Inverse Electromagnetic Scattering for a Locally Perturbed Perfectly Conducting Plate

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Abstract

We consider a simple but fully three-dimensional inverse problem to determine the shape of a local perturbation of a perfectly conducting plate from far-field measurements of time harmonic electromagnetic fields. For this purpose we reformulate the model problem as an exterior Maxwell problem for a symmetric domain, and prove a equivalence between the model problem and its reformulation. Then, linear sampling method is applied to solve the reformulated problem. We illustrate the feasibility of this method by some numerical examples.

Key words: locally rough surface, inverse electromagnetic scattering problem, Maxwell's equations, linear sampling method.

1 Introduction

In recent years, there has been increased interest in inverse electromagnetic scattering from unbounded obstacles. It can be classified roughly in four classes: locally rough surface problem, rough surface problem, cavity problem and grating problem. These problems arise from many applications such as radar, remote sensing, medical imaging and nondestructive testing.

In this paper, our attention is restricted to the discussion of locally rough surface problem. In [9], for the local perturbation is both star-like and above the infinite plane case, the scattering problem is reformulated as an exterior Dirichlet problems for a symmetric planar domain with corners, and then a Newton method is proposed to reconstruct the local rough surface from the far-field pattern. In [10], the scattering

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problem is reformulated as an equivalent boundary value problem in a bounded domain with a DtN map, and a continuation method is applied to reconstruct the local disturbance with multiple frequency data. Both of the paper [11] and [12] are concerned with the near-field imaging of locally rough surface. Based on the plane wave decomposition of the scattered wave and boundary integral equations, an optimization method is introduced in [11] to identify the profile of a reflective local rough surface by the scattered wave measured in the near field. In [12], the surface displacement is reconstructed by solving a minimization problem from the near-field data under the condition that the local surface displacement is above an infinite ground plane. In [15], a integral equation formulation defined on a bounded curve with two corners is designed to solve the direct scattering problem, and a Newton method with multifrequency far field data is applied to reconstruct the local perturbed surface. In contrast to [10], the integral equation formulation does not involve any infinite integral or a DtN map, and therefore it can be solved by using the Nyström method efficiently. Compared with [9], the author in [19] make use of linear sampling method [5,6] to solve the inverse problem under the condition that the symmetric domain with no corners. Then, [16] gives the uniqueness of the inverse scattering problem by using the generalization of Green's theorem to nonsmooth domain. The well-posedness of direct scattering problems are established in [10,13,14,17,18]. For the rough surface problems we refer to [22–24]. For the inverse cavity problems, the reader is referred to [20,25]. For the grating problems, we refer to [21] and references therein.

Motivated by [9] and [19] for the two dimensional case, we are currently trying to extend the technique to a simple but fully three dimensional model for the electromagnetic exploration under the condition that the local perturbation is above an infinite perfectly conducting plate. We reformulate the scattering problem as an exterior Maxwell problem for a symmetric domain, and then use linear sampling method to reconstruct the shape of local perturbation. For the application of linear sampling method in inverse electromagnetic scattering problems, we refer to [3].

This paper is organized as follows. In section 2 we describe the model problem in detail, and transform the model problem for unbounded obstacle into a new scattering problem for bounded obstacle by symmetric continuation. Then, we prove a equivalence between the direct scattering problem and its reformulation. Section 3 formulates the inverse scattering problem, and describes the implementation details of linear sampling method. Finally, we show some numerical examples in this section.

2 Presentation of the forward and inverse scattering problem

In this section, a direct scattering problem is considered for Maxwell's equations within the upper half-space $\mathbb{R}_+^3 = \{\mathbf{x} \in \mathbb{R}^3 : x_3 > 0\}$. We decompose the space $\mathbb{R}^3 = \mathbb{R}_+^3 \cup \mathbb{R}_0^3 \cup \mathbb{R}_-^3$ in a hyperplane $\mathbb{R}_0^3 = \{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\}$ corresponding to the surface where the local perturbation above on, and $\mathbb{R}_-^3 = \{\mathbf{x} \in \mathbb{R}^3 : x_3 < 0\}$

corresponding to the lower half-space. Let $D_+ \subset \mathbb{R}_{+,0}^3 := \{\mathbf{x} \in \mathbb{R}^3 : x_3 \geq 0\}$ be a bounded domain such that $\partial D_+ \subset \mathbb{R}_+^3$ is a smooth surface. Suppose $\Sigma_{D,0} = D_+ \cap \mathbb{R}_0^3$ is also bounded such that the boundary $\partial \Sigma_{D,0}$ is a smooth closed curve.

Fig. 2.1. Geometry of the model problem.

Denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the usual Cartesian basis of $\mathbb{R}^3$, by $\mathbf{x} = (x_1, x_2, x_3)^\top$ a point in $\mathbb{R}^3$, and by ν_+ the unit outward normal to the boundary. Throughout let $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ be the scalar product and the vector product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, respectively, and let $|\mathbf{x}|$ denote the Euclidean norm of $\mathbf{x}$. For $\mathbf{x} \in \mathbb{R}^3$ we denote by

$$T_1 \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x} = (x_1, x_2, -x_3)^\top$$

its reflection about the plane $\mathbb{R}_0^3$, and define $T_2 \mathbf{x} = -T_1 \mathbf{x} = (-x_1, -x_2, x_3)^\top$. For convenience and also for later use in this section, we give that $(T_1 \mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (T_1 \mathbf{y})$, $(T_1 \mathbf{x}) \times (T_1 \mathbf{y}) = T_2(\mathbf{x} \times \mathbf{y})$ and $(T_2 \mathbf{x}) \times (T_1 \mathbf{y}) = T_1(\mathbf{x} \times \mathbf{y})$.

Given as incident fields

$$\begin{aligned} \mathbf{E}^i(\mathbf{x}) &= \frac{i}{k} \nabla \times \nabla \times (\mathbf{p} \exp(ik\mathbf{x} \cdot \mathbf{d})) = ik(\mathbf{d} \times \mathbf{p}) \times \mathbf{d} \exp(ik\mathbf{x} \cdot \mathbf{d}), \\ \mathbf{H}^i(\mathbf{x}) &= \nabla \times (\mathbf{p} \exp(ik\mathbf{x} \cdot \mathbf{d})) = ik\mathbf{d} \times \mathbf{p} \exp(ik\mathbf{x} \cdot \mathbf{d}), \end{aligned}$$

where the wave number k is positive, $\mathbf{d}$ is a unit vector describing the direction of propagation, and $\mathbf{p}$ is the polarization vector. The solution of scattering problem for the half-space, i.e. absence of D_+ , is given by the reflected wave $\mathbf{E}^r(\mathbf{x}) = T_2 \mathbf{E}^i(T_1 \mathbf{x})$, $\mathbf{H}^r(\mathbf{x}) = T_1 \mathbf{H}^i(T_1 \mathbf{x})$. For convenience, we denote $\mathbf{E}^{i,r} = \mathbf{E}^i + \mathbf{E}^r$, $\mathbf{H}^{i,r} = \mathbf{H}^i + \mathbf{H}^r$ and $\mathbf{f}_+ := -\nu_+ \times \mathbf{E}^{i,r} = 0$, for $\mathbf{x} \in \mathbb{R}_0^3$.

Then the direct scattering problem for the scattering of a time harmonic electromagnetic plane wave by a locally perturbed perfectly conducting surface ∂D_+ is to find an electric field $\mathbf{E}_+ = \mathbf{E}^{i,r} + \mathbf{E}_+^s$ and a magnetic field $\mathbf{H}_+ = \mathbf{H}^{i,r} + \mathbf{H}_+^s$ such that

$$\nabla \times \mathbf{E}_+ - ik\mathbf{H}_+ = 0, \quad \text{in } \mathbb{R}_+^3 \setminus \overline{D_+}, \quad (1)$$

$$\nabla \times \mathbf{H}_+ + ik\mathbf{E}_+ = 0, \quad \text{in } \mathbb{R}_+^3 \setminus \overline{D_+}, \quad (2)$$

$$\nu_+ \times \mathbf{E}_+ = 0, \quad \text{on } (\mathbb{R}_0^3 \setminus \overline{\Sigma_{D,0}}) \cup \partial D_+. \quad (3)$$

The unknown scattered fields $\mathbf{E}_+^s$ and $\mathbf{H}_+^s$ are required to satisfy the Silver-Müller radiation condition

$$\lim_{r \rightarrow \infty} r(\mathbf{H}_+^s \times \hat{\mathbf{x}} - \mathbf{E}_+^s) = 0, \quad (4)$$

uniformly in $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}| \in \partial B := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$, $r = |\mathbf{x}|$. The geometry of this model problem is shown in Fig. 2.1.

The corresponding inverse scattering problem can be stated as: for given wave number $k > 0$ and incident field $(\mathbf{E}^i, \mathbf{H}^i)$, reconstruct the unknown boundary ∂D_+ , by use of the electric far field pattern $\mathbf{E}_+^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p})$, $\forall \hat{\mathbf{x}} \in \partial B_+$, $\forall \mathbf{d} \in \partial B_-$, $\forall \mathbf{p} \in \mathbb{R}^3$.

In order to reformulate this direct scattering problem in half-space as an exterior Maxwell problem, we denote by $\partial D_- = \{x \in \mathbb{R}^3 : T_1 \mathbf{x} \in \partial D_+\}$ the reflection of ∂D_+ about $\mathbb{R}_0^3$. Then $\partial D_- \cup \partial D_+$ is the boundary ∂D of a bounded domain D . Let $\mathbf{E}^s, \mathbf{H}^s$ be the radiating solution to the Maxwell equations satisfying

$$\nabla \times \mathbf{E}^s - ik\mathbf{H}^s = 0, \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \quad (5)$$

$$\nabla \times \mathbf{H}^s + ik\mathbf{E}^s = 0, \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \quad (6)$$

$$\nu \times \mathbf{E}^s = \mathbf{f}, \quad \text{on } \partial D, \quad (7)$$

$$\lim_{r \rightarrow \infty} r(\mathbf{H}^s \times \hat{\mathbf{x}} - \mathbf{E}^s) = 0, \quad (8)$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{f}_+(\mathbf{x}), & x_3 \geq 0, \\ T_1 \mathbf{f}_+(T_1 \mathbf{x}), & x_3 < 0, \end{cases} \quad \text{and} \quad \nu(\mathbf{x}) = \begin{cases} \nu_+(\mathbf{x}), & x_3 \geq 0, \\ T_1 \nu_+(T_1 \mathbf{x}), & x_3 < 0. \end{cases}$$

Then we can prove that

Theorem 1 *Problem (1)-(4) is equivalent to problem (5)-(8).*

Proof. Assume that $(\mathbf{E}_+^s, \mathbf{H}_+^s)$ is a radiating solution to the problem (1)-(4), and extend $(\mathbf{E}_+^s, \mathbf{H}_+^s)$ by

$$\mathbf{E}^s(\mathbf{x}) = \begin{cases} \mathbf{E}_+^s(\mathbf{x}), & x_3 \geq 0, \\ T_2 \mathbf{E}_+^s(T_1 \mathbf{x}), & x_3 < 0, \end{cases} \quad \text{and} \quad \mathbf{H}^s(\mathbf{x}) = \begin{cases} \mathbf{H}_+^s(\mathbf{x}), & x_3 \geq 0, \\ T_1 \mathbf{H}_+^s(T_1 \mathbf{x}), & x_3 < 0. \end{cases}$$

Straightforward calculations show that $(\mathbf{E}^s, \mathbf{H}^s)$ is the solution of the problem (5)-(8).

On the other hand, if $(\mathbf{E}^s(\mathbf{x}), \mathbf{H}^s(\mathbf{x}))$ solves the problem (5)-(8), we then obtain that $(T_2 \mathbf{E}^s(T_1 \mathbf{x}), T_1 \mathbf{H}^s(T_1 \mathbf{x}))$ also solves the problem (5)-(8). From the uniqueness for

solutions to exterior Maxwell boundary value problems [1] we conclude that $\mathbf{E}^s(\mathbf{x}) = T_2 \mathbf{E}^s(T_1 \mathbf{x})$, which implies that the first two components of vector $\mathbf{E}^s$ vanish at $\mathbf{x} \in \mathbb{R}_0^3$. Consequently, $\nu_+ \times \mathbf{E}_+^s = 0$, for $\mathbf{x} \in \mathbb{R}_0^3$. Thus it can be verified that $(\mathbf{E}_+^s, \mathbf{H}_+^s)$ is the solution of the problem (1)-(4). $\square$

Theorem 2 *The electric far field pattern $\mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p})$ for the scattering problem (5)-(8) satisfies the following relations*

- (1) $\mathbf{E}^\infty(T_1 \hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}) = T_2 \mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p});$
- (2) $\mathbf{E}^\infty(\hat{\mathbf{x}}, T_1 \mathbf{d}, \mathbf{p}) = -\mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, T_1 \mathbf{p});$
- (3) $\mathbf{E}^\infty(T_1 \hat{\mathbf{x}}, T_1 \mathbf{d}, \mathbf{p}) = T_1 \mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, T_1 \mathbf{p}).$

Proof. (1) Since by the Huygens' principle([1], Th6.24), the far field pattern is given by

$$\mathbf{E}^\infty(\hat{\mathbf{x}}) = \frac{ik}{4\pi} \hat{\mathbf{x}} \times \int_{\partial D} \nu(y) \times \mathbf{H}(y) \times \hat{\mathbf{x}} \exp(-ik\hat{\mathbf{x}} \cdot y) ds(y).$$

Define

$$R(\hat{\mathbf{x}}) = \int_{\partial D} Q(y, \hat{\mathbf{x}}) \exp(-ik\hat{\mathbf{x}} \cdot y) ds(y),$$

where $Q(y, \hat{\mathbf{x}}) = P(y) \times \hat{\mathbf{x}}$, $P(y) = \nu(y) \times \mathbf{H}(y)$. Now the far field pattern is written as

$$\mathbf{E}^\infty(\hat{\mathbf{x}}) = \frac{ik}{4\pi} \hat{\mathbf{x}} \times R(\hat{\mathbf{x}}).$$

Noting that $\mathbf{H}^{i,r}(T_1 \mathbf{x}) = T_1 \mathbf{H}^{i,r}(\mathbf{x})$ and $\mathbf{H}^s(T_1 \mathbf{x}) = T_1 \mathbf{H}^s(\mathbf{x})$, we have $\mathbf{H}(T_1 \mathbf{x}) = T_1 \mathbf{H}(\mathbf{x})$. Using the symmetry of scatterer D , we obtain that $\nu(T_1 \mathbf{x}) = T_1 \nu(\mathbf{x})$. Hence,

$$\begin{aligned} \mathbf{E}^\infty(T_1 \hat{\mathbf{x}}) &= \frac{ik}{4\pi} (T_1 \hat{\mathbf{x}}) \times \int_{\partial D} \left(\nu(y) \times \mathbf{H}(y) \right) \times (T_1 \hat{\mathbf{x}}) \exp\left(-ik(T_1 \hat{\mathbf{x}}) \cdot y\right) ds(y) \\ &= \frac{ik}{4\pi} (T_1 \hat{\mathbf{x}}) \times \int_{\partial D} \left(\left(T_1 \nu(T_1 y) \right) \times \left(T_1 \mathbf{H}(T_1 y) \right) \right) \times (T_1 \hat{\mathbf{x}}) \exp\left(-ik\hat{\mathbf{x}} \cdot (T_1 y)\right) ds(y) \\ &= \frac{ik}{4\pi} (T_1 \hat{\mathbf{x}}) \times \int_{\partial D} \left(T_2 \left(\nu(T_1 y) \times \mathbf{H}(T_1 y) \right) \right) \times (T_1 \hat{\mathbf{x}}) \exp\left(-ik\hat{\mathbf{x}} \cdot (T_1 y)\right) ds(y) \\ &= \frac{ik}{4\pi} (T_1 \hat{\mathbf{x}}) \times \int_{\partial D} \left(T_2 P(T_1 y) \right) \times (T_1 \hat{\mathbf{x}}) \exp\left(-ik\hat{\mathbf{x}} \cdot (T_1 y)\right) ds(y) \\ &= \frac{ik}{4\pi} (T_1 \hat{\mathbf{x}}) \times \int_{\partial D} T_1 Q(T_1 y, \hat{\mathbf{x}}) \exp\left(-ik\hat{\mathbf{x}} \cdot (T_1 y)\right) ds(y) \\ &= \frac{ik}{4\pi} (T_1 \hat{\mathbf{x}}) \times \left(T_1 R(\hat{\mathbf{x}}) \right) \\ &= \frac{ik}{4\pi} T_2 \left(\hat{\mathbf{x}} \times R(\hat{\mathbf{x}}) \right) \\ &= T_2 \mathbf{E}^\infty(\hat{\mathbf{x}}) \end{aligned}$$

(2) In terms of the incident field $\mathbf{E}^i$ and the reflected wave $\mathbf{E}^r$, we have

$$\mathbf{E}^i(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}) = T_1 \mathbf{E}^i(T_1 \mathbf{x}, \mathbf{d}, \mathbf{p}), \quad \mathbf{E}^r(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}) = -\mathbf{E}^i(\mathbf{x}, \mathbf{d}, \mathbf{p}),$$

and

$$T_1 \mathbf{E}^{i,r}(T_1 \mathbf{x}, \mathbf{d}, \mathbf{p}) = \mathbf{E}^{i,r}(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}).$$

Since $\mathbf{E}^{i,r}(T_1 \mathbf{x}, \mathbf{d}, \mathbf{p}) = T_2 \mathbf{E}^{i,r}(\mathbf{x}, \mathbf{d}, \mathbf{p})$, we get $\mathbf{E}^{i,r}(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}) = -\mathbf{E}^{i,r}(\mathbf{x}, \mathbf{d}, \mathbf{p})$. According to the boundary condition (7), one can derive that

$$\begin{aligned} \nu \times \mathbf{E}^s(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}) &= -\nu \times \mathbf{E}^{i,r}(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}) \\ &= -\nu \times \left(-\mathbf{E}^{i,r}(\mathbf{x}, \mathbf{d}, \mathbf{p}) \right) \\ &= \nu \times \mathbf{E}^{i,r}(\mathbf{x}, \mathbf{d}, \mathbf{p}). \end{aligned}$$

Thus,

$$\nu \times \left(-\mathbf{E}^s(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}) \right) = -\nu \times \mathbf{E}^{i,r}(\mathbf{x}, \mathbf{d}, \mathbf{p}) = \nu \times \mathbf{E}^s(\mathbf{x}, \mathbf{d}, \mathbf{p}).$$

From the uniqueness for solutions to exterior Maxwell boundary value problems [1] we get

$$-\mathbf{E}^s(\mathbf{x}, T_1 \mathbf{d}, T_1 \mathbf{p}) = \mathbf{E}^s(\mathbf{x}, \mathbf{d}, \mathbf{p}).$$

Consequently, the corresponding far field pattern satisfies $\mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}) = -\mathbf{E}^\infty(\hat{\mathbf{x}}, T_1 \mathbf{d}, T_1 \mathbf{p})$.

(3) Combining the previous results (1) and (2), we finally obtain the following results

$$\begin{aligned} T_1 \mathbf{E}^\infty(T_1 \hat{\mathbf{x}}, T_1 \mathbf{d}, T_1 \mathbf{p}) &= T_1 \left(-\mathbf{E}^\infty(T_1 \hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}) \right) \\ &= T_2 \mathbf{E}^\infty(T_1 \hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}) \\ &= \mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}). \end{aligned}$$

The proof of the theorem is now completed. $\square$

This theorem tell us that for given the far field pattern $\mathbf{E}_+^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p})$, for $\hat{\mathbf{x}} \in \partial B_+$, $\mathbf{d} \in \partial B_-$, the far field pattern

$$\mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}), \quad \forall \hat{\mathbf{x}} \in \partial B, \quad \forall \mathbf{d} \in \partial B,$$

corresponding the scattered field $\mathbf{E}^s$ of problem (5-8) is known. Hence, we are able to deal with this problem by using of linear sampling method.

3 Linear sampling method and numerical implementation

To analyze the inverse scattering problem we consider the following Hilbert spaces:

$$\begin{aligned}
H(\text{curl}, D) &:= \{\mathbf{a} \in (L^2(D))^3 : \text{curl} \mathbf{a} \in (L^2(D))^3\}, \\
H_t^{-\frac{1}{2}}(\partial D) &:= \{\mathbf{a} \in (H^{-\frac{1}{2}}(\partial D))^3 : \nu \cdot \mathbf{a} = 0\}, \\
L_t^2(\partial B) &:= \{\mathbf{a} \in (L^2(\partial B))^3 : \mathbf{a} \cdot \hat{\mathbf{x}} = 0, \hat{\mathbf{x}} \in \partial B\}, \\
H^{-\frac{1}{2}}(\text{Div}, \partial D) &:= \{\mathbf{a} \in H_t^{-\frac{1}{2}}(\partial D) : \text{Div} \mathbf{a} \in H^{-\frac{1}{2}}(\partial D)\} \text{ with norm}
\end{aligned}$$

$$\|\mathbf{a}\|_{H^{-1/2}(\text{Div}, \partial D)}^2 := \|\mathbf{a}\|_{(H^{-1/2}(\partial D))^3}^2 + \|\text{Div} \mathbf{a}\|_{H^{-1/2}(\partial D)}^2.$$

Furthermore, $H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \bar{D})$ denotes the space of all fields $\mathbf{a} : \mathbb{R}^3 \setminus \bar{D} \rightarrow \mathbb{C}^3$ such that $\mathbf{a} \in H(\text{curl}, (\mathbb{R}^3 \setminus \bar{D}) \cup B_R)$ for all open balls B_R containing the closure of D .

3.1 far-field data

We are now in a position to describe the numerical algorithm. To this end, we introduce the far field operator $F : L_t^2(\partial B) \rightarrow L_t^2(\partial B)$ such that

$$(F\mathbf{g})(\hat{\mathbf{x}}) := \int_{\partial B} \mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{g}(\mathbf{d})) ds(\mathbf{d}), \quad \hat{\mathbf{x}} \in \partial B. \quad (9)$$

The linear sampling method is based on seeking a function $\mathbf{g}(\cdot, \mathbf{z}, \mathbf{p}) \in L_t^2(\partial B)$, such that for fixed $\mathbf{p}$ and $\mathbf{z}$,

$$\begin{aligned}
(F\mathbf{g}(\cdot, \mathbf{z}, \mathbf{p}))(\hat{\mathbf{x}}) &= \mathbf{E}_e^\infty(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{p}) \\
&= \frac{ik}{4\pi}(\hat{\mathbf{x}} \times \mathbf{p}) \times \hat{\mathbf{x}} \exp(-ik\hat{\mathbf{x}} \cdot \mathbf{z}) - \frac{ik}{4\pi}(\hat{\mathbf{x}} \times T_1 \mathbf{p}) \times \hat{\mathbf{x}} \exp(-ik\hat{\mathbf{x}} \cdot T_1 \mathbf{z}),
\end{aligned} \quad (10)$$

where $\mathbf{E}_e^\infty$ is the electric far field pattern of the electric dipole $\mathbf{E}_e$. In contrast to [2], the electric dipole $\mathbf{E}_e$ has to be modified to satisfy $\nu \times \mathbf{E}_e = 0$ on $\mathbb{R}_0^3$. In this case, we have

$$\mathbf{E}_e(\mathbf{x}, \mathbf{z}, \mathbf{p}) = \frac{i}{k} \nabla_x \times \nabla_x \times \left(\mathbf{p} \Phi(\mathbf{x}, \mathbf{z}) \right) - \frac{i}{k} \nabla_x \times \nabla_x \times \left(T_1 \mathbf{p} \Phi(\mathbf{x}, T_1 \mathbf{z}) \right),$$

where

$$\Phi(\mathbf{x}, \mathbf{z}) = \frac{1}{4\pi} \frac{\exp(ik|\mathbf{x} - \mathbf{z}|)}{|\mathbf{x} - \mathbf{z}|}, \quad \mathbf{x}, \mathbf{z} \in \mathbb{R}^3, \quad \mathbf{x} \neq \mathbf{z}.$$

Moreover, we introduce a bounded linear operator $A : H^{-\frac{1}{2}}(\text{Div}, \partial D) \rightarrow L_t^2(\partial B)$ which maps the boundary data $\mathbf{f} \in H^{-\frac{1}{2}}(\text{Div}, \partial D)$ to the far field pattern $\mathbf{E}^\infty$ of the radiating solution $\mathbf{E}^s \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \bar{D})$ of (5)-(8). Then we have that

$$F\mathbf{g} = -A(\nu \times \mathbf{E}_g), \quad (11)$$

where

$$\mathbf{E}_g(\mathbf{x}) = ik \int_{\partial B} e^{ik\mathbf{x} \cdot \mathbf{d}} \mathbf{g}(\mathbf{d}) ds(\mathbf{d}) + ik \int_{\partial B} e^{ik(T_1 \mathbf{x}) \cdot \mathbf{d}} T_2 \mathbf{g}(\mathbf{d}) ds(\mathbf{d}).$$

We can now prove the main result of this paper.

Theorem 3 *Assume that k is not a Maxwell eigenvalue for D and let F be the far field operator (9) for the scattering problem (5)-(8). Then the following hold:*

(1) *For $\mathbf{z} \in D$ and a given $\varepsilon > 0$ there exists a $\mathbf{g}_{\mathbf{z}}^\varepsilon \in L_t^2(\partial B)$ such that*

$$\|F \mathbf{g}_{\mathbf{z}}^\varepsilon - \mathbf{E}_e^\infty(\cdot, \mathbf{z}, \mathbf{p})\|_{L_t^2(\partial B)} < \varepsilon \quad (12)$$

and the corresponding Herglotz function $\mathbf{E}_{\mathbf{g}_{\mathbf{z}}^\varepsilon}$ converges to the solution of interior Maxwell problem

$$\nabla \times \mathbf{E} - ik\mathbf{H} = 0, \quad \nabla \times \mathbf{H} + ik\mathbf{E} = 0 \quad \text{in } D, \quad (13)$$

$$\nu \times (\mathbf{E} + \mathbf{E}_e) = 0, \quad \text{on } \partial D. \quad (14)$$

in $H(\text{curl}, D)$ as $\varepsilon \rightarrow 0$.

(2) *For $z \in \mathbb{R}^3 \setminus \bar{D}$ and a given $\varepsilon > 0$, every $\mathbf{g}_{\mathbf{z}}^\varepsilon \in L_t^2(\partial B)$ that satisfies (12) is such that*

$$\lim_{\varepsilon \rightarrow 0} \|\mathbf{g}_{\mathbf{z}}^\varepsilon\|_{L_t^2(\partial B)} = \infty.$$

Proof. For convenience, we denote

$$\begin{aligned} \mathbf{E}_e^1(\mathbf{x}, \mathbf{z}, \mathbf{p}) &= \frac{i}{k} \nabla_x \times \nabla_x \times \left(\mathbf{p} \Phi(\mathbf{x}, \mathbf{z}) \right), \quad \mathbf{E}_e^2(\mathbf{x}, \mathbf{z}, \mathbf{p}) = -\frac{i}{k} \nabla_x \times \nabla_x \times \left(T_1 \mathbf{p} \Phi(\mathbf{x}, T_1 \mathbf{z}) \right), \\ \mathbf{E}_g^1(\mathbf{x}) &= ik \int_{\partial B} e^{ik\mathbf{x} \cdot \mathbf{d}} \mathbf{g}(\mathbf{d}) ds(\mathbf{d}), \quad \mathbf{E}_g^2(\mathbf{x}) = ik \int_{\partial B} e^{ik(T_1 \mathbf{x}) \cdot \mathbf{d}} T_2 \mathbf{g}(\mathbf{d}) ds(\mathbf{d}). \end{aligned}$$

Then $\mathbf{E}_e = \mathbf{E}_e^1 + \mathbf{E}_e^2$, $\mathbf{E}_g = \mathbf{E}_g^1 + \mathbf{E}_g^2$. Under the assumption on k we have well-posedness of the interior Maxwell problem in $H(\text{curl}, D)$. Given $\varepsilon > 0$, from the denseness of Herglotz operator (Corollary 7.19 in [1]) we can choose $\mathbf{g}_{\mathbf{z}}^\varepsilon \in L_t^2(\partial B)$ such that

$$\left\| \nu(\mathbf{x}) \times \left(\mathbf{E}_{\mathbf{g}_{\mathbf{z}}^\varepsilon}^1(\mathbf{x}, \mathbf{z}, \mathbf{p}) + \mathbf{E}_e^1(\mathbf{x}, \mathbf{z}, \mathbf{p}) \right) \right\|_{H^{-1/2}(\text{Div}, \partial D)} < \frac{\varepsilon}{2\|\mathbf{A}\|}.$$

Define $\mathbf{W}(\mathbf{x}, \mathbf{z}, \mathbf{p}) = \nu(\mathbf{x}) \times \left(\mathbf{E}_{\mathbf{g}_{\mathbf{z}}^\varepsilon}^1(\mathbf{x}, \mathbf{z}, \mathbf{p}) + \mathbf{E}_e^1(\mathbf{x}, \mathbf{z}, \mathbf{p}) \right)$, the above equation implies

$$\|\mathbf{W}(\cdot, \mathbf{z}, \mathbf{p})\|_{H^{-1/2}(\text{Div}, \partial D)} < \frac{\varepsilon}{2\|\mathbf{A}\|}. \quad (15)$$

Straightforward calculations show that

$$\nu(\mathbf{x}) \times \left(\mathbf{E}_{\mathbf{g}_{\mathbf{z}}^\varepsilon}^2(\mathbf{x}, \mathbf{z}, \mathbf{p}) + \mathbf{E}_e^2(\mathbf{x}, \mathbf{z}, \mathbf{p}) \right)$$

$$\begin{aligned}
&= T_1 \nu(T_1 \mathbf{x}) \times \left(T_2 \mathbf{E}_{g_z}^1(T_1 \mathbf{x}, \mathbf{z}, \mathbf{p}) + T_2 \mathbf{E}_e^1(T_1 \mathbf{x}, \mathbf{z}, \mathbf{p}) \right) \\
&= T_2 \left(\nu(T_1 \mathbf{x}) \times \left(\mathbf{E}_{g_z}^1(T_1 \mathbf{x}, \mathbf{z}, \mathbf{p}) + \mathbf{E}_e^1(T_1 \mathbf{x}, \mathbf{z}, \mathbf{p}) \right) \right) \\
&= T_2 \mathbf{W}(T_1 \mathbf{x}, \mathbf{z}, \mathbf{p}).
\end{aligned}$$

With the aid of (15) and the symmetry of scatterer D , it can be verified that

$$\|T_2 \mathbf{W}(T_1 \cdot, \mathbf{z}, \mathbf{p})\|_{H^{-1/2}(\text{Div}, \partial D)} = \|\mathbf{W}(\cdot, \mathbf{z}, \mathbf{p})\|_{H^{-1/2}(\text{Div}, \partial D)} < \frac{\varepsilon}{2\|A\|}.$$

Thus, we get

$$\begin{aligned}
&\left\| \nu(\mathbf{x}) \times \left(\mathbf{E}_{g_z}(\mathbf{x}, \mathbf{z}, \mathbf{p}) + \mathbf{E}_e(\mathbf{x}, \mathbf{z}, \mathbf{p}) \right) \right\|_{H^{-1/2}(\text{Div}, \partial D)} \\
&\leq \|\mathbf{W}(\cdot, \mathbf{z}, \mathbf{p})\|_{H^{-1/2}(\text{Div}, \partial D)} + \|T_2 \mathbf{W}(T_1 \cdot, \mathbf{z}, \mathbf{p})\|_{H^{-1/2}(\text{Div}, \partial D)} < \frac{\varepsilon}{\|A\|}.
\end{aligned}$$

Then (12) follows by observing equation (11). Now if $\mathbf{z} \in D$, from the well-posedness of the interior Maxwell problem,

$$\lim_{\varepsilon \rightarrow 0} \left\| \nu(\mathbf{x}) \times \left(\mathbf{E}_{g_z}(\mathbf{x}, \mathbf{z}, \mathbf{p}) + \mathbf{E}_e(\mathbf{x}, \mathbf{z}, \mathbf{p}) \right) \right\|_{H^{-1/2}(\text{Div}, \partial D)} = 0$$

implies

$$\lim_{\varepsilon \rightarrow 0} \|\mathbf{E}_{g_z} - \mathbf{E}\|_{H(\text{curl}, D)} = 0,$$

where $\mathbf{E}$ is the solution to (13)-(14).

By arguments similar to those in Theorem 7.21 of [1], the second statements can be established by noting (15). $\square$

In the following, we briefly describe the discretization of the far field equation and the details may be found in [2–4]. Let $(\hat{\mathbf{x}}, \mathbf{e}_1(\hat{\mathbf{x}}), \mathbf{e}_2(\hat{\mathbf{x}}))$ denote an orthonormal basis of $\mathbb{R}^3$ with $\hat{\mathbf{x}} \in \partial B$. Note that $\mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{p}) \cdot \hat{\mathbf{x}} = 0$, the far field equation (10) is then equivalent to

$$\begin{cases} \int_{\partial B} \mathbf{e}_1(\hat{\mathbf{x}}) \cdot \mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{g}(\mathbf{d}, \mathbf{z}, \mathbf{p})) ds(\mathbf{d}) = \mathbf{e}_1(\hat{\mathbf{x}}) \cdot \mathbf{E}_e^\infty(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{p}), \\ \int_{\partial B} \mathbf{e}_2(\hat{\mathbf{x}}) \cdot \mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{g}(\mathbf{d}, \mathbf{z}, \mathbf{p})) ds(\mathbf{d}) = \mathbf{e}_2(\hat{\mathbf{x}}) \cdot \mathbf{E}_e^\infty(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{p}). \end{cases} \quad (16)$$

By using of the reciprocity relation

$$\mathbf{p} \cdot \mathbf{E}^\infty(\hat{\mathbf{x}}, \mathbf{d}, \mathbf{q}) = \mathbf{q} \cdot \mathbf{E}^\infty(-\mathbf{d}, -\hat{\mathbf{x}}, \mathbf{p}), \quad \forall \hat{\mathbf{x}}, \mathbf{d} \in \partial B, \quad \forall \mathbf{p}, \mathbf{q} \in \mathbb{C}^3,$$

and the numerical quadrature formula

$$\int_{\partial B} f(\mathbf{d}) ds(\mathbf{d}) \approx \sum_{j=1}^N \omega_j f(\mathbf{d}_j),$$

where ω_j are the weights associated with the mesh triangular meshing of the unit sphere ∂B , $\mathbf{d}_j$ are the vertices of mesh triangles. The system (16) can be transformed into the following discrete version

$$\begin{cases} \sum_{j=1}^N \omega_j \mathbf{E}^\infty(-\mathbf{d}_j, -\hat{\mathbf{x}}, \mathbf{e}_1(\hat{\mathbf{x}})) \cdot \mathbf{g}(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) = \mathbf{e}_1(\hat{\mathbf{x}}) \cdot \mathbf{E}_e^\infty(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{p}), \\ \sum_{j=1}^N \omega_j \mathbf{E}^\infty(-\mathbf{d}_j, -\hat{\mathbf{x}}, \mathbf{e}_2(\hat{\mathbf{x}})) \cdot \mathbf{g}(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) = \mathbf{e}_2(\hat{\mathbf{x}}) \cdot \mathbf{E}_e^\infty(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{p}). \end{cases}$$

In view of the fact that the unknown $\mathbf{g}(\mathbf{d}_j, \mathbf{z}, \mathbf{p})$ is a tangential vector field, we can set

$$\mathbf{g}(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) = g_1(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) \mathbf{e}_1(\mathbf{d}_j) + g_2(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) \mathbf{e}_2(\mathbf{d}_j)$$

with $g_1(\mathbf{d}_j, \mathbf{z}, \mathbf{p})$ and $g_2(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) \in \mathbb{C}$. Thus, we obtain the fully discrete linear equations

$$\begin{cases} \sum_{j=1}^N \sum_{l=1}^2 \omega_j g_l(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) \mathbf{E}^\infty(-\mathbf{d}_j, -\hat{\mathbf{x}}, \mathbf{e}_l(\hat{\mathbf{x}})) \cdot \mathbf{e}_l(\mathbf{d}_j) = \mathbf{e}_1(\hat{\mathbf{x}}) \cdot \mathbf{E}_e^\infty(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{p}), \\ \sum_{j=1}^N \sum_{l=1}^2 \omega_j g_l(\mathbf{d}_j, \mathbf{z}, \mathbf{p}) \mathbf{E}^\infty(-\mathbf{d}_j, -\hat{\mathbf{x}}, \mathbf{e}_l(\hat{\mathbf{x}})) \cdot \mathbf{e}_l(\mathbf{d}_j) = \mathbf{e}_2(\hat{\mathbf{x}}) \cdot \mathbf{E}_e^\infty(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{p}). \end{cases} \quad (17)$$

In our numerical experiments, the far field pattern $\mathbf{E}^\infty(-\mathbf{d}_j, -\hat{\mathbf{x}}, \mathbf{e}_l(\hat{\mathbf{x}}))$, $l = 1, 2$ for N measurement points $\{\hat{\mathbf{x}}_j\}_{j=1}^N$ is computed by boundary integral equation method [8]. These measurement points are roughly uniformly distributed on the unit sphere ∂B , and they will also serve as incident directions $\{\mathbf{d}_j\}_{j=1}^N$. Thus, the quadrature weights ω_j can be seen as a constant. Let $\hat{\mathbf{q}}$ be some unit vector in $\mathbb{R}^3$ satisfying $\hat{\mathbf{q}} \times \hat{\mathbf{x}}_j \neq 0$ for any j . In our examples, we take $\hat{\mathbf{q}} = (1, 1, 1)^\top$. Then we set the two orthogonal polarizations

$$\mathbf{e}_1(\hat{\mathbf{x}}_j) = \hat{\mathbf{q}} \times \hat{\mathbf{x}}_j / |\hat{\mathbf{q}} \times \hat{\mathbf{x}}_j| \quad \text{and} \quad \mathbf{e}_2(\hat{\mathbf{x}}_j) = \hat{\mathbf{q}} \times (\hat{\mathbf{x}}_j \times \hat{\mathbf{q}}) / |\hat{\mathbf{q}} \times (\hat{\mathbf{x}}_j \times \hat{\mathbf{q}})|.$$

Now, the system (17) can be rewritten as

$$A_\infty \vec{g}(\mathbf{z}, \mathbf{p}) = \vec{f}(\mathbf{z}, \mathbf{p}) \quad (18)$$

where A_∞ is a $2N \times 2N$ complex matrix corresponding to the given far field data, vector $\vec{g} = (g_1, g_2)^\top$ represents the unknowns and $\vec{f}$ is a vector given by the right hand side of (17).

In order to avoid inverse crimes, we corrupt the matrix A_∞ with random noise

$$A_\infty := A_\infty(I + \epsilon R_1 + i\epsilon R_2)$$

where R_1 and R_2 are two matrices of real random numbers in $[-1, 1]$ and ϵ is a parameter. In the following examples, we have chosen $\epsilon = 0.01$ and $\epsilon = 0.1$. Since

Fig. 3.1. Reconstruction of the half ball with radius $R = 1$ and the exact surface.

the ill-posedness of this problem, instead of equation (18), we solve the regularized equation

$$\gamma \vec{g} + A_\infty^* A_\infty \vec{g} = A_\infty^* \vec{f}. \quad (19)$$

where A_∞^* is the adjoint of A_∞ and γ is the regularization parameter. Let $\mathbf{p}_1 = (1, 0, 0)^\top$, $\mathbf{p}_2 = (0, 1, 0)^\top$, $\mathbf{p}_3 = (0, 0, 1)^\top$. For each z in the sampling grid, we compute the indicator function

$$G(\mathbf{z}) = \frac{1}{3} \left(\|\vec{g}(\mathbf{z}, \mathbf{p}_1)\|^{-2} + \|\vec{g}(\mathbf{z}, \mathbf{p}_2)\|^{-2} + \|\vec{g}(\mathbf{z}, \mathbf{p}_3)\|^{-2} \right)^{\frac{1}{2}}.$$

The reconstruction corresponds to the isosurfaces of $G(\mathbf{z})$ close to zero. Consequently, we will choose a constant C and plot surfaces where $G(\mathbf{z}) = C$.

In this section, we present and discuss three numerical examples. In each example,

Fig. 3.2. Reconstruction of the half cushion and the exact surface.

we use 50 and 100 incoming waves respectively to reconstruct the shape of local perturbation, and these incoming waves are generated by Gmsh which is a 3D finite element grid generator. Because of the symmetry of this problem, we only need to put sampling points in upper space.

Example 1 In the first example, we consider a specific case in which the shape of local perturbation ∂D_+ is a half sphere with radius $R = 1$. The wave number is $k = 3$. We take $(41)^3$ sampling points within a cube prescribed as $\Omega = [-2, 2] \times [-2, 2] \times [0.01, 1.5]$. As can be seen from Figure 3.1, the local perturbation can be reconstructed when illuminations and measurements $N = 50$ and $N = 100$ respectively, provided a good isosurface of $G(z)$ and regularization parameter are chosen.

Example 2 For the second example, we wish to reconstruct a half cushion-

Fig. 3.3. Reconstruction of the half bean and the exact surface.

shaped surface with the parametrization

$$z(\theta, \varphi) = \sqrt{0.27 + 0.065(\cos 2\varphi - 1)(\cos 4\theta - 1)}r(\theta, \varphi),$$

where spherical coordinates

$$r(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad \theta \in [0, \pi/2], \quad \varphi \in [0, 2\pi].$$

We choose wave number $k = 4$ and $(41)^3$ sampling points within $\Omega = [-1, 1] \times [-1, 1] \times [0.01, 0.8]$. Figure 3.2 shows the results.

Example 3 Our third example is intended to reconstruct a half bean-like surface

with the parametrization

$$z(\theta, \varphi) = \begin{pmatrix} 0.5\sqrt{1 - 0.1 \cos(\pi \cos \theta) \sin \theta \cos \varphi} \\ 0.5 \sin \theta \sin \varphi + 0.3 \cos(\pi \cos \theta) \\ 0.7 \cos \theta \end{pmatrix}, \quad \theta \in [0, \pi/2], \quad \varphi \in [0, 2\pi].$$

We assume wave number $k = 7$ and the sampling domain is $\Omega = [-1, 1] \times [-1, 1] \times [0.01, 1]$ with $(41)^3$ sampling points. Results are shown in Figure 3.3. Moreover, one can observe how the increase of N helps a better resolution of the reconstruction.

Fig. 3.4. Reconstruction for limited aperture case.

The above numerical results illustrate that the linear sampling method gives a feasible reconstruction of the locally perturbed perfectly conducting plate even in the presence of 10% noise in measurements.

3.2 limited aperture data

In this section, we restrict the set of observation points to only half of the upper half sphere $\Gamma \subset \partial B$. This means we know the half of measurement data. In this case the data are $\mathbf{E}^\infty(\hat{\mathbf{x}}_i, \mathbf{d}_j, \mathbf{p})$ for some $(\hat{\mathbf{x}}_i, \mathbf{d}_j, \mathbf{p}) \in \Gamma \times (-\Gamma) \times \mathbb{R}^3$. Fig.3.4 and Fig.3.5 display reconstructions that correspond to the three examples. We observe that the quality of the reconstruction deteriorates as the aperture decreases. However, one

Fig. 3.5. Reconstruction for limited aperture case.

notices in all cases that we have a rather good approximation of the projection of the scatterer on the plane located in the illuminated region.

3.3 near-field data

In this section we consider the linear sampling method for the near field measurements. Let $G_+ \subset \mathbb{R}_{+,0}^3$ be a bounded domain enclosing the local perturbation such that $\partial G_+ \subset \mathbb{R}_+^3$ is a smooth surface. The incident field is considered to be the electric field of an electric dipole located at $\mathbf{y} \in \partial G_+$ with polarization $\mathbf{p} \in \mathbb{R}^3$, and is given by

$$\mathbf{E}_e^i(\mathbf{x}, \mathbf{y}, \mathbf{p}) = \frac{i}{k} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \left(\mathbf{p} \Phi(\mathbf{x}, \mathbf{y}) \right) = \mathbf{E}_e^1(\mathbf{x}, \mathbf{y}, \mathbf{p}).$$

The inverse scattering problem we are interested in is to determine the unknown boundary ∂D_+ from knowledge of the tangential components $\nu_+ \times \mathbf{E}_+^s$ of the scattered electric field measured on ∂G_+ for all incident point sources $\mathbf{E}_e^i(\mathbf{x}, \mathbf{y}, \mathbf{p})$, $\forall \mathbf{y} \in \partial G_+$, $\forall \mathbf{p} \in \mathbb{R}^3$.

Note that the reflected wave is $\mathbf{E}_e^r(\mathbf{x}, \mathbf{y}, \mathbf{p}) = T_2 \mathbf{E}_e^i(T_1 \mathbf{x}, \mathbf{y}, \mathbf{p}) = \mathbf{E}_e^2(\mathbf{x}, \mathbf{y}, \mathbf{p})$. Analogously to the case of far-field data, we can reformulate the scattering problem as an exterior Maxwell problem (5)-(8) for a symmetric domain $\bar{D}$ with $\mathbf{E}^{i,r}$ replaced by $\mathbf{E}_e$. Then, we have the following result.

Theorem 4 For $\mathbf{x}, \mathbf{y} \in \partial G$, the tangential components $\nu \times \mathbf{E}^s$ of the scattered electric field satisfies the following relations

$$(1) (\nu \times \mathbf{E}^s)(T_1 \mathbf{x}, \mathbf{y}, \mathbf{p}) = T_2(\nu \times \mathbf{E}^s)(\mathbf{x}, \mathbf{y}, \mathbf{p});$$

$$(2) (\nu \times \mathbf{E}^s)(\mathbf{x}, T_1 \mathbf{y}, \mathbf{p}) = -(\nu \times \mathbf{E}^s)(\mathbf{x}, \mathbf{y}, T_1 \mathbf{p});$$

$$(3) (\nu \times \mathbf{E}^s)(T_1 \mathbf{x}, T_1 \mathbf{y}, \mathbf{p}) = T_1(\nu \times \mathbf{E}^s)(\mathbf{x}, \mathbf{y}, T_1 \mathbf{p}).$$

Now we reformulate the linear sampling method in terms of the near field data. To this end we consider the near field operator $\mathcal{F} : L_t^2(\partial G) \rightarrow L_t^2(\partial G)$ where $G = G_+ \cup G_-$ is a bounded domain such that $\bar{D}$ is contained in G , and the near field equation

$$(\mathcal{F}\varphi_z)(\mathbf{x}) := \int_{\partial G} \nu(\mathbf{x}) \times \mathbf{E}^s(\mathbf{x}, \mathbf{y}, \varphi_{z,p}(\mathbf{y})) ds(\mathbf{y}) = \nu(\mathbf{x}) \times \mathbf{E}_e(\mathbf{x}, \mathbf{z}, \mathbf{p}), \quad \mathbf{x} \in \partial G. \quad (20)$$

Fig. 3.6. Reconstruction for near-field data.

For convenience, ∂G is chosen to be a sphere with radius $r = 2$ in Example 1 and $r = 1$ in Example 2 and Example 3. By using of the symmetry relation(Theorem 6.32 in[1])

$$\mathbf{q} \cdot \mathbf{E}^s(\mathbf{x}, \mathbf{y}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{E}^s(\mathbf{y}, \mathbf{x}, \mathbf{q}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \setminus \bar{D},$$

the near field equation (20) can be transformed into the fully discrete version

$$\begin{cases} \sum_{j=1}^N \sum_{l=1}^2 \omega_j \varphi_{z,p}^l(\mathbf{y}_j) \mathbf{E}^s(\mathbf{y}_j, \mathbf{x}_i, \mathbf{e}_2(\hat{\mathbf{x}}_i)) \cdot \mathbf{e}_l(\hat{\mathbf{y}}_j) = \mathbf{e}_2(\hat{\mathbf{x}}_i) \cdot \mathbf{E}_e(\mathbf{x}_i, \mathbf{z}, \mathbf{p}), \\ \sum_{j=1}^N \sum_{l=1}^2 \omega_j \varphi_{z,p}^l(\mathbf{y}_j) \mathbf{E}^s(\mathbf{y}_j, \mathbf{x}_i, \mathbf{e}_1(\hat{\mathbf{x}}_i)) \cdot \mathbf{e}_l(\hat{\mathbf{y}}_j) = \mathbf{e}_1(\hat{\mathbf{x}}_i) \cdot \mathbf{E}_e(\mathbf{x}_i, \mathbf{z}, \mathbf{p}), \end{cases}$$

where we have set $\varphi_{z,p}(\mathbf{y}_j) = \varphi_{z,p}^1(\mathbf{y}_j) \mathbf{e}_1(\hat{\mathbf{y}}_j) + \varphi_{z,p}^2(\mathbf{y}_j) \mathbf{e}_2(\hat{\mathbf{y}}_j)$ and the measurement points $\{\mathbf{y}_j\}_{j=1}^N$ are also serve as the locations $\{\mathbf{x}_i\}_{i=1}^N$ of the point sources. Similarly as far-field data case, we plot the isosurfaces of $G(\mathbf{z})$ for $\varphi_{z,p}$.

In the numerical experiments, we keep the settings of wave number, sampling domain and the number of sampling points in each example. Fig.3.6 shows the result of reconstruction.

4 Conclusions

In this paper we have presented a numerical method for solving a three-dimensional inverse problem to determine the shape of a local perturbation of a perfectly conducting plate. We have applied the linear sampling method to the far-field data case, limited aperture case and near-field data case. The numerical experiments show that satisfactory reconstructions can be obtained in the three cases.

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