Wave Propagation in Periodic Buckled Beams. Part II: Experiments

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Abstract

Buckled periodic beams possess a geometrically nonlinear, load-deformation relation and intrinsic length scales such that stable, nonlinear waves are possible. In part I of this paper, a model has been derived that predict compressive/rarefaction, supersonic/supersonic solitary waves, varying the level of compression and the support type (guided or pinned). Although this work has been validated by simulating the structure with finite-elements, in the present paper, investigations are done experimentally, focusing on the guided-supported, slightly-buckled beam.

Keywords: buckled periodic beam, experiment, solitary wave.

1. Introduction

Periodic buckled beams (Fig. 1) are dispersive and geometrically (physically) nonlinear structures that admit solitary waves [1, 2]. In [1], modeling the buckled beam by a chain of masses and nonlinear springs taking into account transverse inertial and coupling effects, it is shown that homogenization of the discretized system leads to the doubly dispersive Boussinesq equation. This equation admits compressive supersonic, rarefaction (tensile) supersonic, compressive subsonic and rarefaction subsonic solitary waves, and these four waves are all possible in the same structure, the buckled beam, simply by varying the pre-compression level and support type (guided or pinned, see Fig. 1). While finite-element (FE) simulations

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of the structure have been performed validating the derived models, the aim of the proposed paper is to repeat this work experimentally.

Figure 1: Guided (only axial displacement allowed) (a) and pinned (both axial displacement and rotation free) (b) support configurations composed of $p = 4$ wavelengths.

Solitary mechanical waves have been observed experimentally in discrete media as non-linear penduli [3, 4], woodpiles [5], complex lattices [6], and granular media [7, 8], and the dispersion arise from the periodicity. However, for buckled beams, the beam thickness and the curvature are also important dispersive sources and have to be taken into account as well.

From the static solution of beam theory, a straight beam compressed by an axial load admits analytically an infinite number of periodic buckling modes [9]. However, due to imperfections (e.g. geometry not perfectly straight), only the first buckling mode is stable, but is not periodic. Whereas stability of high-order modes can be ensured via an elastic foundation [10–13], cooling of thin film fixed to a substrate of different thermal-expansion coefficient [10, 14], or compressing a beam between two fixed walls [15], these additional media bring complex interactions as friction or secondary paths for wave propagation. In the literature, two/three dimensional (2D/3D) self-supported periodic buckled structures are also reported experimentally, such as stretched membranes [16], twisted and stretched strips [17, 18], compressed cylinders [19, 20], sandwich panels [21, 22], reinforced plates [23] and lattices [24, 25].

Here, to ensure the stability of the initially straight beam, the use of additional supports that prevent transverse displacements is proposed, as shown in Fig. 1. While transverse displacements of the supports are forbidden and axial ones are free, no constrain applies on the

rotation such that either pinned or guided supports can be used (Fig. 1), and both configurations are investigated in [1, 26]. Although the support type changes the speed behavior (supersonic or subsonic medium) and the solitary wave type (compressive or rarefaction) [1], in the present paper, only guided supports that are easier to realize experimentally are investigated.

From material considerations, an experimental limitation is the maximum reachable buckling level. While in [1] it is shown that wave characteristics are also dependent on pre-compression, experimentally, only small buckling levels can be tested for metallic structures. Indeed, when the buckling is too high such that the material reaches its yield limit, plastic hinges occur and the structure collapses [27]. Hyperelastic materials can be used to avoid plasticity [28], but are not investigated here.

This article is organized as follows: while the generalized Boussinesq model derived in [1] is shortly reviewed in Sec. 2, the experimental setup is shown in Sec. 3. Results are presented in Sec. 4, investigating first static buckling and focusing then on the wave profile and its characteristics. Discussion and conclusion follow.

2. Analytical Models

In part I of this paper [1], analytical models describing wave propagation in buckled beams have been derived and are shortly reviewed here. Modeling the buckled beam by a chain of masses and nonlinear springs, and accounting for transverse inertial effects [1, 26], homogenization of the momentum equation leads in the case of guided supports to the Boussinesq equation [1]:

$$\xi_{tt} = C_0^2 \xi_{xx} + 2C_0 \gamma \xi_{xxxx} + \sigma (\xi \xi_x)_x, \quad (1)$$

where $C_0^2 = \frac{P'\{\Delta U_0\}L_0^2}{m}$ is the linear speed, $\sigma = \frac{P''\{\Delta U_0\}L_0^3}{m}$ the coefficient of nonlinearity, and $\gamma = -\frac{a_4 L_0^2 C_0}{2}$ the coefficient of dispersion. a_4 is the dimensionless dispersion coefficient that account for all the dispersion sources, and is determined numerically from the semi-analytical dispersion relation [1, 26]. At the static level, $L_0 = L - \Delta U_0$ is the length between two consec-

utive supports after applying pre-compression, $P\{\Delta U_0\}$ represent the load-displacement relation with ΔU_0 the relative displacement between two consecutive supports and $\chi_0 = \Delta U_0/L$ is the initial strain level. At the dynamic level, $\xi = -U_x = -\Delta U/L_0 = -u_x + \chi_0$ is the compressive strain, and U are the total support displacements. Under the long-wave assumption, Eq. (1) admits the solitary-wave solution [1]:

$$\Delta\xi = \Delta\xi_m \text{sech}^2 \{ \Lambda^{-1}(x - Vt) \}, \quad (2)$$

where $\Delta\xi = \xi - \chi_0$, $\Delta\xi_m = \xi_m - \chi_0$, and ξ_m is the maximum strain. The solitary wave phase speed V is:

$$V = \sqrt{C_0^2 + \sigma\Delta\xi_m/3}, \quad (3)$$

and its characteristic width Λ is

$$\Lambda = \sqrt{24C_0\gamma/(\sigma\Delta\xi_m)}. \quad (4)$$

While the Boussinesq model assumes the load-displacement relation ($P\{\Delta U\}$) described by a second-degree polynomial, it is shown in [1] that this model is not accurate for high-amplitude waves and alternative methods using the potential of $P\{\Delta U\}$ described by either a power law or an arbitrary function are proposed. However, since these additional models are more complex to implement and interpret, experimental parameters are restricted here where the Boussinesq model applies.

3. Experimental Setup

A 2 m long continuous steel ribbon is periodically supported by 30 trolleys such that the supports are of guided type (Figs. 1b and 2b). The ribbon (reference number *162830.0500* from *Brüttsch-Rüegger*) is modeled as a beam and the following notation and numerical values are used: the cross-section area is A ($b = 12.7 \text{ mm} \times t = 0.4 \text{ mm}$), and the area-moment of inertia is $I_z = bt^3/12$; $E = 190 \text{ GPa}$ and $\rho = 7600 \text{ Kg m}^{-3}$ are respectively the

measured Young modulus and density. Each trolley (model *SELB10* from *Misumi*) provides an additional mass of 2.4×10^{-2} Kg and the free distance between them is 6×10^{-2} m.

The experiment is divided into two parts; a static one, where the pre-compression is applied and a dynamic one, where a wave is sent through the structure. The static load is applied with a screw (model *NSE5-SN* from *Nozag*) at the left side and carried at the right side by a spring (reference *8101-084928* from *Kubo*, with nominal rigidity of $K_{\text{spring}} = 7.9$ Nmm⁻¹ and length of $L_{\text{spring}} = 41$ mm). The averaged load-displacement relation of one unit cell is then obtained from the displacement between the screw and the spring, measured with two identical linear systems (*S System LIN*, 2 μ m accurate, from *Sylvac*), and the force computed from the spring compression.

To generate a wave at the right extremity, a shaker is used and is directly connected to the extremity of the spring (last trolley). The shaker model is *LDS V406M4* from *Brüel & Kjær* and has internal axial stiffness $K_{\text{Shaker}} = 12.3$ Nmm⁻¹. Note that the shaker is connected after applying the static load such that all the pre-compression is carried by the spring and not the shaker. Its displacement is recorded by a LVDT sensor (model *WI/5mm-T* and a signal conditioner *MP55* from *HBM*).

The wave propagation throughout the structure was originally measured with accelerometers, but due to the problem arising from time integration and low frequency noise, a laser which give directly the speed is used instead (compact laser vibrometer *CLV 2534* from *Polytec*). Since the laser can record the speed at one point only, it is mounted on a positioning system (model *BiSlide* by *Velmex*) which can move along the beam. Note that in order to do so, the experiment is assumed reproducible.

All signals are acquired through *National Instruments PXIe-6368* card managed via *LabView*.

4. Experimental Results

In this section, experimental results are given, starting from static considerations and then analyzing wave propagation.

Figure 2: Experimental model (a) with details view of the buckled beam (b), left extremity (c), right extremity (d), laser mounted on the positioning system (e), right top extremity (f), and trolley (g). Note that the beam alternates with up-down deformations due to small rotational play in the trolley (b). The $(++)$ and $(-)$ indicate respectively large positive and small negative deformations.

4.1. Static buckling pattern

When a beam with equispaced pinned supports is buckled (Fig. 1a), the buckling pattern alternates between positive and negative transverse deformations and is explained by the fact that when one trolley starts to turn, a moment in the opposite direction is created on the next trolley, which turns in the opposite direction and so on. For the guided-support configuration (Fig. 1b), since only axial displacements are allowed, the buckling is expected to be random, depending only on initial imperfections. If gravity is not negligible, the deformations are expected to be unidirectional. However, as shown in Fig. 2b, the buckling direction is neither random nor uniform, and alternates between up and down, similarly to the pinned-support case. This distribution can be explained by the fact that the trolleys possess a small play in rotation, such that a similar deformation mechanism as the pinned-support configuration occurs. Moreover, as shown in Fig. 2b, the buckling in the top direction is more significant in the bottom one, which is opposite to gravity effects but is explained by the residual strain in the beam. Indeed, the beam is originally received coiled in a curved box and once placed on a flat surface, the residual curvature is evident. Note that if the beam is flipped, the preferred direction is also changed.

Since the pre-buckling level is not homogeneous whereas the model assumes a periodic pattern, the averaged value is used, as presented next for the load-displacement relation.

4.2. Static load-displacement relation

The averaged experimental load-displacement ($P\{\Delta U\}$) curve is given in Fig. 3 and compared to the FE results. While results are in good agreement for the weakly buckled beam (Fig. 3a), the beam becomes plastic and collapses around $\chi_0 = 0.05$ (Fig. 3b). For this reasons, only small strain is considered.

For the aforementioned parameters (small initial buckling and guided supports), supersonic rarefaction waves are expected [1].

Figure 3: Normalized load displacement relation ($P\{\Delta U\}/P_C$ where $P_C = 4EI_z\pi^2/L^2$) in terms of the small (a) and large (b) buckling level χ_0 . Experiment in full lines (becoming plastic) and FE simulation with * (assuming linear elastic material). FE results are given for an initial imperfection of sinusoidal form and has its maximum transverse amplitude $w_0 = 10^{-3}L$ such that using this value, both curves in (a) overlaps.

4.3. Wave input

A classical way to see if a structure can hold solitary waves is to excite it with a delta function and observe if there is creation of a solitary train [7]. However, the present setup is limited to 30 trolleys which is not long enough to see generation of stable solitons from an impulse. Hopefully this distance can be long enough to see non-stationary effects. Indeed, non-stationary wave effects (front, tail, etc.) appear promptly compared to the time required for the generation of full stable waves. For this reason, the structure is excited with a guessed solitary solution and the propagating wave is observed. If the resulting wave is unchanged, solitary waves are obtained whereas if a front or a tail is present, the input guess is not the correct one, and the measured wave is discarded. This operation is repeated iteratively keeping the wave amplitude constant and varying only the period input, until no tail or front is present.

However, this procedure is working only for displacement control whereas the shaker works on force control. To work on displacement control, piezo-stack actuator would be a good alternative, but no configuration for the desired displacement and dynamic range

is available. In order to use the shaker in displacement control, the proportional-integral-derivative (PID) method has been implemented, but works only for slow dynamics, far from the desired frequencies. Finally, the input is tuned by hand as follows.

Knowing that the shaker works in force control, its delivered force is of form:

$$F_{Shaker} = M_{eff}\ddot{U} + K_{eff}U + F_{Soliton}\{U\}, \quad (5)$$

where M_{eff} and K_{eff} are effective mass and stiffness. K_{eff} is defined easily and is such that:

$$K_{eff} = K_{Shaker} + K_{Spring}. \quad (6)$$

M_{eff} involves however the mass of the different trolleys which do not constitute a rigid body such that its determination analytically is too difficult. U and its derivatives in time are obtained from the strain solution (Eq. (2)) such that:

$$\begin{aligned} U &= U_m (\tanh \{-tT^{-1}\} + 1), \\ \dot{U} &= -U_m T^{-1} \text{sech}^2 \{-tT^{-1}\}, \\ \ddot{U} &= -2U_m T^{-2} \text{sech}^2 \{-tT^{-1}\} \tanh \{-tT^{-1}\}, \end{aligned} \quad (7)$$

where $U_m = \Delta\xi_m\Lambda$ is the wave amplitude in displacement, and $T = \Lambda V^{-1}$ its characteristic width in time that is referred next as period. $F_{Soliton}(U)$ is assumed proportional to the strain wave that is also proportional to the wave-speed profile such that:

$$F_{Soliton}(U) = F_{eff}\dot{U}. \quad (8)$$

Finally, for each combination of U_m and T , the two unknowns M_{eff} and F_{eff} have to be determined, and they are tuned manually by varying iteratively their values until the displacement of the first trolley, measured by the LVDT sensor, correspond to the desired one.

4.4. Stationary wave profile

For a given initial compression, the initial input is parametrized by U_m and T . However, for solitary waves, given the amplitude U_m , only one solution T results to a stationary wave. In

order to find these different combinations, a grid from the parameters $\{U_m, T\}$ is constructed and experiments are run for each grid combination.

For a given U_m , the corresponding value of T is found when the resulting wave propagates maintaining its shape. For example, in Figs. 4a-c, the wave speed in time of the first 5 even trolleys is shown for three different values of T , fixing χ_0 and U_m . In Fig. 4a, the period T is initially too large and decreases while propagating, meaning that the input is not correct. In Fig. 4c, the period seems to remain constant but a tail appears, which is contrary to stationary waves. Finally, in Fig. 4b, the period is constant and there is a flat part before the reflection meaning that there is no creation of tail and that the solution is approximatively stationary.

Figure 4: Speed of the trolleys V_p in terms of the time t of the first 5 even trolleys, for $\chi_0 = 1.72 \times 10^{-3}$, $U_m = 0.5$ mm and $T^{-1} = 70$ s $^{-1}$ (a), $T^{-1} = 140$ s $^{-1}$ (b), or $T^{-1} = 150$ s $^{-1}$ (c). Speed at the 10th trolley for $U_m = 0.5$ mm and $T^{-1} = \{70, 90, 110, 130, 150, 170, 190, 210, 230\}$ s $^{-1}$, respectively from dark to light gray lines. The experiment resulting to stationary solution is in dotted line ($T^{-1} = 150$ s $^{-1}$).

Since variations of the period are hardly detectable due to limitation in the setup size, selection of the corresponding period T is based on the tail behavior. Indeed, as shown in Fig. 4d, where the 10th trolley speed $V_p\{t\}$ is plotted for U_m constant and several values of T , the retained value of T corresponding to a stationary wave is the first one before the appearance of a tail (see dotted line in Fig. 4d). As a general remark, front has not been observed in this structure, in agreement with the dispersion characteristics of the weakly-buckled guided-supported beam [1].

Once the combinations $\{U_m, T\}$ are determined, the wave profile at each trolley is recorded and profiles are shown in Figs. 5a-c for three pre-compression levels. In Fig. 5d, FE simulations of the structure using the same parameters as in Fig. 5b are given, and comparison shows some discrepancies. In order to carry out a quantitative comparison, the phase speed and its period are investigated next.

4.5. Phase speed

As shown in [1], for guided supports and a slightly-buckled beam, the medium is supersonic meaning that nonlinear waves travel faster than linear ones and the phase speed of the former increases. It is this characteristic that is intended to be demonstrated in this section. The speed between each trolley is computed by measuring the time shift between two consecutive trolleys, knowing the distance between them, and using the maximum amplitude of the signal as reference. However, as shown in Fig. 6a, this speed is not constant and alternates between high and low values. Indeed, it is possible to make a link between the buckling pattern and the speed since when the beam is highly buckled, the slope of the load-displacement curve is low corresponding to small speed, whereas it is the opposite for small pre-compression.

In order to analyze the influence of amplitude on wave speed, the averaged speed is used and results are shown in Fig. 6b in terms of the amplitude for three pre-compression levels. As expected, the phase speed increases with amplitude and decreases with the initial compression. Considering these measurements, results from Boussinesq model are recovered.

Figure 5: Speed V_p of the 15th first trolleys in terms of the time t , for $U_m = 0.5 \times 10^{-3}$ and $\{ \chi_0 = 1.14 \times 10^{-3}, T^{-1} = 240 \text{ s}^{-1} \}$ (a), $\{ \chi_0 = 1.72 \times 10^{-3}, T^{-1} = 150 \text{ s}^{-1} \}$ (b) and $\{ \chi_0 = 3.00 \times 10^{-3}, T^{-1} = 90 \text{ s}^{-1} \}$ (c). FE simulations with the same input as (b) without (d) and with (e) friction (pointwise friction applied at each support location such that the frictional force is 1.5 times the speed).

Figure 6: Phase speed ΔV between the 15th first trolleys for $\chi_0 = 1.72 \times 10^{-3}$, $T^{-1} = 150 \text{ s}^{-1}$ and $U_m = 0.5 \text{ mm}$ (a), and averaged phase speed V in terms of the amplitude U_m (b) for $\chi_0 = 1.14 \times 10^{-3}$ (light gray), $\chi_0 = 1.72 \times 10^{-3}$ (dark gray) and $\chi_0 = 3.00 \times 10^{-3}$ (dark), with experimental data in full lines and Boussinesq model in dashed lines.

4.6. Period and wavelength

After the phase speed, the selected input period T that leads to a stationary wave is compared to the Boussinesq model in Fig. 7. It results that T increases with pre-compression, in agreement with the Boussinesq-type model. However, concerning the experimental values of the input period, T is found too large for small amplitude and too small otherwise. Since the estimation of the phase speed is found acceptable, an explanation of the fact that the wavelength (or period) is badly predicted could be that all the dispersion sources are not correctly captured. Indeed, the phase speed (Eq. (3)) do not depends on the dispersion, contrary to the wavelength and the period (Eq. (4)).

However, since numerical FE simulations of the structure are in agreement with the Boussinesq model [1], the mismatch is explained by experimental limitations. Indeed the phase speed depends mainly on the medium and is almost independent of the input, whereas it is the opposite for the period when measured at the first trolleys. The fact that the input is badly selected can find an explanation in its selection criterion. Indeed, for a given U_m , the selected T is based on the tail creation criterion and as it is shown next, tails can result

from heterogeneities.

In conclusion, in order to measure the period independently of the input, a longer setup would be required.

Figure 7: Inverse of the period (T^{-1}) in terms of the amplitude U_m (b) for $\chi_0 = 1.14 \times 10^{-3}$ (light gray), $\chi_0 = 1.72 \times 10^{-3}$ (dark gray) and $\chi_0 = 3.00 \times 10^{-3}$ (dark), with experimental data in full lines and Boussinesq model in dashed lines.

4.7. Effects of friction

When the wave from experiment (Figs. 5a-c) is compared to FE simulations (Fig. 5d), the reflected wave from FE keeps its amplitude whereas in experiments, the amplitude is at least two times smaller. This can be explained by either the boundary conditions of the left side which might be badly assumed clamped (play in the screw), or dissipations induced by the friction of the trolleys. To this end, a FE simulation including support friction modeled by a pointwise linear relation between support force and speed is shown in Fig. 5e. However, due to the limited setup size, the reflection of the wave appears before the observation of the decay. Moreover, note that for the first supports, the variation of the profile is more complex than a simple decay (Figs. 5a-c) and heterogeneities have to be accounted for.

4.8. Effects of heterogeneities

In Figs. 2b and 6a, it is shown that the buckling is not homogeneous and effects of heterogeneities are investigated here. While the setup does not allow long-wave observations,

investigations are restricted here to numerical FE simulations, and an uniformly distributed random imperfection is considered such that:

$$w_{0,p} = w_0 \pm 5\%, \quad (9)$$

where $w_{0,p}$ is the midspan transverse displacement of the p^{th} beam portion. The buckled beam is modeled with 100 supports and the propagating wave is shown in Fig. 8 and compared to the homogenized structure ($w_{0,p} = w_0, \forall p$). In the presence of heterogeneities, the global trend of the wave profile seems on average to remain unchanged, localized and stationary. Locally, however, effects of periodicity-defects are significant, as observed on the wave amplitude. Note also the presence of a small tail uniquely in the presence of heterogeneities, which confirms that the use of the tail-criterion to select the input (Sec. 4.4) is biased by heterogeneities.

Figure 8: Strain wave propagating in a beam uniformly (a) and randomly (b) buckled (Eq. (9)) in terms of the support number p and the time t for $\chi_0 = 1.72 \times 10^{-3}$ and $U_m = 0.5 \times 10^{-3}$.

In order to carry out a quantitative comparison, the wave phase speed and its period are given in Fig. 9. The simulation is repeated 25 times with each time a new random distribution, and in addition to the mean and the standard deviation, results are compared to the homogeneous case. Concerning the evolution of the speed and period in terms of the support number, they both converge asymptotically meaning that heterogeneities, which

Figure 9: Wave speed V (a) and inverse of the period T^{-1} (b) in terms the support number p for $\chi_0 = 1.72 \times 10^{-3}$ and $U_m = 0.5 \times 10^{-3}$. Mean value and its standard deviation resulting from 25 FE simulations with random imperfections (Eq. (9)) in full lines, compared to the FE simulation without heterogeneities in dashed lines.

are local phenomena do not play a role in the long-wave propagation, validating the use of homogenized models here for the imperfect structure. In addition to this observation, the convergence rate of the phase speed (Fig. 9a) is much faster than the one of the period (Fig. 9b) and is explained again by the fact that the phase speed is weakly dependent on the input, whereas it is the opposite for the period. This confirms why experimentally, the phase speed is correctly captured whereas this is not true for the wavelength (period).

5. Conclusions

Wave propagation in guided-supported, weakly-buckled beams is investigated experimentally and two characteristics predicted by the Boussinesq model are recovered: (i) the medium is supersonic, meaning that the phase speed increases with amplitude and (ii) if the input is not stable, there is creation of a tail but never a front. Moreover, the phase speed decreases with pre-compression and is in agreement with the predicted values. While the wave profile seems stationary, due to limitations in the setup size, the reflected wave appears after few supports and long-wave observations as the wavelength, period, heterogeneity effects, and friction-induced decay are not possible. Future work will focus on self-supported buckled

structures as the longitudinally wrinkled and creased helicoids for which, longer setups can be obtained easily.

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- [1] F. Maurin, A. Spadoni, Wave Propagation in Periodic Buckled Beams. Part I: Analytical Models And Numerical Simulations, Wave Motion, under review.
- [2] F. Maurin, A. Spadoni, Low-frequency wave propagation in post-buckled structures, Wave Motion 51 (2) (2014) 323–334.
- [3] M. Remoissenet, Waves Called Solitons: Concepts and Experiments, Vol. 63, Springer, 1995.
- [4] L. Munteanu, S. Donescu, Introduction to soliton theory: applications to mechanics, Kluwer Academic Publishers, 2004.
- [5] E. Kim, F. Li, C. Chong, G. Theocharis, J. Yang, P. G. Kevrekidis, Highly Nonlinear Wave Propagation in Elastic Woodpile Periodic Structures, Physical Review Letters 114 (11) (2015) 118002.
- [6] G. A. Maugin, Nonlinear Waves in Elastic Crystals, Oxford University Press, 1999.
- [7] V. F. Nesterenko, Dynamics of Heterogeneous Materials, Springer, 2001.
- [8] S. Sen, J. Hong, J. Bang, E. Avalos, R. Doney, Solitary waves in the granular chain, Physics Reports 462 (2) (2008) 21–66.
- [9] Z. P. Bazant, L. Cedolin, J. W. Hutchinson, Stability of Structures: Elastic, Inelastic, Fracture, and Damage Theories, Vol. 60, World Scientific, 1993.

- [10] J. Genzer, J. Groenewold, Soft matter with hard skin: From skin wrinkles to templating and material characterization, *Soft Matter* 2 (4) (2006) 310.
- [11] Y. Sun, W. M. Choi, H. Jiang, Y. Y. Huang, J. A. Rogers, Controlled buckling of semiconductor nanoribbons for stretchable electronics., *Nature nanotechnology* 1 (3) (2006) 201–207.
- [12] F. Brau, H. Vandeparre, A. Sabbah, C. Poulard, A. Boudaoud, P. Damman, Multiple-length-scale elastic instability mimics parametric resonance of nonlinear oscillators, *Nature Physics* 7 (1) (2010) 56–60.
- [13] S. Cai, D. Breid, A. Crosby, Z. Suo, J. Hutchinson, Periodic patterns and energy states of buckled films on compliant substrates, *Journal of the Mechanics and Physics of Solids* 59 (5) (2011) 1094–1114.
- [14] X. Chen, J. W. Hutchinson, A family of herringbone patterns in thin films, *Scripta Materialia* 50 (6) (2004) 797–801.
- [15] G. Domokos, P. Holmes, B. Royce, Constrained euler buckling, *Journal of Nonlinear Science* 7 (3) (1997) 281–314.
- [16] A. Takei, F. Brau, B. Roman, J. Bico, Stretch-induced wrinkles in reinforced membranes: From out-of-plane to in-plane structures, *EPL (Europhysics Letters)* 96 (6) (2011) 64001.
- [17] A. P. Korte, E. L. Starostin, G. H. M. van der Heijden, Triangular buckling patterns of twisted inextensible strips, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 467 (2125) (2010) 285–303.
- [18] J. Chopin, V. Démery, B. Davidovitch, Roadmap to the Morphological Instabilities of a Stretched Twisted Ribbon, *Journal of Elasticity* 119 (1-2) (2014) 137–189.

- [19] A. Goriely, R. Vandiver, M. Destrade, Nonlinear Euler buckling, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 464 (2099) (2008) 3003–3019.
- [20] J. Xiong, R. Ghosh, L. Ma, A. Vaziri, Y. Wang, L. Wu, Sandwich-walled cylindrical shells with lightweight metallic lattice truss cores and carbon fiber-reinforced composite face sheets, *Composites Part A: Applied Science and Manufacturing* 56 (2014) 226–238.
- [21] C. Lainé, P. Le Grogne, S. Comas Cardona, C. Binetruy, Analytical, numerical and experimental study of the bifurcation and collapse behavior of a 3D reinforced sandwich structure under through-thickness compression, *International Journal of Mechanical Sciences* 67 (2013) 42–52.
- [22] E. Lignon, P. Le Tallec, N. Triantafyllidis, Onset of failure in a fiber reinforced elastomer under constrained bending, *International Journal of Solids and Structures* 50 (2) (2013) 279–287.
- [23] B. Audoly, Y. Pomeau, *Elasticity and Geometry: From hair curls to the non-linear response of shells*, Vol. 2, OUP Oxford, 2010.
- [24] J. T. B. Overvelde, S. Shan, K. Bertoldi, Compaction through buckling in 2D periodic, soft and porous structures: effect of pore shape., *Advanced materials* (Deerfield Beach, Fla.) 24 (17) (2012) 2337–42.
- [25] S. H. Kang, S. Shan, A. Košmrlj, W. L. Noorduyn, S. Shian, J. C. Weaver, D. R. Clarke, K. Bertoldi, Complex Ordered Patterns in Mechanical Instability Induced Geometrically Frustrated Triangular Cellular Structures, *Physical Review Letters* 112 (9) (2014) 098701.
- [26] F. Maurin, A. Spadoni, Wave dispersion in periodic post-buckled structures, *Journal of Sound and Vibration* 333 (19) (2014) 4562–4578.

- [27] D. Ehrlich, F. Armero, Finite element methods for the analysis of softening plastic hinges in beams and frames, *Computational Mechanics* 35 (4) (2004) 237–264.
- [28] A. F. Bower, *Applied Mechanics of Solids*, CRC Press, 2009.