New Analytical Solutions to the Two-phase Water Faucet Problem

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ABSTRACT

The one-dimensional water faucet problem is one of the classical benchmark problems originally proposed by Ransom to study the two-fluid two-phase flow model. With simplifications, such as massless gas phase and no wall and interfacial frictions, analytical solutions had been previously obtained for the transient liquid velocity and void fraction distribution. The water faucet problem and its analytical solutions have been widely used for code assessment, benchmark, and numerical verification. In this work, we present a new set of analytical solutions to the water faucet problem at the steady-state condition, with the gas-phase density's effect on pressure distribution considered. This new set of analytical solutions is used in a rigorous numerical verification process from which the anticipated second-order spatial accuracy is achieved for a second-order spatial discretization scheme. On the contrary, the same anticipated order of accuracy could not be obtained using the Ransom solutions as the reference. In addition, extended Ransom transient solutions for the gas-phase velocity and pressure are derived with the assumption of decoupled liquid and gas pressures. Numerical benchmark on the extended Ransom solutions is also presented.

KEYWORDS

Two-phase flow, water faucet problem, analytical solutions, numerical verification

1. INTRODUCTION

The water faucet problem is one of the classical two-phase flow benchmark problems originally proposed by Ransom (Hewitt et al., 1987) to study the two-fluid two-phase flow model. This test problem and its analytical solutions have been widely used for the purposes of code assessment, benchmark, and numerical verification for two-phase flow simulations. The problem was originally proposed to test the interaction of the body force terms with the acceleration terms in the momentum equations of the two-phase flow models. With certain simplifications, such as a massless gas phase, no wall and interfacial frictions, and no pressure variations in the liquid phase, analytical solutions have been previously

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obtained by Ransom for the transient liquid-phase velocity and void fraction distribution (Idaho National Laboratory, 2012). This problem has been chosen for assessing the reactor safety analysis code, such as RELAP5-3D (Idaho National Laboratory, 2012) and MARS (Lim, H., et al., 2003). It has also been widely used in the studies on new numerical algorithms (Saleem and Kozlowski, 2014, 2016; Coquel et al., 1997; García-Cascales, 2001; Benkhaldoun and Quivy, 2006; Jerez-Carrizales et al., 2014; Munkejord 2006b; Ghidaglia et al., 2001; Paillère et al., 2003), on new two-phase models (Gallouet et al., 2004; Loilier and Omgba-Essama, 2005; Munkejord 2006a), and on the study of the non-hyperbolic nature of the single-pressure two-phase two-fluid models (Dinh et al., 2003; Fullmer et al., 2014). In most of these existing studies, the analytical solutions proposed by Ransom were used as references for comparing the numerical results. Only a few of those studies (Saleem and Kozlowski, 2014, 2016; Coquel et al., 1997; Munkejord 2006b; Loilier and Omgba-Essama, 2005; Munkejord 2006a) performed mesh convergence studies, where the Ransom solutions are used as the reference analytical solutions. These convergence studies were performed in transient situations in which spatial discontinuities are still present in the solutions. As expected, the anticipated order of spatial accuracy cannot be obtained due to the discontinuities in the solutions. The discussion on spatial convergence over spatial discontinuity can be found in Section 8.7 of LeVeque's book (LeVeque, 2004). These incomplete mesh convergence studies hide the fact that the original Ransom solutions do not give a very good approximation to the true steady-state solution, although they are very close approximations.

In our previous work (Zou et al., 2015a, b), the Ransom solutions were used for the convergence study of a high-resolution spatial discretization scheme using staggered grids. The high-resolution spatial discretization scheme has a second-order spatial accuracy for solutions in smooth regions (Zou et al., 2015a). However, when the Ransom solutions are used as the reference, it was found that a second order of spatial accuracy could not be obtained, even at the steady-state condition where all solutions are smooth in space. By carefully examining the water faucet problem and the single-pressure two-phase flow model, we concluded that new analytical solutions could be derived for the steady-state condition. In our previous work, highly simplified equations of states were used for both phases, and in this work, the realistic water/steam properties are used for the two phases. In this paper, the derivation of the new analytical solutions for the steady-state conditions will be presented for the water faucet problem for two conditions assuming (1) constant phasic densities and (2) linearized phasic densities. An extension of the Ransom solutions to include the transient distributions of gas-phase velocity and pressure will also be presented. A correction is made on our previous work (Zou et al., 2015b), when evaluating the pressure jump across the spatial discontinuity. Numerical verifications using the high-resolution spatial

discretization scheme will be presented for comparison with both the transient and the steady-state analytical solutions.

2. TWO-PHASE FLOW MODEL AND WATER FAUCET PROBLEM

2.1. Water Faucet Problem Description

Originally proposed by Ransom (Hewitt et al., 1987), the one-dimensional water faucet problem has been widely used in the two-phase flow model studies. The problem consists of a vertical pipe of length L . Initially, the pipe is filled with a uniform column of liquid moving with initial velocity $u_{l,init}$, and an annulus of gas moving with initial velocity $u_{g,init} = 0$. The initial void fraction (gas-phase volume fraction) is $\alpha_{g,init}$, and the initial pressure is p_{init} . Due to the gravity effect, the liquid column accelerates and becomes thinner with time. A schematic drawing of the water faucet problem is shown in Figure 1. The problem can be formulated using the two-fluid two-phase flow model. For the two-phase flow model, we are interested in the single-pressure, two-fluid, two-phase flow equations, which are commonly used in existing system analysis codes, such as RELAP5 (U.S. Nuclear Regulatory Commission, December 2001), TRAC (U.S. Nuclear Regulatory Commission, April 2001), and TRACE (U.S. Nuclear Regulatory Commission, 2010). The water faucet problem is a hydrodynamics problem that the internal energies of both phases do not change. Thus, for simplicity, the energy equations of both phases are omitted in this paper. The water faucet problem also assumes no wall friction and no interfacial friction between the two phases. Based on these simplifications and assumptions, a four-equation set for the two-phase flow could be formulated as follows:

$$\frac{\partial(\alpha_l \rho_l)}{\partial t} + \frac{\partial(\alpha_l \rho_l u_l)}{\partial x} = 0 \quad (1)$$

$$\frac{\partial(\alpha_g \rho_g)}{\partial t} + \frac{\partial(\alpha_g \rho_g u_g)}{\partial x} = 0 \quad (2)$$

$$\frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial x} + \frac{1}{\rho_l} \frac{\partial p}{\partial x} - g = 0 \quad (3)$$

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + \frac{1}{\rho_g} \frac{\partial p}{\partial x} - g = 0 \quad (4)$$

in which subscripts l and g denote the liquid and gas phase, respectively. The variables to be solved from this set of equations are p , α_g , u_l , and u_g , which are pressure, void fraction, liquid-phase velocity, and gas-phase velocity, respectively. Note that $\alpha_l = 1 - \alpha_g$. This set of equations is closed by additional equations of state for both phases. The densities of the liquid water and steam are provided as functions of pressure

and temperature (which is a given constant in this work), using a computer code developed (Zou et al., 2014) based on the International Association for the Properties of Water and Steam (IAPWS) publication (Wagner and Pruss, 2002).

With certain simplifications such as massless gas phase, no wall and interfacial frictions, and no pressure variations in the liquid phase, analytical solutions had been previously obtained for the transient liquid velocity and void fraction distribution (Hewitt et al., 1992) (hereafter referred to as the Ransom solutions). Some details of the derivation can also be found in these references (Idaho National Laboratory, 2012; García-Cascales, 2001; and Idaho National Engineering Laboratory, 1994).

Figure 1. Schematic drawing of the two-phase water faucet problem.

The Ransom solutions provide the transient distributions of liquid velocity and void fraction along the pipe length:

$$u_l(x, t) = \begin{cases} \sqrt{u_{l,init}^2 + 2gx} & \text{if } x \leq x_d \\ u_{l,init} + gt & \text{otherwise} \end{cases} \quad (5)$$

and,

$$\alpha_g(x, t) = \begin{cases} 1 - \frac{(1 - \alpha_{g,init})u_{l,init}}{\sqrt{u_{l,init}^2 + 2gx}} & \text{if } x \leq x_d \\ \alpha_{g,init} & \text{otherwise} \end{cases} \quad (6)$$

where $x_d = u_{l,init}t + gt^2/2$ is the location of the discontinuity in void fraction. Under the steady-state conditions, the Ransom solutions (Equations (5)] and (6)) simply become,

$$u_{l,ss}(x, t) = \sqrt{u_{l,init}^2 + 2gx} \quad (7)$$

and

$$\alpha_{g,ss}(x, t) = 1 - \frac{(1 - \alpha_{g,init})u_{l,init}}{\sqrt{u_{l,init}^2 + 2gx}} \quad (8)$$

in which, the subscript ‘ss’ denotes the steady-state condition.

2.2. Finite Volume Method

In this [paper](#), a traditional first-order upwind and a high-resolution (second-order in smooth regions) upwind spatial discretization scheme on staggered grid mesh are used for the purpose of numerical verification. The spatial discretization scheme used in our previous [papers](#) (Zou et al., 2015a, b, 2016a, b, c, d) for both single- and two-phase flow thermal-hydraulics problems, proved appropriate in solving flow problems. The high-resolution upwind scheme can be seen as an extension to the traditional first order upwind scheme, which is obtained by introducing the linear reconstruction of the variable solutions into the traditional first-order upwind method (see the original discussion in Stelling and Duinmeijer, 2003). For the purpose of completeness, the spatial discretization scheme is briefly discussed in this section.

For the staggered grid mesh commonly used in existing reactor safety system codes, scalar variables (such as pressure and void fraction) are arranged in cell centers, while vector variables (such as phasic velocities) are arranged on cell edges (Figure 2).

Figure 2. Schematic illustration of the staggered grid mesh arrangement.

One major difference between the high-resolution scheme and the first-order scheme is the way in which the advection terms are treated, or more specifically, how edge values are estimated for flux evaluation. Though, in both schemes, the discretization of the advection terms uses the same concept of donor cells, the donor values used are different for these two schemes. Using [the advection term in the gas-phase mass equation as an example for illustration](#), regardless of whether the traditional first-order upwind method or high-resolution method is used, it is discretized as,

$$\left. \frac{\partial(\alpha_g \rho_g u_g)}{\partial x} \right|_i = \frac{1}{\Delta x} [u_{g,i+1/2}(\alpha_g \rho_g)_{i+1/2}^* - u_{g,i-1/2}(\alpha_g \rho_g)_{i-1/2}^*] \quad (9)$$

where, the superscript ‘*’ indicates a donor cell value. For the traditional first-order method, donor cell value is the cell-averaged value from an upstream cell depending on flow direction,

$$(\alpha_g \rho_g)_{i+1/2}^* = \begin{cases} \alpha_{g,i} \rho_{g,i} & \text{if } u_{g,i+1/2} > 0 \\ \alpha_{g,i+1} \rho_{g,i+1} & \text{otherwise} \end{cases} \quad (10)$$

For the high-resolution version of the discretization, the donor cell concept is still used, but cell-averaged values are replaced with linearly reconstructed cell edge values,

$$(\alpha_g \rho_g)_{i+1/2}^* = \begin{cases} \alpha_{g,i+1/2}^- \rho_{g,i+1/2}^- & \text{if } u_{g,i+1/2} > 0 \\ \alpha_{g,i+1/2}^+ \rho_{g,i+1/2}^+ & \text{otherwise} \end{cases} \quad (11)$$

where, the superscript ‘+’ and ‘-’ denote the reconstructed values on cell edges, shown in Figure 3. The discretization of the advection term in the momentum equations is slightly different. [Again, regardless of whether the traditional first-order upwind method or high-resolution method is used](#), it is discretized as,

$$u_g \left. \frac{\partial u_g}{\partial x} \right|_{i+1/2} = u_{g,i+1/2} \frac{u_{g,i+1}^* - u_{g,i}^*}{\Delta x} \quad (12)$$

For the first-order method,

$$u_{g,i+1}^* - u_{g,i}^* = \begin{cases} u_{g,i+1/2} - u_{g,i-\frac{1}{2}} & \text{if } u_{g,i+1/2} > 0 \\ u_{g,i+3/2} - u_{g,i+\frac{1}{2}} & \text{otherwise} \end{cases} \quad (13)$$

and for the high-resolution method,

$$u_{g,i+1}^* - u_{g,i}^* = \begin{cases} u_{i+1}^- - u_i^- & \text{if } u_{g,i+1/2} > 0 \\ u_{i+1}^+ - u_i^+ & \text{otherwise} \end{cases} \quad (14)$$

For high-resolution scheme, the reconstructions of variables are calculated using the slope limiter concept. For example, the reconstruction of scalar variable ϕ in the i^{th} volume can be calculated as,

$$\phi_{i+1/2}^- = \phi_i + \frac{\Delta x}{2} \phi_{x,i} \quad (15)$$

in which, $\phi_{x,i}$ is the reconstructed slope,

$$\phi_{x,i} = \frac{\phi_{i+1} - \phi_i}{\Delta x} \eta(r) \quad (16)$$

in which, $r = (\phi_i - \phi_{i-1})/(\phi_{i+1} - \phi_i)$ is the nonlinear indicator; $\eta(r)$ the limiter function. The limiter function can be obtained by any standard slope limiter schemes, such as the Van Albada slope limiter used in this work (Toro, 2009),

$$\eta(r) = \begin{cases} \frac{r + r^2}{1 + r^2} & \text{if } r \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

Figure 3. Schematic illustration of the first-order (blue) and high-resolution (red) spatial discretization schemes for the staggered grid mesh. In this plot, ϕ represents scalar properties (i.e., p and α). Reconstruction for velocities can be done similarly.

For time integration schemes, two fully implicit methods are used in this paper, including the first-order BDF1 (Backward Euler) and second-order BDF2 schemes. [Here we use the first-order BDF1 method as an example to summarize the discretized equations. Replacing BDF1 scheme with BDF2 scheme would be straightforward.](#) Discretized equations are shown in the following:

$$\frac{\alpha_{l,i}^{n+1} \rho_{l,i}^{n+1} - \alpha_{l,i}^n \rho_{l,i}^n}{\Delta t} + \frac{1}{\Delta x} [u_{l,i+1/2}^{n+1} (\alpha_l \rho_l)_{i+1/2}^{n+1,*} - u_{l,i-1/2}^{n+1} (\alpha_l \rho_l)_{i-1/2}^{n+1,*}] = 0 \quad (18)$$

$$\frac{\alpha_{g,i}^{n+1} \rho_{g,i}^{n+1} - \alpha_{g,i}^n \rho_{g,i}^n}{\Delta t} + \frac{1}{\Delta x} [u_{g,i+1/2}^{n+1} (\alpha_g \rho_g)_{i+1/2}^{n+1,*} - u_{g,i-1/2}^{n+1} (\alpha_g \rho_g)_{i-1/2}^{n+1,*}] = 0 \quad (19)$$

$$\frac{u_{l,i+1/2}^{n+1} - u_{l,i+1/2}^n}{\Delta t} + u_{l,i+1/2}^{n+1} \frac{u_{l,i+1}^{*,n+1} - u_{l,i}^{*,n+1}}{\Delta x} + \frac{1}{\bar{\rho}_{l,i+1/2}^{n+1}} \frac{p_{i+1}^{n+1} - p_i^{n+1}}{\Delta x} - g = 0 \quad (20)$$

$$\frac{u_{g,i+1/2}^{n+1} - u_{g,i+1/2}^n}{\Delta t} + u_{g,i+1/2}^{n+1} \frac{u_{g,i+1}^{*,n+1} - u_{g,i}^{*,n+1}}{\Delta x} + \frac{1}{\bar{\rho}_{g,i+1/2}^{n+1}} \frac{p_{i+1}^{n+1} - p_i^{n+1}}{\Delta x} - g = 0 \quad (21)$$

In these discretized equations, superscripts ‘ n ’ and ‘ $n+1$ ’ denote the old and new time step, respectively; superscript ‘ $*$ ’ denotes the donor cell values that have been described previously in this section. The averaged values on the cell edges, $\bar{\rho}_{l,i+1/2}^{n+1}$ and $\bar{\rho}_{g,i+1/2}^{n+1}$, are the arithmetically averaged values from its two neighbor cells. It is also worth noting that the pressure gradient is discretized in a central difference way. Equations (18)–(21) represent a set of nonlinear equations for the discretized unknown vector, $\mathbf{U} = [\dots, \alpha_g, p_i, u_{l,i+1/2}, u_{g,i+1/2}, \dots]^T$. In this paper, a Jacobian-free Newton–Krylov (JFNK) method was used to solve the nonlinear equations. This method has been successfully used in our previous studies, and more details of this method can be found in these papers (Zou et al., 2016a, b, c, d).

3. DERIVATION OF NEW ANALYTICAL SOLUTIONS

In this section, analytical derivation work is presented in two distinct parts. The first and most important part is the derivation of a set of new analytical solutions to the water faucet problem at the steady-state condition. The derivations strictly follow the single-pressure constraint imposed in the single-pressure equation set, and thus they are the exact solutions to the two-fluid model. A second part is to explore the transient behaviors of pressure and gas-phase velocity by extending the original Ransom solutions, which inherits Ransom’s assumption of decoupled liquid-and gas-phase pressures. The analytical solutions from the first part can be used as a reference in a rigorous verification process, while those from the second part can provide the insight on the complete picture of the water faucet transient behaviors.

3.1. New Analytical Solutions at the Steady State Condition

The derivation of the analytical solutions at the steady state is relatively simple. However, to the authors’ best knowledge, it is not available in literatures. The derivation simply starts from the steady-state gas-phase mass equation (Equation (2)) without the transient term). By integrating from 0 to x , and applying gas-phase velocity boundary condition, $u_g(x = 0) = 0$, it can be obtained that,

$$u_{g,ss}(x) = 0 \quad (22)$$

in which, subscript ‘ss’ denotes the steady-state condition. By substituting Equation (22) into the steady-state gas-phase momentum equation (Equation (4) without transient term), it can be obtained as:

$$\frac{1}{\rho_g} \frac{dp}{dx} - g = 0 \quad (23)$$

Equation (23) can be integrated to obtain the pressure function, $p(x)$. For the purpose of demonstration, the nearly incompressible approximation was used for both the liquid and the gas phase, which only introduces trivial errors. Both phasic densities are evaluated at the outlet pressure condition. Thus, the pressure distribution can be obtained as,

$$p_{ss}(x) = p_{outlet} - \rho_g g (L - x) \quad (24)$$

The liquid-phase velocity at the steady state can be obtained by substituting Equation (24) into the steady-state liquid-phase momentum equation, such that,

$$u_{l,ss}(x) = \sqrt{u_{l,init}^2 + 2g_{eff}x} \quad (25)$$

in which, g_{eff} is the effective gravity considering the effect of gas-phase density on pressure distribution. It is defined as,

$$g_{eff} = g \left(1 - \frac{\rho_g}{\rho_l} \right) \quad (26)$$

The void fraction distribution at the steady state can be obtained from the liquid-phase mass equation:

$$\alpha_{g,ss}(x) = 1 - \frac{(1 - \alpha_{g,inlet})u_{l,inlet}}{u_{l,ss}(x)} \quad (27)$$

Comparing this set of analytical solutions, Equations (25) and (27), with the Ransom solutions, Equations (7) and (8), it can be found that the only difference is in our analytical solutions, the gravity in the Ransom solutions is replaced with the newly defined effective gravity constant. In other words, the massless gas-phase assumption used in Ransom’s derivation is removed in this work. Also, it is noted that the new solution of Equations (25) and (27) exactly reduce to Equations (7) and (8), if the massless gas-phase assumption could be made. In addition, if the incompressible assumption is relaxed, the analytical solutions can be re-derived. As an example, the re-derived analytical solutions using the linearized assumptions are shown in Appendix A.

3.2. Extension to the Original Ransom Solutions

In the previous section, analytical solutions were derived for the steady state condition; however, transient solutions could not be derived if single pressure assumption is followed. On the other hand, the original Ransom solutions provide transient solutions to the void fraction and liquid-phase velocity only, while pressure- and gas-phase velocity are not available. Possibly because of a lack of references, many of the existing work chose not to show their results on these two quantities. In this section, we would like to explore the transient behaviors of these two quantities while still following Ransom's original assumptions. Therefore, the derivation directly starts from the two-phase equations: Equations(1)–(4), and the Ransom solutions: Equations (5) and (6). Following Ransom's original assumptions, the liquid phase admits a 'free fall' motion, and the pressure gradient in the liquid phase is ignored (a uniform pressure distribution). We also assume that the pressures for the liquid phase and gas phase can be different (decoupled pressure assumption), which is [an extremely different constraint than what was assumed in the previous derivations](#). Here, we further assume that the gas phase is also nearly incompressible.

The gas-phase velocity can be obtained by integrating the gas-phase mass equation, from 0 to x ,

$$\int_0^x \left[\frac{\partial(\alpha_g \rho_g)}{\partial t} + \frac{\partial(\alpha_g \rho_g u_g)}{\partial s} \right] ds = 0 \quad (28)$$

With the subsequent manipulations: (1) applying Leibniz's rule on the first term on the left-hand side; (2) applying a gas-phase velocity boundary condition $u_g(0, t) = 0$; and 3) assuming ρ_g is constant (nearly incompressible); the following is obtained,

$$\frac{d}{dt} \left[\int_0^x \alpha_g(s, t) ds \right] + \alpha_g(x, t) u_g(x, t) = 0 \quad (29)$$

By substituting the known void fraction distribution, Equation (6), into Equation (29), the gas-phase velocity is obtained as,

$$u_g(x, t) = \begin{cases} 0 & \text{if } x \leq u_{l,init}t + gt^2/2 \\ -\frac{1 - \alpha_{g,init}}{\alpha_{g,init}} gt & \text{otherwise} \end{cases} \quad (30)$$

The transient solution for the gas-phase pressure can be obtained by substituting Equation (30) into the gas-phase momentum in Equation (4); and then integrating from 0 to x . However, due to the discontinuity in the gas-phase velocity, the integration needs to be carried out in three parts to obtain the entire pressure distribution in the pipe.

The [first part](#) of the integration covers the region beyond the discontinuity point, x_d . Integrating from a position beyond the discontinuity point ($x > x_d$) to L ,

$$\int_x^L \left[\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial s} + \frac{1}{\rho_g} \frac{\partial p}{\partial s} - g \right] ds = 0 \quad (31)$$

yields the following pressure distribution,

$$p(x) = p_{outlet} - \frac{\rho_g g (L - x)}{\alpha_{g,init}} \quad \text{for } x > x_d \quad (32)$$

The [second part](#) of the integration covers the narrow region across the discontinuity point where the pressure jump occurs. The pressure jump can be obtained by integrating the gas-phase momentum equation near the discontinuity, as shown in Figure 4,

$$\lim_{\Delta x \rightarrow 0} \int_{x_d - \Delta x}^{x_d + \Delta x} \left[\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial s} + \frac{1}{\rho_g} \frac{\partial p}{\partial s} - g \right] ds = 0 \quad (33)$$

First, by taking the limit $\Delta x \rightarrow 0$, the fourth term (gravity term) in the integral becomes zero. The contribution of the first term can be obtained by applying the Leibniz rule,

$$\begin{aligned} & \lim_{\Delta x \rightarrow 0} \int_{x_d - \Delta x}^{x_d + \Delta x} \frac{\partial u_g(s, t)}{\partial t} ds \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{d}{dt} \int_{x_d - \Delta x}^{x_d + \Delta x} u_g(s, t) ds - \frac{d(x_d + \Delta x)}{dt} u_g(x_d + \Delta x, t) + \frac{d(x_d - \Delta x)}{dt} u_g(x_d - \Delta x, t) \right] \\ &= 0 - (u_{l,init} + gt) \left(-\frac{1 - \alpha_{g,init}}{\alpha_{g,init}} gt \right) + 0 \\ &= \frac{1 - \alpha_{g,init}}{\alpha_{g,init}} gt (u_{l,init} + gt) \end{aligned} \quad (34)$$

It is noted that, the contribution of this term was incorrectly omitted in our previous work (Zou et al., 2015b). The remaining two terms (i.e., the second and third terms on the left-hand side of Equation [(33)]) yield the following contributions,

$$\lim_{\Delta x \rightarrow 0} \int_{x_d - \Delta x}^{x_d + \Delta x} \left[u_g \frac{\partial u_g}{\partial s} + \frac{1}{\rho_g} \frac{\partial p}{\partial s} \right] ds = \frac{u_{g,2}^2 - u_{g,1}^2}{2} + \frac{p_2 - p_1}{\rho_g} \quad (35)$$

Combining Equations (33), (34), and (35), the pressure jump across the discontinuity can be obtained as,

$$\begin{aligned} p_1 - p_2 &= \frac{1}{2} \rho_g (u_{g,2}^2 - u_{g,1}^2) + \rho_g \frac{1 - \alpha_{g,init}}{\alpha_{g,init}} gt (u_{l,init} + gt) \\ &= \frac{1}{2} \rho_g \left(\frac{1 - \alpha_{g,init}}{\alpha_{g,init}} gt \right)^2 + \rho_g \frac{1 - \alpha_{g,init}}{\alpha_{g,init}} gt (u_{l,init} + gt) \end{aligned} \quad (36)$$

Figure 4. Integration near the discontinuity at x_d .

The [third part](#) of the integration covers the region between the pipe inlet (top) and the discontinuity point. The pressure in this region (e.g. from 0 to $(x < x_d)$), can be obtained the same way by integrating the gas-phase momentum equation from 0 to x . [The integration is omitted here for its simplicity.](#)

[Aggregating the results for these three parts, the final solution for the pressure distribution can be summarized as,](#)

$$p(x, t) = \begin{cases} p_{outlet} - \frac{\rho_g g (L - x_d)}{\alpha_{g,init}} + \frac{1}{2} \rho_g \left(\frac{1 - \alpha_{g,init}}{\alpha_{g,init}} g t \right)^2 + \rho_g \frac{1 - \alpha_{g,init}}{\alpha_{g,init}} g t (u_{l,init} + g t) - \rho_g g (x_d - x) & \text{if } x \leq x_d \\ p_{outlet} - \frac{\rho_g g (L - x)}{\alpha_{g,init}} & \text{otherwise} \end{cases} \quad (37)$$

[It is noted that, although expressions on the right-hand-side of Equation \(37\) do not explicitly contain time \$t\$, the location of the discontinuity does. Thus, the pressure is expressed as a function of both time and space, \$p\(x, t\)\$.](#)

At the steady state, the discontinuity moves out from the domain and the pressure distribution becomes,

$$p_{ss}(x, t) = p_{outlet} - \rho_g g (L - x) \quad (38)$$

Recall that, in these derivations, decoupled liquid and gas-phase pressures are assumed. Equations (37) and (38) represent the gas-phase pressure, and the liquid-phase pressure is constant along the pipe, which is the same as the outlet pressure.

4. NUMERICAL VERIFICATION AND BENCHMARK

[In this section, numerical results are obtained from solving the two-fluid two-phase *single pressure* model presented in Section 2. As stated by Oberkampf and Roy \(2010\), “*solution verification addresses the question of whether a given simulation \(i.e., numerical approximation\) of a mathematical model is*](#)

sufficiently accurate for its intended use". Analytical solutions derived in Section 3.1 strictly follow the same single-pressure condition; thus, a rigorous numerical verification can be done against this set of solutions following Oberkampf and Roy's definition. However, analytical solutions derived in Section 3.2 do not follow the same single-pressure condition constrained in the equation set; thus, only numerical *benchmarking* is performed to compare them to numerical results. For the new analytical solutions at the steady-state condition, simulations were performed in different pressure conditions such that the vapor-to-liquid-density ratio varies significantly. A mesh convergence study is provided for these conditions, using the high-resolution spatial discretization scheme, and with both the Ransom solutions and the new analytical solutions derived in this work as the references. For the extended Ransom solutions, one transient simulation was carried out at the atmospheric pressure condition, using both the first-order and high-resolution spatial discretization schemes. Comparisons between numerical results and the extended Ransom solutions are given.

4.1. Verification on the new steady-state solutions

Verification on the newly proposed steady-state solutions was performed using the high-resolution spatial discretization scheme (second-order accuracy with spatially smooth solutions) described in Section 2.2. Steady-state solutions were obtained from transient simulations using the first-order BDF1 implicit time integration scheme. Mesh refinement study was done for different numbers of cells, ranging from 12 to 192. Both the Ransom solutions, Equations (7) and (8), and the new analytical solutions derived in Section 3.1 were used as the reference solutions. The simulations were performed with three different outlet pressures (i.e., 0.1, 1.0, and 5.0 MPa) corresponding to different vapor to liquid density ratios of 6.15×10^{-4} , 5.80×10^{-3} , and 3.26×10^{-2} , respectively.

The results of the mesh convergence studies are summarized in Table 2 for 0.1 MPa, Table 3 for 1.0 MPa, and Table 4 for 5.0 MPa, respectively. From these tables, it is clear that a second-order of spatial accuracy is obtained by using the analytical solutions derived in this work, which considers the gas phase's density on pressure distribution. On the contrary, using the Ransom solutions, the anticipated second-order of spatial accuracy cannot be obtained due to the simplifications made in Ransom's derivations *such that they do not represent the exact solutions to the equation set*. As indicated in Equation (26), it is expected that, if the vapor to liquid density ratio is large enough, the Ransom solutions give large errors *from the massless gas-phase assumption*. This is evident in Tables 2–4. At first, the Ransom solutions give much larger errors compared to the new analytical solutions proposed in this work. Secondly, the errors from the Ransom solutions increase with pressure (gas to liquid density ratio). It is also worth noting that, in

Table 4, the convergence rate using the new analytical solutions exceeds the anticipated convergence rate of 2.0. This indicates that the constant density assumption made in Section 3.1 starts to produce noticeable errors. For this pressure condition (5.0 MPa), the convergence study is repeated with the new analytical solutions shown in Appendix A. The results are summarized in Table 5 from which an anticipated second-order spatial accuracy is observed.

The numerical simulation results (using 96 cells) are also plotted against our new analytical solutions and the Ransom solutions in Figure 5 and Figure 6 for the 5 MPa outlet pressure case. The numerical results overlap with the new analytical solutions very well, and visible errors are found in the Ransom solutions.

Table 1. Parameters, initial, and boundary conditions for the water faucet problem.

Parameter	Value	Unit
Pipe length (L)	12	[m]
Gravity (g)	9.81	[m/s^2]
$u_{l,init}$	10	[m/s]
$u_{g,init}$	0	[m/s]
$\alpha_{g,init}$	0.2	[-]
p_{init}	$= p_{outlet}$	[Pa]
$u_{l,inlet}$	$= u_{l,init}$	[m/s]
$u_{g,inlet}$	$= u_{g,init}$	[m/s]
$\alpha_{g,inlet}$	$= \alpha_{g,init}$	[-]
p_{outlet}	0.1/1.0/5.0	[MPa]

Figure 5. Numerical results of pressure and void fraction with analytical solutions at the steady state condition.

Figure 6. Numerical results of liquid and gas-phase velocities with analytical solutions at the steady state condition.

4.2. Benchmark of the extended Ransom solutions

In this section, it will be shown that the transient Ransom solutions, as well as our extensions presented in Section 3.2, are only approximated solutions to Equations (1)-(4). In this sense, the comparison between numerical results to Equations (1)-(4) and the extended Ransom solutions is not strictly numerical verification and thus are described as numerical benchmarking in this section. Numerical simulations of Equations (1)-(4) were performed using two different options: (1) first-order method (first-order spatial discretization scheme and first-order Backward Euler time integration scheme), and (2) high-order method (high-resolution spatial discretization scheme and second-order BDF2 time integration scheme). Simulation results were obtained at different times, 0.3 s, 0.4 s, and 0.5 s, using 96 cells and a time step at 10^{-3} s. Simulation results are plotted in Figure 7 - Figure 10 along with the new analytical solutions derived in Section 3.2. The numerical results for the liquid-phase velocity (Figure 7), void fraction (Figure 8), and vapor-phase velocity (Figure 9) all agree well with the extended Ransom solutions, with visible discrepancies shown in the vapor-phase velocity.

Figure 7. Numerical results of liquid-phase velocity with analytical solutions at 0.3, 0.4 and 0.5s.

Figure 8. Numerical results of void fraction with analytical solutions at 0.3, 0.4 and 0.5 s.

Figure 9. Numerical results of gas-phase velocity with analytical solutions at 0.3, 0.4, and 0.5 s.

However, the simulation results of the pressure (Figure 10) differ significantly from the gas-phase pressure derived in the extended Ransom solutions. Between the numerical results and the extended Ransom solutions, the only similarity is the positions of the discontinuities. In existing literatures, the simulation results of the pressure distribution have rarely been reported, with an exception of Munkejord's work (Munkejord, 2006a). The pressure distributions from numerical results shown in Figure 10 are similar to those shown in Munkejord's work (for example, Figure 3.4 of Munkejord, 2006a). The transient pressure change trends are also somewhat different from those observed in our previous work (Zou et al., 2015b), probably due to the different compressibility of the fluids used in this work. The strong disagreement between the numerical results and analytical solutions on pressure distribution again manifests the key difference between the flow model and the derivation of analytical solutions (i.e., single pressure in the flow model versus decoupled two pressures in the analytical solutions). *Here, it has to be made clear that in almost all existing literatures, the significance of this difference was overlooked. For too many times, the *two-pressure* analytical solutions are used to gauge numerical results from *single-pressure* models, and not surprisingly that they do not always match.*

Figure 10. Numerical results of pressure with analytical solutions at 0.3, 0.4 and 0.5s.

However, the transient analytical solutions derived in Section 3.2 can be verified by solving a highly simplified mathematical model that follows Ransom's assumptions. This numerical exercise is presented in Appendix B.

5. DISCUSSIONS AND CONCLUSIONS

In this paper, for the classical two-phase water faucet problem, we have derived a new set of analytical solutions for the single-pressure two-fluid model at the steady state condition by considering the gas phase's density effect on pressure distribution. It shows that when the gas-to-liquid-density ratio is large, the original Ransom solution yields large errors. The newly derived steady-state analytical solutions were used for rigorous numerical verification in form of mesh convergence study using a high-resolution spatial discretization scheme. The anticipated second-order of accuracy was achieved in this numerical verification process. On the contrary, the original Ransom solutions could not give a second-order spatial accuracy from the mesh convergence study.

Extending the original Ransom solutions under transient conditions was also carried out following Ransom's assumptions. Numerical results from transient simulations agreed with the new analytical solutions, with the exception of the pressure distribution. This is due to the difference in the key assumption used between the flow model, which assumed a single pressure, and the derivations of analytical solutions, which assumed decoupled two pressures. This key difference has been largely overlooked in many existing literatures.

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Appendix A: Analytical solutions to the water faucet problem at steady state using linearized equations of state

Assuming the phasic densities could be linearized around the reference pressure, outlet pressure, and let

$$\kappa_g \equiv \left. \frac{d\rho_g}{dp} \right|_{p=p_{outlet}} \quad (\text{A. 1})$$

$$\kappa_l \equiv \left. \frac{d\rho_l}{dp} \right|_{p=p_{outlet}} \quad (\text{A. 2})$$

$$\kappa_{lg} \equiv \frac{\kappa_l}{\kappa_g} \quad (\text{A. 3})$$

The phasic densities can be expressed as,

$$\rho_g = \rho_{g,outlet} + \kappa_g(p - p_{outlet}) \quad (\text{A. 4})$$

$$\rho_l = \rho_{l,outlet} + \kappa_l(p - p_{outlet}) \quad (\text{A. 5})$$

in which,

$$\rho_{g,outlet} = \rho_g(p_{outlet}) \quad (\text{A. 6})$$

$$\rho_{l,outlet} = \rho_l(p_{outlet}) \quad (\text{A. 7})$$

Following the derivations presented in Section 3.1, the steady-state solutions for the water faucet problem can be obtained as,

$$u_{g,ss}(x) = 0 \quad (\text{A. 8})$$

$$p_{ss}(x) = p_{outlet} + \frac{\rho_{g,outlet} [1 - e^{\kappa_g g(L-x)}]}{\kappa_g e^{\kappa_g g(L-x)}} \quad (\text{A. 9})$$

$$u_{l,ss}(x) = \sqrt{u_{l,inlet}^2 + 2 \left\{ gx - g \left[\frac{x}{\kappa_{lg}} + \frac{1}{g\kappa_l} \ln \frac{(\rho_{l,outlet} - \kappa_{lg}\rho_{g,outlet})e^{\kappa_g g(L-x)} + \kappa_{lg}\rho_{g,outlet}}{(\rho_{l,outlet} - \kappa_{lg}\rho_{g,outlet})e^{\kappa_g gL} + \kappa_{lg}\rho_{g,outlet}} \right] \right\}} \quad (\text{A.10})$$

$$\alpha_{g,ss}(x) = 1 - \frac{(1 - \alpha_{g,inlet})u_{l,inlet}\rho_{l,ss}(x=0)}{\rho_{l,ss}(x)u_{l,ss}(x)} \quad (\text{A.11})$$

in which,

$$\rho_{l,ss}(x) = \rho_{l,outlet} + \frac{\kappa_{lg}\rho_{g,outlet} [1 - e^{\kappa_g g(L-x)}]}{e^{\kappa_g g(L-x)}} \quad (\text{A.12})$$

Appendix B: A simple numerical experiment on water faucet problem

We performed a simple numerical experiment to demonstrate that the water faucet problem can be indeed described with a highly simplified mathematical model. Following Ransom's assumptions (1) pressures are decoupled between the two phases; (2) both phases are incompressible; (3) the liquid phase admits a 'free fall' motion, so liquid-phase pressure gradient is zero; a highly simplified mathematical model can be formulated to describe the water faucet problem,

$$\frac{\partial \alpha_l}{\partial t} + \frac{\partial(\alpha_l u_l)}{\partial x} = 0 \quad (39)$$

$$\frac{\partial \alpha_g}{\partial t} + \frac{\partial(\alpha_g u_g)}{\partial x} = 0 \quad (40)$$

$$\frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial x} - g = 0 \quad (41)$$

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + \frac{1}{\rho_{g, const}} \frac{\partial p_g}{\partial x} - g = 0 \quad (42)$$

Using the same parameters presented in Table 1, and numerical methods presented in Section 2.2. Numerical results are obtained and compared with the transient solutions obtained in Section 3.2. The results are shown in Figure 11 for gas-phase velocity, and Figure 12 for gas-phase pressure, respectively. Results of void fraction and liquid-phase velocity are omitted, since they are almost identical to Figure 7 and Figure 8. Very good agreements can be observed from these figures.

Figure 11. Numerical results of gas-phase velocity to the modified equation set with analytical solutions.

Figure 12. Numerical results of pressure to the modified equation set with analytical solutions.

Table 2. Spatial convergence rate when outlet pressure, $p_{outlet} = 0.1$ MPa, using the Ransom's solutions and the new analytical solutions proposed in this work as the references.

N_{cell}	$\Delta x=L/N_{cell}$	Using the Ransom Solutions				Using New Solutions in this Work			
		a_g		u_l		a_g		u_l	
		L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate
12	1	2.0025E-02	-	2.7630E-01	-	1.9265E-02	-	2.4665E-01	-
24	0.5	6.2828E-03	1.67	9.4643E-02	1.55	5.4293E-03	1.83	6.5958E-02	1.90
48	0.25	2.3486E-03	1.42	4.5266E-02	1.06	1.4480E-03	1.91	1.7070E-02	1.95
96	0.125	1.2982E-03	0.86	3.2285E-02	0.49	3.7410E-04	1.95	4.3347E-03	1.98
192	0.0625	1.0335E-03	0.33	2.8911E-02	0.16	9.4925E-05	1.98	1.0835E-03	2.00

Table 3. Spatial convergence rate when outlet pressure, $p_{outlet} = 1.0$ MPa, using the Ransom's solutions and the new analytical solutions proposed in this work as the references.

N_{cell}	$\Delta x=L/N_{cell}$	Using the Ransom solutions				Using new solutions in this work			
		a_g		u_l		a_g		u_l	
		L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate
12	1	2.6350E-02	-	5.2412E-01	-	1.9174E-02	-	2.4451E-01	-
24	0.5	1.3795E-02	0.93	3.3580E-01	0.64	5.3975E-03	1.83	6.5292E-02	1.90
48	0.25	1.0245E-02	0.43	2.8272E-01	0.25	1.4356E-03	1.91	1.6818E-02	1.96
96	0.125	9.2805E-03	0.14	2.6777E-01	0.08	3.6716E-04	1.97	4.1931E-03	2.00
192	0.0625	9.0289E-03	0.04	2.6339E-01	0.02	8.9509E-05	2.04	9.7064E-04	2.11

Table 4. Spatial convergence rate when outlet pressure, $p_{outlet} = 5.0$ MPa, using the Ransom's solutions and the new analytical solutions proposed in this work as the references.

N_{cell}	$\Delta x=L/N_{cell}$	Using the Ransom Solutions				Using New Solutions in this Work			
		a_g		u_l		a_g		u_l	
		L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate
12	1	6.7432E-02	-	1.8121E+0	-	1.8694E-02	-	2.3337E-01	-
24	0.5	5.5565E-02	0.28	1.5890E+0	0.19	5.2208E-03	1.84	6.1706E-02	1.92
48	0.25	5.2101E-02	0.09	1.5166E+0	0.07	1.3580E-03	1.94	1.5360E-02	2.01
96	0.125	5.1162E-02	0.03	1.4915E+0	0.02	3.1852E-04	2.09	3.3028E-03	2.22
192	0.0625	5.0917E-02	0.01	1.4818E+0	0.01	6.3419E-05	2.33	4.1731E-04	2.98

Table 5. Spatial convergence rate when outlet pressure, $p_{outlet} = 5.0$ MPa, using new analytical solutions with linearized density assumption (shown in Appendix A) as the references.

N_{cell}	$\Delta x=L/N_{cell}$	Using New Analytical Solutions (Appendix A)			
		a_g		u_l	
		L-1 Norm	Convergence Rate	L-1 Norm	Convergence Rate
12	1	1.8730E-02	-	2.3422E-01	-
24	0.5	5.2621E-03	1.83	6.2534E-02	1.91
48	0.25	1.4014E-03	1.91	1.6178E-02	1.95
96	0.125	3.6218E-04	1.95	4.1156E-03	1.97
192	0.0625	9.2319E-05	1.97	1.0380E-03	1.99