Novel Sign Subband Adaptive Filter Algorithms with Individual Weighting Factors

Yi Yu, and Haiquan Zhao

Abstract : This paper presents a new sign subband adaptive filter (SSAF) algorithm with an individual-weighting-factor (IWF) for each subband, instead of a common weighting factor in the original SSAF algorithm, called the IWF-SSAF. Each individual weighting factor only depends on the corresponding subband input signal power. Compared with the SSAF algorithm, the proposed approach fully utilizes the inherent decorrelating property of subband adaptive filter for colored inputs, leading to a better convergence performance. After that, to further enhance the performance of the IWF-SSAF in sparse system, an improved proportionate IWF-SSAF (IWF-IPSSAF) algorithm is proposed. The proposed algorithms not only inherit the good robustness of sign algorithm against impulsive interferences, but also obtain a significant improvement in the performance as compared to their counterparts (i.e., SSAF and IPSSAF), in terms of the convergence rate and tracking capability. Besides, the IWF-IPSSAF algorithm has faster convergence rate than the IWF-SSAF algorithm for sparse impulse responses. Finally, the performances of two proposed algorithms are demonstrated in the system identification and the acoustic echo cancellation with double-talk.

Keywords: Sign subband adaptive filter, individual weighting factor, proportionate sign subband adaptive filter, impulsive interference, system identification, acoustic echo cancellation.

1 Introduction

Adaptive filtering techniques have been utilized in a wide variety of applications such as system identification, active noise control, channel equalization, and acoustic echo cancellation (AEC) [1]. The normalized least mean square (NLMS) is a very well-known and widely used algorithm because of its low complexity and robustness, but it converges slowly for colored input signals. In order to address this problem, one of the methods is to use the affine projection algorithm (APA) [1] as well as its modified versions [2, 3]. However, they require large computational burden. Another wonderful way to speed up convergence is the multiband-structured subband adaptive filter, because it can whiten the colored input signals through analysis filter bank [4]. That is to say, it possesses the decorrelating property for colored inputs in the subband domain. Benefiting from the principle of

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minimum disturbance, Lee and Gan proposed the normalized subband adaptive filter (NSAF) algorithm by using subband input signals and subband output error signals to update the tap-weight vector [5]. In fact, it generalizes the NLMS algorithm along the subband axis, namely, it reduces to the NLMS algorithm when the number of subbands is equal to one. Apart from fast convergence for colored input signals, the computational complexity of the NSAF algorithm is comparable to that of the NLMS algorithm for a high-order adaptive filter, and is far less than that of the APA. Afterwards, the authors did analyze the inherent decorrelating and least perturbation properties of the NSAF algorithm [6]. Following their works, the statistical models of the NSAF algorithm (including the first and the second moments of the tap-weight error vector) were discussed in [7]. Furthermore, many modified versions have been developed to further improve the performance of the NSAF algorithm [8-12].

Unfortunately, the aforementioned algorithms suffer from large steady-state misalignment or even divergence when the impulsive interference is present in the environment, because they are derived based on the optimization of L_2 -norm. In echo cancellation scenario with double-talk, moreover, the near-end speech signal can also be viewed as the impulsive interference. Although the sign algorithm (SA) [13] based on the L_1 -norm optimization can efficiently suppress the impulsive interferences, its convergence rate is very slow for colored input signals. Inspired by the presented the affine projection sign algorithm (APSA) [14], Ni *et al.* proposed a sign subband adaptive filter (SSAF) algorithm by extending the SA to the multiband-structured subband adaptive filter, and then proposed a variable regularization parameter SSAF (VRP-SSAF) algorithm to further reduce the steady-state error [15]. Recently, many works from the following aspects have been devoted to improve the performance of the SSAF algorithm in terms of the convergence rate and the steady-state error. On the one hand, to overcome a trade-off problem between fast convergence and low steady-state error caused by the fixed step-size, several variable step size SSAF algorithms were proposed in [16-18]. On the other hand, in the light of the advantage of the APA for colored input signals, the affine projection SSAF (AP-SSAF) algorithm [19] was developed which provides faster convergence rate. Nevertheless, in many practical applications such as AEC, the impulse response of the echo path that is identified by an adaptive filter is usually sparse, i.e., only a small number of coefficients of the impulse response have large magnitude (called *active coefficients*) while the rest coefficients (called *inactive coefficients*) are close or equal to zero [2, 3, 20, 21]. Aiming to this class of impulse responses, therefore, Ni and his co-authors proposed a class of proportionate SSAF (PSSAF) algorithms such as the improved proportionate PSSAF (IPSSAF) and the PSSAF with individual-activation-factors (IAF-PSSAF) [22]. In the SSAF, however, there is a key problem that all subbands are interconnected by a common weighting factor. This directly impacts its convergence performance, and some modified SSAF algorithms such as the IPSSAF also have the same problem.

In this paper, we propose a new SSAF algorithm, called the individual-weighting-factors SSAF (IWF-SSAF), by re-examining the way to compute the weighting factor in the SSAF algorithm, aiming to improve the filter's performance. In addition, the mean-square transient behavior of the IWF-SSAF algorithm is analyzed. In contrast to the original SSAF algorithm, the new

algorithm has the following characteristics:

- 1) An individual weighting factor is used for each subband;
- 2) Each weighting factor is computed only using the corresponding subband input signal power. Moreover, the subband with larger input signal power receives lower weight, vice versa.

As a result, the proposed algorithm is of significant improvement in the convergence performance. For sparse systems, we also develop an improved proportionate IWF-SSAF (IWF-IPSSAF) algorithm by incorporating the proportionate idea into the IWF-SSAF, which speeds up the convergence rate.

The remainder of this paper is organized as follows. Section II briefly reviews the original SSAF algorithm. In Section III, we firstly discuss the problem of the SSAF algorithm, and then develop the IWF-SSAF algorithm and its mean-square analysis. In Section IV, the proportionate idea is introduced into the IWF-SSAF algorithm. In Section V, computer simulations in the context of the system identification and AEC verify the excellent performance of the proposed algorithms. Finally, Section VI gives our conclusions.

2 Review SSAF Algorithm

Let us consider that the observed output data $d(n)$ from the unknown system is given by

$$d(n) = \mathbf{u}^T(n) \mathbf{w}_o + J(n), \quad (1)$$

where $\mathbf{w}_o$ denotes an unknown M -dimensional vector that we want to estimate with an adaptive filter, the superscript T indicates transpose operator, $\mathbf{u}(n) = [u(n), u(n-1), \dots, u(n-M+1)]^T$ is the input signal vector, and $J(n)$ is the additive noise. In many practical situations, the additive noise consists of the background noise $v(n)$ and the impulsive interference $h(n)$, i.e., $J(n) = v(n) + h(n)$. Fig. 1 shows the multiband-structure of subband adaptive filter [4] which has been used to design the NSAF and SSAF algorithms, where N represents the number of subbands. The observed output data $d(n)$ and the input signal $u(n)$ are partitioned into N subband signals $d_i(n)$ and $u_i(n)$ through analysis filter bank $\{H_i(z), i \in [0, N-1]\}$, respectively. Then, subband signals $y_i(n)$ and $d_i(n)$ for $i \in [0, N-1]$ are critically decimated to yield $y_{i,D}(k)$ and $d_{i,D}(k)$, respectively, where n and k indicate the original sequences and the decimated sequences. Thereby, the subband output error signals can be calculated by

$$e_{i,D}(k) = d_{i,D}(k) - \mathbf{u}_i^T(k) \mathbf{w}(k) \quad \text{for } i \in [0, N-1] \quad (2)$$

where $\mathbf{w}(k) = [w_1(k), w_2(k), \dots, w_M(k)]^T$ denotes the tap-weight vector of the adaptive filter, $d_{i,D}(k) = d_i(kN)$ and $\mathbf{u}_i(k) = [u_i(kN), u_i(kN-1), \dots, u_i(kN-M+1)]^T$.

Rewriting (2) in matrix form yields

$$\mathbf{e}_D(k) = \mathbf{d}_D(k) - \mathbf{U}^T(k)\mathbf{w}(k) \quad (3)$$

$$@ \hat{\mathbf{e}}_{0,D}(k), e_{1,D}(k), \dots, e_{N-1,D}(k) \mathbf{u}^T$$

where $\mathbf{d}_D(k) = [\hat{d}_{0,D}(k), d_{1,D}(k), \dots, d_{N-1,D}(k)]^T$ and $\mathbf{U}(k) = [\mathbf{u}_0(k), \mathbf{u}_1(k), \dots, \mathbf{u}_{N-1}(k)]$.

Fig. 1 Multiband structure of subband adaptive filter.

As reported in [15], the tap-weight vector of the original SSAF algorithm is updated as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \frac{\mathbf{U}(k)\text{sign}(\mathbf{e}_D(k))}{\sqrt{\sum_{i=0}^{N-1} \mathbf{u}_i^T(k)\mathbf{u}_i(k) + \varepsilon}} \quad (4)$$

where μ is the step size, ε is a small positive number to avoid division by zero, and $\text{sign}(\cdot)$ indicates the sign function and $\text{sign}(\mathbf{e}_D(k))$ takes the sign operation of each element of $\mathbf{e}_D(k)$.

3 Proposed IWF-SSAF algorithm

3.1 Problems of SSAF

In this part, the problems of the SSAF algorithm are provided by analyzing the contribution of each subband on the overall convergence performance of the algorithm, aiming to get some insights for devising an improved SSAF version. According to weighting criterion's viewpoint [4-6], the SSAF algorithm can also be derived by minimizing a weighting cost function which is defined as

$$J(k) = \sum_{i=0}^{N-1} l_i |e_{i,D}(k)| \quad (5)$$

where l_i denotes the weighting factor for the i th subband.

Using the gradient descent theory, we have

$$\begin{aligned} \mathbf{w}(k+1) &= \mathbf{w}(k) - \mu \frac{\partial J(k)}{\partial \mathbf{w}(k)} \\ &= \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \lambda_i \mathbf{u}_i(k) \text{sign}(e_{i,D}(k)) \end{aligned} \quad (6)$$

Apparently, (4) is a special case of (6). In other words, both (4) and (6) are equivalent when

$$l_1 = l_2 = \dots = l_{N-1} = \frac{1}{\sqrt{\sum_{i=0}^{N-1} \mathbf{u}_i^T(k) \mathbf{u}_i(k) + e}} \quad (7)$$

Remark 1: From (5)-(7), it is easy to find the disadvantage of the SSAF algorithm. For two arbitrary analysis filters $H_i(z)$ and $H_l(z)$, where $i, l \in [0, N-1]$, used for partitioning the input signal $u(n)$ and the observed output data $d(n)$, their transfer functions are different [5], so that signals of each subband (i.e., $\mathbf{u}_i(k)$ and $e_{i,D}(k)$) used to update the tap-weight vector characterize their unique information. In other words, each of the subbands should provide an irreplaceable contribution on the overall convergence performance of the algorithm. Along this line of thought, in (5) and (6), each subband should receive an individual weighting factor l_i to obtain fast convergence. However, the SSAF algorithm uses the same weighting factor for all subbands, see (7). From another point of view, in subband adaptive filtering, the colored input signal is divided into multiple approximate white subband signals which are almost mutually orthogonal [5]. This is the reason why the subband adaptive filter can speed up convergence for the colored input signals. For the SSAF algorithm, nevertheless, it can be seen from (7) that the weighting factor assigned to each subband is controlled by other subband input signals, thus damaging the inherent decorrelating property of subband adaptive filter for the colored input signals. In summary, the SSAF algorithm does not make full use of the advantages of subband adaptive filtering.

3.2 IWF-SSAF algorithm

Considering the previous discussion of the SSAF algorithm, in this section, a new strategy for computing the weighting factor is proposed to improve the convergence rate of the SSAF algorithm. Namely, each subband is assigned an individual weighting factor l_i . In the meanwhile, the band-dependent weighting factor is only associated with the corresponding subband input signal. With these two principles, therefore, (7) can be changed as

$$l_i = \frac{1}{\sqrt{\mathbf{u}_i^T(k) \mathbf{u}_i(k) + e}} \quad \text{for } i \in [0, N-1] \quad (8)$$

Substituting (8) into (6), we have

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \frac{\mathbf{u}_i(k) \text{sign}(e_{i,D}(k))}{\sqrt{\mathbf{u}_i^T(k) \mathbf{u}_i(k) + \varepsilon}}. \quad (9)$$

The formula (9) expresses the update of the tap-weight vector for the proposed IWF-SSAF algorithm, which is summarized in Table 1.

Remark 2: As can be seen from (8), each subband in the proposed IWF-SSAF algorithm has an individual weighting factor l_i , such that the tap-weight vector update of the IWF-SSAF algorithm has similar computation procedure to that of the NSAF algorithm, i.e., each subband input signal is normalized by its own variance. In other words, the subband decomposition and the normalization features in (9) fully reveal the decorrelating property of the IWF-SSAF algorithm. Moreover, the weighting factors given by (8) are inversely proportional to the corresponding subband input signal power. This signifies that in the IWF-SSAF, the subband with smaller (larger) input signal power obtains larger (smaller) weight, instead of the same weight for all subbands in the SSAF. This result can also be observed from Fig. 2 that shows the difference of weighting factors between the SSAF and IWF-SSAF algorithms. Consequently, the proposed IWF-SSAF algorithm can obtain a significant improvement in the convergence rate in comparison with the SSAF algorithm.

Fig. 2 The weighting factors used in the SSAF and IWF-SSAF, with number of subbands $N = 4$ and AR(1) input.

Table 1. Summary of the proposed IWF-SSAF and IWF-IPSSAF algorithms

Initialization	$\mathbf{w}_{M-1}(0) = \mathbf{0}_{M-1}$
Parameters	m, e, z
General formulations	$\mathbf{u}_i(k) = [u_i(kN), u_i(kN-1), \dots, u_i(kN-M+1)]^T$ $e_{i,D}(k) = d_{i,D}(k) - \mathbf{u}_i^T(k) \mathbf{w}(k)$ $\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \frac{\mathbf{G}(k) \mathbf{u}_i(k) \text{sign}(e_{i,D}(k))}{\sqrt{[\mathbf{G}(k) \mathbf{u}_i(k)]^T [\mathbf{G}(k) \mathbf{u}_i(k)] + \varepsilon}}$

IWF-SSAF	$\mathbf{G}(k) = \mathbf{I}_{M' \times M}$
IWF-IPSSAF	$g_m(k) = \frac{1-a}{2M} + (1+a) \frac{ w_m(k) }{2\ \mathbf{w}(k)\ _1 + z}$ $\mathbf{G}(k) = \text{diag}\{g_1(k), g_2(k), \dots, g_M(k)\}$

3.3 Mean-square Behavior of the IWF-SSAF

In this section, we provide a theoretical model to predict the mean-square behavior of the IWF-SSAF algorithm. In order to make the analysis tractable, we use the following assumptions:

- 1) The background noise $v(n)$ is Gaussian with zero-mean and variance s_v^2 . The impulsive interference is usually modeled as $h(n) = z(n)A(n)$ [14-18, 27], where $z(n)$ is a Bernoulli process with the probability density function described by $p\{z(n)=1\} = P_r$ and $p\{z(n)=0\} = 1 - P_r$, and $A(n)$ is Gaussian with zero-mean and variance $s_A^2 = k s_v^2$, $k \geq 1$. Hence, the probability density function (pdf) of the additive noise $J(n)$ is also Gaussian with zero-mean and variance

$$s_J^2 = s_v^2 + P_r s_A^2 = (1 - P_r) s_v^2 + P_r (k + 1) s_v^2. \quad (10)$$

- 2) Assuming that the analysis filter bank is paraunitary [7, 27]. This assumption is to guarantee that the subband noises $J_i(k)$ for $i \in [0, N-1]$ (that are obtain by partitioning the additive noise $J(n)$) are also Gaussian with zero-mean and variance

$$s_{J_i}^2 = (1 - P_r) \frac{1}{N} s_v^2 + P_r \frac{1}{N} (k + 1) s_v^2. \quad (11)$$

- 3) The filter tap-weight vector $\mathbf{w}(k)$, the subband input vector $\mathbf{u}_i(k)$, and the subband noise $J_i(k)$ are statistically independent.

This is commonly used independent assumption for the analysis of adaptive filters, i.e., [1, 7, 8, 23, 27].

- 4) The i th subband input signal is approximately white. Thus, we have $E\{\mathbf{u}_i(k)\mathbf{u}_i^T(k)\} = \mathbf{I}_{M' \times M} s_{\mathbf{u}_i}^2(k)$ and $E\{\mathbf{u}_i^T(k)\mathbf{u}_i(k)\} = M s_{\mathbf{u}_i}^2(k)$ [23, 27], where $\mathbf{I}_{M' \times M}$ is the $M' \times M$ identity matrix.

- 5) When adaptive filter is long, i.e., $M \gg 1$, it can assume that the fluctuation of $\|\mathbf{u}_i(k)\|^2$ from one iteration to the next is small enough [1, 23, 27].

Defining the tap-weight error vector, $\boldsymbol{\theta}(k) = \mathbf{w}_o - \mathbf{w}(k)$ and substituting it into (9) neglecting e yields

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k) - \mu \sum_{i=0}^{N-1} \frac{\mathbf{u}_i(k) \text{sign}(e_{i,D}(k))}{\sqrt{\mathbf{u}_i^T(k)\mathbf{u}_i(k)}}. \quad (12)$$

By pre-multiplying (12) by its transpose and then taking the statistical expectation of both sides of the equation, we obtain

$$\text{MSD}(k+1) = \text{MSD}(k) - \frac{2\mu \sum_{i=0}^{N-1} E \left[\boldsymbol{\theta}^T(k) \mathbf{u}_i(k) \text{sign}(e_{i,D}(k)) \right]}{E \left[\sqrt{\mathbf{u}_i^T(k) \mathbf{u}_i(k)} \right]} + \mu^2 N \quad (13)$$

where $\text{MSD}(k) = E \left[\boldsymbol{\theta}^T(k) \boldsymbol{\theta}(k) \right]$ denotes the mean-square deviation (MSD) of the algorithm. For obtaining (13), we use the assumption 5) and an approximation $\mathbf{u}_i^T(k) \mathbf{u}_j(k) \gg 0, i \neq j$ which has been reported in [4, 5, 27].

Since the small fluctuation of $\|\mathbf{u}_i(k)\|^2$, we can get an approximation $E \left[\sqrt{\mathbf{u}_i^T(k) \mathbf{u}_i(k)} \right] \approx \sqrt{E \left[\mathbf{u}_i^T(k) \mathbf{u}_i(k) \right]}$ [27]. Then, (13) is changed as

$$\text{MSD}(k+1) = \text{MSD}(k) - \frac{2\mu \sum_{i=0}^{N-1} E \left[\boldsymbol{\theta}^T(k) \mathbf{u}_i(k) \text{sign}(e_{i,D}(k)) \right]}{\sqrt{M} \sigma_{\mathbf{u}_i}(k)} + \mu^2 N \quad (14)$$

Using the assumptions above and Price's theorem [13], we can obtain the following equation (proved in Appendix)

$$E \left[\boldsymbol{\theta}^T(k) \mathbf{u}_i(k) \text{sign}(e_{i,D}(k)) \right] = \sqrt{\frac{2}{\pi}} \sigma_{\mathbf{u}_i}(k) \left\{ \frac{1 - P_r}{\sqrt{\frac{1}{N} \sigma_v^2 + \sigma_{\mathbf{u}_i}^2(k) \text{MSD}(k)}} + \frac{P_r}{\sqrt{\frac{1}{N} (\kappa + 1) \sigma_v^2 + \sigma_{\mathbf{u}_i}^2(k) \text{MSD}(k)}} \right\} \text{MSD}(k) \quad (15)$$

Substituting (15) into (14), the MSD behavior of the IWF-SSAF algorithm is given by

$$\text{MSD}(k+1) = \text{MSD}(k) - \frac{2\mu}{\sqrt{M}} \sqrt{\frac{2}{\pi}} \sigma_{\mathbf{u}_i}(k) \left\{ \frac{1 - P_r}{\sqrt{\frac{1}{N} \sigma_v^2 + \sigma_{\mathbf{u}_i}^2(k) \text{MSD}(k)}} + \frac{P_r}{\sqrt{\frac{1}{N} (\kappa + 1) \sigma_v^2 + \sigma_{\mathbf{u}_i}^2(k) \text{MSD}(k)}} \right\} \text{MSD}(k) + \mu^2 N \quad (16)$$

Generally, $\sigma_{\mathbf{u}_i}^2(k)$ in (16) can be estimated by

$$\sigma_{\mathbf{u}_i}^2(k) = b \sigma_{\mathbf{u}_i}^2(k) + (1 - b) u_i^2(kN) \quad (17)$$

where b is the forgetting factor which is determined by $b = 1 - N/gM$ with a typical range of $1 \leq g \leq 6$ [8, 11].

Assuming that the algorithm has gone into the steady-state, i.e., $\text{MSD}(k+1) = \text{MSD}(k)$, $k \rightarrow \infty$. From (16), therefore, we obtain an equation on the steady-state MSD, i.e., $\text{MSD}(\infty)$, given in (18).

$$\frac{2\mu}{\sqrt{M}} \sqrt{\frac{2}{\pi}} \sigma_{\mathbf{u}_i}(\infty) \left\{ \frac{1 - P_r}{\sqrt{\frac{1}{N} \sigma_v^2 + \sigma_{\mathbf{u}_i}^2(\infty) \text{MSD}(\infty)}} + \frac{P_r}{\sqrt{\frac{1}{N} (\kappa + 1) \sigma_v^2 + \sigma_{\mathbf{u}_i}^2(\infty) \text{MSD}(\infty)}} \right\} \text{MSD}(\infty) = \mu N \quad (18)$$

Evidently, it is difficult to find a closed-form solution of $\text{MSD}(\mu)$ from (18), since it is a transcendental equation. Hence, we only can obtain its numerical solution by using the following ways. 1) Since both (16) and (18) are equivalent during the steady-state, we can obtain $\text{MSD}(\mu)$ from (16) in a recursive way; and 2) based on some numerical computation methods such as the Newton method and the Chebyshev iteration method [28], we can directly solve (18) for $\text{MSD}(\mu)$ when other quantities are given.

Interestingly, we can use an assumption to solve (18), i.e., $s_{u_i}^2(\mu)\text{MSD}(\mu) = s_v^2/N$ for a very small step size [8]. In this case, the steady-state MSD of the IWF-SSAF algorithm is approximated as

$$\text{MSD}(\mu) = \frac{mN\sqrt{Mp/8}}{\sum_{i=0}^{N-1} s_{u_i}^2(\mu)} \frac{1-P_r}{\sqrt{\frac{1}{N}s_v^2}} + \frac{P_r}{\sqrt{\frac{1}{N}(k+1)s_v^2}} \quad (19)$$

The relation (19) shows that the steady-state error of the IWF-SSAF algorithm will increase as the step size increases.

4 Proportionate version of the IWF-SSAF

4.1 IWF-IPSSAF algorithm

In this section, in order to deal with the slow convergence of the IWF-SSAF algorithm in sparse system environments, a unified proportionate frame is proposed by introducing a diagonal matrix into (9), referred as the IWF-PSSAF. Inspired by the work in [22], the tap-weight vector in the IWF-PSSAF is updated in an analogical fashion as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \frac{\mathbf{G}(k)\mathbf{u}_i(k)\text{sign}(e_{i,D}(k))}{\sqrt{[\mathbf{G}(k)\mathbf{u}_i(k)]^T [\mathbf{G}(k)\mathbf{u}_i(k)] + \varepsilon}} \quad (20)$$

where the role of the $M \times M$ diagonal matrix $\mathbf{G}(k) = \text{diag}\{g_1(k), g_2(k), \dots, g_M(k)\}$, called the *proportionate matrix*, is to assign an individual step-size governing by $mg_m(k)$ to each filter tap-weight $w_m(k)$, thereby improving the global convergence rate for sparse systems. Obviously, devising different strategies to calculate the diagonal elements $g_m(k)$ for $m \in [1, M]$ will result in different types of the IWF-PSSAF algorithm. At present, several approaches of computing $g_m(k)$ have been proposed in the literature (e.g., see [12] and the references therein). Here, we employ the improved proportionate method presented in [21], owing mainly to its robustness to the unknown system (e.g., echo path) with different sparseness degrees. In this case, the diagonal elements of $\mathbf{G}(k)$ are evaluated as

$$g_m(k) = \frac{1-a}{2M} + (1+a) \frac{|w_m(k)|}{2\|\mathbf{w}(k)\|_1 + z} \quad (21)$$

where $\|\cdot\|_1$ indicates L_1 -norm of vector, z is also a small positive constant to avoid division by zero, and $a \in [-1, 1]$. In practice, good choices of a are 0 or -0.5. Together with (11), the proposed IWF-PSSAF algorithm is named the IWF-IPSSAF hereafter, as summarized in Table 1.

Remark 3: Likewise, let us rewrite the proposed IWF-PSSAF algorithm as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \lambda_i \mathbf{G}(k) \mathbf{u}_i(k) \text{sign}(e_{i,D}(k)). \quad (22)$$

where the weighting factors λ_i for $i \in [0, N-1]$ are given by

$$\lambda_i = \frac{1}{\sqrt{[\mathbf{G}(k) \mathbf{u}_i(k)]^T [\mathbf{G}(k) \mathbf{u}_i(k)] + \varepsilon}} \quad (23).$$

If the common weighting factor for all subbands is expressed by

$$\lambda_1 = \lambda_2 = \dots = \lambda_{N-1} = \frac{1}{\sqrt{[\mathbf{G}(k) \mathbf{U}(k) \text{sign}(\mathbf{e}_D(k))]^T [\mathbf{G}(k) \mathbf{U}(k) \text{sign}(\mathbf{e}_D(k))] + \varepsilon}} \quad (24).$$

then the PSSAF algorithm presented in [22] is obtained. Comparing (23) and (24), one can find that the IWF-PSSAF algorithm is a modified version of the PSSAF with an individual weighting factor λ_i for each subband. In conclusion, as long as exploiting the same way to calculate the *proportionate matrix* $\mathbf{G}(k)$, e.g., (21), the proposed IWF-PSSAF has faster convergence rate than the PSSAF in [22] because of using an individual weighting factor for each subband rather than the common one in the PSSAF.

Note that: 1) if $\mathbf{G}(k) = \mathbf{I}_{M \times M}$, then the IWF-PSSAF algorithm reduces to the IWF-SSAF algorithm; 2) if $\lambda_1 = \lambda_2 = \dots = \lambda_{N-1}$, then the IWF-PSSAF algorithm reduces to the PSSAF algorithm; and 3) if $\mathbf{G}(k) = \mathbf{I}_{M \times M}$ and $\lambda_1 = \lambda_2 = \dots = \lambda_{N-1}$, then the IWF-PSSAF algorithm reduces to the SSAF algorithm.

4.2 Computational complexity

Table 2 gives the computational complexity of the proposed IWF-SSAF and IWF-IPSSAF algorithms in terms of the total number of additions, multiplications, divisions and square-roots for each fullband input sample, where the integer L is the length of the analysis filters $H_i(z)$ and synthesis filters $G_i(z)$. Compared with the SSAF and IPSSAF algorithms, the proposed IWF-SSAF and IWF-IPSSAF algorithms require the additional computational amount, mainly resulted from the accumulation operation in (9) and (18), respectively. Specifically, since M is greater than the product NL in AEC applications [5], the IWF-SSAF algorithm increases about $M - 2M/N$ multiplications than the SSAF algorithm, and the IWF-IPSSAF algorithm increases about $3M - M/N$ multiplications and $2M - M/N$ additions than the IPSSAF algorithm. Fortunately, the increases of both proposed algorithms in the computational complexity are worth, since they enhance significantly convergence rate.

Table 2 Computational complexity of various sign subband algorithms for each fullband input sample

Algorithms	Additions	Multiplications	Square-roots	Divisions
SSAF	$2M + M/N + 3NL - 2N - 1$	$M + 2M/N + 3NL$	$1/N$	$1/N$
IPSSAF	$M + (3M-1)/N + 3NL - 2N - 1$	$M + 3M/N + 3NL$	$1/N$	$(M+1)/N$
IWF-SSAF	$2M + 3NL - N + 1$	$2M + 2N + 3NL$	1	1
IWF-IPSSAF	$3M + 2M/N + 3NL - 3N$	$4M + M/N + 3NL$	1	$M/N + 1$

5 Simulations

In this section, the proposed IWF-SSAF and IWF-IPSSAF algorithms are evaluated in a system identification setup with the impulsive interferences and an AEC with double-talk. In our simulations except Section 5.3, the unknown impulse responses are true acoustic echo paths with $M = 512$ taps, shown in Fig. 3. Also, in order to compare the tracking capability of the algorithms, the impulse responses are multiplied by a factor of -1 at 4×10^4 th input sample.

Fig. 3 Two acoustic echo paths. (a) Sparse impulse response. (b) Dispersive impulse response.

The colored input signal $u(n)$ is either an AR(1) process generated by filtering a zero-mean white Gaussian signal through a first-order autoregressive system with a pole at 0.8 or a speech signal. A four-subband (*i.e.*, $N = 4$) cosine modulated filter bank is used, with 60 dB stopband attenuation, whose prototype has 32 tap-weights [4]. The background noise $v(n)$ is Gaussian white and added to the output of the unknown system with a signal-to-background-noise rate (SBNR) = 30dB, which is defined as

$$\text{SBNR} = \frac{E[y^2(n)]}{s_v^2}, \quad y(n) = \mathbf{u}^T(n)\mathbf{w}_o. \quad (25)$$

The normalized MSD (NMSD) defined by $10\log_{10} \frac{\sum_{k=0}^K \text{MSD}(k)}{\|\mathbf{w}_o\|_2^2} \text{ (dB)}$, is used to measure the performance of the algorithms,

and all results are the ensemble average of 10 independent trials unless otherwise specified.

5.1 System identification under impulsive interference

In the system identification, we set $P_r = 0.001$ (i.e., the probability of the occurrence of the impulsive interference) and $s_A^2 = 1000E[y^2(n)]$. Note that, for a fair performance comparison, parameters' values are selected in such a way that all algorithms have the same steady-state NMSD.

Fig. 4 shows the NMSD curves of the NSAF, SSAF and IWF-SSAF algorithms for dispersive impulse response given by Fig. 3(b). As we can see that only the NSAF algorithm based on the L_2 -norm optimization is divergent under impulsive interferences, while the remaining algorithms are convergent because they inherit the robustness of the L_1 -norm optimization against impulsive interferences. Moreover, the proposed IWF-SSAF algorithm exhibits faster convergence rate and better tracking capability than the SSAF algorithm.

Fig. 4 The NMSD curves of several subband algorithms for dispersive impulse response in the system identification. NSAF: $m=0.4$, $d=0.1$; SSAF: $m=0.021$, $e=0.01$; IWF-SSAF: $m=0.01$, $e=0.01$.

Fig. 5 The NMSD curves of several subband algorithms for sparse impulse response in the system identification. SSAF: $m=0.01$, $e=0.01$; IPSSAF: $m=0.052$, $e=0.01$, $a=0$; IWF-SSAF: $m=0.0048$, $e=0.01$; IWF-IPSSAF: $m=0.02$, $e=0.01$, $a=0$.

Fig. 5 provides the NMSD curves of the SSAF, IPSSAF, IWF-SSAF and IWF-IPSSAF algorithms for sparse impulse response shown in Fig. 3(a). As we expected, all of the sign subband algorithms have strong capability combating impulsive interference. Also, the proportionate algorithms (i.e., IPSSAF and IWF-IPSSAF) are more suitable than their non-proportionate counterparts (i.e., SSAF and IWF-SSAF) for sparse impulse response. More importantly, the proposed IWF-SSAF and IWF-IPSSAF algorithms obviously outperform the SSAF and IPSSAF algorithms, respectively, in terms of the convergence rate and tracking capability. The reason behind this result is that the proposed algorithms assign an individual weighting factor to each subband.

5.2 AEC with double-talk

In this section, we again test and verify the performance of the proposed algorithms in an AEC system with double-talk (one trial). As we all know, the fundamental idea in echo cancellation application is to identify the impulse response of echo path, i.e., system identification problem. Here, the only difference is that both the far-end input signal $u(n)$ and the near-end signal $h(n)$ are speech signals which are non-stationary and strongly correlated, as shown in Fig. 6.

Fig. 6 The speech signals. (a) Far-end input. (b) Near-end input.

In this case, Figs. 7 and 8 describe the comparisons of the performance of the proposed algorithms with that of other existing subband algorithms for dispersive and sparse impulse responses, respectively. In Figs. 7 and 8, all algorithms except the NSAF are robust against double-talk happened in the period with sample index of $[2, 3] \times 10^4$ and $[6, 7] \times 10^4$. Compared with the original SSAF (or IPSSAF) algorithm, the proposed IWF-SSAF (or IWF-IPSSAF) algorithm obtains a large improvement in the convergence rate as well as tracking capability, respectively. This result is reasonable because an individual weighting factor for each subband is utilized in the proposed algorithms, i.e., (8) and (23). Likewise, for sparse impulse response, the IWF-IPSSAF algorithm has better convergence performance than the IWF-SSAF algorithm.

Fig. 7 The NMSD curves of several subband algorithms for dispersive impulse response in the AEC. The values of parameters are the same as those in Fig. 3.

Fig. 8 The NMSD curves of several subband algorithms for sparse impulse response in the AEC. SSAF: $m = 0.009$, $e = 0.01$; IPSSAF: $m = 0.04$, $e = 0.01$, $a = 0$; IWF-SSAF: $m = 0.0045$, $e = 0.01$; IWF-IPSSAF: $m = 0.02$, $e = 0.01$, $a = 0$.

5.3 Validation of (16) and (19)

In the examples, we compare the theoretical MSD models given by (16) and (19) of the IWF-SSAF algorithm with simulated results. The unknown impulse response with length of M , is randomly generated and normalized by $\mathbf{w}_o^T \mathbf{w}_o = 1$. According to the definition of the MSD, its initial value in (16) is set to $\text{MSD}(0) = 1$. The input signal $x(n)$ is an AR(1) process and we also consider the presence of impulsive interferences with $k = 1000000$, $P_r = 0.001$.

Fig. 9 shows the results of comparison for step sizes $m = 0.01$, 0.006 , and 0.003 . Fig. 10 shows the results of comparison for $M = 128$, 256 and 512 taps. In Figs. 9 and 10, one can see that the theory values provided by (16) are slightly less than simulated values. The main reasons giving rise to the difference are to utilize some approximations and assumptions to derive the theory model. Therefore, the statistical MSD model from (16) for the IWF-SSAF algorithm is basically valid to predict the evolution of

the algorithm. In addition, we can also see from Fig. 9 that a large step size (e.g., $m=0.01$) leads to faster convergence rate but increases the steady-state NMSD, conversely, a small step size (e.g. $m=0.003$) gives lower steady-state NMSD but slows convergence rate. In a word, like most of other recursive algorithms (e.g., SSAF), the IWF-SSAF algorithm also exists a tradeoff problem of choosing the step size. Similarly, the IWF-IPSSAF algorithm has the same tradeoff problem. Therefore, to obtain both fast convergence and low steady-state NMSD, we will attempt to apply the existing variable step size schemes in [16-18] as well as the convex combination method [26] to the proposed algorithms in future works.

Finally, Fig. 11 checks the theoretical steady-state result given by (19). To verify (19), the step size must be very small, e.g., $m=0.0002$. As can be seen, theory results can match well with simulation results of the algorithm in the steady-state. Therefore, it is concluded that (19) can be used to predict the steady-state behavior of the IWF-SSAF algorithm when the step size is very small.

Fig. 9. A comparison of the theoretical model given by (16) of the IWF-SSAF algorithm with MC simulations for step sizes $m=0.01, 0.006, 0.003$. Length: $M=512$; parameter: $g=2$.

Fig. 10. A comparison of the theoretical model given by (16) of the IWF-SSAF algorithm with MC simulations for lengths $M=128, 256, 512$. Step size:

$m = 0.006$; parameter: $g = 2$.

Fig. 11. A verification of the theoretical model given by (19) for the IWF-SSAF algorithm for lengths $M = 128, 256, 512$.

5.4 Range of Step Size for stability

In this part, we provide the range of step size for the proposed algorithms to ensure convergence in a simulated way, due to the fact that finding a theoretical range for the algorithms based on the L_1 -norm optimization is very difficult. Figs. 12 and 13 show the steady-state NMSD values of the IWF-SSAF and IWF-IPSSAF algorithms for dispersive and sparse impulse responses, respectively, increased with the step size in a similar manner for different intensities of impulsive interference. The results were obtained by averaging over 500 instantaneous NMSD values in the steady-state.

From Figs. 12 and 13, to ensure convergence of the algorithms (achieve a steady-state NMSD less than 0 dB), a smaller step size has to be chosen for the impulsive interference with larger intensity (e.g., $P_r = 0.05$). The ranges of $0 < m \leq 0.02$ and $0 < m \leq 0.1$ are recommended for the proposed IWF-SSAF and IWF-IPSSAF algorithms, respectively.

Fig. 12 The steady-state NMSD values of the IWF-SSAF algorithm versus various step sizes and levels of impulsive interference. The values of P_r are chosen as 0.05, 0.01, and 0.001.

Fig. 13 The steady-state NMSD values of the IWF-IPSSAF algorithm versus various step sizes and levels of impulsive interference. The values of P_r are chosen as 0.05, 0.01, and 0.001.

6 Conclusions

This work first develops an individual-weighting-factor SSAF (IWF-SSAF) algorithm, in which each weighting factor is inversely proportional to the corresponding subband input signal power. The algorithm provides better performance than the original SSAF algorithm in term of the convergence rate as well as tracking capability. Then, we propose a proportionate IWF-SSAF algorithm, namely, IWF-IPSSAF, which obtains faster convergence rate in comparison with the IWF-SSAF and the existing SSAF and IPSSAF algorithms for sparse impulse responses. Also, two proposed algorithms also exhibit good robustness combating impulsive interferences. Simulation results in the context of system identification and AEC have verified the superiority of the proposed algorithms.

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Appendix: Proof of (15)

Before verifying (15), the following lemma is necessary [24, 25, 27].

Lemma: Let w_1 and w_2 be jointly Gaussian zero-mean random variables with variances $s_{w_1}^2$ and $s_{w_2}^2$, and let $j = w_2 + J_i$, and

J_i with variance given in (11) be independent of w_1 and w_2 . Let $c_1 = w_2 + h_1$ and $c_2 = w_2 + h_2$, where h_1 and h_2 are zero-mean Gaussian variables with variances $s_{h_1}^2 = \frac{1}{N}(k+1)s_v^2$ and $s_{h_2}^2 = \frac{1}{N}s_v^2$ and independent of w_1 and w_2 . Thus,

$$E[\text{sign}(\varphi)\omega_1] = P_r E[\text{sign}(c_1)\omega_1] + (1-P_r)E[\text{sign}(c_2)\omega_1]. \quad (\text{A1})$$

Using Price's theorem in [13] and (A1), we have

$$\begin{aligned} & E[\text{sign}(e_{i,D}(k))\boldsymbol{\theta}^T(k)\mathbf{u}_i(k)|\boldsymbol{\theta}(k)] \\ &= \sqrt{\frac{2}{\pi}}\boldsymbol{\theta}^T(k) \left\{ \frac{1-P_r}{\sqrt{E[e_{i_2,D}^2(k)]}} E[e_{i_2,D}(k)\mathbf{u}_i(k)|\boldsymbol{\theta}(k)] + \frac{P_r}{\sqrt{E[e_{i_1,D}^2(k)]}} E[e_{i_1,D}(k)\mathbf{u}_i(k)|\boldsymbol{\theta}(k)] \right\} \end{aligned} \quad (\text{A2})$$

with

$$e_{i,D}(k) = \boldsymbol{\theta}^T(k)\mathbf{u}_i(k) + J_i(k), \quad (\text{A3})$$

$$e_{i_1,D}(k) = \boldsymbol{\theta}^T(k)\mathbf{u}_i(k) + h_i(k), \quad (\text{A4})$$

and

$$e_{i_2,D}(k) = \boldsymbol{\theta}^T(k)\mathbf{u}_i(k) + v_i(k) \quad (\text{A5})$$

where $v_i(k)$ and $h_i(k)$ are zero-mean Gaussian sequences with variances $s_{h_i}^2 = \frac{1}{N}(k+1)s_v^2$ and $s_{v_i}^2 = \frac{1}{N}s_v^2$.

According to the assumption 4), we have

$$E[\dot{\xi}_{i_1,D}^2(k)] = s_{\mathbf{u}_i}^2(k)\text{MSD}(k) + \frac{1}{N}(k+1)s_v^2 \quad (\text{A6})$$

and

$$E[\dot{\xi}_{i_2,D}^2(k)] = s_{\mathbf{u}_i}^2(k)\text{MSD}(k) + \frac{1}{N}s_v^2. \quad (\text{A7})$$

Taking expectation of both sides of (A2) with respect to $\boldsymbol{\theta}(k)$ and applying (A6) and (A7), we can obtain

$$\begin{aligned} & E[\text{sign}(e_{i,D}(k))\boldsymbol{\theta}^T(k)\mathbf{u}_i(k)] = E[E[\text{sign}(e_{i,D}(k))\boldsymbol{\theta}^T(k)\mathbf{u}_i(k)|\boldsymbol{\theta}(k)]] \\ &= \sqrt{\frac{2}{\pi}}\sigma_{\mathbf{u}_i}^2(k) \left\{ \frac{1-P_r}{\sqrt{\frac{1}{N}\sigma_v^2 + \sigma_{\mathbf{u}_i}^2(k)\text{MSD}(k)}} + \frac{P_r}{\sqrt{\frac{1}{N}(\kappa+1)\sigma_v^2 + \sigma_{\mathbf{u}_i}^2(k)\text{MSD}(k)}} \right\} \text{MSD}(k). \end{aligned} \quad (\text{A8})$$

To derive (A8), we also employ the following equation

$$E[\dot{\xi}_{i_1,D}(k)\mathbf{u}_i(k)|\boldsymbol{\theta}(k)] = E[\dot{\xi}_{i_2,D}(k)\mathbf{u}_i(k)|\boldsymbol{\theta}(k)] = \boldsymbol{\theta}(k)s_{\mathbf{u}_i}^2(k). \quad (\text{A9})$$

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