Green's function retrieval in a field of random water waves

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Abstract

It has recently been demonstrated using a variety of wave types that cross-correlating time series of apparently random waves measured at two locations yields an estimate of the Green's function that describes the wave field generated at one of those locations and measured at the other. This procedure can be described as random wave interferometry. In this paper random surface gravity wave interferometry is described theoretically, demonstrated using numerical simulations, and investigated experimentally using both wavetank measurements and ocean wave measurements. Simulations and wavetank measurements are in good agreement with theoretical predictions, but the ocean-measurement-based cross correlations do not yield the predicted structure. Possible explanations are discussed.

Keywords: Green's function retrieval, random wave interferometry, random wave field cross-correlations, water waves

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1. Introduction

In recent years it has become appreciated that deterministic wave propagation information can be extracted from an apparently random wave field via a process that can be described as random wave interferometry; the cross-

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5 correlation of measurements at two locations of a random wave field yields an approximation to the Green's function that describes propagation between those locations[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Random wave interferometry has been widely investigated in the context of elastic waves in solids[14, 15, 16], including seismic[17, 18, 19, 20, 21, 22] and helioseismic[23, 24] applications,
10 and sound waves in fluids, including applications to ocean acoustics[25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35] and atmospheric acoustics[36, 37, 38]. The underlying theory is widely applicable, as it can be applied to any type of linear wave propagation.

In this paper we study, both theoretically and experimentally, the application
15 of these ideas to water waves, i.e., surface gravity waves. This topic can be referred to as random water wave interferometry. Previous work on random water wave interferometry is described in [39] and [40]. [39] focuses on theory relating to tsunamis (very low frequency ocean surface gravity waves for which $kh \ll 1$), while [40] presents theory and data analysis relating to infragravity
20 waves (low frequency ocean surface gravity waves for which $kh = O(1)$ in the deep ocean). The latter work builds on the earlier work [41] which demonstrated that seafloor measurements of infragravity-wave-induced pressure fluctuations separated by a few 10's of km are coherent, and the work reported in [42], which demonstrated the utility of cross-correlating measurements of that type.

25 The basic result on which we focus in this paper is

$$\gamma C_{AB}(t) = D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t)]. \quad (1)$$

Here γ is a constant, $C_{AB}(t)$ is the correlation function of records of surface elevation $\eta(t)$ at locations $\mathbf{x}_A$ and $\mathbf{x}_B$, $D(t)$ is a band-limited approximation to a delta function, and $G_\eta(\mathbf{x}|\mathbf{x}_0, t)$ is the transient surface elevation at point $\mathbf{x}$ due to an impulsive point source at $\mathbf{x}_0$. Two derivations of Eq. (1), which holds
30 in both open and closed systems, are given below. It is important to appreciate that two results underlie Eq. (1) – an exact identity involving Green's functions, and a strong approximate statistical assumption. The Green's function identity holds provided the underlying fluid dynamical assumptions (adequacy

of linear theory, including the assumption that the flow remains irrotational)
 35 are satisfied. The statistical assumptions underlying Eq. (1) are strong and
 might reasonably be questioned, but we note that experience in other fields has
 shown that relationships similar to Eq. (1) are good approximations even when
 the underlying statistical assumptions are only approximately satisfied.

The remainder of this paper is organized as follows. The theory underlying
 40 Eq. (1) is presented in section 2. Derivations of Eq. (1) for both open and
 closed systems are included. In section 3 simulations, in both open and closed
 systems, are presented and shown to be consistent with Eq. (1). In section 4
 the analysis of data collected in a wavetank for the purpose of testing Eq. (1)
 is described. Agreement between theory and measurements is good. In section
 45 5 the analysis of a set of ocean wave measurements, again to test Eq. (1), is
 described. Those results are not in good agreement with theory. In section 6
 our results are discussed and summarized.

2. Theory

In this section we present two derivations of Eq. (1), which holds in both
 50 open and closed systems. Each derivation makes use of an identity involving
 Green's functions, Eq. (12) or (22), and a strong statistical assumption, Eq.
 (14) or (24).

2.1. Preliminaries

We begin with comments on two topics. First, it is important to distin-
 55 guish between two different Green's functions: G_ϕ is the velocity potential
 Green's function evaluated at the undisturbed free surface; and G_η is the sur-
 face displacement Green's function. Both of these quantities have time- and
 frequency-domain representations. Second, convolutions and cross-correlations
 play important roles in the derivations below. Consider real-valued functions
 60 $f_1(t)$ and $f_2(t)$. Convolution of $f_1(t)$ and $f_2(t)$ is defined as $f_1(t) * f_2(t) =$
 $\int_{-\infty}^{\infty} d\tau f_1(\tau) f_2(t - \tau)$. The cross-correlation of $f_1(t)$ and $f_2(t)$ is defined as

$f_1(t) * f_2(-t) = \int_{-\infty}^{\infty} d\tau f_1(\tau) f_2(t + \tau)$. Let $\bar{f}(\omega) = F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$ denote the Fourier transform of $f(t)$, and let $f(t) = F^{-1}[\bar{f}(\omega)]$ denote the inverse transform. Because these variables are real-valued, it is assumed that
65 $\bar{f}(-\omega) = \bar{f}^*(\omega)$, where the superscript $*$ denotes complex conjugation. The Fourier transform of the convolution $f_1(t) * f_2(t)$ is $\bar{f}_1(\omega) \bar{f}_2(\omega)$, and the Fourier transform of the cross-correlation $f_1(t) * f_2(-t)$ is $\bar{f}_1(\omega) \bar{f}_2^*(\omega)$. For the purpose of computing the cross-correlation of two measured time series it is necessary to replace the unbounded integration domain with a finite domain. This leads a
70 modified definition of the cross-correlation, $f_1(t) \star f_2(t) = \int_0^T d\tau f_1(\tau) f_2(\tau + t)$. The $*$ and $\star$ notation introduced here is used below. It is often convenient to normalize $f_1(t) \star f_2(t)$ by dividing by T ; when this is done, division by T will be shown explicitly.

The starting point of our analysis is the linearized water wave equations of
75 motion (see, e.g., [43]). Let $\mathbf{x} = (x, y)$ denote lateral position vector, and assume that the vertical coordinate z increases upwards, with $z = 0$ at the undisturbed free surface. Let ∇^2 denote the three dimensional Laplacian operator, and let $\nabla_{\perp}$ and $\nabla_{\perp}^2$ denote the two-dimensional – in (x, y) – gradient and Laplacian operators, respectively. We assume that the water depth h is constant. Let
80 $\Phi(\mathbf{x}, z, t)$ denote the velocity potential and $\bar{\Phi}(\mathbf{x}, z, \omega)$ its Fourier transform. The problem is to solve $\nabla^2 \Phi(\mathbf{x}, z, t) = f(\mathbf{x}, z, t)$ subject to $\partial^2 \Phi / \partial t^2 + g \partial \Phi / \partial z = 0$ at $z = 0$, and $\partial \Phi / \partial z = 0$ at $z = -h$. The free surface displacement $\eta(\mathbf{x}, t)$ is equal to $(-1/g) \partial \Phi / \partial t$ evaluated at $z = 0$. $f(\mathbf{x}, z, t)$ is a source function; ρf is the rate at which mass per unit volume is injected. The problem as defined
85 so far is poorly posed. The difficulty is that most functions $f(\mathbf{x}, z, t)$ don't couple naturally to small amplitude surface gravity waves, and therefore don't allow the specified free surface and bottom boundary conditions to be satisfied. Instead, most functions $f(\mathbf{x}, z, t)$ lead to a localized transient non-wave-like response. To avoid this difficulty, we shall assume that the depth dependence
90 of the source function $f(\mathbf{x}, z, t)$ is chosen to couple naturally to small amplitude surface gravity waves. (It is worth noting that the same assumption is made below in a less obvious way when we find the surface displacement Green's

function as the solution to an initial value problem; in both cases the depth dependence of the velocity field is constrained to be dynamically consistent
 95 with the free surface displacement.)

In the frequency domain, the problem is to solve $\nabla^2 \bar{\Phi}(\mathbf{x}, z, \omega) = \bar{f}(\mathbf{x}, z, \omega)$ subject to the boundary conditions $\omega^2 \bar{\Phi} = g \partial \bar{\Phi} / \partial z$ at $z = 0$, and $\partial \bar{\Phi} / \partial z = 0$ at $z = -h$. We shall assume that $\bar{f}(\mathbf{x}, z, \omega) = \bar{p}(\mathbf{x}, \omega) \cosh k(z + h) / \cosh kh$ and that $\bar{\Phi}(\mathbf{x}, z, \omega) = \bar{\phi}(\mathbf{x}, \omega) \cosh k(z + h) / \cosh kh$. With these assumptions
 100 the boundary conditions are satisfied provided the dispersion relation $\omega^2 = gk \tanh kh$ is satisfied, and the problem is reduced to solving $(\nabla_{\perp}^2 + k^2) \bar{\phi}(\mathbf{x}, \omega) = \bar{p}(\mathbf{x}, \omega)$. The solution can be written $\bar{\phi}(\mathbf{x}, \omega) = \iint d\mathbf{x}' \bar{G}_{\phi}(\mathbf{x}|\mathbf{x}', \omega) \bar{p}(\mathbf{x}', \omega)$ where $\bar{G}_{\phi}(\mathbf{x}|\mathbf{x}', \omega)$ is the Green's function for the Helmholtz equation in two space dimensions,

$$(\nabla_{\perp}^2 + k^2) \bar{G}_{\phi}(\mathbf{x}|\mathbf{x}', \omega) = \delta(\mathbf{x} - \mathbf{x}'). \quad (2)$$

105 In anticipation of our focus below on the free surface displacement, we introduce the free surface displacement Green's function $\bar{G}_{\eta}(\mathbf{x}|\mathbf{x}', \omega)$. Consistent with the free-surface boundary condition, $\bar{\eta}(\mathbf{x}, \omega) = (i\omega/g) \bar{\phi}(\mathbf{x}, \omega)$, we set $\bar{G}_{\eta}(\mathbf{x}|\mathbf{x}', \omega) = (i\omega/g) \bar{G}_{\phi}(\mathbf{x}|\mathbf{x}', \omega)$ and $\bar{\eta}(\mathbf{x}, \omega) = \iint d\mathbf{x}' \bar{G}_{\eta}(\mathbf{x}|\mathbf{x}', \omega) \bar{p}(\mathbf{x}', \omega)$. In the time domain this corresponds to

$$\eta(\mathbf{x}, t) = \iint d\mathbf{x}' G_{\eta}(\mathbf{x}|\mathbf{x}', t) * p(\mathbf{x}', t), \quad (3)$$

110 where $G_{\eta}(\mathbf{x}|\mathbf{x}', t) = (-1/g)(\partial/\partial t) G_{\phi}(\mathbf{x}|\mathbf{x}', t)$. The preceding discussion serves to define the relevant Green's functions and their relationship to measurable quantities.

2.2. General derivation of Eq. (1)

We now present a derivation of Eq. (1) that holds in both open and closed
 115 systems. We begin with a derivation of the Green's function identity, Eq. (12), that underlies Eq. (1). In the presence of weak dissipation, which is assumed to be proportional to ω times the parameter ϵ , the Green's function $\bar{G}_{\phi}(\mathbf{x}|\mathbf{x}_0, \omega)$ at $\mathbf{x}$, corresponding to point source excitation at $\mathbf{x}_0$, satisfies

($\nabla_{\perp}^2 + k^2 + i\omega\epsilon$) $\bar{G}_{\phi}(\mathbf{x}|\mathbf{x}_0, \omega) = \delta(\mathbf{x} - \mathbf{x}_0)$. Thus, $\bar{G}_{\phi}$ at point $\mathbf{x}$ excited by a point
 120 source at $\mathbf{x}_A$ satisfies

$$(\nabla_{\perp}^2 + k^2 + i\omega\epsilon)\bar{G}_{\phi}(\mathbf{x}|\mathbf{x}_A, \omega) = \delta(\mathbf{x} - \mathbf{x}_A). \quad (4)$$

Similarly, $\bar{G}_{\phi}$ at point $\mathbf{x}$ excited by a point source at $\mathbf{x}_B$ satisfies

$$(\nabla_{\perp}^2 + k^2 + i\omega\epsilon)\bar{G}_{\phi}(\mathbf{x}|\mathbf{x}_B, \omega) = \delta(\mathbf{x} - \mathbf{x}_B). \quad (5)$$

Due to reciprocity $\bar{G}_{\phi}(\mathbf{x}|\mathbf{x}_A, \omega)$ is equal to $\bar{G}_{\phi}(\mathbf{x}_A|\mathbf{x}, \omega)$, and $\bar{G}_{\phi}(\mathbf{x}|\mathbf{x}_B, \omega)$ is equal to $\bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)$. Multiplication of (4) by $\bar{G}_{\phi}^*(\mathbf{x}_B|\mathbf{x}, \omega)$, followed by integration over $\mathbf{x}$, and complex conjugation of left- and right-hand sides gives

$$\int \int d\mathbf{x} \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)(\nabla_{\perp}^2 + k^2 - i\omega\epsilon)\bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega) = \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}_A, \omega). \quad (6)$$

125 Multiplication of (5) by $\bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega)$, followed by integration over $\mathbf{x}$ gives

$$\int \int d\mathbf{x} \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega)(\nabla_{\perp}^2 + k^2 + i\omega\epsilon)\bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega) = \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}_B, \omega). \quad (7)$$

Subtracting (7) from (6) gives

$$Q - 2i\omega\epsilon \int \int d\mathbf{x} \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)\bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega) = \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}_A, \omega) - \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}_B, \omega). \quad (8)$$

where

$$\begin{aligned} Q &= \int \int d\mathbf{x} \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)\nabla_{\perp}^2 \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega) - \int \int d\mathbf{x} \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega)\nabla_{\perp}^2 \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega) \\ &= \oint dl [\bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)\nabla_{\perp} \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega) - \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega)\nabla_{\perp} \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)] \cdot \hat{n} \end{aligned}$$

By reciprocity $\bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}_A, \omega) = \bar{G}_{\phi}(\mathbf{x}_A|\mathbf{x}_B, \omega)$, so the rhs of (8) can be written $2 \operatorname{Im} \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}_A, \omega)$. The second form of Q given above follows from applying the divergence theorem to $\int \int d\mathbf{x} \nabla_{\perp} \cdot [\bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)\nabla_{\perp} \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega) - \bar{G}_{\phi}^*(\mathbf{x}_A|\mathbf{x}, \omega)\nabla_{\perp} \bar{G}_{\phi}(\mathbf{x}_B|\mathbf{x}, \omega)]$. The integral in the latter form of Q is a line integral over the boundary of the $\mathbf{x}$ -domain and $\hat{n}$ is a unit outward normal. We consider two choices for the domain of $\mathbf{x}$. First, appropriate for open ocean conditions, consider the domain enclosed within a large circle of radius R centered at the midpoint between $\mathbf{x}_A$ and $\mathbf{x}_B$. Owing to dissipation, for sufficiently
 135

large R , $\bar{G}_\phi$ and $\nabla_\perp \bar{G}_\phi \cdot \hat{n}$ approach zero faster than $R^{-1/2}$, so Q approaches 0. Second, consider an $\mathbf{x}$ -domain bounded by rigid walls that enclose $\mathbf{x}_A$ and $\mathbf{x}_B$, as in a wavetank. Because $\nabla_\perp \bar{G}_\phi \cdot \hat{n} = 0$ at rigid walls, each term in Q vanishes so $Q = 0$. Setting $Q = 0$ in (8) gives

$$-2i\omega\epsilon \iint d\mathbf{x} \bar{G}_\phi(\mathbf{x}_B|\mathbf{x}, \omega) \bar{G}_\phi^*(\mathbf{x}_A|\mathbf{x}, \omega) = \bar{G}_\phi(\mathbf{x}_B|\mathbf{x}_A, \omega) - \bar{G}_\phi^*(\mathbf{x}_A|\mathbf{x}_B, \omega). \quad (9)$$

140 Transforming to the time domain gives

$$2\epsilon \frac{d}{dt} \iint d\mathbf{x} G_\phi(\mathbf{x}_B|\mathbf{x}, t) * G_\phi(\mathbf{x}_A|\mathbf{x}, -t) = G_\phi(\mathbf{x}_B|\mathbf{x}_A, t) - G_\phi(\mathbf{x}_B|\mathbf{x}_A, -t). \quad (10)$$

where $*$ (not a superscript) denotes convolution. Recall that the convolution of $G_\phi(\mathbf{x}_B|\mathbf{x}, t)$ and $G_\phi(\mathbf{x}_A|\mathbf{x}, -t)$ is equivalent to the cross-correlation of $G_\phi(\mathbf{x}_B|\mathbf{x}, t)$ and $G_\phi(\mathbf{x}_A|\mathbf{x}, t)$.

The above considerations focus on the Green's function for the velocity potential evaluated at the free surface, G_ϕ . From a measurement point of view it is more natural to focus on the surface elevation Green's function, G_η . Recall that $\bar{\eta}(\mathbf{x}) = (i\omega/g)\bar{\phi}(\mathbf{x})$, $\bar{G}_\eta = (i\omega/g)\bar{G}_\phi$, and $G_\eta = (-1/g)\partial G_\phi/\partial t$. Using these relationships, Eqs. (9) and (10) are equivalent to

$$2\epsilon g \iint d\mathbf{x} \bar{G}_\eta(\mathbf{x}_B|\mathbf{x}, \omega) \bar{G}_\eta^*(\mathbf{x}_A|\mathbf{x}, \omega) = \bar{G}_\eta(\mathbf{x}_B|\mathbf{x}_A, \omega) + \bar{G}_\eta^*(\mathbf{x}_A|\mathbf{x}_B, \omega) \quad (11)$$

and

$$2\epsilon g \iint d\mathbf{x} G_\eta(\mathbf{x}_B|\mathbf{x}, t) * G_\eta(\mathbf{x}_A|\mathbf{x}, -t) = G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_B|\mathbf{x}_A, -t), \quad (12)$$

150 respectively. Equation (12) is the Green's function identity that we set out to prove.

The dissipation ϵ plays a subtle role in Eqs. (9 - 12). On an unbounded domain the integrals over $\mathbf{x}$ in those equations diverge if $\epsilon = 0$, so the introduction of a small but nonzero dissipation prevents the occurrence of indeterminate products (zero times infinity). For numerical purposes, e.g., to test (12), one might evaluate $2g \iint d\mathbf{x} G_\eta(\mathbf{x}_B|\mathbf{x}, t) * G_\eta(\mathbf{x}_A|\mathbf{x}, -t)$ using the large circular domain described above and compute the Green's functions assuming

no dissipation. To a very good approximation the result will be $P(R)$ times $G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_B|\mathbf{x}_A, -t)$, where $P(R)$ grows with increasing domain radius R and has the dimensions of ϵ^{-1} , or m^2/s . For the purposes described in this paper, the relevant spatial domains and integration times are finite, and proportionality of the left- and right-hand sides of (12) is all that is required. With these comments in mind, for our purposes there is no need to account for dissipation when computing the relevant Green's functions.

In an unbounded domain and in the limit $\epsilon = 0$, $\bar{G}_\phi(\mathbf{x}|\mathbf{x}_0, \omega) = (-i/4)H_0^{(1)}(kr)$ where $r = |\mathbf{x} - \mathbf{x}_0|$ and $H_0^{(1)}$ is the Hankel function of the first kind. Thus, $G_\phi(\mathbf{x}|\mathbf{x}_0, t) = F^{-1}[(-i/4)c_g^{-1}(k(\omega))]H_0^{(1)}(k(\omega)r)$. (The factor $c_g^{-1}(k(\omega)) = (\partial\omega/\partial k)^{-1}$ is the Jacobian of the transformation from k to ω . Also, when evaluating the Fourier integral, the constraint on negative frequencies noted above insures that $G_\phi(\mathbf{x}|\mathbf{x}_0, t)$ is real). $G_\phi(\mathbf{x}|\mathbf{x}_0, t)$ constructed in this way is well-defined provided $k(\omega)$ is single-valued. A caveat is discussed below.

We now complete the derivation of the correlation function result (1). Equations (11) and (12) are mathematical identities. In contrast, the corresponding correlation function result is an approximate result whose validity depends on Eq. (12) and additional statistical assumptions, as described below. It follows from Eq. (3) that the surface displacement at $\mathbf{x}_A$ can be written $\eta(\mathbf{x}_A, t) = \iint d\mathbf{x} G_\eta(\mathbf{x}_A|\mathbf{x}, t) * p(\mathbf{x}, t)$ where $p(\mathbf{x}, t)$ is the source function. Similarly, $\eta(\mathbf{x}_B, t) = \iint d\mathbf{x} G_\eta(\mathbf{x}_B|\mathbf{x}, t) * p(\mathbf{x}, t)$. Let $C_{AB}(t)$ denote the cross-correlation of $\eta(\mathbf{x}_A, t)$ and $\eta(\mathbf{x}_B, t)$. Then

$$C_{AB}(t) = \iint d\mathbf{x} \iint d\mathbf{x}' G_\eta(\mathbf{x}_A|\mathbf{x}, t) * G_\eta(\mathbf{x}_B|\mathbf{x}', -t) * p(\mathbf{x}, t) * p(\mathbf{x}', -t). \quad (13)$$

We shall assume that the source function is delta-correlated in space and nearly so in time,

$$p(\mathbf{x}, t) * p(\mathbf{x}', -t) = a^2 \delta(\mathbf{x} - \mathbf{x}') D(t), \quad (14)$$

where $D(t)$ is a band-limited approximation to a delta function. If the assumed random sources have time history $d(t)$, then $D(t) = d(t) * d(-t)$. Note that $\bar{D}(\omega) = \bar{d}(\omega)\bar{d}^*(\omega) = |\bar{d}(\omega)|^2$, so any phase information contained in $d(t)$ is lost in $D(t)$. The quantity a has dimension m. Both $d(t)$ and $D(t)$ have dimension

s^{-1} ; in the limiting case of infinite bandwidth, $D(t) \rightarrow \delta(t)$. Making use of Eq. (14) in Eq. (13), the latter equation reduces to

$$C_{AB}(t) = a^2 D(t) * \iint d\mathbf{x} G_\eta(\mathbf{x}_A|\mathbf{x}, t) * G_\eta(\mathbf{x}_B|\mathbf{x}, -t). \quad (15)$$

Comparison of Eqs. (12) and (15) then yields

$$\frac{2\epsilon g}{a^2} C_{AB}(t) = D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_B|\mathbf{x}_A, -t)], \quad (16)$$

which is seen to be identical to Eq. (1) for a particular choice of the constant γ .

190 In practice, the correlation function $C_{AB}(t)$ is evaluated as $\eta(\mathbf{x}_A, t) \star \eta(\mathbf{x}_B, t)$. The comments made above about expecting only proportionality of the lhs and rhs of Eq. (12) apply also to Eq. (16). Note that Eq. (12) is exact (within a linear theory framework), while the validity of Eq. (16) rests on the idealized statistical assumption, Eq. (14), that we have made about the source function
 195 $v(\mathbf{x}, t)$. Experiments with other wave types suggest that Eq. (16) should be a good approximation even when those assumptions are relaxed.

2.3. Alternate derivation of Eq. (1) for a closed system

We consider now an alternative derivation of Eq. (1), appropriate for a bounded domain, such as the wavetank geometry described above. Although
 200 this alternate derivation is less general than the one presented above, it is useful for two reasons. First, it gives additional insight into Eq. (1). Second, some of the results presented in this subsection are used as the basis for numerical results presented in the two sections that follow. Unlike the derivation presented above, the following derivation (see also [2]) requires that we first explicitly solve
 205 for the Green's function. In a rectangular basin, $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$, whose depth is h , the velocity potential

$$\Phi(x, y, z, t) = g \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\cosh k_{lm}(z+h)}{\cosh k_{lm}h} \cos\left(\frac{l\pi x}{L_x}\right) \cos\left(\frac{m\pi y}{L_y}\right) (a_{lm} \cos \omega_{lm}t + b_{lm} \sin \omega_{lm}t)$$

satisfies the relevant lateral boundary conditions and $\nabla^2 \Phi = 0$ provided $(l\pi/L_x)^2 + (m\pi/L_y)^2 = k_{lm}^2$. The linearized free surface boundary condition is satisfied provided $\omega_{lm}^2 = gk_{lm} \tanh k_{lm}h$. The vertical component of fluid velocity $\partial\Phi/\partial z$

210 evaluated at $z = 0$ is equal to $\partial\eta/\partial t$, so

$$\eta(x, y, t) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \omega_{lm} \cos\left(\frac{l\pi x}{L_x}\right) \cos\left(\frac{m\pi y}{L_y}\right) (a_{lm} \sin \omega_{lm} t - b_{lm} \cos \omega_{lm} t). \quad (17)$$

The $l = m = 0$ term can be neglected in (17) because $\omega_{00} = 0$. It follows from Eq. (3) that $G_\eta(\mathbf{x}|\mathbf{x}_0, t)$ is $1/A_0$ times the particular choice of $\eta(\mathbf{x}, t)$ for which the source function $p(\mathbf{x}, t) = A_0 \delta(\mathbf{x} - \mathbf{x}_0) \delta(t)$. (A_0 has dimension m^2 , but dependence on A_0 cancels in the resulting expression for $G_\eta(\mathbf{x}, t)$.)
 215 Physically, this source function corresponds to an instantaneous injection of a volume $V_0 = L_0 A_0$ of water at $\mathbf{x} = \mathbf{x}_0$, $t = 0$. The Green's function $G_\eta(\mathbf{x}|\mathbf{x}_0, t)$ can be constructed from Eq. (17) as $1/A_0$ times the solution, $\eta(\mathbf{x}, t)$, of the initial value problem, $\eta(\mathbf{x}, 0) = V_0 \delta(\mathbf{x} - \mathbf{x}_0)$ and $(\partial\eta(\mathbf{x}, t)/\partial t)(\mathbf{x}, 0) = 0$. The constants a_{lm} and b_{lm} , whose units are $\text{m} \times \text{s}$, are found by making use of the
 220 orthogonality of the modes. After some algebra one finds

$$G_\eta(\mathbf{x}|\mathbf{x}', t) = \frac{L_0}{L_x L_y} H(t) \sum_l \sum_m \beta_l \beta_m \cos\left(\frac{l\pi x}{L_x}\right) \cos\left(\frac{m\pi y}{L_y}\right) \cos\left(\frac{l\pi x'}{L_x}\right) \cos\left(\frac{m\pi y'}{L_y}\right) \cos \omega_{lm} t \quad (18)$$

where $\beta_l = 1$ for $l = 0$, $\beta_l = 2$ for $l = 1, 2, \dots$, and $H(t)$ is the Heaviside step function. Equation (18) is a special case of a more general modal representation of the Green's function,

$$G_\eta(\mathbf{x}|\mathbf{x}', t) = \frac{L_0}{A} H(t) \sum_n \psi_n(\mathbf{x}) \psi_n(\mathbf{x}') \cos \omega_n t \quad (19)$$

where A is the lateral area of the domain. The dimensionless orthogonal modes
 225 satisfy the relevant boundary conditions and are assumed to be normalized, $\iint d\mathbf{x} \psi_n(\mathbf{x}) \psi_m(\mathbf{x}) = A \delta_{nm}$. In the following we shall make use of the more general and more convenient expression (19), noting that Eq. (18) is a special case.

We now demonstrate that the Green's function representation (19) is con-
 230 sistent with the Green's function identity (12). After some reordering of terms and making use of the modal orthogonality condition, it follows from Eq. (19)

that

$$\int \int d\mathbf{x} G_\eta(\mathbf{x}_A|\mathbf{x}, t) \star G_\eta(\mathbf{x}_B|\mathbf{x}, t) = \frac{L_0}{A} \sum_n \psi_n(\mathbf{x}_A) \psi_n(\mathbf{x}_B) [H(t) \cos \omega_n t \star H(t) \cos \omega_n t]. \quad (20)$$

Evaluation of the cross-correlation $H(t) \cos \omega_n t \star H(t) \cos \omega_n t$ requires some care due to the one-sided nature of the functions involved:

$$\begin{aligned} H(t) \cos \omega_n t \star H(t) \cos \omega_n t &= \int_0^T d\tau \cos \omega_n \tau [\cos \omega_n (\tau + t) + \cos \omega_n (\tau - t)] \\ &= 2 \cos \omega_n t \int_0^T d\tau \cos^2 \omega_n \tau \\ &= T \cos \omega_n t \left(1 + \frac{\sin \omega_n T \cos \omega_n T}{\omega_n T} \right). \end{aligned} \quad (21)$$

235 We shall assume that $\omega_n T$ is large so the second term in square brackets in (21) can be neglected. Noting the one-sided nature of the Green's function (19), substituting (21) into (20) gives

$$\frac{L_0}{T} \int \int d\mathbf{x} G_\eta(\mathbf{x}_A|\mathbf{x}, t) \star G_\eta(\mathbf{x}_B|\mathbf{x}, t) = G_\eta(\mathbf{x}_A|\mathbf{x}_B, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t). \quad (22)$$

Apart from an unimportant constant, Eq. (22) is seen to be identical to Eq. (12). Recall that the cross-correlation $G_\eta(\mathbf{x}_A|\mathbf{x}, t) \star G_\eta(\mathbf{x}_B|\mathbf{x}, t)$ is an integral
240 over a finite time interval T . Had we considered the limit $T \rightarrow \infty$, we would have encountered an indeterminate form (zero times infinity) on the left hand side of Eq. (22), as discussed earlier.

The argument involving Eqs. (13) and (15) that leads to Eq. (16) is independent of the form of $G_\eta(\mathbf{x}|\mathbf{x}', t)$, so that argument applies when the Eq. (19)
245 representation of $G_\eta(\mathbf{x}|\mathbf{x}', t)$ is valid. To supplement that argument, we now provide an alternate derivation of the correlation result, Eq. (16), that makes use of (19). A random wavefield can be described as a superposition of modes with random phases,

$$\eta(\mathbf{x}, t) = d(t) * \sum_n (c_n \cos \omega_n t + d_n \sin \omega_n t) \psi_n(\mathbf{x}). \quad (23)$$

The random phase assumption dictates that

$$\langle c_n c_m \rangle = \langle d_n d_m \rangle = a^2 \delta_{nm}, \quad \langle c_n d_m \rangle = 0 \quad (24)$$

250 where $\langle \rangle$ denotes ensemble average. Convolution with $d(t)$ is included in Eq. (23) to account for the fact that any physically realizable wave field must have finite bandwidth. As was the case following Eq. (14), we assume that $d(t)$ has dimension s^{-1} . Then, because the normalized modes $\psi_n(\mathbf{x})$ are dimensionless, c_n , d_n and a have dimension m. Although specification of $d(t)$ includes phase
 255 information, that information is lost in $\eta(\mathbf{x}, t)$ because c_n and d_n have random phases. This observation is consistent with Eqs. (26) and (27) below, because, as noted in the discussion following Eq. (14), $D(t)$ does not depend on the phase information contained in $d(t)$. It is natural to define the cross-correlation of measurements of surface displacement at locations $\mathbf{x}_A$ and $\mathbf{x}_B$ as an ensemble
 260 average,

$$C_{AB}(t) = \langle \int_0^T d\tau \eta(\mathbf{x}_A, \tau) \eta(\mathbf{x}_B, \tau + t) \rangle. \quad (25)$$

Substituting Eq. (23) into Eq. (25) and making use of the assumed statistical properties of the c_n 's and d_n 's, Eq. (24), gives, after some algebra,

$$C_{AB}(t) = D(t) * a^2 T \sum_n \psi_n(\mathbf{x}_A) \psi_n(\mathbf{x}_B) \cos \omega_n t, \quad (26)$$

where $D(t) = d(t) * d(-t)$. Comparison of this expression with Eq. (19) gives

$$\frac{L_0}{A a^2 T} C_{AB}(t) = D(t) * [G_\eta(\mathbf{x}_B | \mathbf{x}_A, t) + G_\eta(\mathbf{x}_B | \mathbf{x}_A, -t)]. \quad (27)$$

Like Eq. (16), Eq. (27) is seen to be identical to Eq. (1) for a particular choice
 265 of the constant γ .

3. Simulations

In this section we present simulations to test the validity of Eq. (1) in both open and closed systems. First we consider an open system, consisting of a laterally unbounded domain. We consider both deep and nearly-shallow
 270 conditions. Recall that $\bar{G}_\eta = (i\omega/g)\bar{G}_\phi$, and that in an unbounded domain $\bar{G}_\phi(\mathbf{x}|\mathbf{x}_0, \omega) = (-i/4)H_0^{(1)}(k(\omega)r)$ where $r = |\mathbf{x} - \mathbf{x}_0|$. It follows that

$$G_\eta(\mathbf{x}|\mathbf{x}_0, t) = \frac{1}{4g} F^{-1} \left[\frac{\omega}{c_g(k(\omega))} H_0^{(1)}(k(\omega)r) \right]. \quad (28)$$

More generally, we wish to compute the response, $d(t) * G_\eta(\mathbf{x}|\mathbf{x}_0, t)$, to a band-limited point source. For numerical purposes, we set (neglecting an unimportant dimensional constant) $\bar{d}(\omega) = (\sin(0.5 \omega / f_{max}))^{1/2}$ with wave frequencies, $f = \omega/2\pi$, limited to the band $0 < f < f_{max}$. The convolution $d(t) * G_\eta(\mathbf{x}|\mathbf{x}_0, t)$ has the general form $F^{-1}[\bar{b}(\omega)]$; to insure that $d(t) * G_\eta(\mathbf{x}|\mathbf{x}_0, t)$ is real, the condition $\bar{b}(-\omega) = \bar{b}^*(\omega)$ is applied in our numerical simulations.

To test Eq. (1) a circular domain with 1500 m radius, centered at $x = y = 0$, was used. The field points of interest were assumed to be at $\mathbf{x}_A = (-25 \text{ m}, 0)$ and $\mathbf{x}_B = (25 \text{ m}, 0)$. A random field of waves was constructed by simulating the response to approximately $\pi/4 \times 10^6$ quasi-impulsive point sources, at positions $\mathbf{x}_j$ and times t_j , that were randomly distributed, with a uniform distribution, throughout the circular domain and throughout a 409600 s (about 114 hours) time window. $\eta(\mathbf{x}_A, t)$ was computed as the sum (over j) of contributions, $d(t) * G_\eta(\mathbf{x}_A|\mathbf{x}_j, t - t_j)$, and similarly for $\eta(\mathbf{x}_B, t)$. $\eta(\mathbf{x}_A, t)$ and $\eta(\mathbf{x}_B, t)$ computed in this fashion look like band-limited noise (as expected). In two different environments the cross-correlation $C_{AB}(t)$ of $\eta(\mathbf{x}_A, t)$ and $\eta(\mathbf{x}_B, t)$ was computed by coherently averaging approximately 5000 realizations of the correlation function, each based on 160 s blocks of simulated surface elevation at $\mathbf{x}_A$ and $\mathbf{x}_B$; the resulting $C_{AB}(t)$ estimates are shown in Figs. 1 and 2. In those figures $C_{AB}(t)$ is compared to $D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t)]$, where $D(t) = d(t) * d(-t)$. Like the random wave field just described, the Green's function $G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) = G_\eta(\mathbf{x}_A|\mathbf{x}_B, t)$ was constructed using Eq. (28). Figure 1 corresponds to a water depth of 1.0 m with $f_{max} = 0.3$ Hz. To account for weak dispersion in this frequency band, the approximate dispersion relation, $\omega^2 = ghk^2(1 - (kh)^2/3)$, was used. $C_{AB}(t)$ in Fig. 1 is seen to be in very good agreement with $D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t)]$, as predicted by Eq. (16) or (1). Figure 2 corresponds to deep water conditions, $\omega^2 = gk$, with $f_{max} = 0.5$ Hz. Again, agreement between $C_{AB}(t)$ and $D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t)]$ is seen to be very good. A caveat is discussed below. Note that, owing to dispersion, the deep water Green's function shown in Fig. 2 consists of a simple frequency-modulated sweep with frequency increasing with increasing

time.

We consider now a closed system consisting of a model of a narrow rectangular wave tank with rigid walls. Let x and y denote position coordinates in the
305 along-tank and cross-tank directions, and l and m the corresponding mode numbers. We shall assume that the tank is sufficiently narrow that only the gravest ($m = 0$) cross-tank mode is excited. Then the Green's function $G_\eta(\mathbf{x}|\mathbf{x}', t)$ is independent of y , and Eq. (18) reduces to (with $L_0 = L_y$, for convenience)

$$G_\eta(\mathbf{x}|\mathbf{x}', t) = \frac{1}{L_x} H(t) \sum_l \beta_l \cos\left(\frac{l\pi x}{L_x}\right) \cos\left(\frac{l\pi x'}{L_x}\right) \cos \omega_l t \quad (29)$$

310 where $\omega_l^2 = gk_l \tanh k_l h$ with $k_l = l\pi/L_x$. In the same geometry, a random band-limited wave field is described by a special case of Eq. (23),

$$\eta(\mathbf{x}, t) = d(t) * \sum_l \beta_l (c_l \cos \omega_l t + d_l \sin \omega_l t) \cos\left(\frac{l\pi x}{L_x}\right), \quad (30)$$

where c_l and d_l satisfy conditions (24). To test Eq. (1) the following procedure was used: two positions, x_A and x_B were chosen; $\eta(x_A, t)$ and $\eta(x_B, t)$ were computed in each of 240 realizations of a random wave field described by Eq.
315 (30); the cross-correlation $C_{AB}(t)$ of $\eta(x_A, t)$ and $\eta(x_B, t)$ was computed for each realization of the random wave field; estimates of $C_{AB}(t)$ based on individual realizations of the random wave field were coherently averaged to give the final estimate of $C_{AB}(t)$; the result was compared to $D(t) * [G_\eta(x_B|x_A, t) + G_\eta(x_A|x_B, t)]$, with $D(t) = d(t) * d(-t)$ and $G_\eta(x_A|x_B, t) = G_\eta(x_B|x_A, t)$ computed using Eq.
320 (29). Note that the steps leading to the final estimate of $C_{AB}(t)$ are independent of the calculation of $G_\eta(x_B|x_A, t)$. The procedure just described was followed using $L_x = 11.9$ m, $L_y = 1$ m, $h = 0.42$ m, $x_A = 2.38$ m and $x_B = 9.52$ m. The random variables c_l and d_l were Gaussian random variables with zero mean and unit variance; an independent (c_l, d_l) pair was used in each of 240
325 realizations of the random wave field. $\bar{d}(\omega)$ was assumed to have the form (neglecting an unimportant dimensional constant) $3\pi\omega \exp(-\omega/3\pi)$ for $\omega < 3\pi$, and $\omega^2 \exp(-(\omega/3\pi)^2)$ for $\omega \geq 3\pi$. Results are shown in Fig. 3. In that figure only positive time lags are shown; the negative lag structure is essentially identical to the positive lag structure shown. Agreement between the average

330 $C_{AB}(t)$ and $D(t) * G(x_B|x_A, t)$ is seen to be excellent, as predicted by Eq. (27) or (1).

4. Wave tank experiment

An experiment was performed in a wave tank to test Eq. (1). In the experiment, the parameters L_x, L_y, h, x_A, x_B were identical to those used in the simulations just described. The calculation of $G_\eta(x_A|x_B, t)$ is thus identical to that just described, based on Eq. (29). The effective source function $d(t)$ for the wave tank experiment was different from that used in the simulations, however, as described below.

In the experiment, a programmable wave paddle at one end ($x = 0$) of the wave tank was driven by 88 repetitions of a pseudo-random signal $s_i(t)$. The pseudo-random signal had the form $s_i(t) = F^{-1}[\bar{s}_i(\omega)]$, where $\bar{s}_i(\omega)$ was equal to $3\pi\omega \exp(-\omega/3\pi) \exp(i\vartheta_i)$ for $\omega < 3\pi$, and $\omega^2 \exp(-(\omega/3\pi)^2) \exp(i\vartheta_i)$ for $\omega \geq 3\pi$. The phase ϑ_i was a random variable, uniformly distributed between 0 and 2π . In each of the 88 repetitions of $s_i(t)$, an independent ϑ_i was used. Each realization of $s_i(t)$ had a duration of 204.8 s. Note that $|\bar{s}_i(\omega)|^2$ is equal to $|\bar{d}(\omega)|^2$ that was used in the closed system simulations described in the previous section, but, unlike $d(t)$, $s_i(t)$ includes a random phase. That was done to generate a random wave field in the wave tank; in the simulations, the random variables c_l and d_l insured that phases were random. Time histories of the surface elevation, $\eta(x_A, t)$ and $\eta(x_B, t)$, sampled at 547 Hz, were measured using a laser-based system over the 5-hour duration of the experiment. For analysis purposes, the raw data were resampled at 50 Hz. The data were block-processed in overlapping 204.8 s windows, with 50% overlap. Each 204.8 s block of data was treated as an independent realization of the random wave field; coherent averaging of estimates of $C_{AB}(t)$ over all realizations of the wave field was performed, as in the simulations described above. The measured spectra $\bar{\eta}(x_A, \omega)$ and $\bar{\eta}(x_B, \omega)$ were nearly identical (as predicted by Eq. (30) because in our experiment $L_x - x_B = x_A$), but those spectra were slightly different than the

corresponding spectra in the simulations just described. This difference can be
 360 attributed to the input-voltage-to-surface-displacement transfer function. Be-
 cause that transfer function is poorly known, we used an empirical approach
 to estimate $D(t)$ in Eq. (1) or (27). $\bar{D}(\omega) = \bar{d}(\omega)\bar{d}^*(\omega)$ is independent of the
 phase information in $\bar{d}(\omega)$, so we may, without loss of generality, assume that
 $\bar{d}(\omega)$ is real. Using a simple curve-fitting procedure, $\bar{d}(\omega)$ was chosen so that
 365 the theoretical spectrum $\sum_l \bar{d}(\omega_l) |\cos(l\pi x_A/L_x)|$ matches the measured aver-
 age spectrum $(|\bar{\eta}(x_A, \omega)| + |\bar{\eta}(x_B, \omega)|)/2$; the resulting fit is shown in Fig. 4.
 With this $\bar{d}(\omega)$, $D(t) * [G_\eta(x_B|x_A, t) + G_\eta(x_A|x_B, -t)]$ was computed and com-
 pared to the measurement-based estimate of the average $C_{AB}(t)$. To slightly
 improve signal-to-noise, $C_{AB}(t)$ and $C_{AB}(-t)$ were averaged and compared to
 370 $D(t) * G_\eta(x_B|x_A, t)$ in the figure. The result is shown in Fig. 5; agreement is
 seen to be good, consistent with Eq. (27) or (1).

5. Ocean measurements

A natural extension of the analysis of simulations and wave tank measure-
 ments described in the previous two sections is the analysis of ocean waves. The
 375 minimum requirement that a candidate data set must satisfy is that the data
 must include two or more sets of concurrent wave measurements made in prox-
 imity to each other. Two data sets that satisfied this criterion were identified
 and analyzed. Both resulted in very disappointing results. Only one of those
 analysis efforts is described here. (The data set not described here was inferior
 380 in many respects to the one whose analysis is described here.)

The data set analyzed here was collected as part of the Shallow Water 2006
 (SW06) experiment [44], which was a large multidisciplinary oceanographic field
 project conducted on the New Jersey continental shelf between July and Septem-
 ber 2006. The surface wave data analyzed here was collected on two moored spar
 385 buoys [45] at locations (39.0739° N, 73.1641° W) and (39.0192° N, 73.0536° W).
 These were named ‘Romeo’ and ‘Yankee’, respectively. The ocean depth at
 both measurement locations was 80 m, and the separation between measure-

ment buoys was approximately 11 km. Although approximately 10 days of data was collected, beginning on 2 August, only one 11 hour block of data is analyzed here. That 11 hour block was chosen because that was the longest available continuous block of data corresponding to very low wind conditions; during that 11 hour window the average wind speed was 1.9 m/s and the maximum wind speed was 3.0 m/s. The wave data was sampled at 20 Hz, and decimated to 10 Hz for the analysis shown here. The data at both buoys included some small gaps (see Fig. 6) that were filled with zeros. The wave spectra measured at both locations had energy concentrated in the frequency band $0.3 \text{ Hz} < f < 0.5 \text{ Hz}$, with a peak at 0.35 Hz. This frequency band corresponds to wavelengths in the range $17 \text{ m} > \lambda > 6 \text{ m}$ and group speeds in the range $2.6 \text{ m/s} > c_g > 1.5 \text{ m/s}$. During the measurement period, the wave field was highly directional, with wave energy propagating shoreward, from Yankee to Romeo. Waves in the excited frequency band take between approximately 70 min and 120 min to travel from Yankee to Romeo.

Before presenting results, we note that directional wave spectra at the two measurement locations have implications for the anticipated structure of the cross-correlation functions that we compute. If the cross correlation function is computed as $C_{AB}(t) = \langle \int_0^T d\tau \eta_A(\tau) \eta_B(\tau + t) \rangle$, and wave energy is propagating predominantly from A to B , then deterministic structure is expected to be observed in $C_{AB}(t)$ for $t > 0$, but not for $t < 0$. That is the case here with A and B corresponding to Yankee and Romeo, respectively. Thus we expect, after coherent stacking is performed, to see a cross-correlation function in which almost all energy is at positive lag.

These comments lead us to expect deterministic structure in the stacked correlation function at positive lag between approximately 70 min and 120 min, with maximum absolute amplitude at approximately 82 min. Furthermore, the correlation function is expected to consist of a smooth frequency-modulated sweep with frequency increasing with increasing lag – like a one-sided version of Fig. 2, but with many more wiggles. Realizations of the correlation function were computed by sliding 210 min subsequences of the Romeo sea surface

elevation time series relative to the Yankee time series. Adjacent subsequences
420 were staggered by 30 min. This gave 14 realizations (which are not fully independent of each other) of the cross-correlation function that were subsequently coherently averaged. Results are shown in Fig. 7. Unfortunately, there is no hint that the anticipated structure – or, indeed, any deterministic structure – is emerging in the average cross-correlation function shown in Fig. 7. (Note
425 that using a subsequence length of 210 min leads to a maximum lag of 105 min, which is not quite long enough to capture the entire anticipated deterministic signal. We have also processed the data using a subsequence length of 264 min, corresponding to a maximum lag of 132 min. Those results are equally disappointing. The advantage of using the shorter subsequence length is that a few
430 more realizations of the correlation function are produced.) Potential causes for the disappointing results described here are discussed below.

6. Summary and discussion

In this paper we have: 1) presented theoretical results that underlie random water wave interferometry; 2) tested the theoretical predictions using simulations in both open and closed systems; 3) tested the theoretical predictions using
435 wave tank measurements; and 4) tested the theoretical predictions using ocean wave measurements.

Simulations were shown to be in very good agreement with theoretical predictions, but a caveat should be noted: Green’s functions computed using Eq. (28)
440 are causal only in the nondispersive limit. Note, however, that non-causal precursors in the Green’s functions used to construct Figs. 1 and 2 are too small to be seen. In those figures non-causal precursors are negligibly small both because the precursors have negligible amplitude compared to the statistical uncertainty associated with our estimates of $C_{AB}(t)$, and because the temporal extent of the precursors is negligible compared to the temporal smearing associated with
445 convolution with $D(t)$. In spite of the smallness of non-causal precursors in the Green’s functions used to construct Figs. 1 and 2, it should be stated clearly

that non-causal Green’s functions are not physical, and that this shortcoming should be corrected. It is noteworthy that the Green’s functions on which our
450 closed domain simulations are based are causal and correspond to intermediate depth, i.e., dispersive, conditions. (Those simulations are shown in Fig. 3 and are also in very good agreement with theoretical predictions.) Thus dispersion need not lead to non-causal Green’s functions. It has previously been recognized [46, 47] that, in general, the construction of causal water wave Green’s
455 functions requires special care. That topic is worthy of a renewed investigation in the context of water wave interferometry. A possibility is to assume weak dissipation and make use of a Kramers-Kronig relationship when constructing Green’s functions.

More significant than our simulations, wavetank measurements were shown
460 to be in good agreement (recall Fig. 5) with the fundamental result of water wave interferometry, Eq. (1).

The ocean measurements were not in agreement with theoretical predictions. As noted in the previous section, the correlation functions based on ocean measurements were expected to have a different character than the correlation func-
465 tions based on simulations or wave tank measurements. Even with this caveat, our attempt to produce a band-limited estimate of a water wave Green’s function using the SW06 data must be regarded as a failure. Several likely factors that contributed to this failure will now be discussed.

An important factor that likely contributed to the failure of our SW06 data
470 analysis is a shortcoming of the data. The relative drift of the spar buoys that made the ocean wave measurements was probably sufficiently large over the 11 hour data window that phase coherence between instruments was lost. The spar buoys were moored to anchors at 80 m depth. The expected maximum horizontal drift of each buoy relative to its mean position is about 90 m (N.
475 Williams, personal communication); relative drift between the two instruments is expected to be much less, but that quantity is more difficult to estimate. Temporal changes in buoy positions during the measurement period were likely due primarily to tidal currents. Note, in this regard, that the 11 hour data window

is comparable to the M2 tidal period, 12.42 hours. The measured wave spectra
480 had peaks near 0.35 Hz, corresponding to a wavelength of about 12 m. It is
probable that the relative position between the two buoy positions drifted by an
amount that is comparable to or larger than a wavelength over the data window,
leading to a loss of phase coherence between instruments at the peak frequency.
Loss of phase coherence leads to a catastrophic loss of information contained
485 in the correlation function. It is noteworthy that the successful demonstrations
of wave interferometry using ocean infragravity waves reported in [41, 42, 38]
made use of fixed (bottom-mounted) pressure measurements. We suspect that a
measurement-based demonstration of the feasibility of random water wave inter-
ferometry for deep-water ($kh \gg 1$) waves will be possible only if measurements
490 are made from fixed platforms, e.g., nearby oil drilling platforms.

Wind-induced modifications to the wave field between the Yankee and Romeo
moorings may have contributed to the failure of our ocean wave correlation func-
tions to yield the anticipated structure. The effect of wind can be quantified
using an empirical wind wave growth rate formula [48]. That formula predicts
495 that the reciprocal of the e-folding time for wave energy growth is proportional
to $\omega(U/c)^2 \cos \theta$ where U is wind speed, c is the wave phase speed, ω is the wave
frequency, and θ is the angle between the wave propagation direction and the
wind direction. Setting $\cos \theta = 1/2$ and the e-folding time equal to 11 hours,
that formula predicts that $U = 1.9$ m/s is the maximum allowable wind speed
500 to keep wave energy enhancement less than a factor of e . Coincidentally, this
was the average wind speed during the analysis window, suggesting that our
assumption of negligible wind influence on the wave field was only marginally
satisfied. It is quite possible that wind effects contributed to a loss of coherence
of the wave fields at the Yankee and Romeo moorings.

505 Shortness of the surface elevation records used in our analysis likely also
contributed to the failure of our ocean wave correlation functions to yield the
anticipated structure. We have doubled the duration of the data analysis time
window, from 11 to 22 hours, and repeated the analysis; like Fig. 7, the result-
ing stacked correlation function had none of the anticipated features that we

510 have described. This might be explained by the higher average winds, 4.0 m/s,
 during the second 11 hours of the 22 hour window; the argument just presented
 suggests that wind effects over the second 11 hour block of data had a non-
 negligible influence the wave field. We return now to the result shown in Fig.
 7, corresponding to processing the first 11 hours only. In [5] and [13] it is esti-
 515 mated that convergence of the correlation function requires that the product of
 the minimum integration time T and the coherent bandwidth exceeds a constant
 times $(kr)^{d-1}$ where $d = 2$ for propagation in two space dimensions and $d = 3$
 for propagation in three space dimensions. It is typical in noise interferometry
 applications, including the work reported here, that the coherent bandwidth is
 520 approximately equal to the center frequency of the band of coherent energy. In
 the simulations shown in Figs. 1 and 2 the minimum integration time to obtain
 a rough estimate of the average correlation function was about 5 hours. Using
 that estimate and the differences in center frequency and kr , the formula just
 presented with $d = 2$ predicts that for the SW06 data the minimum integration
 525 time required to yield a usable estimate of the average correlation function is
 about 30 days. This strongly suggests that the 11 hours of data that we used was
 far too short. But the validity of the assumption that $d = 2$ might be questioned
 because of the strong directionality of the wave field, with waves propagating
 from Yankee to Romeo. For $d = 1$ the formula just presented predicts that T
 530 is independent of kr . If the $d = 1$ limit of that formula is applicable, and if
 the SW06 measurements are described by $d = 1$, the minimum integration time
 for the SW06 data can be estimated from the minimum integration time in our
 1-d wavetank experiment – about 0.5 hour. That time interval corresponds to
 approximately 15 independent realizations of the correlation function. Assum-
 535 ing that 15 independent realizations of the correlation function are required for
 $d = 1$, independent of kr , leads to the conclusion that, if $d = 1$ for the SW06
 data, about 20 hours of data is the minimum required for convergence. We ex-
 pect that minimum required integration time for windless SW06-like conditions
 lies somewhere between the $d = 1$ and $d = 2$ estimates just given, leading to the
 540 conclusion that shortness of the SW06 data records used in our analysis was a

significant factor contributing to the failure of our analysis.

A entirely different – and potentially more interesting – possible explanation for the failure of ocean-measurement-based estimates of correlation functions to reveal the predicted structure is inadequacy of the underlying theory. Specifically, the underlying theory that we have presented is based on linear theory, whose validity requires that ka , where a is wave amplitude, is small. It is possible that finite amplitude effects that are neglected in the theory led to sufficiently large wave field changes between measurement locations that the predicted linear theory information contained in the cross-correlations was lost. In this regard, it is noteworthy that the most striking manifestation of nonlinear effects in nonbreaking ocean waves is the phase locking of short waves to longer waves, causing the short waves to propagate faster than is predicted by linear theory. (This accounts for a basic feature of ocean waves – wave crests are relatively sharp, while wave troughs are flat.) But this effect should not prevent one from obtaining stable estimates of correlation functions. Thus, we don't expect that this was the dominant cause of the failure of our ocean wave analysis. We note also that ka was approximately 0.023 in our wavetank experiment, and 0.007 in the SW06 data that we used; this also suggests that breakdown of linear theory was not the principal cause of the failure of our SW06 analysis. More generally, the breakdown of random water wave interferometry due to nonlinear effects can – and should – be systematically investigated in a wave tank by performing several sets of experiments of the type described in section 4, with varying ka .

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Figure captions

Figure 1. Solid curve: cross-correlation function $C_{AB}(t)$ of $\eta(\mathbf{x}_A, t)$ and $\eta(\mathbf{x}_B, t)$ in an open weakly dispersive water wave system (assuming $\omega^2 = ghk^2(1 - (kh)^2/3)$ with $h = 1$ m), computed by coherently averaging approximately 5000
705 realizations of the correlation function based on 160 s blocks of simulated surface elevation. Dashed curve: $D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t)]$, where $G_\eta(\mathbf{x}_B|\mathbf{x}_A, t)$ was computed using Eq. (28). Each curve is normalized to have maximum absolute amplitude equal to 1.

710 Figure 2. Solid curve: cross-correlation function $C_{AB}(t)$ of $\eta(\mathbf{x}_A, t)$ and $\eta(\mathbf{x}_B, t)$ in an open dispersive water wave system (assuming $\omega^2 = gk$), computed by coherently averaging approximately 5000 realizations of the correlation function based on 160 s blocks of simulated surface elevation. Dashed curve: $D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t)]$, where $G_\eta(\mathbf{x}_B|\mathbf{x}_A, t)$ was computed using Eq.
715 (28). Each curve is normalized to have maximum absolute amplitude equal to 1.

Figure 3. Solid blue curve: cross-correlation function $C_{AB}(t)$ for $t > 0$ of $\eta(x_A, t)$ and $\eta(x_B, t)$, computed by coherently averaging 240 realizations of the correlation function based on 240 s blocks of simulated surface elevation in a
720 wave tank. Dashed red curve: $D(t) * G_\eta(x_B|x_A, t)$, where $G_\eta(x_B|x_A, t)$ was computed using Eq. (29). Each curve is normalized to have maximum absolute amplitude equal to 1.

Figure 4. Blue curve: average measured spectrum $(|\bar{\eta}(x_A, \omega)| + |\bar{\eta}(x_B, \omega)|)/2$ in a wave tank. Red vertical lines: theoretical spectrum $\sum_l \bar{d}(\omega_l) |\cos(k_l x_A)|$,
725 where $\omega_l^2 = gk_l \tanh k_l h$ and $k_l = l\pi/L_x$, based on an empirical smooth curve $\bar{d}(\omega)$. Each spectrum is normalized to have a peak value equal to 1.

Figure 5. Solid blue curve: average cross-correlation function $(C_{AB}(t) + C_{AB}(-t))/2$ of $\eta(x_A, t)$ and $\eta(x_B, t)$ in a closed system, computed by coherently averaging
730 240 realizations of the correlation function based on 240 s blocks of measured surface elevation in a wave tank. Dashed red curve: $D(t) * G_\eta(x_B|x_A, t)$, where

$G_\eta(x_B|x_A, t)$ was computed using Eq. (29). The two curves are normalized so that both have the same integrated energy between 3 and 60 s.

Figure 6. Concurrent eleven hour segments of time series of sea surface elevation measured at buoys Romeo and Yankee.

735 Figure 7. Cross-correlation function $C_{AB}(t)$ of $\eta(\mathbf{x}_A, t)$ and $\eta(\mathbf{x}_B, t)$ in an open system, computed by coherently averaging 16 realizations of the correlation function based on 210 min blocks of measured surface elevation in the ocean. $C_{AB}(t)$ is normalized to have maximum absolute amplitude equal to 1.

Figure 1: Solid curve: cross-correlation function $C_{AB}(t)$ of $\eta(\mathbf{x}_A, t)$ and $\eta(\mathbf{x}_B, t)$ in an open weakly dispersive water wave system (assuming $\omega^2 = ghk^2(1 - (kh)^2/3)$ with $h = 1$ m), computed by coherently averaging approximately 5000 realizations of the correlation function based on 160 s blocks of simulated surface elevation. Dashed curve: $D(t) * [G_\eta(\mathbf{x}_B|\mathbf{x}_A, t) + G_\eta(\mathbf{x}_A|\mathbf{x}_B, -t)]$, where $G_\eta(\mathbf{x}_B|\mathbf{x}_A, t)$ was computed using Eq. (28). Each curve is normalized to have maximum absolute amplitude equal to 1.

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