}{6(1-\nu)}. \quad (19)$$

235 Comparing (18) with (19), one may replace the elastic parameter E_2 and viscous
 236 parameter ξ_2 in (18) by G and $\rho_1 \nu$ respectively. The corresponding dispersion equation is

$$\left[\frac{h^3 \left(\frac{G}{\rho_2 g} - \frac{\sigma \rho_1 \nu}{\rho_2 g} i \right)}{6(1-\nu)} k^4 + 1 - \frac{\rho_1 h \sigma^2}{\rho_2 g} \right] k \tanh kH - \frac{\sigma^2}{g} = 0. \quad (20)$$

238 where Poisson's ratio $\nu = 0.3$ in the rest of the study. The advantage of this dispersion
 239 relation is its simplicity. It does not have the mode switching phenomenon mentioned in
 240 VEWS. Furthermore, VEWS relies on using the stream function to solve for the velocity
 241 field, hence the solution is restricted to two-dimensional problems where the wave front
 242 curvature is small relative to the wavelength.

In what follows, we will use VEWS to illustrate the procedure of the statistical analysis and interpretation of the results. The same analysis will be applied to VESA with the two model results compared in section 6.

3. The ANOVA analysis

We now perform the ANOVA analysis on the viscoelastic dispersion relation. Latin Hypercube (LH) sampling is used for the input parameters: wave frequency, water depth, ice thickness, shear modulus and viscosity. The concept of the LH sampling (McKay et al., 1979) is based on the Monte Carlo simulation but uses a stratified approach that allows efficient estimation of the output. It subdivides the space of each parameter into M strata with a probability of occurrence equal to $1/M$. In this paper, M equals to ten thousand. Random values of the parameters are generated for each of the five parameters in the dispersion relation. In total, two hundred thousand samples are chosen to do further analysis.

The normalized wave number κ and the attenuation coefficient q are two outputs of the viscoelastic dispersion relation. In this analysis, we will group our samples by dividing the input parameters (factors) into subsets (levels) and determine the output complex wave number (response) within each subset. The variance of each response variable is partitioned into components attributable to different sources of variation, including main factor sum of squares, interaction sum of squares and error sum of squares. Details of the partition of each factor into its levels are shown in Table 1. Here, frequency is converted to wave period T ; H is water depth; G is an elastic parameter; h is ice thickness; and ν is a

viscous parameter. For various ice covers, the mechanical behavior is characterized by the equivalent shear modulus represented by G and the equivalent kinematic viscosity represented by ν . The maximum value of the elasticity is based on the physical knowledge of solid ice covers (Timco & Weeks, 2010). We expect that the maximum equivalent elasticity of an ice aggregate is the elasticity of an individual ice floe. The kinematic viscosity of grease ice estimated from laboratory experiment (Newyear & Martin, 1999) and theoretical arguments (De Carolis et al., 2005) is on the order of 10^4 times water viscosity. We thus tentatively choose an upper bound of ν for ice aggregates to be $1\text{m}^2/\text{s}$. The partition of each factor at this point is on the premise of balancing the resolution and computational effort.

ANOVA can capture a range from the first order (main effects from single parameters) to the total order of effects (i.e., all parameter impacts including all interactions). However, for most applications beyond the second order there are not much significant interactions among parameters (Box et al., 1978; Liang & Guo, 2003).

For each parameter and pairs of parameters, two key values: p and F are calculated. The p value is used to test the null hypothesis that the mean values of responses have no differences between different levels of each factor or interaction of factors. In this study, a p value < 0.001 is considered indicative of a significant difference. The F value ranks the relative importance of each factor or interaction of factors. The higher the F value is, the more significant the differences among levels are, and therefore the response is more sensitive to the corresponding factor or the interaction of factors. Keeping in mind that

F itself is a statistical result influenced by the random sampling, small differences in F cannot be used to do this ranking of significance.

Table 1. Intervals of five factors for different levels for VEWS.

Levels	$T(s)$	$H(m)$	$h(m)$	$G(Pa)$	$\nu(m^2/s)$
1	(4,6)	(5,55)	(0.02,0.25)	(5,50)	(0,0.1)
2	(6,8)	(55,105)	(0.25,0.5)	(50,500)	(0.1,0.2)
3	(8,10)	(105,255)	(0.5,0.75)	(500,5000)	(0.2,0.3)
4	(10,12)	(255,505)	(0.75,1.02)	$(5000, 5 \times 10^4)$	(0.3,0.4)
5	(12,14)	(505,755)	(1.02,1.25)	$(5 \times 10^4, 5 \times 10^5)$	(0.4,0.5)
6	(14,16)	(755,1005)	(1.25,1.5)	$(5 \times 10^5, 5 \times 10^6)$	(0.5,0.6)
7	(16,18)	(1005,1505)	(1.5,1.75)	$(5 \times 10^6, 5 \times 10^7)$	(0.6,0.7)
8	(18,20)	(1505,2005)	(1.75,2.02)	$(5 \times 10^7, 5 \times 10^8)$	(0.7,0.8)
9	(20,22)	(2005,2505)		$(5 \times 10^8, 5 \times 10^9)$	(0.8,0.9)
10	(22,24)	(2505,3005)			(0.9,1.0)

4. Statistical results

In this section we examine the data obtained using the ANOVA analysis. Table 2 is the analysis of variance for the viscoelastic model VEWS. As in many prior studies, beyond the second order the p values are large hence our analysis focuses on the first and second order effects only. They are referred to as the main effect and the (pair) interaction effects, respectively. Statistically these results are the most important.

Focusing on the wave number κ first, the p values of T , H , h , G all show some effects. To rank the importance of each factor and their interactions we use the F values. Table 2 shows that for κ the elasticity G is the most influential and water depth H can be regarded as insignificant. Interactions between $T * h$, $T * G$ and $h * G$ pairs are present.

For the attenuation coefficient q , due to the large F value wave period T is the most significant factor. The elasticity G ranks second, ice thickness h ranks third. Water depth

His insignificant to q . There are also many obvious interaction effects, such as $T * G$ and $T * v$ pairs.

For both κ and q , the interaction effects are sometimes stronger than some main effects. Such significant interactions imply that the influence of one factor on κ or q depends highly on the value of the other factor. For example, for κ , interaction $T * G$ is stronger than the main effect of T . Hence how κ varies with T is a strong function of G .

Table 2. The analysis of variance table for viscoelastic model VEWS (three decimal places for p value).

Response: Normalized wave number κ			Response: Attenuation coefficient q		
Source	F	p	Source	F	p
T	1498.60	0.000	T	1364.29	0.000
H	7.00	0.000	H	0.85	0.492
h	107.33	0.000	h	162.44	0.000
G	3641.88	0.000	G	292.48	0.000
v	0.63	0.643	v	108.64	0.000
$T * H$	0.76	0.731	$T * H$	1.50	0.090
$T * h$	186.33	0.000	$T * h$	197.59	0.000
$T * G$	1983.73	0.000	$T * G$	644.96	0.000
$T * v$	0.99	0.469	$T * v$	150.48	0.000
$H * h$	2.13	0.012	$H * h$	1.55	0.098
$H * G$	1.88	0.018	$H * G$	1.50	0.090
$H * v$	0.55	0.924	$H * v$	1.31	0.182
$h * G$	371.02	0.000	$h * G$	35.08	0.000
$h * v$	0.60	0.848	$h * v$	10.68	0.000
$G * v$	0.51	0.946	$G * v$	38.28	0.000

With the knowledge of strong interaction among factors shown in Table 2, main effect may not represent the trend of outcome in response to specific parameter combinations. Such phenomenon is shown in Fig.3a, where κ is seen to increase with h for small G , but

for large G , opposite trend is found. Fig.3b shows the same phenomenon for q where opposite trends may be found in different parameter combinations. Fig.3 is an example which is similar to the interaction plots of ANOVA. Detailed interaction plots for all pairs of factors and in each factor range are given in the Appendix. In statistical terms, when curves in the interaction plots are parallel, there is no interaction between the pair of factors. The less parallel the curves are, the stronger the interactions are.

Fig.3. Ice thickness (h) effect with different wave periods (T) for VEWS: (a) normalized wave number; (b) attenuation coefficient. Circles are mean values of κ and q within each level. They assess how, in general, changing the level of h influences the responses (i.e. main effect plots). Parameters used are: $H = 100m$, $\nu = 0.05m^2/s$.

Interactions among factors are characteristics of a model. These interactions are ranked by the F values given in Table 2. Knowing that two factors have a strong interaction may not be sufficient. It is also helpful to identify over which level of factors the interactions

are the strongest. To obtain this information, we define the mean square for interactions as follows:

$$MSI_{ij} = \sqrt{\frac{\sum_{n=1}^N (X_{(n,ij)} - \bar{X}_i - \bar{X}_j + \bar{X})^2}{N}}, \quad (21)$$

where i and j are the levels of two factors, $X_{(n,ij)}$ is the sample in the (i,j) 'cell', $\bar{X}_i$ and $\bar{X}_j$ are level averages for two factors, $\bar{X}$ is the overall mean value of samples and N is the sample size for each 'cell'. Fig.4 shows the MSI distribution. Darker cells are associated with higher MSI values. Fig.4 provides additional information to the F values in Table 2 by telling at which level of these parameters the interaction is the strongest.

Fig.4. ANOVA second order parameter interactions for VEWS based on MSI (Horizontal and vertical axis are the level numbers). Interactions of factors that are insignificant are omitted.

Figs.4a,b,d,e, and g all show high MSI for level 1 of T which indicate stronger parameter interactions for short waves. Therefore for short waves more accurate data are required to determine G and ν .

5. Implications to the inverse method

In this section we study the implications of the model behavior on inversely determining the model parameters.

For short waves, Figs.A1a, b and Figs. A2 a, b, c show large variability at small T for both κ and q . Figs.4a,b show strong $T * h$ and $T * G$ interactions for κ and Figs. 4d, e, g show strong $T * h$, $T * G$ and $T * \nu$ interactions for q . Hence for short waves it is much easier to inversely determine G and ν , but the confidence interval will depend on the accuracy of T and h .

For long waves, due to the weak $T * h$ interaction and h effect for κ shown in Fig.A1c (and supported by Fig.4a), κ is not affected by h . At the same time, $T * G$ interaction and G effect shown in Fig.4b and Fig. A1d are also too weak to significantly affect κ . Therefore the open water dispersion relation $\kappa \approx 1$ should hold for all long waves ($T \geq \text{level } 8$) under all conditions. Next we look at the relation between the wave number κ and the viscosity parameter ν under long wave conditions. Due to the low sensitivity of κ to ν , ν cannot be determined from the measured κ . However, q is sensitive to both G and ν for long waves as Figs.A2e and f show. It is expected that using measured q to estimate G and ν is feasible. The interaction of $T * h$, $T * G$ and $T * \nu$ are significant from Figs.A2d, e and

f. Hence in general for long waves to inversely determine G and ν from q requires accurate estimates of T and h .

For any wave period, attenuation is due to the viscous effect. Pure elastic ice covers do not lose energy, but elasticity can still influence the strength of viscous damping. Figs. A2b,e show that the variations of q with respect to G are orders of magnitude greater than the variation of q with respect to ν in Figs. A2c,f. This means viscous damping is strongly enhanced or depressed by the elasticity. This conclusion holds for $\nu \leq 1 \text{ m}^2/\text{s}$, the upper limit in Table 1. For larger values of ν , its influence on q may become more significant.

Fig. A3a indicates strong $h * G$ interaction for κ because these curves are far from being parallel. It also shows that it is easier to inversely determine G in thick ice based on the measured κ . For the attenuation q , as shown in Fig. A3b, at high levels of h the curves nearly coincide, suggesting that h has little effect on q . Further evidence from Fig. 4f shows that $h * G$ interaction is indeed negligible, but only for high levels of G . Thus a rough estimate of h is sufficient if q is used to inversely determine G for thicker and more rigid ice ($h \geq \text{level } 6, G \geq \text{level } 5$). For ν , its main effect on κ is negligible as Table 2 shows. But its main effect on q is comparable to h . To inversely determine the viscosity parameter ν accuracy of ice thickness is required.

To illustrate the inverse method we use two field measurements reported in Wadhams et al., 1988. These observations took place in the Greenland Sea on Sept. 16, 1978 and Bering Sea on Feb. 22, 1983. To date, this report represents the most complete data sets available. Both data sets were obtained using vertical accelerometer packages placed at

several locations from the ice edge to roughly 100km into the ice covered region. Only the attenuation coefficient q was measured.

Table 3a. Inversely determined viscoelastic parameters for VEWS in Bering Sea (B) and Greenland Sea (Gr).

	Measured parameters				Inversely determined parameters		
	q ($\text{m}^{-1} \times 10^{-4}$)	T (s)	h (m)	H (m)	G (Pa)	ν (m^2/s)	κ
B	1.04 ± 0.15	8.8~9.7	0.4~1.7	119	$1.95 \times 10^4 \sim 2.43 \times 10^5$	0.04~1.00	0.68~1.08
Gr	1.20 ± 0.16	11.6~13.8	3.1	202	$2.26 \times 10^5 \sim 4.09 \times 10^5$	0.00~1.00	0.83~1.45
Test 1	1.04 ± 0.15	8.8~9.7	0.4~1.7	800	$1.93 \times 10^4 \sim 2.39 \times 10^5$	0.03~1.00	0.65~1.08
Test 2	1.04 ± 0.15	9.2	0.4~1.7	119	$2.37 \times 10^4 \sim 2.24 \times 10^5$	0.05~1.00	0.68~1.08
Test 3	1.04 ± 0.15	9.2	1.05	119	$2.96 \times 10^4 \sim 1.71 \times 10^5$	0.08~1.00	0.69~1.07
Test 4	1.04 ± 0.01	8.8~9.7	0.4~1.7	119	$2.70 \times 10^4 \sim 1.89 \times 10^5$	0.60~1.00	0.66~1.07
Test 5	1.04 ± 0.01	9.2	1.05	119	$3.59 \times 10^4 \sim 1.62 \times 10^5$	0.73~0.99	0.71~1.06
Test 6	1.20 ± 0.16	11.6~13.8	2.0	202	$2.36 \times 10^5 \sim 4.30 \times 10^5$	0.00~1.00	0.83~1.43

Table 3b. Inversely determined viscoelastic parameters for VEWS in Bering Sea (B) assuming κ is available.

	Measured parameters				
	q ($\text{m}^{-1} \times 10^{-4}$)	T (s)	h (m)	H (m)	κ
Test 7	1.04 ± 0.15	8.8~9.7	0.4~1.7	119	0.96~1.04
	Inversely determined parameters				
	G (Pa)	ν (m^2/s)			
Test 7	$1.54 \times 10^5 \sim 2.43 \times 10^5$	0.39~0.99			

Data in the first two rows of Table 3a named B and Gr are obtained from the Bering Sea and Greenland Sea respectively. Measured parameters are q , T , h and H . Inversely estimated parameters are G and ν using the dispersion relation (15). Last column is the range of κ calculated from the resulting G and ν . Recognizing that the measured data have uncertainties, the procedure for this inverse method is also based on a statistical method, as explained next. LH sampling method is used to populate T , h , G and ν . The water depth is assumed precise. For measured input parameters, T and h , the ranges of samples are from

404 Wadhams et al., 1988. For G and v , they are from the entire ranges shown in Table 1. We
405 then use the dispersion relation (15) to solve for κ and q for each set of T , h , G , H , and
406 v . We collect all cases of G and v which yield a q that falls into the measured data range.
407 These ranges of the obtained G and v are given in Tables 3a and 3b.

408 Test 1 to Test 6 shown in Table 3 are different ways we tried to either change the
409 parameters entirely or to ‘narrow down’ the uncertainties of measured parameters. For
410 each ‘Test’, fixed or narrowed down parameters can be regarded as improvements of
411 measurement accuracy. We use these tests to determine how inverted G and v respond to
412 these changes, and if the responses are consistent with the ANOVA analysis. In Table
413 3, inverted G and v intervals of B and Test 1 are almost the same. It verifies the
414 insignificance of H . Comparing Test 2 with B and Test 3 with Test 2, the precision of the
415 inversely determined G and v improves by increasing the accuracy of T or h . Improving
416 the accuracy of q also narrows the inverse intervals as Test 4 shows. If T , h and q are all
417 more precise than the measurement shown in B, Test 5 indicates a significant increase of
418 precision of the estimated G and v . For thick rigid ice as in the Gr case, Test 6 shows that
419 the inversely determined G change little even though the ice thickness changes more than
420 1m, hence ice thickness does not need to be accurately measured for thick rigid ice.

421 This inverse exercise is only based on the knowledge of the attenuation coefficient
422 q . From discussions shown earlier, the normalized wave number κ is sensitive to G , hence if
423 the wave number k_n is available, both the estimated G and v intervals from the inverse
424 method should be narrower. To verify this, we assume that κ is known and conducted a

test as shown in Table 3b, κ is arbitrarily chosen within the range in Table 3a. Comparison between Test 1 in Table 3a and Test 7 in Table 3b proves this point. The above example only used one case each from the Bering Sea and Greenland Sea experiments. Same tests have also been performed using other measurements reported in Wadhams et al., 1988. The results are consistent with the example shown. To gain some overview of the performance of the inverse method, Fig. 5a shows the uncertainty of the inversely determined G from all measurements up to 18 s waves in Bering Sea on Feb. 22, 1983. The large uncertainty of the inverse method comes from the uncertainty of the measured parameters. Due to different strength of interactions at different levels of parameters, the level of uncertainties may vary with different field experiments. Using an order of magnitude estimate for G and ν from this inverse method, and the mean of measured ice thickness h , Fig. 5b shows the comparison between the average q from field data and from VEWS.

Fig. 5.(a) Inversely determined G of all measurements in Bering Sea on Feb. 22, 1983. Error bars represent uncertainties of G and T for each measurement; (b) comparison

between average measured and simulated q from VEWS, parameter values are $h = 1.05\text{m}$, $G = 10^5\text{Pa}$, $\nu = 1\text{m}^2/\text{s}$.

In the inverse analysis above, only averaged ice properties in each observation are used. Williams and Squire (2010) showed that ice thickness distribution influenced the results of an inverse analysis from the scattering attenuation. This level of sophistication is not included here.

6. Comparison between two viscoelastic models

The above analysis is based on the viscoelastic theory VEWS shown in Section 2. We now discuss another model, VESA as shown in (20), and compare it with VEWS. We first repeat the inverse method using the data from Wadhams et al., 1988 to guide our selection of the range of viscoelastic parameters to be studied. We found that within the parameter ranges shown in Table 1, there is no $\{G, \nu\}$ combination that can produce the measurements. We thus increase the upper bound of ν to $20000\text{ m}^2/\text{s}$ to get the inverse results shown in Table 4a. With these different sets of G and ν values VESA is also able to explain the measured data. For ν , the inversely determined upper limit is close to the upper bound. (The same situation also occurs in Table 3 which used VEWS.) Thus larger ν values than the given upper limit are still possible in order to fit the measurements. While for G , the range of inversely determined values is very large even if k is given, as Test 8 shows. This phenomenon is different from VEWS. It appears that in this simpler model the sensitivity of the complex wave number to G is lower than the more complicated viscoelastic model.

Table 4a. Inversely determined viscoelastic parameters for VESA in Bering Sea (B) and Greenland Sea (Gr).

	Measured parameters				Inversely determined parameters		
	q ($\text{m}^{-1} \times 10^{-4}$)	T (s)	h (m)	H (m)	G (Pa)	ν (m^2/s)	κ
B	1.04 ± 0.15	8.8~9.7	0.4~1.7	119	$5 \times 10^0 \sim 6.45 \times 10^8$	2248~19994	0.86~1.09
Gr	1.20 ± 0.16	11.6~13.8	3.1	202	$5 \times 10^0 \sim 1.84 \times 10^8$	7638~19998	0.99~1.09

Table 4b. Inversely determined viscoelastic parameters for VESA in Bering Sea (B) assuming κ is available.

	Measured parameters				
	q ($\text{m}^{-1} \times 10^{-4}$)	T (s)	h (m)	H (m)	κ
Test 8	1.04 ± 0.15	8.8~9.7	0.4~1.7	119	0.96~1.04
	Inversely determined parameters				
	G (Pa)	ν (m^2/s)			
Test 8	$12 \times 10^0 \sim 3.60 \times 10^8$	4110~19994			

In order to further understand this simpler model, we apply the ANOVA analysis with the enlarged ν range. Partition of each factor into its levels is shown in Table 5 and the key results are shown in Table 6. *MSI* results that depict interactions among different pairs of factors are shown in Fig. 6. These results are all qualitatively the same as the more complicated viscoelastic model given by (15). However, if we focus on level 1 to level 6 of G in Figs. A4b, d, A5b, e and A6 and compare them with Figs. A1b, d, A2b, e and A3, it is clear that outputs of this simpler model are much less sensitive to G .

From the above statistical analysis, when we perform the inverse method with these two viscoelastic models, the situations may be different. With VEWS, a narrow range of acceptable G is expected, especially when we have both the measured κ and q . While, with VESA, inversed G can be within a large range due to its insignificance to both κ and q . From another perspective, for VESA in less rigid ice ($G \leq \text{level } 5$), G does not affect the

values of κ and q , the influence of ice properties on wave will only depend on v . However, in the same situation, both G and v play important roles in VEWS. Hence the most significant difference between these two viscoelastic models appears to be in the less rigid materials. The stronger dependence on G for VEWS may be linked to the mode switching characteristics of this model. This issue awaits further study.

Table 5. Intervals of five factors for different levels for VESA.

Levels	$T(s)$	$H(m)$	$h(m)$	$G(Pa)$	$v(m^2/s)$
1	(4,6)	(5,55)	(0.02,0.25)	(5,50)	(0,1000)
2	(6,8)	(55,105)	(0.25,0.5)	(50,500)	(1000,3000)
3	(8,10)	(105,255)	(0.5,0.75)	(500,5000)	(3000,5000)
4	(10,12)	(255,505)	(0.75,1.02)	$(5000, 5 \times 10^4)$	(5000,7000)
5	(12,14)	(505,755)	(1.02,1.25)	$(5 \times 10^4, 5 \times 10^5)$	(7000,9000)
6	(14,16)	(755,1005)	(1.25,1.5)	$(5 \times 10^5, 5 \times 10^6)$	(9000,11000)
7	(16,18)	(1005,1505)	(1.5,1.75)	$(5 \times 10^6, 5 \times 10^7)$	(11000,13000)
8	(18,20)	(1505,2005)	(1.75,2.02)	$(5 \times 10^7, 5 \times 10^8)$	(13000,15000)
9	(20,22)	(2005,2505)		$(5 \times 10^8, 5 \times 10^9)$	(15000,17000)
10	(22,24)	(2505,3005)			(17000,20000)

Table 6. The analysis of variance table for viscoelastic model VESA (three decimal places for p value).

Response: Normalized wave number κ			Response: Attenuation coefficient q		
Source	F	p	Source	F	p
T	3806.69	0.000	T	8734.49	0.000
H	6.07	0.000	H	0.68	0.730
h	114.64	0.000	h	627.75	0.000
G	3705.61	0.000	G	481.35	0.000
v	79.57	0.000	v	76.13	0.000
$T * H$	1.71	0.000	$T * H$	0.99	0.517
$T * h$	873.26	0.000	$T * h$	886.23	0.000
$T * G$	1816.90	0.000	$T * G$	848.79	0.000
$T * v$	107.69	0.000	$T * v$	80.27	0.000
$H * h$	1.66	0.001	$H * h$	0.79	0.893
$H * G$	1.47	0.006	$H * G$	0.97	0.552
$H * v$	1.26	0.056	$H * v$	1.08	0.286
$h * G$	289.72	0.000	$h * G$	62.82	0.000

$h * \nu$	14.87	0.000	$h * \nu$	4.80	0.000
$G * \nu$	11.35	0.000	$G * \nu$	5.10	0.000

Fig.6. ANOVA second order parameter interactions based on *MSI* for VESA (Horizontal and vertical axis are the level numbers).

While the present paper was under review, one of the reviewers pointed out a recent paper by Mosig et al. (2015). In which they compared VEWS and VESA, in addition to another viscoelastic model presented by Robinson and Palmer (1990). Unlike the random sampling used here, in their study, they used a pure deterministic approach, assuming that q is the mean value of measured attenuation, and $\kappa = 0.59$, to exactly solve the dispersion relation for G, ν . Despite the fact that their inverse method was not identical to the present one, and despite the fact that they forced a κ much different from what the present method yields, their main findings were the same: both models were able to

match observations with similar level of agreement for the attenuation, while the parameter values were orders of magnitude different.

7. Discussion and conclusions

We first summarize common characteristics of the two viscoelastic models studies here, which are likely shared by more complicated models.

1. Wave period and elasticity are the strongest factors that influence the change of wave number or wave speed once it enters a viscoelastic material from open water. Attenuation is most strongly dependent on the wave period. These phenomena agree with observation to date.
2. The importance of water depth is ranked very low because H appears only in the $\tanh kH$ term. Hence in deep water it is effectively nonexistent. According to Tables 1 and 5, most of the samples are in deep water cases. Hence the statistical results are biased towards the deep water case. For coastal conditions or ultra-long waves in the open sea we need to redo the same analysis by a different sampling scheme that focuses on shallow water waves only.
3. There are strong interactions between physical parameters. For normalized wave number, the variability is mainly associated with wave period-ice thickness interaction, wave period-elasticity interaction and ice thickness-elasticity interaction. For wave attenuation additional variabilities come from the interaction of viscosity with other variables. Due to these interactions, trends of wave characteristics with respect to one parameter under a specific combination of other parameters may not be generalized.

Next we summarize key implications of this study onusing the inverse method to determine parameter values in the two viscoelastic models examined here.

4. Short wave conditions are preferred to inversely determine elasticity and viscosity because of the high sensitivity of the complex wave number on small wave periods. It is also logistically much easier to measure attenuation of short waves because their high attenuationallows shorter distance between instruments.
5. For long wave conditions ($T \geq \text{level } 8$), $\kappa \approx 1$ should hold under all conditions, thus a poor parameter for inversely determining the viscoelastic parameter. On the other hand, attenuation is sensitive to both elasticity and viscosity,hence still effective for the inverse method under long wave conditions.
6. Due to interactions between wave period, ice thickness, elasticity and viscosity, accurate wave period and ice thickness are required to inversely determine the viscoelastic parameters.
7. For rigid ice ($G \geq \text{level } 5$), as ice thickness increases,inversely determined viscoelastic parameters become less sensitive to the variation of ice thickness. Hence it is not crucial to have accurate ice thickness data for thick and rigid ice.

As far as the similarities and differences between the two viscoelastic models examined, both models can reproduce a wide range of κ and q . The parameter values corresponding to a given model output are different in different models, as should be expected. Beyond which, for less rigid ice covers, normalized wave number and attenuation are insensitive to elasticity for VESA, but sensitive to elasticity for VEWS. In this study, we fixed the

548 Poisson's ratio in VESA at 0.3. This parameter can be absorbed into G and ν in (20)
549 without changing subsequent findings.

550 In conclusion, we studied the sensitivity of a viscoelastic parameterization for gravity
551 wave propagation under various ice covers. The analysis is performed with ANOVA. As
552 a 'global' sensitivity analysis method, it not only provides information on all factors'
553 main effects, but also their interactions. In this study, ANOVA results are also expanded,
554 by using the MSI values, to depict the second order interaction between individual levels
555 of different pairs of factors.

556 The result of ANOVA analysis depends on the parameter ranges and the classification of
557 levels. For the same viscoelastic model, results of ANOVA in a laboratory situation are
558 different from field situations. When using ANOVA to do model tests, we need to make
559 sure that the levels are set properly to the physical situation.

560 The VESA model has the advantage of being simple to invert. The VEWS model appears
561 to provide viscoelastic parameter values that are closer to physically measured values.
562 However, the complexity of the mathematical structure in VEWS as discussed in Wang
563 and Shen (2010, 2011) has invoked further study about the physical meaning of its
564 multiple modes. Some discussions have appeared in Wang and Shen (2011) where the
565 amplitude of horizontal/vertical motions at the free surface and the water/viscoelastic
566 material interface were compared, and in Mosig et al. (2015) where the phases of the
567 modes at the free surface and interface was mentioned. In this study as well as in all our
568 previous studies relating to this model, we have consistently chosen the mode closest to
569 the open water case as the dominant mode, the same as what was done in studying wave

propagation over a mud-layer (Dalrymple and Liu, 1978). We wait for further study of this model, which covers a broad range of other floating materials that may be modeled in the same way, to advance our knowledge of the mathematical structure and the physical meaning of its multiple modes. All of the results derived are based on the assumption that the chosen viscoelastic model does describe the ice cover. This assumption itself needs to be scrutinized. It is likely that grease/pancake ice covers, or broken ice covers that have been pulverized by storms, may be adequately modeled by a viscoelastic continuum. In the MIZ, ice covers often consist of clearly separated large floes. For such ice covers scattering is a much more plausible process (e.g. Williams et al. 2013a,b). Much dedicated study is required to determine the proper model(s) for different ice covers. The present investigation focuses only on the viscoelastic type of models. Knowing the characteristics of the viscoelastic type models helps to determine whether discrepancies between measurements and predictions may be explained by uncertainties of data, or are from the modelsthemselves.

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