On Additive Generators of Overlap Functions

Graçaliz Pereira Dimuro^a, Benjamín Bedregal^{b,*}, Humberto Bustince^c, Maria José Asiáin^c, Radko Mesiar^d

^a*Programa de Pós-Graduação em Modelagem Computacional
Programa de Pós-Graduação em Computação
Centro de Ciência Computacionais
Universidade Federal do Rio Grande*

Av. Itália km 08, Campus Carreiros, 96201-900 Rio Grande, Brazil

^b*Departamento de Informática e Matemática Aplicada
Universidade Federal do Rio Grande do Norte
Campus Universitário s/n, 59072-970 Natal, Brazil*

^c*Universidad Pública de Navarra
Campus Arrosadia s/n, P.O. Box 31006, Pamplona, Spain*

^d*Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering
Slovak University of Technology in Bratislava
Radlinského 11, 813 68 Bratislava, Slovak Republic*

Abstract

In this paper we introduce the notion of additive generator pair for overlap functions. We also study how some of the properties that can be demanded to an overlap function can be expressed in terms of its generator pair. In particular, given a continuous and positive t-norm T , we analyze the overlap functions which are obtained by the distortion of T by a pseudo-automorphism, showing that the additive generator pair of such overlap function may be given in terms of the considered pseudo-automorphism and the additive generator of T . Finally, we also consider the influence of the migrativity, homogeneity and idempotency properties for the overlap functions obtained by such distortion in terms of their respective additive generator pairs.

Keywords: Overlap functions, additive generators, t-norms distorted by pseudo-automorphisms, migrativity, homogeneity, idempotency

*Corresponding author. Phone: +55 53 32336685

Email addresses: gracaliz@furg.br; gracaliz@gmail.com (Graçaliz Pereira Dimuro), bedregal@dimap.ufrn.br (Benjamín Bedregal), bustince@unavarra.es (Humberto Bustince), (Maria José Asiáin), mesiar@math.sk (Radko Mesiar)

URL: <http://www.gracalizdimuro.com> (Graçaliz Pereira Dimuro), <http://www.dimap.ufrn.br/~bedregal> (Benjamín Bedregal), <http://giara.unavarra.es> (Humberto Bustince), (Maria José Asiáin), <http://www.math.sk/mesiar/> (Radko Mesiar)

1. Introduction

Overlap functions arise from some problems in image processing, classification or decision making for which t-norms are usually considered but where associativity is not needed in fact. Moreover, and contrary to the case of t-norms, the class of overlap functions is convex.

Overlap functions are non-decreasing continuous commutative bivariate functions defined over the unit square $[0, 1] \times [0, 1]$, which satisfy appropriate boundary conditions (see Definition 3.1). They are a particular case of *aggregation functions*. As already said, overlap functions play an important role in classification, image processing or decision making problems, when a given object (an instance to be classified or a pixel, for instance) must be assigned to one of two different classes which are not clearly distinct. In these cases, overlapping between the two classes may be checked in order to incorporate to the structure of problem some measure of the lack of information. Overlap functions provide a functional way of carrying out such measurement [1].

In image processing, Bustince et al. [1] presented the basic properties that must be demanded to an overlap functions in order to assign a given pixel either to the object or to the background of a given image. More recently, Jurio et al. [2] analyzed several properties of overlap functions. In particular, they proposed a segmentation algorithm based on the use of a convex combination of overlap functions, which, contrary to the case of t-norms, is again an overlap function.

Since overlap functions generalize continuous t-norms without divisors of zero, they can also be applied in decision making based on fuzzy preference relations, where the associativity property is not strongly required, so the use of t-norms or t-conorms as the combination/separation operators is not necessary. In [3], Bustince et al. applied overlap functions in fuzzy preference modeling and decision making, presenting an algorithm to elaborate on an alternative preference ranking that penalizes the alternatives for which the expert is not sure about his/her preference. Moreover, and also in the setting of fuzzy preference relations, the concepts of indifference and incomparability may also be defined in terms of overlap functions, since they do not require associativity.

Finally, several studies have also been presented in recent years from a theoretical point of view, analyzing properties such as migrativity, homogeneity, idempotency or Lipschitzianity (see, e.g., the works by Bustince et al. [1, 2, 3, 4, 5]).

As we have already stated, there exists a close relation between overlap functions and some particular class of t-norms. Since a very important way for building the latter is the use of additive generators [6, 7, 8, 9, 10, 11, 12, 13] (see, also, [14, 15] for an extension to the interval-valued setting), it seems natural to us to study the conditions under which such construction is possible for overlap functions. Moreover, and with an eye kept on application, the use of additive generators would simplify the choice of an appropriate overlap function for a given problem, since we only will need to consider single-variable functions instead of a bivariate one, reducing in this way the computational complexity [16].

For this reason, the main objective of this paper is to analyze the construction of overlap functions by means of additive generators and to study which properties are preserved by such construction. In particular, this paper introduces the notion of ad-

ditive generator pair of overlap functions and related results, based on some preliminary ideas discussed in [17]. We also introduce the concept of pseudo-automorphism, in order to construct a class of overlap functions obtained by the so-called $(\mathcal{F}, T)$ -distortion¹ of continuous and positive t-norms T by the application of pseudo-automorphisms $\mathcal{F}$, so that it is possible to define the additive generator pair of such overlap functions in terms of the additive generators of T and $\mathcal{F}$. Then, we study the influence of some important properties (namely, migrativity, homogeneity and idempotency) in the overlap functions obtained by such $(\mathcal{F}, T)$ -distortion operation and, consequently, in the determination of the additive generator pair. Several interesting and important results are stated, and different examples are discussed throughout the paper.

The paper is organized as follows. Section 2 summarizes preliminary concepts related to aggregation functions, t-norms and additive generators of t-norms. In Section 3, we present the main concepts and results related to overlap functions, introducing some examples. Section 4 introduces the concept of additive generator pair of overlap functions, with some important results and examples. In Section 5, we introduce the notion of pseudo-automorphism, studying overlap functions as t-norms distorted by pseudo-automorphisms, and their additive generator pairs. Sections 6, 7 and 8 studies additively generated overlap functions considering, respectively, the migrativity, homogeneity and idempotency properties. Section 9 is the Conclusion, with some final considerations, and pointing to future work and further potential applications.

2. Preliminary Concepts on Aggregation Operators and T-norms: some properties and additive generators of t-norms

One important class of fuzzy operators are the *aggregation functions* [18, 19], which were proposed to aggregate several values in just one. A function $A : [0, 1]^n \rightarrow [0, 1]$ is said to be an aggregation function if it satisfies the following two conditions:

- (A1) A is increasing in each argument: for each $i \in \{1, \dots, n\}$, if $x_i \leq y$, then $A(x_1, \dots, x_n) \leq A(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n)$;
- (A2) Boundary conditions: $A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$.

In this paper, we are interested in *overlap functions* [5, 1, 3, 2], one important class of aggregation functions that are related in some sense to *triangular norms* (t-norms).

Triangular norms were introduced by Menger [20] to model distance in probabilistic metric spaces. Schweizer and Sklar [21, 22] redefined the t-norm axioms proposed by Menger into the form used today. Nowadays, t-norms are often applied as a generalization of the classical conjunction.

Definition 2.1. Consider $\alpha \in [0, 1]$. A bivariate aggregation function $A : [0, 1]^2 \rightarrow [0, 1]$ is said to be α -migrative if, for all $x, y \in [0, 1]$, it holds that

$$A(\alpha x, y) = A(x, \alpha y). \quad (1)$$

¹The $(\mathcal{F}, T)$ -distortion operation is in fact the function composition operation $\mathcal{F} \circ T$.

Definition 2.2. A bivariate aggregation function $A : [0, 1]^2 \rightarrow [0, 1]$ is said to be migrative if and only if, for all $x, y \in [0, 1]$ and $\alpha \in [0, 1]$, it holds that

$$A(\alpha x, y) = A(x, \alpha y). \quad (2)$$

Definition 2.3. A function $f : [0, 1]^n \rightarrow [0, 1]$ is said to be homogeneous of order $k \in]0, \infty[$ if, for any $\alpha \in]0, \infty[$ and $x_1, \dots, x_n \in [0, 1]$ such that $\alpha x, \dots, \alpha x_n \in [0, 1]$, it holds that

$$f(\alpha x, \dots, \alpha x_n) = \alpha^k f(x, \dots, x_n). \quad (3)$$

An aggregation function f is said to be homogeneous if it is homogeneous of order k , for some $k \in]0, \infty[$.

Definition 2.4. A bivariate aggregation function $A : [0, 1]^2 \rightarrow [0, 1]$ is said to be homogeneous of order $k \in]0, \infty[$ if, for any $\alpha \in]0, \infty[$ and $x, y \in [0, 1]$ such that $\alpha x, \alpha y \in [0, 1]$, it holds that²

$$A(\alpha x, \alpha y) = \alpha^k A(x, y). \quad (4)$$

A bivariate aggregation function A is said to be homogeneous if it is homogeneous of order k , for some $k \in]0, \infty[$.

Remark 2.1. Note that, in general, for the definition of homogeneity also the parameter value $k = 0$ makes sense. However, A is a 0-homogeneous aggregation function if and only if there exist $a, b \in [0, 1]$ (which may be the same) such that:

$$A(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ a & \text{if } x = 0 \text{ and } y \neq 0 \\ b & \text{if } x \neq 0 \text{ and } y = 0 \\ 1 & \text{in other case.} \end{cases}$$

as an easy calculation shows.

Definition 2.5. A bivariate aggregation function $T : [0, 1]^2 \rightarrow [0, 1]$ is said to be a t -norm if, for all $x, y, z \in [0, 1]$, it satisfies the following conditions:

- (T1) Commutativity: $T(x, y) = T(y, x)$;
- (T2) Associativity: $T(x, T(y, z)) = T(T(x, y), z)$;
- (T3) Boundary condition: $T(x, 1) = x$.

Some other properties may be required for t -norms, such as the following:

- (T4) Continuity: T is continuous in both arguments;

²Observe that since $A(0, 0) = 0$, for any bivariate aggregation function $A : [0, 1]^2 \rightarrow [0, 1]$, it is possible to require that $\alpha \geq 0$ when defining homogeneous bivariate aggregation functions.

(T5) Idempotency: $T(x, x) = x$, for all $x \in [0, 1]$;

(T6) Positivity: if $T(x, y) = 0$ then either $x = 0$ or $y = 0$.

Example 2.1. Typical examples of t-norms are:

(i) Gödel or minimum: $G(x, y) = \min\{x, y\}$;

(ii) Product: $P(x, y) = x \cdot y$;

(iii) Łukasiewicz: $L(x, y) = \max\{x + y - 1, 0\}$.

Notice that $L \leq P \leq G$, that is, for each $x, y \in [0, 1]$, it holds that $L(x, y) \leq P(x, y) \leq G(x, y)$. In fact G is the greatest t-norm. G and P are continuous and positive t-norms. L is continuous but not positive. In particular, G is the unique idempotent t-norm.

An element $x \in]0, 1]$ is said to be a *non-trivial zero divisor* of a t-norm T , if there exists $y \in]0, 1]$ such that $T(x, y) = 0$. Clearly, a t-norm is *positive* if and only if it has no non-trivial zero divisor. See [6], for more details about t-norms.

Definition 2.6. [23, Definition 7] Let $B : [0, 1]^2 \rightarrow [0, 1]$ be a bivariate aggregation function. A bivariate aggregation function $A : [0, 1]^2 \rightarrow [0, 1]$ is said to be *B-migrative* if, for any $x, y, z \in [0, 1]$, it holds that

$$A(B(x, y), z) = A(x, B(y, z)). \quad (5)$$

Notice that a bivariate aggregation function A is *A-migrative* if and only if it is associative. Thus, any t-norm T is *T-migrative*. Moreover, a bivariate migrative aggregation function is, in particular, *P-migrative*, where P is the product t-norm presented in Example 2.1.

Example 2.2. Consider the greatest aggregation function $A^* : [0, 1]^2 \rightarrow [0, 1]$, defined by:

$$A^*(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

and the arithmetic mean aggregation function $AM : [0, 1]^2 \rightarrow [0, 1]$, given by

$$AM(x, y) = \frac{x + y}{2}.$$

Then, it is immediate that A^* is *AM-migrative*, since

$$\begin{aligned} A^*(AM(x, y), z) &= A^*\left(\frac{x + y}{2}, z\right) = \begin{cases} 0 & \text{if } x = y = z = 0 \\ 1 & \text{otherwise} \end{cases} \\ &= A^*\left(x, \frac{y + z}{2}\right) = A^*(x, AM(y, z)). \end{aligned}$$

Definition 2.7. [11] Let $f : [a, b] \rightarrow [c, d]$ be an increasing or decreasing function.³ The function $f^{(-1)} : [c, d] \rightarrow [a, b]$ defined by

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{if } f(a) < f(b), \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{if } f(a) > f(b), \\ a & \text{if } f(a) = f(b) \end{cases} \quad (6)$$

is called the pseudo-inverse of f .

Notice that when f is an increasing (but not constant) function then

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) < y\}. \quad (7)$$

Analogously, when f is a decreasing (but not constant) function then

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) > y\}. \quad (8)$$

In the following, we denote the *range* or *image* of a function $f : A \rightarrow B$ by $\text{Ran}(f)$.

Remark 2.2. [12, page 1632] If a function $f : [a, b] \rightarrow [c, d]$ is increasing (decreasing) then $f^{(-1)}$ is also increasing (decreasing). If f is strictly increasing (decreasing) then $f^{(-1)}$ is continuous, $f^{(-1)} \circ f = \text{Id}_{[a, b]}$ and $f \circ f^{(-1)}(x) = x$ if and only if $x \in \text{Ran}(f)$.

Example 2.3. [14] The functions $p, l : [0, 1] \rightarrow [0, \infty]$, defined, respectively, by

$$p(x) = \begin{cases} -\ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and $l(x) = 1 - x$, are strictly decreasing and $p(1) = l(1) = 0$. In this case, for each $y \in [0, \infty]$, it holds that

$$p^{(-1)}(y) = \begin{cases} e^{-y} & \text{if } y \neq \infty \\ 0 & \text{if } y = \infty \end{cases}$$

and

$$l^{(-1)}(y) = \begin{cases} 1 - y & \text{if } y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

that is, $l^{(-1)}(y) = \max\{1 - y, 0\}$.

In the literature [6, 7, 8, 9, 10, 12, 13, 16], there exist several definitions for additive generators. In this section, we present the one extracted from [11], which was also applied in [14, 15].

³In this paper, a increasing (decreasing) function does not need to be strictly increasing (decreasing).

Definition 2.8. Consider a function $f : [0, 1] \rightarrow [0, \infty]$ and a t -norm $T : [0, 1]^2 \rightarrow [0, 1]$. The function f is said to be an additive generator of T if f is a strictly decreasing function such that, for all $x, y \in [0, 1]$,

$$T(x, y) = f^{(-1)}(f(x) + f(y)), \quad (9)$$

where $f^{(-1)}$ is the pseudo-inverse of f .

A t -norm $T : [0, 1]^2 \rightarrow [0, 1]$ is said to be an *additively generated function* if there exists an additive generator of T . A t -norm T is said to be *additively generated by* $f : [0, 1] \rightarrow [0, \infty]$ if f is an additive generator of T .

Theorem 2.1. [24, Theorem 2.6.3] Let $t : [0, 1] \rightarrow [0, \infty]$ be a strictly decreasing function with $t(1) = 0$, such that t is right-continuous in 0 and, for each $x, y \in [0, 1]$, it holds that

$$t(x) + t(y) \in \text{Ran}(t) \cup [t(0), \infty]. \quad (10)$$

Then, the function $T_t : [0, 1]^2 \rightarrow [0, 1]$, defined by

$$T_t(x, y) = t^{(-1)}(t(x) + t(y)), \quad (11)$$

is a t -norm.

The t -norm T_t given in Equation (11) is said to be additively generated by the function t . In this case, t is an additive generator of the t -norm T_t .

Example 2.4. [14] Let $p, l : [0, 1] \rightarrow [0, \infty]$ be the functions defined in Example 2.3 and consider $x, y \in [0, 1]$. Then, defining

$$T_l(x, y) = l^{(-1)}(l(x) + l(y)) = l^{(-1)}((1 - x) + (1 - y)),$$

one has that:

$$T_l(x, y) = \begin{cases} x + y - 1 & \text{if } 1 \leq x + y \\ 0 & \text{otherwise} \end{cases}$$

and, then, it holds that $T_l(x, y) = \max\{x + y - 1, 0\}$ and so $T_l = L$ (the Łukasiewicz t -norm given in Example 2.1). Analogously, we have that if $x \neq 0$ and $y \neq 0$ then

$$T_p(x, y) = p^{(-1)}(p(x) + p(y)) = p^{(-1)}((-\ln x) + (-\ln y)) = e^{(\ln x \cdot y)} = x \cdot y.$$

Otherwise (suppose that $x = 0$, the case $y = 0$ being analogous), one has that

$$T_p(x, y) = p^{(-1)}(p(x) + p(y)) = p^{(-1)}(\infty + p(y)) = 0.$$

It follows that $T_p = P$ (the product t -norm given in Example 2.1).

Lemma 2.1. Let $t : [0, 1] \rightarrow [0, \infty]$ be a strictly decreasing function. For any $k \in]0, \infty[$, if t is homogeneous of order k then its pseudo-inverse $t^{(-1)} : [0, \infty] \rightarrow [0, 1]$ is homogeneous of order $\frac{1}{k}$.

PROOF. For $\alpha \in]0, \infty]$, if $t : [0, 1] \rightarrow [0, \infty]$ is homogeneous of order k (Definition 2.3), then it holds that:

$$\begin{aligned}
t^{(-1)}(\alpha y) &= \sup\{x \in [0, 1] \mid t(x) > \alpha y\} \quad \text{by Equation (8)} \\
&= \sup\{x \in [0, 1] \mid \frac{1}{\alpha} t(x) > y\} \\
&= \sup\{x \in [0, 1] \mid ((\alpha^{-1})^{\frac{1}{k}})^k t(x) > y\} \\
&= \sup\{x \in [0, 1] \mid t((\alpha^{-1})^{\frac{1}{k}} x) > y\} \quad \text{since } t \text{ is homogeneous of order } k \\
&= \sup\left\{x \in [0, 1] \mid t\left(\frac{x}{\alpha^{\frac{1}{k}}}\right) > y\right\} \\
&= \sup\{\alpha^{\frac{1}{k}} z \in [0, 1] \mid t(z) > y\} \\
&= \alpha^{\frac{1}{k}} \sup\left\{z \in \left[0, \min\left\{1, \frac{1}{\alpha^{\frac{1}{k}}}\right\}\right] \mid t(z) > y\right\} \\
&= \alpha^{\frac{1}{k}} \sup\{z \in [0, 1] \mid t(z) > y\} \quad \text{since } \alpha \leq 1 \\
&= \alpha^{\frac{1}{k}} t^{(-1)}(y)
\end{aligned}$$

Therefore, $t^{(-1)}$ is homogeneous of order $\frac{1}{k}$. $\square$

Proposition 2.1. *Let $T_t : [0, 1]^2 \rightarrow [0, 1]$ be a t -norm additively generated by a function $t : [0, 1] \rightarrow [0, \infty]$. For any $k \in]0, \infty]$, if t is homogeneous of order k (according to Definition 2.3) then T_t is homogeneous of order 1.*

PROOF. If t is homogeneous of order k , then, by Lemma 2.1, $t^{(-1)}$ is homogeneous of order $\frac{1}{k}$. It follows that:

$$\begin{aligned}
T_t(\alpha x, \alpha y) &= t^{(-1)}(t(\alpha x) + t(\alpha y)) \quad \text{by Equation (11)} \\
&= t^{(-1)}(\alpha^k t(x) + \alpha^k t(y)) \\
&= t^{(-1)}(\alpha^k (t(x) + t(y))) \\
&= (\alpha^k)^{\frac{1}{k}} t^{(-1)}(t(x) + t(y)) \\
&= \alpha T_t(x, y) \quad \text{by Equation (11)}
\end{aligned}$$

which proves that T_t is homogeneous of order 1. $\square$

3. Overlap Functions

In this section, we consider the concept of *overlap functions* introduced by Bustince et al. [1, 3, 5]. Some properties of overlap functions (e.g., migrativity, homogeneity, idempotency, convex combination) were also studied by Bedregal et al. [4] and Jurio et al. [2].

Definition 3.1. *A bivariate function $O : [0, 1]^2 \rightarrow [0, 1]$ is said to be an overlap function if it satisfies the following conditions:*

(O1) O is commutative;

(O2) $O(x, y) = 0$ if and only if $xy = 0$;

(O3) $O(x, y) = 1$ if and only if $xy = 1$;

(O4) O is non-decreasing;

(O5) O is continuous.

Example 3.1. It is possible to find several examples of overlap functions, such as any continuous t -norm with no zero divisors (property (O2)). On the other hand, the function $O_{mM} : [0, 1]^2 \rightarrow [0, 1]$, given by

$$O_{mM}(x, y) = \min(x, y) \max(x^2, y^2),$$

is a non associative overlap function having 1 as neutral element, and, thus, it is not a t -norm. This is also the case of the overlap functions $O_{DB}, O_2 : [0, 1]^2 \rightarrow [0, 1]$, defined by

$$O_{DB}(x, y) = \begin{cases} \frac{2xy}{x+y} & \text{if } x + y \neq 0 \\ 0 & \text{if } x + y = 0; \end{cases}$$

and

$$O_2(x, y) = x^2 y^2$$

or, more generally,

$$O_p(x, y) = x^p y^p, \text{ with } p > 0, p \neq 1,$$

which are neither associative nor have 1 as neutral element.

Notice that whenever an overlap function has a neutral element, then, by (O3), this element is necessarily equal to 1. Moreover, the following proposition proves that associative overlap functions always have 1 as neutral element and, thus, they are continuous t -norms with no zero divisors.

Proposition 3.1. [25, Lemma 2.1.1],[6, Proposition 2.41] Let $F : [0, 1]^2 \rightarrow [0, 1]$ be a continuous bivariate function. If F is associative, $F(1, 1) = 1$ and $F(0, 1) = F(1, 0) = 0$, then 1 is a neutral element of F , that is, $F(x, 1) = F(1, x) = x$, for each $x \in [0, 1]$.

As a corollary, we have the Theorem 6 in [1]:

Corollary 3.1. [1, Theorem 6] $O : [0, 1]^2 \rightarrow [0, 1]$ is an associative overlap function if and only if O is a continuous and positive t -norm.

Nevertheless, the converse of Proposition 3.1 does not hold, since, as shown in Example 3.1, the overlap function $O_{mM}(x, y) = \min(x, y) \max(x^2, y^2)$ has 1 as neutral element but is not associative.

4. Additive Generators of Overlap Functions

In this section, we introduce the concept of additive generator pair of an overlap function, allowing the definition of overlap functions (as two-place functions) by means of one-place functions (their additive generator pair).⁴

Notice that the concept of additive generator pair of overlap functions introduced here is inspired by Věcník's work [10, 11, 12, 13] related to additive generators of (non-continuous) t-norms, based on a pair of functions composed by a (non-continuous) one-place function and its pseudo-inverse. In this paper, since overlap functions are continuous, their additive generator pair is composed by continuous functions. Moreover, pseudo-inverses are not required here: it is sufficient to consider a pair of functions satisfying certain properties that allow them to play analogous role to the Věcník's additive generator (and its pseudo-inverse).

Lemma 4.1. *Let $\theta : [0, 1] \rightarrow [0, \infty]$ be a decreasing function such that*

1. $\theta(x) + \theta(y) \in \text{Ran}(\theta)$, for $x, y \in [0, 1]$ and
2. if $\theta(x) = \theta(0)$ then $x = 0$.

Then $\theta(x) + \theta(y) \geq \theta(0)$ if and only if $x = 0$ or $y = 0$.

PROOF. ($\Rightarrow$) Since θ is decreasing and $\theta(x) + \theta(y) \in \text{Ran}(\theta)$, for each $x, y \in [0, 1]$, then one has that $\theta(x) + \theta(y) \leq \theta(0)$. Therefore, if $\theta(x) + \theta(y) \geq \theta(0)$, then it holds that $\theta(x) + \theta(y) = \theta(0)$. Suppose that $\theta(0) = 0$. Then, since θ is decreasing, one has that $\theta(x) = 0$, for each $x \in [0, 1]$, which is contradiction with condition 2, and, therefore, it holds that $\theta(0) > 0$. Now, suppose that $\theta(0) \neq 0$ and $\theta(0) \neq \infty$. Then, since $\theta(0) \neq 0$, one has that $\theta(0) + \theta(0) > \theta(0)$, which is also a contradiction. So, it follows that $\theta(0) = \infty$ and, therefore, since $\theta(x) + \theta(y) = \theta(0)$, we have that $\theta(x) = \infty$ or $\theta(y) = \infty$. Hence, by condition 2, one has that $x = 0$ or $y = 0$. ($\Leftarrow$) It is straightforward. $\square$

Lemma 4.2. *Consider functions $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ such that, for each $x_0 \in [0, 1]$, if it holds that*

$$\vartheta(\theta(x)) = x_0 \text{ if and only if } x = x_0, \quad (12)$$

then $\theta(x) = \theta(x_0)$ if and only if $x = x_0$.

PROOF. ($\Rightarrow$) Observe that whenever $\vartheta(\theta(x)) = x_0 \Leftrightarrow x = x_0$, so in particular, $\vartheta(\theta(x_0)) = x_0$. Now, if $\theta(x) = \theta(x_0)$ then $\vartheta(\theta(x)) = \vartheta(\theta(x_0)) = x_0$, and, thus $x = x_0$. ($\Leftarrow$) It is straightforward. $\square$

Theorem 4.1. *Let $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ be continuous and decreasing functions such that*

1. $\theta(x) + \theta(y) \in \text{Ran}(\theta)$, for $x, y \in [0, 1]$;

⁴Observe that any migrative overlap function can be naturally defined as an one-place function (see, e.g., [1, 4]).

2. $\vartheta(\theta(x)) = 0$ if and only if $x = 0$;
3. $\vartheta(\theta(x)) = 1$ if and only if $x = 1$;
4. $\theta(x) + \theta(y) = \theta(1)$ if and only if $x = 1$ and $y = 1$.

Then, the function $O_{\theta, \vartheta} : [0, 1]^2 \rightarrow [0, 1]$, defined by

$$O_{\theta, \vartheta}(x, y) = \vartheta(\theta(x) + \theta(y)), \quad (13)$$

is an overlap function.

PROOF. We show that the conditions of Definition 3.1 holds. The proofs of the commutativity property (condition **(O1)**) and the continuity property (condition **(O5)**) are immediate. Moreover, it follows that

$$\begin{aligned}
O_{\theta, \vartheta}(x, y) = 0 &\Leftrightarrow \vartheta(\theta(x) + \theta(y)) = 0 \quad \text{by Equation (13)} \\
&\Leftrightarrow \vartheta(\theta(z)) = 0 \text{ for some } z \in [0, 1] \quad \text{by condition 1;} \\
&\Leftrightarrow z = 0 \quad \text{by condition 2;} \\
&\Leftrightarrow \theta(z) = \theta(0) \quad \text{by Lemma 4.2;} \\
&\Leftrightarrow \theta(x) + \theta(y) = \theta(0) \\
&\Leftrightarrow x = 0 \text{ or } y = 0 \quad \text{by Lemma 4.1.}
\end{aligned}$$

which proves the condition **(O2)**. Also, one has that:

$$\begin{aligned}
O_{\theta, \vartheta}(x, y) = 1 &\Leftrightarrow \vartheta(\theta(x) + \theta(y)) = 1 \text{ by Equation (13)} \\
&\Leftrightarrow \vartheta(\theta(z)) = 1 \text{ for some } z \in [0, 1] \quad \text{by condition 1;} \\
&\Leftrightarrow z = 1 \quad \text{by condition 3;} \\
&\Leftrightarrow \theta(z) = \theta(1) \quad \text{by Lemma 4.2;} \\
&\Leftrightarrow \theta(x) + \theta(y) = \theta(1) \\
&\Leftrightarrow x = 1 \text{ and } y = 1 \quad \text{by condition 4,}
\end{aligned}$$

which proves the condition **(O3)**. Finally, to prove the condition **(O4)**, considering $z \in [0, 1]$ with $y \leq z$, then $\theta(y) \geq \theta(z)$. It follows that

$$O_{\theta, \vartheta}(x, y) = \vartheta(\theta(x) + \theta(y)) \leq \vartheta(\theta(x) + \theta(z)) = O_{\theta, \vartheta}(x, z),$$

since ϑ and θ are decreasing. □

Corollary 4.1. Let $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ be continuous and decreasing functions such that

1. $\theta(x) = \infty$ if and only if $x = 0$;
2. $\theta(x) = 0$ if and only if $x = 1$;
3. $\vartheta(x) = 1$ if and only if $x = 0$;
4. $\vartheta(x) = 0$ if and only if $x = \infty$.

Then, the function $O_{\theta, \vartheta} : [0, 1]^2 \rightarrow [0, 1]$, defined by

$$O_{\theta, \vartheta}(x, y) = \vartheta(\theta(x) + \theta(y)), \quad (14)$$

is an overlap function.

PROOF. It follows from Theorem 4.1. $\square$

Proposition 4.1. Let $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ be continuous and decreasing functions such that

1. $\vartheta(x) = 1$ if and only if $x = 0$;
2. $\vartheta(x) = 0$ if and only if $x = \infty$;
3. $0 \in \text{Ran}(\theta)$;
4. $O_{\theta, \vartheta}(x, y) = \vartheta(\theta(x) + \theta(y))$ is an overlap function.

Then, the following conditions also hold:

5. $\theta(x) = \infty$ if and only if $x = 0$;
6. $\theta(x) = 0$ if and only if $x = 1$;

PROOF. It follows that:

5. ($\Rightarrow$) If $O_{\theta, \vartheta}$ is an overlap function, then it holds that:

$$\begin{aligned} \theta(x) = \infty &\Rightarrow \theta(x) + \theta(y) = \infty \text{ for some } y \neq 0; \\ &\Rightarrow \vartheta(\theta(x) + \theta(y)) = 0 \text{ by condition 2} \\ &\Rightarrow O_{\theta, \vartheta}(x, y) = 0. \\ &\Rightarrow x = 0. \end{aligned}$$

($\Leftarrow$) Consider $y \in [0, 1]$ such that $\theta(y) \neq \infty$. Then, one has that:

$$\begin{aligned} x = 0 &\Rightarrow O_{\theta, \vartheta}(x, y) = 0 \\ &\Rightarrow \vartheta(\theta(x) + \theta(y)) = 0 \\ &\Rightarrow \theta(x) + \theta(y) = \infty \\ &\Rightarrow \theta(x) = \infty. \end{aligned}$$

6. ($\Rightarrow$) If $O_{\theta, \vartheta}$ is an overlap function, then it holds that:

$$\begin{aligned} x = 1 &\Rightarrow O_{\theta, \vartheta}(x, 1) = 1 \\ &\Rightarrow \vartheta(\theta(x) + \theta(1)) = 1 \\ &\Rightarrow \theta(x) + \theta(1) = 0 \text{ by condition 1} \\ &\Rightarrow \theta(x) = \theta(1) = 0 \end{aligned}$$

($\Leftarrow$) Consider $y \in [0, 1]$ such that $\theta(y) = 0$, which is possible due to condition 3. Then, one has that:

$$\begin{aligned} \theta(x) = 0 &\Rightarrow \theta(x) + \theta(y) = 0 \\ &\Rightarrow \vartheta(\theta(x) + \theta(y)) = 1 \text{ by condition 1} \\ &\Rightarrow O_{\theta, \vartheta}(x, y) = 1 \\ &\Rightarrow x = y = 1. \end{aligned}$$

□

(θ, ϑ) is called an *additive generator pair* of the overlap function $O_{\theta, \vartheta}$, and $O_{\theta, \vartheta}$ is said to be additively generated by the pair (θ, ϑ) .

Example 4.1. Consider the functions $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$, defined, respectively by:

$$\theta(x) = \begin{cases} -2 \ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and

$$\vartheta(x) = \begin{cases} e^{-x} & \text{if } x \neq \infty \\ 0 & \text{if } x = \infty, \end{cases}$$

which are continuous and decreasing functions, satisfying the conditions 1-4 of Corollary 4.1. Then, whenever $x \neq 0$ and $y \neq 0$, one has that:

$$O_{\theta, \vartheta}(x, y) = \vartheta(\theta(x) + \theta(y)) = e^{-(2 \ln x + 2 \ln y)} = e^{\ln x^2 y^2} = x^2 y^2.$$

Otherwise, if $x = 0$, it holds that

$$O_{\theta, \vartheta}(0, y) = \vartheta(\theta(0) + \theta(y)) = \vartheta(\infty + \theta(y)) = 0,$$

and, similarly, if $y = 0$, then $O_{\theta, \vartheta}(x, 0) = 0$. It follows that

$$O_{\theta, \vartheta}(x, y) = x^2 y^2,$$

and so we recover the non associative overlap function O_2 , given in Example 3.1, for which 1 is not a neutral element.

Corollary 4.2. Considering the same conditions of Theorem 4.1, whenever θ is strictly decreasing and $\vartheta = \theta^{(-1)}$ then $O_{\theta, \vartheta}$ is a positive t-norm.

PROOF. By Theorem 2.1, $O_{\theta, \vartheta}$ is a t-norm (additively generated by θ), and by Theorem 4.1, $O_{\theta, \vartheta}$ is positive. □

Theorem 4.2. Let $O : [0, 1]^2 \rightarrow [0, 1]$ be an overlap function having 1 as neutral element. Then, if O is additively generated by a pair (θ, ϑ) , with $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ satisfying the conditions of Theorem 4.1, then O is associative.

PROOF. If 1 is the neutral element of O , then, since $\theta(1) = 0$, one has that:

$$y = O(1, y) = \vartheta(\theta(1) + \theta(y)) = \vartheta(0 + \theta(y)) = \vartheta(\theta(y)). \quad (15)$$

Moreover, since $\theta(y) + \theta(z) \in \text{Ran}(\theta)$, for $y, z \in [0, 1]$, then there exists $w \in [0, 1]$ such that

$$\theta(w) = \theta(y) + \theta(z). \quad (16)$$

It follows that

$$\begin{aligned}
O(x, O(y, z)) &= \vartheta(\theta(x) + \theta(O(y, z))) && \text{by Equation (9)} \\
&= \vartheta(\theta(x) + \theta(\vartheta(\theta(y) + \theta(z)))) && \text{by Equation (9)} \\
&= \vartheta(\theta(x) + \theta(\vartheta(\theta(w)))) && \text{by Equation (16)} \\
&= \vartheta(\theta(x) + \theta(w)) && \text{by Equation (15)} \\
&= \vartheta(\theta(x) + \theta(y) + \theta(z)) && \text{by Equation (16)} \\
&= \vartheta((\theta(x) + \theta(y)) + \theta(z)) && \text{by the associativity of the addition} \\
&= \vartheta(\theta(u) + \theta(z)) && \text{Analogously to Equation (16)} \\
&= \vartheta(\theta(\vartheta(\theta(u))) + \theta(z)) && \text{by Equation (15)} \\
&= \vartheta(\theta(\vartheta(\theta(x) + \theta(y))) + \theta(z)) && \text{by Equation (16)} \\
&= \vartheta(\theta(O(x, y) + \theta(z))) && \text{by Equation (9)} \\
&= O(O(x, y), z) && \text{by Equation (9)}.
\end{aligned}$$

□

The following result is immediate:

Corollary 4.3. *Let $O : [0, 1]^2 \rightarrow [0, 1]$ be an overlap function additively generated by a pair (θ, ϑ) . O is a t-norm if and only if 1 is a neutral element of O .*

Notice that whenever T is a positive continuous t-norm (that is, an overlap function) that is additively generated by a function $t : [0, 1] \rightarrow [0, \infty]$ in the sense of Theorem 2.1, then it is also additively generated by a pair (θ, ϑ) in the sense of Theorem 4.1, where $\theta = t$ and $\vartheta = t^{(-1)}$, and vice-versa.

5. Overlap Functions as T-norms Distorted by Pseudo-Automorphisms and their Additive Generators

In this section, we discuss on which conditions an overlap function may be obtained by the distortion of a t-norm T by the application of the so-called pseudo-automorphism $\mathcal{F}$, showing that, in this case, the additive generator pair of such overlap function may be given in terms of the additive generator of the t-norm T and the pseudo-automorphism $\mathcal{F}$.

Since every automorphism φ is, in fact, a pseudo-automorphism, we show some important results concerning an overlap function when defined as a distortion of a t-norm T by an automorphism φ .

Definition 5.1. *A function $\mathcal{F} : [0, 1] \rightarrow [0, 1]$ is said to be a pseudo-automorphism if the following conditions hold:*

(PA1) $\mathcal{F}$ is increasing;

(PA2) $\mathcal{F}$ is continuous;

(PA3) $\mathcal{F}(x) = 1$ if and only if $x = 1$;

(PA4) $\mathcal{F}(x) = 0$ if and only if $x = 0$.

An automorphism $\varphi : [0, 1] \rightarrow [0, 1]$ is a strictly increasing pseudo-automorphism.

Example 5.1. The function $\mathcal{F}_{a,b,\alpha} : [0, 1] \rightarrow [0, 1]$, for $a, b, \alpha \in]0, 1[$ and $a < b$, defined by

$$\mathcal{F}_{a,b,\alpha}(x) = \begin{cases} \frac{\alpha x}{a} & \text{if } x < a \\ \alpha & \text{if } a \leq x \leq b \\ \frac{(1-\alpha)x + \alpha - b}{1-b} & \text{if } x > b \end{cases}$$

is an example of a family of pseudo-automorphisms that are not automorphisms, since they are strictly increasing.

Theorem 5.1. Let $\mathcal{F} : [0, 1] \rightarrow [0, 1]$ be a pseudo-automorphism. Then, for every positive and continuous t-norm $T : [0, 1]^2 \rightarrow [0, 1]$, the function $O_{\mathcal{F},T} : [0, 1]^2 \rightarrow [0, 1]$, given by:

$$O_{\mathcal{F},T}(x, y) = \mathcal{F}(T(x, y)), \quad (17)$$

is an overlap function.

PROOF. Since $\mathcal{F}$ and T are continuous, then it is immediate that **(O5)** $O_{\mathcal{F},T}$ is continuous, and, by the commutativity of T , **(O1)** $O_{\mathcal{F},T}$ is commutative. Also, since $\mathcal{F}$ is increasing, then **(O4)** $O_{\mathcal{F},T}$ is increasing, and since T is positive, then **(O2)** $O_{\mathcal{F},T}(x, y) = 0$ if and only if $\mathcal{F}(T(x, y)) = 0$ if and only if $T(x, y) = 0$ if and only if $xy = 0$. One also has that **(O3)** $O_{\mathcal{F},T}(x, y) = 1$ if and only if $\mathcal{F}(T(x, y)) = 1$ if and only if $T(x, y) = 1$ if and only if $xy = 1$. Therefore, $O_{\mathcal{F},T}$ is an overlap function. $\square$

$O_{\mathcal{F},T}$ is called the overlap obtained by the distortion of the t-norm T by the pseudo-automorphism $\mathcal{F}$, or the overlap function obtained by a $(\mathcal{F}, T)$ -distortion.

Theorem 5.2. Let $\varphi : [0, 1] \rightarrow [0, 1]$ be an automorphism and $T : [0, 1]^2 \rightarrow [0, 1]$ be a t-norm. The function $O_{\varphi,T} : [0, 1]^2 \rightarrow [0, 1]$, defined by

$$O_{\varphi,T}(x, y) = \varphi(T(x, y)), \quad (18)$$

is an overlap function if and only if T is positive and continuous.

PROOF. $(\Rightarrow)$ Consider that $O_{\varphi,T}$ is an overlap function. Then, by **(O2)** $O_{\varphi,T}(x, y) = 0$ if and only if $xy = 0$. Then, since φ is an automorphism (strictly increasing pseudo-automorphism), it holds that $T(x, y) = 0$ if and only if $\varphi(T(x, y)) = 0$ if and only if $O_{\varphi,T}(x, y) = 0$ if and only if $xy = 0$, which shows that T is positive. On the other hand, by **(O5)**, $O_{\varphi,T}(x, y) = \varphi(T(x, y))$ is continuous. Then, one has that $\varphi^{-1} \circ O_{\varphi,T}(x, y) = T(x, y)$ is also continuous.

$(\Leftarrow)$ It follows directly from Theorem 5.1, since every automorphism is a pseudo-automorphism. $\square$

$O_{\varphi,T}$ is called the overlap obtained by the distortion of the t-norm T by the automorphism φ , or the overlap function obtained by a (φ, T) -distortion.

Lemma 5.1. Let $O_{\varphi,T}$ be the overlap function obtained by the distortion of a t -norm $T : [0, 1]^2 \rightarrow [0, 1]$ by a (pseudo) automorphism $\varphi : [0, 1] \rightarrow [0, 1]$. Then, it holds that

$$\varphi(x) = O_{\varphi,T}(x, 1). \quad (19)$$

PROOF. It follows that $O_{\varphi,T}(x, 1) = \varphi(T(x, 1)) = \varphi(x)$. $\square$

Remark 5.1. Whenever O is an overlap function obtained by the distortion of a t -norm T by a (pseudo) automorphism φ , then, Lemma 5.1 provides a way for defining such (pseudo) automorphism, that is, $\varphi(x) = O(x, 1)$. In the case that φ is an automorphism, then the t -norm can be trivially obtained as $T = \varphi^{-1} \circ O$.

Example 5.2. Consider the overlap function $O_2(x, y) = x^2 y^2$ given in Example 3.1. Then, one can obtain the automorphism $\varphi(x) = O_2(x, 1) = x^2$ and the t -norm

$$T(x, y) = (\varphi^{-1} \circ O_2)(x, y) = \varphi^{-1}(x^2 y^2) = xy = P(x, y),$$

the product t -norm given in Example 2.1. Then, O_2 is an overlap function obtained by the distortion of the product t -norm by the automorphism φ .

Proposition 5.1. There exists an overlap function $O : [0, 1]^2 \rightarrow [0, 1]$ which is not a $(\mathcal{F}, T)$ -distortion of a positive and continuous t -norm $T : [0, 1]^2 \rightarrow [0, 1]$ for any automorphism $\mathcal{F}$.

PROOF. Consider $0 < \alpha < \beta < \gamma < 1$ and the function $O_{\alpha\beta\gamma} : [0, 1]^2 \rightarrow [0, 1]$, defined by:

$$O_{\alpha\beta\gamma}(x, y) = \begin{cases} \min\{x, y\} & \text{if } \min\{x, y\} \leq \alpha \text{ or } \min\{x, y\} > \gamma; \\ \alpha & \text{if } \alpha < \min\{x, y\} < \beta; \\ \alpha + (\gamma - \alpha) \frac{\min\{x, y\} - \beta}{\gamma - \beta} & \text{if } \beta \leq \min\{x, y\} \leq \gamma. \end{cases} \quad (20)$$

In fact, $O_{\alpha\beta\gamma}$ is an overlap function, since it is immediate that $O_{\alpha\beta\gamma}$ **(O1)** is commutative, **(O4)** non-decreasing and **(O5)** continuous. Moreover, one has that **(O2)** $O_{\alpha\beta\gamma}(x, y) = 0$ if and only if $\min\{x, y\} = 0$ if and only if $xy = 0$, and **(O3)** $O_{\alpha\beta\gamma}(x, y) = 1$ if and only if $\min\{x, y\} = 1$ if and only if $xy = 1$. Now, suppose that there exist an automorphism $\varphi : [0, 1] \rightarrow [0, 1]$ and a positive and continuous t -norm $T : [0, 1]^2 \rightarrow [0, 1]$ such that $O_{\alpha\beta\gamma}$ can be determined by (φ, T) -distortion. Therefore, by Lemma 5.1, it follows that $O_{\alpha\beta\gamma}(x, 1)$ is an automorphism. However, this is a contradiction, since

$$O_{\alpha\beta\gamma}(x, 1) = \begin{cases} x & \text{if } x \leq \alpha \text{ or } x > \gamma; \\ \alpha & \text{if } \alpha < x < \beta; \\ \alpha + (\gamma - \alpha) \frac{x - \beta}{\gamma - \beta} & \text{if } \beta \leq x \leq \gamma. \end{cases}$$

is not strict, and, thus, it is not an automorphism. In fact, $O_{\alpha\beta\gamma}(x, 1)$ is a pseudo-automorphism. $\square$

Example 5.3. Consider the Hamacher t-norm $T_H : [0, 1]^2 \rightarrow [0, 1]$, defined by

$$T_H(x, y) = \frac{xy}{2 - x - y + xy},$$

and the function $O_H^{\alpha\beta} : [0, 1]^2 \rightarrow [0, 1]$, given, for $\alpha, \beta \in]0, 1[$ and $\alpha \leq \beta$, by:

$$O_H^{\alpha\beta}(x, y) = \begin{cases} T_H(x, y) & \text{if } T_H(x, y) \leq \alpha \\ \alpha & \text{if } \alpha < T_H(x, y) < \beta \\ \alpha + \frac{T_H(x, y) - \beta}{1 - \beta}(1 - \alpha) & \text{if } T_H(x, y) \geq \beta. \end{cases} \quad (21)$$

Since T_H is commutative, increasing and continuous then it is immediate that $O_H^{\alpha\beta}$ is **(O1)** commutative, **(O4)** non-decreasing and **(O5)** continuous. Moreover, one has that **(O2)** $O_H^{\alpha\beta}(x, y) = 0$ if and only if $T_H(x, y) = 0$ if and only if $xy = 0$. Also, it holds that **(O3)** $O_H^{\alpha\beta}(x, y) = 1$ if and only if $\alpha + \frac{T_H(x, y) - \beta}{1 - \beta}(1 - \alpha) = 1$ if and only if $T_H(x, y) = 1$ if and only if $xy = 1$, since $\alpha \neq 1$. Thus, $O_H^{\alpha\beta}$ is an overlap function. On the other hand, one has that:

$$O_H^{\alpha\beta}(x, 1) = \begin{cases} x & \text{if } x \leq \alpha \\ \alpha & \text{if } \alpha < x < \beta \\ \alpha + \frac{x - \beta}{1 - \beta}(1 - \alpha) & \text{if } x \geq \beta \end{cases} \quad (22)$$

is not an automorphism, since it is not bijective nor strictly increasing. Now, let $\mathcal{F} : [0, 1] \rightarrow [0, 1]$ be the pseudo-automorphism defined by $\mathcal{F}(x) = O_H^{\alpha\beta}(x, 1)$. Then, one has that $O_H^{\alpha\beta}(x, y) = \mathcal{F}(T_H(x, y))$ is an example of an overlap function for which there is no automorphism φ determining $O_H^{\alpha\beta}$ by (φ, T_H) -distortion, but it is determined by a $(\mathcal{F}, T_H)$ -distortion, for the pseudo-automorphism $\mathcal{F}$.

Proposition 5.2. Not every overlap function $O : [0, 1]^2 \rightarrow [0, 1]$ is determined by a $(\mathcal{F}, T)$ -distortion, for a pseudo-automorphism $\mathcal{F} : [0, 1] \rightarrow [0, 1]$ and a positive and continuous t-norm $T : [0, 1]^2 \rightarrow [0, 1]$.

PROOF. Consider the overlap function $O_{mM}(x, y) = \min(x, y) \max(x^2, y^2)$, given in Example 3.1. Suppose that O_{mM} can be determined by a $(\mathcal{F}, T)$ -distortion, for a pseudo-automorphism $\mathcal{F} : [0, 1] \rightarrow [0, 1]$ and a positive and continuous t-norm $T : [0, 1]^2 \rightarrow [0, 1]$. Then, by Lemma 5.1, one has that $\mathcal{F}(x) = O_{mM}(x, 1) = x$, which is an automorphism. It follows that $O_{mM}(x, y) = \mathcal{F}(T(x, y)) = T(x, y)$, which is a contradiction, since O_{mM} is not associative, and, thus, it is not a t-norm. $\square$

Proposition 5.3. Every overlap function $O : [0, 1]^2 \rightarrow [0, 1]$ that can not be obtained by a $(\mathcal{F}, T)$ -distortion is non associative.

PROOF. If O is associative, then, by Corollary 3.1, O is a positive continuous t-norm. Therefore, O is obtained by a (Id, O) -distortion, where $Id : [0, 1] \rightarrow [0, 1]$ is the identity automorphism. $\square$

The following proposition shows that the additive generators of an overlap function obtained by a (φ, T) -distortion, related to an additively generated, positive, continuous t-norm T , can be given in terms of the additive generator of T .

Proposition 5.4. Let $T_t : [0, 1]^2 \rightarrow [0, 1]$ be a positive and continuous t -norm generated by a strictly decreasing function $t : [0, 1] \rightarrow [0, \infty]$ satisfying the conditions of Theorem 2.1, and $O_{\varphi, T_t} : [0, 1]^2 \rightarrow [0, 1]$ be the overlap function obtained by a (φ, T_t) -distortion, for a (pseudo) automorphism $\varphi : [0, 1] \rightarrow [0, 1]$. Then O_{φ, T_t} is additively generated by the pair $(t, \varphi \circ t^{(-1)})$.

PROOF. From Theorems 5.1 (5.2) and 2.1, it follows that

$$O_{\varphi, T_t}(x, y) = \varphi(T_t(x, y)) = \varphi(t^{(-1)}(t(x) + t(y))) = (\varphi \circ t^{(-1)})(t(x) + t(y)).$$

□

Example 5.4. Consider the positive and continuous product t -norm $P(x, y) = xy$, which, as shown in Example 2.4, has an additive generator given by the strictly decreasing function $p : [0, 1] \rightarrow [0, \infty]$, defined by

$$p(x) = \begin{cases} -\ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0, \end{cases}$$

satisfying the conditions of Theorem 2.1, and having as pseudo-inverse the function $p^{(-1)} : [0, \infty] \rightarrow [0, 1]$, given by

$$p^{(-1)}(y) = \begin{cases} e^{-y} & \text{if } y \neq \infty \\ 0 & \text{if } y = \infty, \end{cases}$$

as defined in Example 2.3. For the automorphism $\varphi_1(x) = x^2$, then, according to Proposition 5.4, one has that $(p, \varphi_1 \circ p^{(-1)})$ is an additive generator pair of the overlap function $O_{\varphi_1, P}$. In fact, it follows that

$$O_{\varphi_1, P}(x, y) = (\varphi_1 \circ p^{(-1)})(p(x) + p(y)) = \varphi_1(xy) = x^2 y^2 = O_2(x, y),$$

where O_2 is the overlap function given in Example 3.1. Now, for the automorphism $\varphi_2(x) = \sin \frac{\pi x}{2}$, one obtains the following additively generated overlap function:

$$O_{\varphi_2, P}(x, y) = (\varphi_2 \circ p^{(-1)})(p(x) + p(y)) = \varphi_2(xy) = \sin \frac{\pi}{2} xy.$$

Example 5.5. Consider the overlap function $O_H^{\alpha\beta} : [0, 1]^2 \rightarrow [0, 1]$, defined by Equation (21). According to Example 5.3, there is no automorphism φ determining $O_H^{\alpha\beta}$ by (φ, T_H) -distortion, considering the Hamacher t -norm T_H . However, $O_H^{\alpha\beta}$ can be additively generated by the pair (θ, ϑ) , where $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ are defined, respectively, by:

$$\theta(x) = \begin{cases} \log \frac{2-x}{x} & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and

$$\vartheta(x) = \begin{cases} 0 & \text{if } x = \infty \\ h(x) & \text{if } x \neq \infty \text{ and } h(x) \leq \alpha \\ \alpha & \text{if } \alpha < h(x) < \beta \\ \alpha + \frac{h(x)-\beta}{1-\beta}(1-\alpha) & \text{if } h(x) \geq \beta, \end{cases}$$

where $\alpha, \beta \in]0, 1[$, $\alpha \leq \beta$ and $h(x) : [0, \infty[\rightarrow [0, 1]$ is given by $h(x) = \frac{2}{10^x + 1}$. It is immediate that θ and ϑ satisfy the conditions 1-4 of Corollary 4.1.

Theorem 5.3. *Let $O : [0, 1]^2 \rightarrow [0, 1]$ be an overlap function and $\varphi : [0, 1] \rightarrow [0, 1]$ be an automorphism. O is obtained by a (φ, T) -distortion if and only if $T = \varphi^{-1} \circ O$, with $\varphi(x) = O(x, 1)$, is associative.*

PROOF. ($\Rightarrow$) By Theorem 5.2, there exist a positive and continuous t-norm T and an automorphism φ such that $O(x, y) = O_{\varphi, T}(x, y) = \varphi(T(x, y))$. It is immediate that $T = \varphi^{-1} \circ O$, with $\varphi(x) = O(x, 1)$, according to Lemma 5.1. Moreover, as t-norm, T is associative.

($\Leftarrow$) Consider the function $T = \varphi^{-1} \circ O$, with $\varphi(x) = O(x, 1)$, which is **(T2)** associative. First, we show that T is a t-norm. It is immediate that T is increasing, since φ^{-1} is increasing and O is non-decreasing. Since O is commutative, then it is immediate that T is also **(T1)** commutative. Moreover, one has that **(T3)** $T(x, 1) = \varphi^{-1}(O(x, 1)) = \varphi^{-1}(\varphi(x)) = x$. Therefore, T is a t-norm. Also, since O and φ are continuous, then T is continuous. Finally, since φ is an automorphism (i.e., bijective and increasing), then $T(x, y) = \varphi^{-1}(O(x, y)) = 0$ if and only if $O(x, y) = 0$ if and only if $xy = 0$, which means that T is positive. Then, it is immediate that $O = \varphi \circ T$ is an overlap function obtained by a (φ, T) -distortion. $\square$

Remark 5.2. *A question that arises here is the following: “Is there an additive generator pair for an overlap function that can not be obtained by a $(\mathcal{F}, T)$ -distortion, for a positive and continuous t-norm T and a pseudo-automorphism $\mathcal{F}$?”*

6. Additively Generated Overlap Functions and the Migrativity Property

In this section, we study the influence of the migrativity property on overlap functions obtained by (φ, T) -distortions, and the consequences to their additive generator pairs.

Theorem 6.1. *Let $O : [0, 1]^2 \rightarrow [0, 1]$ be an overlap function and $\varphi : [0, 1] \rightarrow [0, 1]$ be a (pseudo) automorphism. O can be defined by (φ, T) -distortion, by φ and a t-norm $T : [0, 1]^2 \rightarrow [0, 1]$, if and only if O is T -migrative and φ is given by $\varphi(x) = O(x, 1)$.*

PROOF. ($\Rightarrow$) If O can be defined by (φ, T) -distortion, then, by Theorem 5.2 (5.1), one has that $O(x, y) = \varphi(T(x, y))$. It follows that

$$\begin{aligned} O(T(x, y), z) &= \varphi(T(T(x, y), z)) \\ &= \varphi(T(x, T(y, z))) \quad \text{by the associativity of } T \\ &= O(x, T(y, z)), \end{aligned}$$

and, thus, by Definition 2.6, O is T -migrative. On the other hand, by Lemma 5.1, since O is a (φ, T) -distortion, then it holds that $\varphi(x) = O(x, 1)$.

($\Leftarrow$) Consider that O is T -migrative, for a given t-norm T , and the (pseudo) automorphism $\varphi(x) = O(x, 1)$. It follows that:

$$O(x, y) = O(x, T(y, 1)) = O(T(x, y), 1) = \varphi(T(x, y)),$$

and, thus, O is a (φ, T) -distortion. $\square$

Example 6.1. Consider the positive and continuous product t-norm $P(x, y) = xy$ and the overlap function $O_2(x, y) = x^2y^2$, given in Example 3.1. Then, by Theorem 6.1, O_2 is P -migrative, since, as shown in Example 5.4, O_2 is a (φ_1, P) -distortion, that is, $O_2 = O_{\varphi_1, P}$, for $\varphi_1(x) = x^2$. In fact, observe that

$$O_2(P(x, y), z) = O_2(xy, z) = x^2y^2z^2 = O_2(x, yz) = O_2(x, P(y, z)).$$

Corollary 6.1. Let $T : [0, 1]^2 \rightarrow [0, 1]$ be a t-norm additively generated by a continuous function $t : [0, 1] \rightarrow [0, \infty]$ and $O : [0, 1]^2 \rightarrow [0, 1]$ be an overlap function. If O is T -migrative and $\varphi : [0, 1] \rightarrow [0, 1]$, defined by $\varphi(x) = O(x, 1)$, is a (pseudo) automorphism, then O is additively generated by the pair $(t, \varphi \circ t^{(-1)})$.

PROOF. It follows from Proposition 5.4 (Theorem 5.1) and Theorem 6.1. $\square$

Example 6.2. As discussed in Example 6.1, $O_2(x, y) = x^2y^2$ is P -migrative, and, since $\varphi(x) = O_2(x, 1) = x^2$ is an automorphism, then, by Corollary 6.1, O is additively generated by the pair $(t, \varphi \circ t^{(-1)})$, as previously shown in Example 5.4.

Proposition 6.1. [4, Corollary 3.2] An overlap function $O : [0, 1]^2 \rightarrow [0, 1]$ is migrative if and only if there exists a pseudo-automorphism $\mathcal{F} : [0, 1] \rightarrow [0, 1]$ such that O is a $(\mathcal{F}, P)$ -distortion, where P is the product t-norm.

Proposition 6.2. Let $O : [0, 1]^2 \rightarrow [0, 1]$ be a migrative overlap function. Then there exists a pseudo-automorphism $\mathcal{F} : [0, 1] \rightarrow [0, 1]$ such that O is additively generated by the pair $(p, \mathcal{F} \circ p^{(-1)})$, where $p : [0, 1] \rightarrow [0, \infty]$, is defined by

$$p(x) = \begin{cases} -\ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0, \end{cases}$$

and its pseudo-inverse $p^{(-1)} : [0, \infty] \rightarrow [0, 1]$ is given by

$$p^{(-1)}(y) = \begin{cases} e^{-y} & \text{if } y \neq \infty \\ 0 & \text{if } y = \infty, \end{cases}$$

PROOF. If O is migrative, then, by Proposition 6.1, there exists a pseudo-automorphism $\mathcal{F}$ such that O is a $(\mathcal{F}, P)$ -distortion, where P is the product t-norm. From Theorem 5.4, it follows that O is additively generated by the pair $(t, \mathcal{F} \circ t^{(-1)})$, where t is the additive generator of P , with pseudo-inverse $t^{(-1)}$, defined in Example 2.4.

Example 6.3. Consider the overlap function $O : [0, 1]^2 \rightarrow [0, 1]$, given by $O(x, y) = \sin \frac{\pi}{2} xy$, which is migrative, since, for any $\alpha \in [0, 1]$, it holds that

$$O(\alpha x, y) = \sin \frac{\pi}{2} (\alpha x)y = \sin \frac{\pi}{2} x(\alpha y) = O(x, \alpha y).$$

Then, by Proposition 6.1, one has that $O = O_{\varphi, P}$, for a certain automorphism $\varphi : [0, 1] \rightarrow [0, 1]$. It is immediate that such automorphism is given by $\varphi(x) = \sin \frac{\pi}{2}(x)$. Then, by Proposition 6.2, O is additively generated by the pair (θ, ϑ) , with

$$\theta(x) = p(x) = \begin{cases} -\ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0, \end{cases}$$

and

$$\vartheta(x) = (\varphi \circ p^{(-1)})(x) = \begin{cases} \sin \frac{\pi}{2}(e^{-x}) & \text{if } x \neq \infty \\ 0 & \text{if } x = \infty. \end{cases}$$

7. Additively Generated Overlap Functions and the Homogeneity Property

In this section, we analyse the influence of the homogeneity property on overlap functions obtained by (φ, T) -distortions, and the consequences for their additive generator pairs.

Lemma 7.1. *An automorphism $\varphi : [0, 1] \rightarrow [0, 1]$ is homogeneous of order $k \in]0, \infty[$ if and only if $\varphi(x) = x^k$.*

PROOF. $(\Rightarrow)$ If φ is homogeneous of order $k \in]0, \infty[$ then $\varphi(\alpha) = \varphi(\alpha \cdot 1) = \alpha^k \varphi(1) = \alpha^k$, for all $\alpha \in [0, 1]$.

$(\Leftarrow)$ If $\varphi(x) = x^k$, for $k \in]0, \infty[$, then $\varphi(\alpha x) = (\alpha x)^k = \alpha^k x^k = \alpha^k \varphi(x)$, and, then, φ is homogeneous of order k . $\square$

Remark 7.1. *Observe that Lemma 7.1 guaranties that every homogeneous pseudo-automorphism is, in fact, an automorphism.*

Proposition 7.1. *Let $O_{\varphi, T} : [0, 1]^2 \rightarrow [0, 1]$ be a (φ, T) -distortion overlap function, for a given t -norm $T : [0, 1]^2 \rightarrow [0, 1]$ and a k_1 -homogeneous automorphism $\varphi : [0, 1] \rightarrow [0, 1]$, with $k_1 \in]0, \infty[$. T is homogeneous of order $k_2 \in]0, \infty[$ if and only if $O_{\varphi, T}$ is homogeneous of order $k_1 k_2$.*

PROOF. If $O_{\varphi, T}$ is a (φ, T) -distortion overlap function, then one has that $O_{\varphi, T}(x, y) = \varphi(T(x, y))$. Then: $(\Rightarrow)$ If φ and T are homogeneous of order k_1 and k_2 , respectively, then, for all $\alpha \in [0, \infty[$, it holds that $T(\alpha x, \alpha y) = \alpha^{k_1} T(x, y)$ and $\varphi(\alpha x) = \alpha^{k_2} \varphi(x)$. It follows that

$$\begin{aligned} O_{\varphi, T}(\alpha x, \alpha y) &= \\ \varphi(T(\alpha x, \alpha y)) &= \varphi(\alpha^{k_1} T(x, y)) = \alpha^{k_1 k_2} \varphi(T(x, y)) = \alpha^{k_1 k_2} O_{\varphi, T}(x, y), \end{aligned}$$

and, thus, $O_{\varphi, T}$ is homogeneous of order $k_1 k_2$.

$(\Leftarrow)$ Consider that $O_{\varphi, T}$ is homogeneous of order $k_1 k_2$. Then, since O is a (φ, T) -distortion overlap function, one has that $T(x, y) = \varphi^{-1}(O_{\varphi, T}(x, y))$. On the other hand, since φ is homogeneous of order k_1 , then, by Lemma 7.1, $\varphi(x) = x^{k_1}$. Thus, it is immediate that φ^{-1} is homogeneous of order $\frac{1}{k_1}$. It follows that

$$\begin{aligned} T(\alpha x, \alpha y) &= \varphi^{-1}(O_{\varphi, T}(\alpha x, \alpha y)) = \\ \varphi^{-1}(\alpha^{k_1 k_2} O_{\varphi, T}(x, y)) &= \alpha^{\frac{k_1 k_2}{k_1}} \varphi^{-1}(O_{\varphi, T}(x, y)) = \alpha^{k_2} T(x, y). \end{aligned}$$

Therefore, T is homogeneous of order $k_2 \in]0, \infty[$. $\square$

Theorem 7.1. *Let $T_t : [0, 1]^2 \rightarrow [0, 1]$ be a t -norm additively generated by a k_1 -order homogenous function $t : [0, 1] \rightarrow [0, \infty]$, and consider a k_2 -order homogeneous automorphism $\varphi : [0, 1] \rightarrow [0, 1]$. If an overlap function $O_{\varphi, T_t} : [0, 1]^2 \rightarrow [0, 1]$ is a (φ, T_t) -distortion, then O_{φ, T_t} is homogeneous of order k_2 .*

PROOF. If O_{φ, T_t} is a (φ, T_t) -distortion for a t-norm T_t and an automorphism φ , where t is homogenous of order k_1 and φ is homogenous of order k_2 , then it follows that:

$$\begin{aligned} O_{\varphi, T_t}(\alpha x, \alpha y) &= \varphi(T_t(\alpha x, \alpha y)) \\ &= \varphi(\alpha T_t(x, y)) \quad \text{by Proposition 2.1} \\ &= \alpha^{k_2} \varphi(T_t(x, y)) \\ &= \alpha^{k_2} O_{\varphi, T_t}(x, y), \end{aligned}$$

which shows that O_{φ, T_t} is homogeneous of order k_2 . $\square$

Theorem 7.2. Let $O_{\theta, \vartheta} : [0, 1]^2 \rightarrow [0, 1]$ be an overlap function additively generated by the pair (θ, ϑ) , with $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$. If θ and ϑ are homogenous of order k_1 and k_2 , respectively, then $O_{\theta, \vartheta}$ is homogeneous of order $k_1 k_2$.

PROOF. If $O_{\theta, \vartheta} : [0, 1]^2 \rightarrow [0, 1]$ is an overlap function additively generated by the pair (θ, ϑ) , and θ and ϑ are homogenous of order k_1 and k_2 , respectively, then it follows that:

$$\begin{aligned} O_{\theta, \vartheta}(\alpha x, \alpha y) &= \vartheta(\theta(\alpha x) + \theta(\alpha y)) \quad \text{by Equation 13} \\ &= \vartheta(\alpha^{k_1}(\theta(x) + \theta(y))) \\ &= (\alpha^{k_1})^{k_2} \vartheta(\theta(x) + \theta(y)) \\ &= \alpha^{k_1 k_2} \vartheta(\theta(x) + \theta(y)) \\ &= \alpha^{k_1 k_2} O_{\theta, \vartheta}(x, y) \quad \text{by Equation 13} \end{aligned}$$

which shows that $O_{\theta, \vartheta}$ is homogeneous of order $k_1 k_2$. $\square$

Proposition 7.2. If an overlap function $O : [0, 1]^2 \rightarrow [0, 1]$ is migrative and homogeneous of order k then it is additively generated by the pair (θ, ϑ) , where $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ are defined, respectively, by:

$$\theta(x) = \begin{cases} -\ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and

$$\vartheta(x) = \begin{cases} e^{\frac{-kx}{2}} & \text{if } x \neq \infty \\ 0 & \text{if } x = \infty \end{cases}$$

PROOF. If O is migrative and homogenous of order k then, as shown in [3], it holds that $O(x, y) = (xy)^{\frac{k}{2}}$. Then, it follows that O is a (φ, P) -distortion, where P is the product t-norm and $\varphi(x) = x^{\frac{k}{2}}$. Then, by Proposition 6.2, it follows that O is additively generated by the pair (θ, ϑ) , where $\theta : [0, 1] \rightarrow [0, \infty]$ is defined by:

$$\theta(x) = \begin{cases} -\ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and $\vartheta : [0, \infty] \rightarrow [0, 1]$ is such that $\vartheta = \varphi \circ p^{(-1)}$, that is,

$$\vartheta(x) = \begin{cases} e^{\frac{-kx}{2}} & \text{if } x \neq \infty \\ 0 & \text{if } x = \infty \end{cases}.$$

$\square$

8. Additively Generated Overlap Functions and the Idempotency Property

In this section, we study the relation of the idempotency property with overlap functions obtained by (φ, T) -distortions, and the consequences to their additive generator pairs.

Proposition 8.1. *The unique idempotent (φ, T) -distortion overlap function $O_{\varphi, T} : [0, 1]^2 \rightarrow [0, 1]$ for an idempotent t-norm $T : [0, 1]^2 \rightarrow [0, 1]$ and a given automorphism $\varphi : [0, 1] \rightarrow [0, 1]$ is the minimum overlap, given by $O_{\varphi, T}(x, y) = \min\{x, y\}$.*

PROOF. As it is well known, there exists just a unique idempotent t-norm, namely, the minimum t-norm. For an automorphism φ , if $O_{\varphi, \min}$ is idempotent, then

$$\varphi(x) = \varphi(\min\{x, x\}) = O_{\varphi, \min}(x, x) = x,$$

that is, φ is the identity function Id , and, thus, $O_{I, \min} = \min$. $\square$

Corollary 8.1. *The overlap function $O_{I, \min}$ is not additively generated.*

PROOF. It is immediate, since $O_{I, \min}(x, y) = \min(x, y)$, and it is well known that the minimum t-norm is not additively generated. $\square$

Example 8.1. *There exist idempotent overlap functions that are (φ, T) -distortions of non idempotent t-norms. For example, the overlap function $O(x, y) = \sqrt{xy}$ is a (φ, P) -distortion for the automorphism $\varphi(x) = \sqrt{x}$ and the product t-norm $P(x, y) = xy$. It is immediate that P is not idempotent. However, it holds that $O(x, x) = \sqrt{xx} = x$, and, thus, O is idempotent. In fact, O is also migrative and homogeneous of order 1. From Proposition 7.2, O is additively generated by the pair (θ, ϑ) , where $\theta : [0, 1] \rightarrow [0, \infty]$ and $\vartheta : [0, \infty] \rightarrow [0, 1]$ are defined, respectively, by:*

$$\theta(x) = \begin{cases} -\ln x & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and

$$\vartheta(x) = \begin{cases} e^{-\frac{x}{2}} & \text{if } x \neq \infty \\ 0 & \text{if } x = \infty \end{cases}.$$

Proposition 8.2. *Not every idempotent overlap function $O : [0, 1]^2 \rightarrow [0, 1]$ is determined by a (φ, T) -distortion, for a (pseudo) automorphism $\varphi : [0, 1] \rightarrow [0, 1]$ and a positive and continuous t-norm $T : [0, 1]^2 \rightarrow [0, 1]$.*

PROOF. Consider the overlap function $O_{DB} : [0, 1]^2 \rightarrow [0, 1]$, defined by:

$$O_{DB}(x, y) = \begin{cases} \frac{2xy}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0; \end{cases}$$

which is the idempotent overlap function introduced in Example 3.1. Suppose that O can be determined by a (φ, T) -distortion, for a (pseudo) automorphism $\varphi : [0, 1] \rightarrow [0, 1]$ and a positive and continuous t-norm $T : [0, 1]^2 \rightarrow [0, 1]$. Then, by Lemma 5.1,

one has that $\varphi(x) = O(x, 1) = \frac{2x}{x+1}$, which is clearly an automorphism, with $\varphi^{-1}(x) = \frac{x}{2-x}$. Then, one has that $O_{DB} = \varphi \circ T$, and, as in Remark 5.1, it follows that

$$T(x, y) = (\varphi^{-1} \circ O_{DB})(x, y) = \begin{cases} \frac{xy}{x+y-xy} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0 \end{cases}.$$

However,

$$T(0.9, T(0.5, 0.8)) \approx 0.7347 \neq 0.4236 \approx T(T(0.9, 0.5), 0.8),$$

and, thus, this is not a t-norm. $\square$

9. Conclusion

In this paper, we introduced the notion of additive generator pair of overlap functions, presenting results and examples. We hope that, for applications, the theoretical developments in this paper may help to reduce the computational complexity of algorithms which make use of overlap functions as well as to provide a more systematic methodology for their choice in a given problem.

Given a continuous and positive t-norm T and a pseudo-automorphism $\mathcal{F}$, we have presented an extensive study of the overlap function obtained by the distortion of T by the application of $\mathcal{F}$, showing that the additive generator pair of such overlap function may be given in terms of $\mathcal{F}$ and the additive generators of T . We have also analysed the influence of the migrativity, homogeneity and idempotency properties in the overlap functions obtained by such distortion and their respective additive generator pairs.

Future theoretical work is concerned with the study of the concept of additive generator pair of a grouping function [2, 3, 4], extending the introductory work presented in [26], analysing how some of the properties that can be demanded to a grouping function can be expressed in terms of its generator pair. We also intend to extend the concepts of additive generator pairs of overlap and grouping functions to the interval-valued setting, following the approach adopted in [27, 28, 29, 30, 31, 32].

Then, we are aiming at applications in the context of hybrid BDI-fuzzy [33]⁵ agent models, commonly used in social simulation [35, 36, 37, 38], where the evaluation of social values and exchanges are of a qualitative and subjective nature [39, 40, 41]. We noticed that overlap and grouping functions can be used for dealing with indifference and incomparability when reasoning on the agent's fuzzy belief base, where a kind of weak preference relation may be defined. The computational complexity in this kind of problem is crucial when dealing with large belief bases.

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⁵BDI stands for "Beliefs, Desires, Intentions", a particular cognitive agent model introduced in [34].

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