1 Hierarchical Bayesian fatigue data analysis

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6 Abstract

The problem minimizing the number of specimens required for fatigue data analysis is considered in this research. Assuming unknown hyperparameters described via prior distributions, a hierarchical Bayesian model with accumulated prior information was proposed to deal with this issue. One of the main advantages of hierarchical Bayesian model over the empirical Bayesian model is that the prior distributions with hierarchical structure can incorporate structural prior and subjective prior simultaneously. The probabilistic stress-cycle (P - S - N) curves are generated from the predictive distributions, involving both the randomness of parameters and the scatter of observations, and calculated by an identical hierarchical structure. The numerical calculation is done via the Gibbs sampler, which makes the whole process simple and intuitive.

7 *Keywords:*

8 Fatigue data analysis, P - S - N curves, Hierarchical Bayesian model, Bayesian
9 inference, Markov chain Monte Carlo

1. Introduction

The laboratory fatigue tests data, always displaying a large scatter observation, are often presented in the form of a median stress-cycle (S - N) curve in high cycle fatigue life prediction with a series of stress amplitudes, while P - S - N curves are used for studying the fatigue life reliability. There is no doubt that the estimation of unknown parameters of fatigue design curves relies heavily on statistical analysis. Miner [1] suggested a cumulative damage rule in 1945, as the well-known Palmgren-Miner's rule:

$$D = \sum_i^n \frac{n_i}{N_i} \text{ for } i = 1, \dots, n. \quad (1)$$

Equation (1) describes a linear damage rule, D is the cumulative damage of n_i cycles at each stress level S_i , and N_i is the number of cycles to failure. The number of stress cycles of a certain material is determined experimentally by applying constant or variable amplitude load and recording the number of cycles to failure. N_i is usually assumed to depend on the load amplitude S_i by Basquin relation [2]

$$N_i = AS_i^{-B} \text{ for } i = 1, \dots, n. \quad (2)$$

where $A > 0$, $B > 0$ are fatigue curve coefficients. In a more complex three parameters model [3],

$$N_i = A(S_i - S_0)^{-B} \text{ for } i = 1, \dots, n. \quad (3)$$

where S_0 is a material constant too, which can be the mean life of the specimen [4] or the fatigue limit [5]. Random effects of the mechanical properties of structure and material, and even environmental factors lead to scattered fatigue life cycle data. The fatigue lives of similar specimens or structures under the same fatigue load can be significantly different. Some research [6, 7] proposed to utilize the regularized fatigue life to deal with the random nature of the fatigue life data and to examine the continuous dependence of the fatigue life curve on the measured data. In order to describe material fatigue precisely, fatigue life dispersion characteristics usually are expressed by P - S - N curves. In fatigue reliability analysis, Equation (2) can be extended to represent P - S - N curves at survival probability p as

$$N_{ip} = A_p S_i^{-B_p} \text{ for } i = 1, \dots, n. \quad (4)$$

37 Similarly, in the case of three parameters,

$$N_{ip} = A_p(S_i - S_{0p})^{-B_p} \text{ for } i = 1, \dots, n. \quad (5)$$

38 where A_p , B_p and S_{0p} are material constants, and N_{ip} is the life correspond-
39 ing to survival probability p at the stress level S_i . The scatter of material
40 parameters can be expressed by the lognormal distribution or a three pa-
41 rameters Weibull distribution, stated by Weibull [8] in 1939. A zero fatigue
42 life of the lognormal distribution is physically impossible, so this discrepancy
43 does not occur in the three parameters Weibull distribution function. In fact,
44 the lognormal distribution can fit fatigue data well at the high or low stress
45 levels, while Weibull distribution achieves better results at medium stress
46 levels [9]. Unfortunately, the distribution function of fatigue life hardly can
47 be derived on the basis of physical arguments. In a reason, the validation
48 of the two above distribution functions requires large test data [10], which is
49 time-consuming and costly.

50 Research works have been carried out for P - S - N curves estimation from
51 small data sets, often involving complex data structures, by statistical meth-
52 ods, such as maximum likelihood method with a reference stress level [11],
53 empirical Bayesian model [12], the random fatigue limit model [5], backwards
54 statistical inference method [13], etc., driven by data analysis or physical
55 models. The notional fatigue reliability on the design curve only reflects the
56 observed physical uncertainty associated with the fatigue process itself, but
57 for the small fatigue data set the statistical uncertainties concerning the point
58 parameter estimates could be equally significant. Bayesian analysis has been
59 presented for establishing design S - N curves from small censored data sets to
60 solve underlying statistical uncertainties [14]. Although a number of papers
61 have appeared which exploited Bayesian inference in the analysis of the prop-
62 agation of fatigue cracks, very few attempts have been made in the past to use
63 the Bayesian approach in the context of S - N fatigue tests [12]. The Bayesian
64 inference approach focuses on updating the probability for a hypothesis of
65 the parameters on the basis on observations [15]. Bayesian linear regression
66 has been considered in fatigue data analysis where the posterior distribution
67 of model parameters was then used to predict the fatigue life [16]. The em-
68 pirical prior knowledge was derived from material parameters [12], maximum
69 likelihood estimates [17], or test series with similar components of different
70 geometry [16]. Different residual stress and strain data measured from vari-
71 ous techniques are analyzed using a Bayesian statistical approach and then
72 interpolated utilizing modified Shepard method [18], which has been used for

fatigue life measurements analysis indirectly [6]. All of the aforementioned Bayesian inference applications of fatigue data analysis focused on using empirical Bayesian model, while hierarchical Bayesian models with structural and subjective hierarchical priors, both or one of them, have been applied to describe the fatigue crack growth [19] or the crack growth rate [20].

The present research primary focuses on the fatigue curves estimation method by using hierarchical Bayesian model, for both the two parameters and three parameters methods. Then the predictive distributions in the same hierarchical structure are proposed to estimate the P - S - N curves. Moreover, one of the main advantages of the hierarchical Bayesian model is that it allows the use of both structural prior information and subjective prior information simultaneously. As a result, the choice of noninformative prior distributions is discussed in detail, as well as the model checking and the convergence of Gibbs sampling. Finally, numerical examples of hierarchical Bayesian models for estimating the S - N and P - S - N curves from the collection of real data under study are presented, along with their comparisons of the maximum likelihood estimation (MLE).

2. Hierarchical Bayesian models for estimating S - N curves

2.1. Fundamentals of hierarchical Bayesian models

From the perspective of Bayesian statistics, the parameters within models are regarded as random variables, and thereby having probability distributions, which are known as prior distributions. An important type of prior distribution is a hierarchical prior, since it is often convenient to model structural knowledge in stages. Distinguishing feature of the hierarchical Bayesian approach to empirical Bayesian analysis is the hierarchical nature in which information is accumulated. As a result, the hierarchical approach is most commonly used approach to building complex models by specifying a series of more simple conditional distributions [21]. Generally, hierarchical models are more flexible than the typical nonhierarchical models since a more complicated structure is accommodated in the model. After ignoring the normalizing constant $f(\mathbf{y})$, a hierarchical Bayesian model is defined by

$$f(\mathbf{v}, \boldsymbol{\theta} | \mathbf{y}) \propto f(\mathbf{y} | \boldsymbol{\theta}, \mathbf{v}) f(\boldsymbol{\theta} | \mathbf{v}) f(\mathbf{v}) \quad (6)$$

with a first stage likelihood, $f(\mathbf{y} | \boldsymbol{\theta}, \mathbf{v})$, and second stage density, $f(\boldsymbol{\theta} | \mathbf{v})$. $\boldsymbol{\theta}$ and $\mathbf{v}$ represent likelihood parameters and hyperparameters of prior distributions respectively, and $f(\cdot)$ are distribution functions.

Comparing with the empirical Bayesian model, for which the prior distribution is fixed before any data are observed, the hierarchical Bayesian model offers many advantages [21, 22, 23]. Empirical Bayesian model is failure to consider hyperparameter estimation error and also does not indicate how to incorporate the hyperparameter estimation error in the analysis by itself, while the hierarchical Bayesian analysis incorporates such errors automatically. Another advantage is that the hierarchical structure leads to a more robust analysis, since it reduces the arbitrariness of the hyperparameter choice. Moreover, the hierarchical Bayesian model can incorporate actual subjective prior information at the second stage, so that it allows the use of both structural prior information and subjective prior information simultaneously, random fatigue limit model in fatigue data analysis (Subsection 2.2), for example. Finally, empirical Bayes theory requires the solution of likelihood equations, while the hierarchical Bayes approach requires numerical integration, Markov Chain Monte Carlo (MCMC) algorithms for instance, and resulting in conditional distributions. The increasing applications and practical implementations of Bayesian models have owed much to the development of MCMC algorithms, such as Gibbs sampling [24], Metropolis-Hastings algorithm, etc. for estimation [25, 26] relatively recently. The Gibbs sampler, also called alternating conditional sampling, the proposal density of which can be generated by iteratively sampling from the full conditional posterior distribution $p_{\theta|y}(\theta_j|\boldsymbol{\theta}_{\setminus j}, \mathbf{y})$, where $\boldsymbol{\theta}_{\setminus j} = (\theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \dots, \theta_d)^T$. As a special case of the Metropolis-Hastings algorithm, the Gibbs sampler meets the detailed balance condition and leads to a stationary distribution, but samples one random variable at a time and does not require the check for the proposed sample acceptance.

The Bayesian approach expands the class of models to fit data, enabling one to handle complex correlations, unbalanced or missing data, etc. In addition to the advantages described above, the hierarchical Bayesian models, in a sense, all Bayesian models are hierarchical [15], provide a formal framework for analysis with a complexity of structure that matches the system being studied.

2.2. Models for estimating S - N curves

If the random error is taken into consideration, Equation (2) can be expressed as:

$$N_i = AS_i^{-B}\epsilon_i \quad (7)$$

142 where ε_i indicates the randomness in stress level S_i , comprising the random
 143 effect of materials and random error in observations. Take the natural loga-
 144 rithm transformation for Equation (7),

$$\log_{10} N_i = \log_{10} A - B \log_{10} S_i + \delta_i \quad (8)$$

145 The random variables δ_i in Equation (8) can be a normal distribution, Weibull
 146 distribution, or Gaussian mixture distribution, of which the mixing mea-
 147 sure, which uses a unfixed number of parameters or accounts for uncertainty
 148 about distributional shape [27, 28], can be a Dirichlet process in a Bayesian
 149 nonparametric approach. For the purpose of introducing the hierarchical
 150 Bayesian models in this paper, the normal distribution is chosen in the sub-
 151 sequent study. Therefore, we assume $\delta_i = \sigma_i e_i$, σ_i is the standard deviation
 152 of the logarithmic fatigue life under the stress level S_i and $e_i \sim \mathcal{N}(0, 1)$. Fol-
 153 lowing Guida's [12] suggestion, let $u_i = \log_{10} S_i$ and $\bar{u} = (1/n) \sum_i^n \log_{10} S_i$,
 154 then

$$y_i = \alpha + \beta x_i + \mathcal{N}(0, \sigma_i^2) \quad (9)$$

155 where $y_i = \log_{10} N_i$, $x_i = u_i - \bar{u}$, $\alpha = \log_{10} A - B\bar{u}$ and $\beta = -B$. For
 156 statistical analysis it is convenient to rewrite the Equation (9) to a normal
 157 distribution with mean $\mu = \alpha + \beta x_i$ and variance σ_i^2 [12]. which can be
 158 expressed as:

$$y_{ij} \sim \mathcal{N}(\alpha + \beta x_i, \sigma_i^2) \quad (10)$$

159 or

$$y_{ij} \sim \mathcal{N}(\mu_i, \sigma_i^2) \text{ for } i = 1, \dots, n \text{ and } j = 1, \dots, m. \quad (11)$$

160 where $\mu_i = \alpha + \beta x_i$, y_{ij} is the j^{th} observation of the i^{th} stress level. The prior
 161 distributions in the second stage are assumed that $\alpha | \mu_\alpha, \sigma_\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$,
 162 $\beta | \mu_\beta, \sigma_\beta \sim \mathcal{N}(\mu_\beta, \sigma_\beta^2)$ (see Subsection 4.1 for more information on the choice
 163 of prior distributions). The directed acyclic graph (DAG) of the hierarchical
 164 sturcture is shown in Figure 1. In this representation, variables are arranged
 165 in a series of levels, with data in the innermost and hyperparameters in
 166 the outermost. The arrows represent dependencies of variables, which are
 167 assumed to be independent conditional on each level. Known data are placed
 168 in the box and unknown variables are put in the circles. Similarly, the three
 169 parameters expression for S - N curves is assumed as

$$N_i = A(S_i - S_0)^{-B} \varepsilon_i \quad (12)$$

Figure 1: The DAG of the hierarchical Bayesian model for estimating S - N curve or P - S - N curves. The arrows represent dependencies of variables, which are assumed to be independent conditional on each level. These nodes with fixed quantities in the analysis are represented by rectangles, while unobserved random variables are represented by circles.

where ε_i is also assumed to be distributed as a log-normal random variable,
 same as the two parameters method:

$$\log_{10} N_i = \log_{10} A - B \log_{10} (S_i - S_0) + \sigma_i e_i \quad (13)$$

Let $\alpha = \log_{10} A$ and $\beta = -B$, then

$$y_i = \alpha + \beta \log_{10} (x_i - \gamma) + \sigma_i e_i \quad (14)$$

where $x_i = S_i$, $\gamma = S_0$, and

$$y_{ij} \sim \mathcal{N}(\alpha + \beta \log_{10} (x_i - \gamma), \sigma_i^2) \quad (15)$$

OR

$$y_{ij} \sim \mathcal{N}(\mu_i, \sigma_i^2) \text{ for } i = 1, \dots, n \text{ and } j = 1, \dots, m. \quad (16)$$

where $\mu_i = \alpha + \beta \log_{10} (x_i - \gamma)$, y_{ij} is the j^{th} observation of the i^{th} stress level. Note that the x_i in the three parameters model is very different from the two parameters model. $\alpha | \mu_\alpha, \sigma_\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$, $\beta | \mu_\beta, \sigma_\beta \sim \mathcal{N}(\mu_\beta, \sigma_\beta^2)$, $\gamma | \mu_\gamma, \sigma_\gamma \sim \mathcal{N}(\mu_\gamma, \sigma_\gamma^2)$ are still assumed in the second stage. Suppose that, γ is the fatigue limit of the specimen, and $f_V(v; \mu_\gamma, \sigma_\gamma)$ is probability density function of V by letting $V = \log(\gamma)$

$$f_V(v; \mu_\gamma, \sigma_\gamma) = \frac{1}{\sigma_\gamma} \phi V \left(\frac{v - \mu_\gamma}{\sigma_\gamma} \right) \quad (17)$$

181 where $\phi V(\cdot)$ is either the standardized smallest extreme value or normal
 182 probability density function. When $f_V(v; \mu_\gamma, \sigma_\gamma)$ is the prior of γ , that is
 183 $V \sim f_V(v; \mu_\gamma, \sigma_\gamma)$, then Equation (15) can be considered as a kind of random
 184 fatigue-limit model [5].

185 **3. P - S - N curves estimation by using predictive distributions**

186 The fatigue P - S - N curve with the given survival probability p is easily
 187 generated via the parameters α_p and β_p . Equation (4) can represent P - S - N
 188 curve at survival probability p as

$$\log_{10} N_{ip} = \log_{10} A_p - B_p \log_{10} S_i \quad (18)$$

189 The logarithm of fatigue life is assumed to normally distribute previously, so
 190 that the logarithm of fatigue life at survival probability p , $\log_{10} N_{ip}$, can be
 191 expressed as

$$\log_{10} N_{ip} = \log_{10} N_i - \mu_p \sigma_i \quad (19)$$

192 where μ_p is the standard normal deviate corresponding to survival probability
 193 p . Substituting Equations (8) and (18) into Equation (19) yields

$$\mu_p \sigma_i = \alpha - \alpha_p + (\beta - \beta_p) x_i \quad (20)$$

194 where $\alpha_p = \log_{10} A_p - B_p \bar{u}$ and $\beta_p = -B_p$. However, the model complexity
 195 and computational cost are greatly increased, therefore another approach
 196 will be discussed to obtain the P - S - N curves using prediction of observations
 197 $\mathbf{y}$ in Subsection 3.2 within the same hierarchical structure.

198 *3.1. The predictive distributions*

199 The predictive distributions of observations can be presented within the
 200 same hierarchical structure. The Gibbs sampler takes advantages of hierar-
 201 chical structures [23], when a Bayesian model can be written as

$$f_y(\mathbf{y}^{\text{pred}}) = \int f_{y|\theta}(\mathbf{y}^{\text{pred}}|\boldsymbol{\theta}) f_\theta(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (21)$$

202 where $f_y(\cdot)$ are predictive distributions or prior predictive distributions. It
 203 is easy to generate the prediction $\mathbf{y}^{\text{pred}}$ by using the predictive distributions.
 204 The $f_y(\cdot)$ is the stationary distribution of this Markov chain [29], expressed
 205 as follows:

$$f_y(\mathbf{y}^{\text{pred}}) = \int \mathcal{K}(\mathbf{y}^{\text{pred}}|\mathbf{y}) f_y(\mathbf{y}) d\mathbf{y} \quad (22)$$

206 where $\mathcal{K}(\mathbf{y}^{\text{pred}}|\mathbf{y})$ is the transition kernel for the chains of $\mathbf{y}^{\text{pred}}$, given by:

$$\mathcal{K}(\mathbf{y}^{\text{pred}}|\mathbf{y}) = \int f_{\mathbf{y}|\theta}(\mathbf{y}^{\text{pred}}|\theta, \mathbf{y})f_{\theta|\mathbf{y}}(\theta|\mathbf{y})d\theta \quad (23)$$

207 3.2. *P-S-N curves estimation by using predictive distributions*

208 The fatigue life cycles $\mathbf{y}^{\text{pred}}$ of all the stress levels can be easily generated
 209 from the posterior predictive distributions by adding a single simple step
 210 within the Gibbs sampler, that is the expected life values in all stress levels
 211 $E(\mathbf{y}^{\text{pred}}|\mathbf{y}, \mathbf{x})$ can be estimated by using the predictive distributions

$$f(\mathbf{y}^{\text{pred}}|\mathbf{y}) = \int f(\mathbf{y}^{\text{pred}}|\boldsymbol{\mu}, \boldsymbol{\sigma}, \mathbf{y})f(\boldsymbol{\mu}, \boldsymbol{\sigma}|\mathbf{y})d\boldsymbol{\mu}d\boldsymbol{\sigma} \quad (24)$$

212 where $\boldsymbol{\sigma}$ is the standard deviation, $\boldsymbol{\mu} = \alpha + \beta\mathbf{x}$, the same as previously
 213 mentioned, $\mathbf{y}^{\text{pred}}$ denotes the future data for predicting, and $\mathbf{y}$ is the com-
 214 plete test data. When the observations of some stress levels is insufficient,
 215 the observations of fatigue life can be treated as incomplete data, and the
 216 insufficient data is considered as missing data. That is, the complementary
 217 data $\mathbf{y}$ include the observations and the missing data $\mathbf{y}^{\text{miss}}$ in this case. The
 218 survival function $S(y_i) = 1 - F(y_i) = 1 - P(Y_i \leq y_i)$, the probability of
 219 surviving at the fatigue life y_i at stress level S_i , can be expressed using the
 220 predictive distributions naturally, so that the *P-S-N* curves can be obtained
 221 by using prediction of observations $\mathbf{y}$ easily.

222 Follow the discussion in Subsection 2.2, the same effect as Ling's maxi-
 223 mum likelihood method [11] can be achieved by making $\sigma_i = \sigma_r$ in Equations
 224 11 and 15 at each stress level S_i , where σ_r is the various standard deviations
 225 of fatigue life at reference stress, i.e., choosing a stress level S_r as a reference
 226 stress level and a group of specimens are tested. Note that the reference stress
 227 used here is very different from the reference prior in the objective Bayesian.
 228 It will greatly reduce the number of test samples when the reference stress is
 229 used, consistent with the maximum likelihood method, of course, the same
 230 limitation too. It is based on the assumption that the statistical nature of the
 231 life data is the same at all applied stress levels of interest. In the Bayesian
 232 hierarchical model, an unknown σ can be assumed to characterize the dis-
 233 persion of fatigue life at each stress level, that is $\sigma_i = \sigma$, without providing
 234 a reference stress level. It might be more reasonable because the fatigue life
 235 cycle scattered nature of the stress levels of all available data are taken into
 236 account.

237 All of the above discussions are based on the assumptions that there
238 are similarities in the distributions of fatigue life at different stress levels.
239 An assumption of “similarity” here is not equivalent to the assumption of
240 “exchangeability”, which was shown to be equivalent to assuming that the
241 observations of each stress level were independent and identically distributed
242 from one distribution with the hyperparameters unknown, and the priors
243 of those hyperparameters are given. These hyperparameters should arise
244 from a common “population” distribution whose parameters are unknown
245 and assigned appropriate prior distributions [30]. Hierarchical models tend
246 to recognise that it is unlikely that all stress levels have the same underlying
247 survival rate by giving different distribution parameter σ_i , with the same
248 prior distribution, for the various stress levels.

249 In addition, in dealing with the S - N or P - S - N estimate problem, the
250 fatigue measured data often have missing data problems or there are not as
251 many observations at different stress levels. Less data means less expense and
252 test time, especially at lower stress levels. The hierarchical Bayesian model
253 can easily deal with these situations by using the predictive distributions
254 since the missing y_i^{miss} of x_i can be considered as an additional parameter
255 under estimation. Obviously, more missing data also means more uncertainty
256 in the estimation and therefore results in more conservative estimates.

257 4. Prior choice and model checking

258 4.1. The choice of prior distributions

259 The first step of Bayesian inference is to determine the prior distributions
260 of parameters θ . Undoubtedly, one of the most critical and most criticized
261 points of Bayesian analysis is that it deals with the choice of the prior dis-
262 tribution, so that the prior distribution is the key to Bayesian inference and
263 its determination is, therefore, the most important step in drawing this in-
264 ference. There are two kinds of prior, the informative prior and the nonin-
265 formative prior. The informative prior could be based upon some or all of
266 the following: (1) mathematical or physical models, (2) engineering, physics,
267 etc. information, (3) expert’s judgements, (4) historical data of the same or
268 similar circumstances, and other reasonable information.

269 The noninformative prior can be directly derived from the sampling dis-
270 tribution. Jeffreys [31] described a method to derive the prior distribution
271 directly from the sampling distribution. For normal distribution, consider
272 $y \sim N(\mu, \sigma^2)$, with μ, σ unknown. In the Jeffreys noninformative prior case,

273 the corresponding noninformative prior distribution is $\pi \propto 1/\sigma^2$. Gamma(ϵ ,
 274 ϵ), with ϵ small and positive, is ‘just proper’ form of Jeffreys prior [32]. In
 275 the particular case of conjugate distributions in the usual exponential fam-
 276 ilies, the posterior expectations of the natural parameters can obviously be
 277 expressed analytically. The normal distribution, for example, the conjugate
 278 prior of its mean is still normal distribution, while the variance of it has
 279 an inverse gamma conjugate prior [23]. If the prior information is weak,
 280 $\theta_i \sim \mathcal{N}(\mu_{\theta_i}, \sigma_{\theta_i}^2)$, in which μ_{θ_i} , σ_{θ_i} are hyperparameters, can be chosen, and
 281 μ_{θ_i} is often set to 0, while σ_{θ_i} is often set to 10^k with a sufficiently large k in
 282 Bayesian model [33]. However, $\theta_i \sim \mathcal{N}(\mu_{\theta_i}, \sigma_{\theta_i}^2)$ is set in hierarchical Bayesian
 283 model, and the hyperparameter θ_i are assumed to have noninformative or
 284 weakly-informative prior distributions. In addition, to assess sensitivity to
 285 prior assumptions, the analysis may be repeated over a limited range of alter-
 286 native priors. Some researchers [34, 35] suggested a gamma prior on inverse
 287 variance, $1/\sigma^2$, governing random walk effects (e.g., baseline hazard rates in
 288 survival analysis), namely, $1/\sigma^2 \sim \mathcal{G}(a, b)$, where a is set at 1, but b is varied
 289 over choices such as 0.05 or 0.0005. Unfortunately, Gelman [32] shows that
 290 inferences become very sensitive to a , b , especially for problems where the
 291 group-level variance σ_{θ_i} is close to zero and is crucially affected by the choice
 292 the scale of σ_{θ_i} . Thus a , b must be set to a reasonable value, so that the
 293 prior distribution hardly looks noninformative. Gelman also recommended
 294 starting with a noninformative uniform prior density $\mathcal{U}(0, u)$ on a wide range
 295 of standard deviation parameters in fitting hierarchical models. As a result,
 296 uniform distributions and the inverse gamma distributions are selected as a
 297 prior distribution in Section 5.

298 The physical model can not only be expressed as a likelihood function, but
 299 also, as described in Subsection 2.2, can be incorporated into a hierarchical
 300 Bayesian model as an informative prior conveniently. Moreover, Statistical
 301 uncertainties of historical data can also be easily added to the hierarchical
 302 Bayesian model as an informative prior. In fact, for a large sample size,
 303 MLE of a scalar parameter θ , say $\hat{\theta}$, is approximately normally distributed
 304 with mean θ and variance equal to the negative reciprocal of the observed
 305 information, $I(\theta)$. Similarly, because the weight given to the prior mean
 306 decreases to 0 as the number of experimental becomes large, the posterior
 307 distribution will converge to a normal distribution centered on the MLE,
 308 and the variance of the posterior distribution converges to the inverse of
 309 the $I(\theta)$. Therefore the asymptotic (large sample) properties of the MLE
 310 and the posterior distribution are similar in this sense [33]. As a result, it

is reasonable that the asymptotic distribution is selected as an informative prior distribution.

4.2. Convergence of the MCMC algorithm and model checking

As discussed in Subsection 2.1, the target posterior distributions and the hyperparameters noninformative prior distributions are easily obtained by Gibbs sampling. MCMC methods broadly applicable, but require care in parametrization and convergence diagnosis. In order to obtain correct target posterior distributions, there are many ways to monitor convergence, including the MC error (the simplest way), the trace plots, the ergodic mean, plotting autocorrelations etc. Multiple-chain comparisons with different initial values are also efficient in practice, and BrooksGelmanRubin (BGR) diagnostics [36] also can be employed in this case.

The deviance information criterion (DIC), which can be applied to compare the fatigue estimation models, is obtainable as the expected deviance plus the effective model dimension, and was introduced by Spiegelhalter *et al.* [37]. The DIC can be seen as a Bayesian version of Akaike information criterion (AIC),

$$\text{DIC} = 2\bar{D} - D(\bar{\theta}) = D(\bar{\theta}) + 2p_D \quad (25)$$

where $D(\bar{\theta})$ is the usual deviance measure, which is equal to minus twice the log-likelihood $D(\bar{\theta}) = -2\log f(y|\bar{\theta})$ and $\bar{D}$ is its posterior mean, p_D can be interpreted as the number of effective parameters given by $p_D = \bar{D} - D(\bar{\theta})$ and $\bar{\theta}$ is the posterior mean of the parameters. A probability density $f(y|\theta)$ can be greater than 1 for a small standard deviation, hence a deviance can be negative, and then generate a negative DIC. In any case, smaller DIC values always indicate a better-fitting model. In fact, Kelly proposed using BIC of non-hierarchical models while preferring DIC if hierarchical models are used [38].

For hierarchical models, expectation posterior predictive p -values can be calculated [39, 40] under current posterior distribution by new data $\mathbf{y}^{\text{new}}$. Equivalently, the p -values can be estimated by posterior probability that $P(\mathbf{y}^{\text{pred}} < \mathbf{y}^{\text{new}})$, $\mathbf{y}^{\text{pred}}$ is the predictive observation, which was described in detail in Section 3. However, it is questionable for model checking because of the repeated use of data. As a result, several researchers [41, 42, 43, 44] have proposed the use of cross-validation predictive densities [22]. Geisser *et al.* [45] proposed using the leave-one-out cross-validation predictive density

$$f(y_i|\mathbf{y}_{\setminus i}) = \int f(y_i|\boldsymbol{\theta})f(\boldsymbol{\theta}|\mathbf{y}_{\setminus i})d\boldsymbol{\theta} \quad (26)$$

where $\mathbf{y}_{\setminus i}$ is $\mathbf{y}$ after omitting y_i .

5. Numerical Examples

In order to verify the hierarchical Bayesian model in the fatigue data analysis, two numerical examples are carried out to assess hierarchical Bayesian model performances with respect to the MLE, and each example is discussed in two cases: the variances of different stress levels are same in Case 1 and the variances are different but they distribute from a same prior in Case 2, as described in Subsection 3.2. The data used in these examples are from the published literature [13, 46]. For Example 1, fatigue test was conducted with standard plate specimens of alloy 2524-T3 under four stress levels with about 15 observations each by Xie *et al.* [13], see Table B.1. Since the sequence of the selected specimen's life data randomly, the asymptotic result of the parameters can be calculated, as shown in Figures 2(a) and 2(b). The asymptotic distributions of parameters can be seen as normal distributions due to the small-scale of vertical axis, so that it is reasonable to choose normal distributions as prior distributions for parameters α and β . For Example 2, durability data for flat specimens cut from the S420MC steel plate is given by Klemenc *et al.* [46], and these standard specimens had a standard shape in accordance with the ASTM E606-92 standard. In these fatigue tests, 65 specimens were broken before 2 million load cycles but 15 specimens survived. These broken specimens were used to estimate the S - N curve and its scatter, see Table B.2, and they are also employed for cross-validation here.

5.1. Technical details

Gibbs sampling is employed to simulate previously mentioned distributions. WinBUGS [47] is used to achieve the purpose of Gibbs sampling in this paper, and all computations are done on a desktop PC. Note that the initial values given in these numerical examples, for both cases of the hierarchical Bayesian model and the MLE, are identical. The Gibbs sampling of two parameters case steps for the distributions are as follows:

- (1) Specifying the likelihood $y_{ij} \sim \mathcal{N}(\mu_i, \sigma_i^2)$, where $\mu_i = \alpha + \beta x_i$;
- (2) Sample α, β , given the other parameters, from $\mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$ and $\mathcal{N}(\mu_\beta, \sigma_\beta^2)$ respectively;

Figure 2: QQ plot of parameters to be estimated versus standard normal, the asymptotic distributions of parameters can be seen as normal distributions for the small-scale of vertical axis.

- 379 (3) Sample μ_α, μ_β from $\mathcal{N}(0, 10^6)$ and sample τ_α, τ_β from $\mathcal{G}(0.001, 0.001)$,
 380 Sample τ and τ_i from $\mathcal{G}(0.001, 0.001)$ in Case 1 and Case 2 respectively;
 381
- 382 (4) For prediction or P - S - N curves estimation, sample y_i^{pred} from predictive
 383 distribution $\mathcal{N}(\mu_{y_i^{\text{pred}}}, \sigma_{y_i^{\text{pred}}}^2)$, where $\mu_{y_i^{\text{pred}}} = \hat{\alpha} + \hat{\beta}x_i$, $\sigma_{y_i^{\text{pred}}} = \hat{\sigma}$ in Case
 384 1 and $\sigma_{y_i^{\text{pred}}} = \hat{\sigma}_i$ in Case 2.

385 where $\tau_* = 1/\sigma_*^2$ for using inverse gamma distributions, and $*$ denotes sub-
 386 scripts. Table 1 shows the settings of prior distributions which are used for
 387 hierarchical Bayesian model to estimate the curves in these examples. In
 388 order to verify the convergence of MCMC, two chains with different initial
 389 values are used in the examples, both of which burnin period are 1000 iter-
 390 ations and then sample 10000 times for each chain. Recalling the material
 391 parameter $B > 0$, for the two parameters model, the initial values of one of
 392 the chains are set as follows: $\mu_\beta = \beta = -B = -1$, τ_i and y_i^{pred} are given a
 393 value close to the observed values y_{ij} , and the remaining variables are set to 1.
 394 The other group of initial values is completely arbitrary and all of which are
 395 set to 1 here. Concerning the sensitivity of the prior of hyperparameter σ_{θ_i}
 396 to inferences, the hierarchical model is calculated by setting $\tau_{\theta_i} \sim \text{U}(0, 10^4)$,
 397 and obtain a consistent result.

Table 1: The settings of prior distributions in Case 1.

	Second stage	Third stage
$\mu \alpha, \beta$	$\alpha \mu_\alpha, \tau_\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$	$\frac{\mu_\alpha \sim \mathcal{N}(0, 10^6)}{\tau_\alpha \sim \mathcal{G}(0.001, 0.001)}$
	$\beta \mu_\beta, \tau_\beta \sim \mathcal{N}(\mu_\beta, \sigma_\beta^2)$	$\frac{\mu_\beta \sim \mathcal{N}(0, 10^6)}{\tau_\beta \sim \mathcal{G}(0.001, 0.001)}$
	$\tau, \tau_i \sim \mathcal{G}(0.001, 0.001)$	—

Since Example 2 is also employed for cross-validation, so that fatigue test data is divided into two parts as shown in Figure 4. The data used for model estimation are divided into 5 groups, by putting the data of close stress levels together. Finally, Example 1 contains four sets of data, each 15, while Example 2 contains five sets, each 12, in both of which the lack of data is set as missing data.

5.2. Results and discussions

The convergence of MCMC in Example 1 is monitored as shown in Figures A.1. Figures A.1(a) and A.1(d) give smooth density curves of the material parameters α and β . A low degree of autocorrelation in the sample as shown in Figures A.1(b) and A.1(e), BGR diagnostics are shown in Figures A.1(c) and A.1(f), in which the dashed line denotes the reference value of one, and as the number of iterations increase, the evolution of the pooled posterior variance (in green) and the mean of the variances within each sample tends to stabilization, and their ratio (in red) tends to one. All of these monitors, including the MC error results in Table 2, indicate the good convergence of the algorithm in Example 1.

For Case 1, with common variance, the results of Example 1 and Example 2 are given in Tables 2 and 3 respectively, while the MLE is calculated with results $\alpha = 4.9740$, $\beta = -3.4708$ of Example 1 and $\alpha = 5.5723$, $\beta = -8.0075$, of Example 2. The hierarchical Bayesian model fitting results of the examples are shown in Figures 3 and 4, with 97.5% survival probability or 95% prediction intervals. Credible intervals, which are used for interval estimation, are intervals in the domain of the posterior probability distribution in Bayesian statistics, and they are analogous to confidence intervals in frequentist statistics. Although the predictive distribution is a kind of posterior distribution, but the prediction intervals here refers the predictive distribution domain of observations specifically. Obviously, the infimum of the 95%

426 prediction interval corresponds to the 97.5% survival probability. The hi-
 427 erarchical Bayesian model with common variance (Case 1) results are also
 428 compared with the MLE, as shown in Figures 7 and 8, and the hierarchical
 429 Bayesian model with independent variance (Case 2), as shown in Figures 5
 430 and 6, respectively. Figures 5 and 6 illustrate that there is little difference
 431 between the estimation results of the one with common variance σ^2 and the
 432 one with several different variances σ_i^2 , so it means that good enough results
 433 are obtained by the model of Case 1. The cross-validation is used for model
 434 checking in Example 2, the results show that the hierarchical Bayesian model
 435 used for fatigue data estimation is credible.

436 Table 4 shows a model comparison of between Case 1 and Case 2 via DIC
 437 values, for Example 1 and 2 respectively. The negative DIC values is easy
 438 to understand for the logarithm of the data narrow the scale, resulting in a
 439 small variance. As Subsection 4.2 discussed, smaller DIC values indicate a
 440 better-fitting model. Case 2 in Example 1 and Case 1 in Example 2 have
 441 smaller DIC and achieve better fitting models. Corresponding to the dis-
 442 tributed characteristics of the data, the different variances σ_i^2 settings prefer
 443 to suit large differences in distribution of life in different stress levels, while
 444 a common variance σ^2 setting for small difference is recommended.

445 Compared with the MLE, the advantage of the hierarchical Bayesian
 446 model for the S - N curve fitting is inconspicuous, while the hierarchical Bayesian
 447 approach highlight the advantages in the P - S - N curves estimation, see Fig-
 448 ures 7 and 8. Moreover, consistent with the results in Tables 5 and 6, as
 449 well as our expectations, the P - S - N curves estimating results of the hierar-
 450 chical Bayesian model are more conservative than the MLE results. This is
 451 because hierarchical Bayesian model incorporates parameters and hyperpa-
 452 rameter estimation error into the analysis and then meets more reasonable
 453 results, even for a common variance setting. Its performance for the vari-
 454 ances is that $\sigma_{y_i^{\text{pred}}} > \sigma_i > \sigma_i^{\text{MLE}}$ always happened, as shown in Tables 5
 455 and 6. Additionally, the calculation of the missing data contained in the
 456 hierarchical Bayesian model is another influencing factor for its conservative
 457 results. To summarize, the hierarchical Bayesian model shows a significant
 458 advantage for the P - S - N curves estimation, while the likelihood-based meth-
 459 ods are considerably faster in computational efficiency than the hierarchical
 460 Bayesian model with the MCMC algorithm involved.

461 Due to the limitations of the regression models, both the MLE and the
 462 hierarchical Bayesian model do not have very good performance at the high-

Table 2: Estimation results of hierarchical Bayesian model in Example 1 Case 1.

Node	Posterior			Posterior percentiles		
	Mean	SD	MC error	2.5%	Median	97.5%
α	4.983	0.01263	9.77E-5	4.958	4.983	5.008
β	-3.387	0.1102	9.123E-4	-3.603	-3.387	-3.171
σ	0.09617	0.009221	6.866E-5	0.08028	0.09547	0.1162
μ_α	8.369	320.2	2.125	-697.1	4.983	769.0
σ_α	1098.0	28920.0	206.8	0.04523	11.2	3941.0
μ_β	-4.047	294.4	2.069	-661.8	-3.381	624.2
σ_β	5006.0	605100.0	4289.0	0.0418	7.023	3199.0

est stress levels for the S - N curve fitting in Example 1. In fact, the scatter is less at high-stress amplitudes, while larger at low-stress amplitudes [10]. Considering that the logarithmic coordinate display is non-linear, the display magnifies the error at high-stress levels (low lifetime) is understandable (Figure 5). Furthermore, it seems that there is a knee point between stress levels 350 and 400 in Example 1. Thus, it is hard to obtain a better fitting result without a more complex model than Basquins relation, Kohout and Vechet function [48], for example. Since model selection is not the main focus of this paper, it will not be presented a detailed analysis here.

The three parameters model with structural noninformative prior also has been attempted to estimate the fatigue curves, as an extra trial. The prior of the third parameter, γ , is set to $\gamma \sim \mathcal{N}(\mu_\gamma, \tau_\gamma)$, and the prior of hyperparameters are set as conjugate priors, that is $\mu_\gamma \sim \mathcal{N}(0, 10^6)$, $\tau_\gamma \sim \mathcal{G}(0.001, 0.001)$. Because there is a logarithm scale of the parameter γ , the three parameters model is difficult to set a reasonable initial value for avoiding numerical overflow. This problem has been solved by setting $\tau_\gamma \sim \mathcal{G}(1, 1)$, limiting the variance of γ to a smaller range. Since the estimate of γ is very close to zero, the result of the three parameters model is close to the two parameters model. The smooth density curves, good BGR diagnostics result are obtained by the MCMC convergence monitors of the three parameters model, but a high autocorrelation. High autocorrelation leads to smaller effective sample size, it means that a subjective prior of γ with informative prior must be given.

Table 3: Estimation results of hierarchical Bayesian model Example 2 Case 1.

Node	Posterior			Posterior percentiles		
	Mean	SD	MC error	2.5%	Median	97.5%
α	5.582	0.02203	1.701E-4	5.538	5.582	5.626
β	-7.925	0.5109	0.006555	-8.917	-7.93	-6.916
σ	0.1602	0.01636	1.542E-4	0.1319	0.1588	0.1963
σ_α	1212.0	59100.0	417.4	0.04438	10.13	4016.0
μ_α	3.357	302.9	2.074	-690.1	5.584	670.6
σ_β	1546.0	76900.0	547.6	0.0437	9.57	3712.0
μ_β	-9.968	310.6	1.848	-733.3	-7.952	687.0

Table 4: The DIC values of Examples

Examples	Modle	$\bar{D}$	$\hat{D}$	p_D	DIC
Example 1	Case 1	-105.665	-108.670	3.005	-102.660
	Case 2	-145.930	-152.139	6.209	-139.721
Example 2	Case 1	-41.812	-44.867	3.055	-38.757
	Case 2	-40.804	-48.004	7.200	-33.603

Figure 3: The $S-N$ curve fitting and $P-S-N$ curves estimation, with 97.5% survival probability or 95% prediction intervals, in Example 1.

Figure 4: The $S-N$ curve fitting and $P-S-N$ curves estimation, with 97.5% survival probability or 95% prediction intervals, in Example 2.

Figure 5: A comparison of the hierarchical Bayesian model Case 1 and Case 2 in Example 1.

Figure 6: A comparison of the hierarchical Bayesian model Case 1 and Case 2 in Example 2.

Figure 7: A comparison of the hierarchical Bayesian model (Case 1) and MLE to estimate $S-N$ curve and $P-S-N$ curves in Example 1.

Figure 8: A comparison of the hierarchical Bayesian model (Case 1) and MLE to estimate S - N curve and P - S - N curves in Example 2.

Table 5: The comparison of the standard deviations in Example 1.

Cases	SD	$\log_{10} N_1$	$\log_{10} N_2$	$\log_{10} N_3$	$\log_{10} N_4$
Case 1	σ	0.09617			
	$\sigma_{y_i^{\text{pred}}}$	0.0996	0.09588	0.09891	0.09777
Case 2	σ_i	0.05881	0.06053	0.04056	0.1939
	$\sigma_{y_i^{\text{pred}}}$	0.06243	0.06239	0.04202	0.1983
MLE	σ_i^{MLE}	0.05395	0.05485	0.03676	0.08711

Table 6: The comparison of the standard deviations in Example 2.

Cases	SD	$\log_{10} N_1$	$\log_{10} N_2$	$\log_{10} N_3$	$\log_{10} N_4$	$\log_{10} N_5$
Case 1	σ	0.1602				
	$\sigma_{y_i^{\text{pred}}}$	0.1664	0.1622	0.1607	0.1629	0.1643
Case 2	σ_i	0.1519	0.2106	0.1794	0.1413	0.1472
	$\sigma_{y_i^{\text{pred}}}$	0.1635	0.2188	0.1846	0.1483	0.1557
MLE	σ_i^{MLE}	0.1323	0.1968	0.1731	0.1339	0.1352

486 6. Conclusions

487 The hierarchical Bayesian model for estimating $S-N$ curve have been pre-
488 sented in order that the test time and the number of specimens can be mini-
489 mized, getting benefit from the hierarchical structure robustifies the usual
490 Bayesian analysis. Following this, a $P-S-N$ curves estimation method is
491 proposed by using the predictive distributions, with the same hierarchical
492 structure. The following conclusions can be drawn by numerical examples:

- 493 (1) Compared with the MLE, the hierarchical Bayesian model obtains more
494 conservative results for $P-S-N$ curves estimation, because the param-
495 eters and hyperparameter estimation errors are incorporated into the
496 analysis, as well as the uncertainty effects of missing data. As a result,
497 the hierarchical Bayesian model meets a safer design curve from small
498 censored data sets.
- 499 (2) Benefit from the advantage of the hierarchical Bayesian model that can
500 incorporate actual subjective prior information at the second stage, the
501 physical models, such as random fatigue limit model, can be incorpo-
502 rated into a hierarchical Bayesian model conveniently for reducing the
503 uncertainties. Besides, it allows the use of both structural prior infor-
504 mation and subjective prior information simultaneously in fatigue data
505 analysis. In addition, the physical model also can be set as a likelihood
506 function for updating.
- 507 (3) Missing data can be easily handled in the hierarchical Bayesian model,
508 it is helpful when the data is insufficient. Because of the more missing
509 data means more uncertainties, and the results will achieve in larger
510 prediction intervals.
- 511 (4) MCMC methods are broadly applicable, but require care in parametriza-
512 tion and convergence diagnosis.

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Figure A.1: Results of MCMC convergence monitoring in Example 1, where sub-graph (a), (b), (c) for α and (d), (e), (f) for β .

Table B.1: Fatigue life test data of aluminum alloy 2524-T3 [13].

S_i (MPa)	$\log_{10} N_i$
200	5.603, 5.544, 5.528, 5.630, 5.594, 5.540, 5.581, 5.548, 5.426, 5.567, 5.554, 5.627, 5.630, 5.596, 5.626
300	5.028, 5.074, 5.016, 4.894, 4.993, 5.071, 5.024, 5.035, 4.954, 5.039, 5.098, 5.057, 5.092, 5.082, 5.005
350	4.784, 4.842, 4.776, 4.813, 4.813, 4.860, 4.798, 4.776, 4.758, 4.770, 4.755, 4.837, 4.736, 4.842, 4.796
400	4.477, 4.400, 4.426, 4.462, 4.592, 4.411, 4.447, 4.402, 4.665, 4.475, 4.458, 4.551, 4.525, 4.641

519 Appendix A. Figures of MCMC convergence monitoring

520 Appendix B. The original data

521 References

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Table B.2: List of experimental durability data for the S420MC steel [46].

S_i (MPa)	N_i	S_i (MPa)	N_i	S_i (MPa)	N_i
204	946200	229	1447200	268	120800
207	1851500	232	488400	268	139800
210	1281700	232	380500	268	159100
211	1215000	232	567000	268	187100
214	628100	232	701800	268	219600
214	1307600	232	553000	268	238600
214	1316000	232	630000	271	259500
214	1410600	248	286700	271	313000
214	851900	248	376900	271	346100
214	1566600	248	488300	286	61600
214	959900	248	650100	286	119400
214	1159400	248	585900	286	81600
219	1095800	248	698500	286	132000
219	1499200	250	313700	286	130000
221	1926800	250	256900	286	104300
224	1999500	250	238800	286	97400
224	997600	250	323500	286	175600
229	690600	250	213700	286	136500
229	730500	250	389000	295	136800
229	1009600	267	199400	295	129900
229	1555800	267	194000	295	151400
229	1358000	267	224800		

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