Large Displacement Behavior of Double Parallelogram Flexure Mechanisms with Underconstraint Eliminators

Robert M. Panas¹

Lawrence Livermore National Laboratory, L-229

7000 East Avenue, Livermore, CA 94551

This paper presents a new analytical method for predicting the large displacement behavior of flexural double parallelogram (DP) bearings with underconstraint eliminator (UE) linkages. This closed-form perturbative Euler analysis method is able to- for the first time- directly incorporate the elastomechanics of a discrete UE linkage, which is a hybrid flexure element that is linked to ground as well as both stages on the bearing. The models are used to understand a nested linkage UE design, however the method is extensible to other UE linkages. Design rules and figures-of-merit are extracted from the analysis models, which provide powerful tools for accelerating the design process. The models, rules and figures-of-merit enable the rapid design of a UE for a desired large displacement behavior, as well as providing a means for determining the limits of UE and DP structures. This will aid in the adoption of UE linkages into DP bearings for precision mechanisms. Models are generated for a nested linkage UE design, and the performance of this DP with UE structure is compared to a DP-only bearing. The perturbative Euler analysis is shown to match existing theories for DP-only bearings with distributed compliance within $\approx 2\%$, and Finite Element Analysis for the DP with UE bearings within an average 10%.

Keywords: underconstraint elimination; four-bar flexure; folded flexure; double parallelogram flexure; nested linkage; nonlinear beam analysis; elastokinematic effect; large displacement axial stiffness; large stroke; flexure mechanism

¹ Corresponding Author: Phone 925-423-5569; E-mail: panas3@llnl.gov

Nomenclature

Symbol	Units	Definition	Symbol	Units	Definition
Motion			Forces		
θ_{ld}	rad	Angle of flexures at large displacement	F_{mt}	N	Transverse reaction force of type m flexure
x_b	m	Transverse displacement of individual flexure blade	F_{ma}	N	Axial reaction force of type m flexure
x_f	m	X-axis displacement of final stage	F_{mx}	N	X-axis reaction force of type m flexure
δ_{xi}	m	X-axis perturbation displacement of intermediate stage	F_{my}	N	Y-axis reaction force of type m flexure
δ_{xf}	m	X-axis perturbation displacement of final stage	F_x	N	X-axis reaction force to hold final stage
δ_{yi}	m	Y-axis perturbation displacement of intermediate stage	F_y	N	Y-axis applied load on final stage
δ_{yf}	m	Y-axis perturbation displacement of final stage	Stiffness		
δ_{mt}	m	Transverse displacement of the type m flexure in the flexure reference frame	k_{ma}	N/m	Axial stiffness for type m flexures
δ_{ma}	m	Axial displacement of the type m flexure in the flexure reference frame	k_{mt}	N/m	Transverse (in-plane) stiffness for type m flexures
X	N/A	X-axis displacement of final stage, nondimensionalized via h_m	k_{pa}	N/m	Axial stiffness for type p flexures
Large Displacement Effect Scale Figure-of-Merit			k_{pb}	N/m	Rigid body relaxation stiffness for type p flexures
τ_b	N/A	LDES of single beam transverse motion	$k_{j\theta}$	N-m/rad	Clamped-free z-axis moment-to-rotation stiffness for type j flexures
τ_{DP}	N/A	LDES of DP bearing with no UE main DOF	k_{jt}	N-m/m	Clamped-free z-axis moment to in-plane transverse displacement stiffness
τ_{11}	N/A	LDES of type 11 flexures in main DOF	k_{ue}	N/m	UE stiffness for intermediate stage underconstrained DOF motion
τ_{12}	N/A	LDES of type 12 flexures in main DOF	k_{fue}	N/m	UE stiffness for final stage main translational DOF motion
τ_2	N/A	LDES of type 2 flexures in main DOF	k_{ba}	N/m	Axial stiffness of a single beam
τ_m	N/A	LDES of type m flexures in main DOF	k_{bae}	N/m	Euler axial stiffness of a single beam
τ_{yf}	N/A	LDES of final stage y-axis elastokinematics	k_{DP}	N/m	Axial (y-axis) stiffness of a DP bearing with no UE
τ_{d2}	N/A	LDES of 1 st order term, DP + UE net effect	k_{yi}	N/m	Y-axis stiffness for intermediate stage deformation
τ_{d4}	N/A	LDES of 2 nd order term, DP + UE net effect	k_{yf}	N/m	Y-axis stiffness for final stage
τ_{nUE}	N/A	LDES of final stage y-axis net effect	r_k	N/A	Underconstrained DOF Stiffness ratio
τ_{nDP}	N/A	LDES of final stage y-axis net effect	Subscript		
Geometry			m	N/A	Index for type m flexures
E	Pa	Young's modulus for all flexure types	1	N/A	Index for type 1 flexures
A	m ²	Axial-normal cross-sectional area for all flexure types	11	N/A	Index for type 11 flexures
I	m ⁴	Axial-normal second moment of area for all flexure types	12	N/A	Index for type 12 flexures
L	m	Length for all flexure types	2	N/A	Index for type 2 flexures
h	m	Bending direction thickness for all flexures	p	N/A	Index for type {11, 12, 2} flexures
γ	N/A	PRBM flexure length fraction	j	N/A	Index for type {1, 2} flexures
θ_2	rad	Angle of type 2 flexures off the y-axis			
r_1	m	Lever arm for type 1 flexures			
r_2	m	Lever arm for type 2 flexures			
d	m	Lever arm for UE rotation			
d_3	m	Separation between serial type m flexures			
a_0	N/A	Compliance distribution ratio			

1 Introduction

The intent of this work is to model the effect of a nested linkage underconstraint eliminator [1] on the large displacement y-axis stiffness of a flexural double parallelogram linear bearing. A new method is presented for the closed-form analysis of the large displacement axial stiffness which can easily incorporate the complex elastomechanics of a UE linkage, unlike existing large displacement models [2]–[18]. Design rules and new figures-of-merit are defined to further simplify the design process. These tools are

demonstrated on a nested-linkage UE design as presented in [1], but they are generalizable to other UE designs. These tools enable the rapid, direct design of the UE for the desired large displacement behavior, in addition to mapping the performance limitations intrinsic to the DP bearing as well as those of the UE. This work will accelerate the adoption of UE linkages into DP bearings, as designers can now quickly analyze the tradeoffs associated with their use, even for large displacement applications. This offers the potential for improving bearing performance via reduced error motion, improved dynamics and greater bearing stiffness for a range of applications including their use as MEMS [5], [6], [19]–[27], single-/multi-axis precision positioning stages [19], [28]–[32], and macroscale bearings [8], [33]–[35].

1.1 Double Parallelogram Flexure Bearing

Double parallelogram flexure bearings, as shown in Fig. 1, are commonly used in precision devices [8], [13], [18], [34], [36], [37] for several reasons. The flexure component of the bearing can provide motion with very fine resolution, no stiction, and high repeatability [8], [34] among other features. The double parallelogram structure provides increased range and reduced kinematic errors for its main translation Degree-of-Freedom (DOF) over that of a single parallelogram flexure bearing [2], [18]. The kinematic errors of the single parallelogram are shown as elliptical arcs in Fig. 1b. At large displacements, these arcs begin to show a y-axis component. The second stage of the bearing shows these same errors in reverse, so the net effect is to cancel out the kinematic error at the final stage [2], [18] given that the final and intermediate stages are allowed to move in a 2:1 ratio. This is shown in Fig. 1b with solid arrows.

Fig. 1. a) Double parallelogram flexure bearing, with b) effective linkage model of the bearing undergoing displacement along its main translational DOF. The solid arrows show the main translation DOF motion, while the dotted arrows show the y-axis DOF of the final stage at large displacements enabled by the underconstrained DOF of the intermediate stage. Image from [1].

The intermediate stage possesses the same translational DOF as the final stage [1], [2], [8], [18], [28], [37], [38]. The underconstraint is observable in two forms [1], [18]. The freedom of the intermediate stage to vibrate appears as a relatively low frequency collocated resonance which can negatively affect the dynamics and control of the bearing [1], [18]. Axial loading (y-axis) on the final stage at large displacements drives the intermediate stage back along its arcuate path towards equilibrium, while the flexures holding the final stage are further deflected, as shown in Fig. 1b via the dotted arrows. This elastokinematic effect allows the final stage to translate axially, further than allowed by pure axial compression [1], [2], [18]. The bearing performance with regards to the y-axis axial stiffness is thus compromised by the underconstraint. The flexural DP is often used as a bearing for microscale positioning stages driven by capacitive drives. It must resist pull-in effects via the DP axial stiffness (y-axis), over a range of displacements, so this large displacement axial stiffness reduction is of direct concern to device performance. This has been a common focus for bearing designers [3]–[6], [14].

1.2 Nested Linkage Underconstraint Eliminator

The author previously introduced the design of a new, compact linkage intended to remove the underconstrained DOF from DP bearings [1]. This underconstraint eliminator design is shown in Fig. 2.

Fig. 2. Nested UE linkage shown with a) main geometries identified, including the three flexure types: main (m), type 1 (1) and type 2 (2). The effective linkage model for the design is shown in b), where the UE linkage is replaced with a bar that guides the final and intermediate stage to a coupled 2:1 motion ratio as shown with the solid arrows. Image from [1].

The design is inserted within a standard DP flexure bearing whose four type m parallel blade flexures, labeled m , guide the intermediate and final stages. The triangular UE linkage connects to ground via two angled type 2 flexures, labeled 2, which generate an instant center above the bearing around which the rigid body rotates. The instant center is located along the plane of symmetry of the device which is identified via the dotted line bisecting the structure in Fig. 2a, and is located distance d above the UE-intermediate stage flexure joint. The same distance separates the UE-intermediate stage joint from the UE-final stage joint, enforcing a 2:1 lever arm ratio for the final vs intermediate stages when they translate in the x-axis as shown in Fig. 2b. The arcuate motion of the UE linkage is decoupled from both of the stage linear motions by the two symmetric sets of type 1 flexures, which allow the UE linkage to move in the y-axis as it rotates around its instant center, without forcing the stages to do likewise. The type 1 flexures are located at two locations, and serve generally the same task. The type 1 flexures have the same geometry, but are broken into subtypes 11 and 12, labeled 11 and 12 , to account for differences in their rigid body relaxation stiffnesses, k_{11b} and k_{12b} . The UE couples the intermediate and final stage DOFs with a 2:1 ratio, ensuring kinematic error cancellation and removing the underconstrained DOF. Further detail on the design is found in [1].

2 Large Displacement Analysis

The nested linkage UE performance with regards to the small displacement stiffnesses and bearing dynamics has previously been analyzed [1], but no established closed-form analytical methods exist to model the effect of a UE on the y-axis stiffness of the DP at large displacements. Instead, the typical solution is to use computational analysis [3], [4], [39], which like Finite Element Analysis (FEA), is non-invertible and thus difficult to use as a design synthesis tool. This work provides such needed models and design rules, so that the y-axis stiffness of UE DP bearings can be calculated in closed form and inverted for use in design synthesis.

A large displacement analysis of static axial performance is presented to provide insight into the capability of the new UE nested linkage design to remove underconstraint. The models are specifically developed for the nested linkage. The method is, however, directly generalizable to the closed form modeling of the effect of any UE design on the y-axis stiffness of a DP flexure bearing at large displacements.

The large displacement behavior of beams in flexural bearings has been the subject of comprehensive analysis in literature [2]–[18]. These analyses typically include several features: i) non-simplified Euler-Bernoulli theorem to link beam curvature to applied moment with axial loaded included, ii) kinematics of beam arc-length conservation, and iii) elliptic integrals or transcendental functions. The resulting general models are good for a wide range of elastokinematic behaviors and are accurate over wide displacement ranges. The tradeoff for this modeling capability is rather complex mathematics [13], that can quickly grow cumbersome for complex flexure mechanisms.

The axial stiffness can alternatively be viewed as a small displacement perturbative effect, occurring around an equilibrium defined by large displacement elastokinematics. We present a new method of analysis based on this understanding. The perturbative approach provides a means for reducing the complexity of the problem. The addition of the triply-linked (ground, intermediate stage, and final stage) UE flexure mechanism adds significant complexity, so it is desired that the mathematical framework be kept as simple as possible, such as the use of Euler stiffness formations for the analysis. The perturbative approach formulation is able to provide such a methodology, and thus enables the simple inclusion of a triply-linked UE into the closed form analysis of the DP. A closed form analysis of a DP bearing with a discrete UE mechanism (as

compared to the UE effect generated by angling the DP beams, which generates significant bearing performance issues as noted in [1]) has not previously been demonstrated in literature. Efforts have been limited to computational analysis for DP bearings with discrete UE linkages [3], [4], [39].

We also introduce a new figure-of-merit that enables designers to simply include the large displacement effect into design tradeoffs. The characteristic Large Displacement Effect Scale (LDES) figure-of-merit, τ , will be introduced in this work to describe the nondimensional scale of displacement at which axial stiffening effects begin to significantly influence bearing stiffness. The nondimensionalization is carried out by dividing the transverse displacement at the guided end of the beam by the beam bending thickness, as has been done in several previous analyses [4], [18], [39]. The beam bending thickness is the thickness of the beam in the dimension parallel to the DOF deflection. This direction is transverse to the beam axis, and in the x-y plane. Simple equations are developed to approximately calculate the LDES for DP bearings with (and without) UEs; these expressions can then be used as constraints or objective functions in the design process, which was not previously feasible. The method is shown to be independent of the exact UE topology, but rather relies only on the contribution of the UE linkage to the x-axis stiffness of both the intermediate and final stages. The methodology can thus be extended to the closed-form analysis of other UE designs.

3 Perturbative Euler Analysis

The perturbative model of axial stiffness at large displacements is generated via three steps. First, the large displacement equilibrium is established. The type m flexures are modeled as at an angle to the y-axis. Pseudo-rigid body modeling (PRBM) [13] is used to determine the effective beam angle which accounts for the flexures' elastokinematics. Second, a set of linear, small displacement perturbation free-body analyses are carried out at this large displacement equilibrium, with the beams stiffness vectors rotated to the effective beam angle. This is able to capture the elastokinematic effect of the underconstraint on the axial stiffness. Third, the beam profile-based axial stiffness variation is included into the axial stiffness expression for each beam. The axial stiffness effect is driven by the warped shape of the flexure at large displacements, so is not captured in the small displacement analysis in step two. Rather, the deformed profile alters the axial loading to axial-only displacement relation [2], [5]. Each beam axial stiffness term is replaced with a displacement dependent function.

3.1 Elastokinematic Equilibrium

The large displacement equilibrium for the overall structure is first defined by the effective beam angle off the y-axis, θ_{ld} , as shown in Fig. 2b, where positive x-axis stage displacement produces positive angle. The type m flexures are subjected to the large displacement in the structure, and their profile is altered by the displacement of the final stage. Each set of parallelogram flexures sees approximately half the final stage x-axis displacement, x_f , as shown in Fig. 2b. The length of the flexure is attenuated by beam-length conservation. This can be modeled for a distributed compliance beam using PRBM [13], where the fixed-guided flexure is broken into three rigid segments and two joints. The segments on either end are of equal length, and are attached to their respective stages (eliminating rotation). The center element has a length that is a fraction, $\gamma \approx 0.8517$, of the total beam length, L_m . The beam-length fraction shows a slight dependency on axial load, and can drop about 5% to 0.815 under high axial loading [13]. For simplicity, the no-load value is used. Over the majority of the beam length in the PRBM model the beam is at angle θ_{ld} off the y-axis, as defined in Eq. (1), so this is assumed to be the effective beam angle.

$$\theta_{ld} = \sin^{-1} \left(\frac{x_f}{2L_m\gamma} \right) \quad (1)$$

3.2 Small Displacement Perturbation

A small displacement perturbative free-body analysis is carried out using the equilibrium conditions. This is generated by combining the individual beam stiffness expressions into force balances on both the intermediate and final stages which account for the rotation of the beams.

3.2.1 Individual Beam Stiffnesses

The free-body analysis first requires several stiffnesses to be defined. These have been derived and described in detail previously [1], but will be repeated here for clarity. The stiffnesses of the individual beams under multiple conditions are presented in Eq. (2) [1], where E , A , I , L and h , are the Young's Modulus, 2nd moment of area, cross-sectional area, length, and bending thickness respectively for the flexure types (m, 1, 11, 12, 2) as noted by the parameter's subscript (m , I , 11 , 12 , 2). The geometry and material properties for the type 11 and type 12 flexures are equivalent in this design and are set to be equal a single value, covered under the type 1 label.

$$\begin{aligned}
k_{ma} &= \frac{A_m E_m}{L_m} & k_{mt} &= \frac{12 E_m I_m}{L_m^3} & k_{pa} &= \frac{A_p E_p}{L_p} \square k_{pb} \\
E_{11} &= E_{12} = E_1 & A_{11} &= A_{12} = A_1 & L_{11} &= L_{12} = L_1 \\
k_{j\theta} &= \frac{E_j I_j}{L_j} & k_{jt} &= \frac{12 E_j I_j}{L_j^3} & k_{j\tau} &= \frac{2 E_j I_j}{L_j^2} \\
p &= \{11, 12, 2\} & j &= \{1, 2\}
\end{aligned} \tag{2}$$

The parallel summation symbol is used to indicate the operation of the inverse of the sum of the inverses. The main DP flexures' axial force-displacement stiffness is k_{ma} , and transverse force-displacement stiffness is k_{mt} . The term k_{pa} describes the p-type flexures' axial force to displacement stiffness with p value 11 for the type 1 top set, 12 for the type 1 bottom set, and 2 for the type 2 set. The axial stiffness is composed of the flexure axial stiffness in parallel with a rigid body relaxation stiffness, k_{pb} , which is unique for each flexure set, to account for the failure of the rigid body assumption. The rigid body relaxation stiffness for each flexure is a function of the local geometry of the rigid bodies at the location where the flexure is attached. The rigid body relaxation stiffness acts to slightly reduce the effective stiffness of the flexure via series compliance. The terms are defined and explained in [1]; a full redefinition lies outside of the scope of this paper. The term $k_{j\theta}$ describes the j-type flexures' clamped-free z-axis moment to rotation stiffness at the free end with j value 1 for the type 1, and 2 for the type 2 flexures. The term k_{jt} describe the j-type flexures' clamped-guided in-plane transverse force-to-displacement stiffness at the guided end. The terms $k_{j\tau}$ describe the j-type flexures' clamped-free z-axis moment to in-plane transverse displacement stiffness at the free end.

3.2.2 UE Linkage Stiffness

The contributions of the UE linkage to the stiffness of the underconstrained DOF, k_{iue} , and to the main translational DOF, k_{jue} , are also required for this analysis. These two parameters capture the effect of the UE on the large displacement behavior. Any UE design can be analyzed via the perturbative Euler analysis method once these two terms are calculated for the design.

The UE contribution to the underconstrained DOF is measured by anchoring the final stage and driving the intermediate stage in the x-axis. The UE resists this (as intended) via the axial loading of its flexures, as shown in Eq. (3) and derived in [1], where θ_2 is the angle of the type 2 flexures off of the y-axis, as shown in Fig. 2a.

$$k_{iue} = 2 \left[\frac{1}{4 \sin(\theta_2)^2 k_{2a}} + \frac{1}{k_{11a}} + \frac{1}{4k_{12a}} \right]^{-1} \quad (3)$$

The UE stiffness contribution to the intermediate stage underconstrained DOF motion is a combination of two parallel modes. First, the UE linkage rotates around the center of stiffness of the type 12 flexures, forcing the type 2 flexures to deflect. This generates the left-most term on the right-hand side of Eq. (3). Second, the UE linkage rotates around the center of stiffness of the type 2 flexures, driving the type 11 and 12 flexures as sets in series. This generates the second and third term in the right hand side of Eq. (3). These are derived and described in detail in [1].

The UE contribution to the main translational DOF is measured by driving the final stage in the x-axis. The UE allows this motion to occur since the intermediate stage is free to move at a 1:2 ratio with the final stage. The UE motion does, however, add extra stiffness to the main DOF which can be calculated as shown in Eq. (4) and derived in [1], where r_1 and r_2 are the lever arms for the type 1 and 2 flexures respectively, as shown in Fig. 2. The type 1 flexures are not split into subtypes 11 and 12 here because the axial stiffness (the source of the subtype split) is not utilized for this DOF-stiffness level analysis.

$$k_{fue} = \frac{k_{1\theta} + (k_{1\theta}/k_{1t\tau} + r_1)^2 k_{1t}}{d^2} + \frac{k_{2\theta} + (k_{2\theta}/k_{2t\tau} + r_2)^2 k_{2t}}{2d^2} \quad (4)$$

3.2.3 Force Balance

The x-axis (first row) and y-axis (second row) force balances are carried out for both the intermediate and final stages. This is derived assuming Eulerian small displacement stiffnesses for the main beams, but assuming the beams are at an angle to their equilibrium orientation. The small perturbations of the intermediate (i) and final (f) stages are captured in the x-axis with δx_i , δx_f and in the y-axis with δy_i , δy_f . The stage motions can be transferred to the flexure reference frame via a rotation matrix as shown in the first line in Eq. (5). The flexure frame motions are transformed to reaction forces in the second line, and the reaction forces can be transformed back to generate the x-axis and y-axis forces via the final rotation matrix. These rotations are the key element of PE analysis which allows the linear elastomechanics analysis to capture the large deflection stiffness reduction Eq. (5) corresponds to the matrix transforms for the type m flexures connecting the intermediate stage to ground.

$$\begin{aligned}
\begin{bmatrix} \delta_{mt} \\ \delta_{ma} \end{bmatrix} &= \begin{bmatrix} \cos(\theta_{ld}) & -\sin(\theta_{ld}) \\ \sin(\theta_{ld}) & \cos(\theta_{ld}) \end{bmatrix} \begin{bmatrix} \delta x_i \\ \delta y_i \end{bmatrix} \\
\begin{bmatrix} F_{mt} \\ F_{ma} \end{bmatrix} &= - \begin{bmatrix} 2k_{mt} & 0 \\ 0 & 2k_{ma} \end{bmatrix} \begin{bmatrix} \delta_{mt} \\ \delta_{ma} \end{bmatrix} \\
\begin{bmatrix} F_{mx} \\ F_{my} \end{bmatrix} &= \begin{bmatrix} \cos(\theta_{ld}) & -\sin(\theta_{ld}) \\ \sin(\theta_{ld}) & \cos(\theta_{ld}) \end{bmatrix}^T \begin{bmatrix} F_{mt} \\ F_{ma} \end{bmatrix}
\end{aligned} \tag{5}$$

The transverse displacement δ_{mt} and transverse reaction force F_{mt} point in the positive x-axis direction when the DP is undeflected. The axial displacement δ_{ma} and axial reaction force F_{ma} likewise point in the positive y-axis direction when the DP is undeflected. The same analysis applies to the type m flexures from the intermediate stage to the final stage, but with two changes: i) the deflection is now the difference between the intermediate and final stage motion, and ii) the flexure angle θ_{ld} is replaced by the negative angle $-\theta_{ld}$ to account for the flexures' flipped orientation.

Fig. 3. Free body diagram of nested linkage UE effective linkage model, with loads and displacements shown for each of the stages. An expanded view of the type m flexure joint with the intermediate stage is used to clarify the orientation of the forces and displacements defining the type m flexures' elastomechanics.

A free body diagram of the various forces on the intermediate stage and final stage is shown in Fig. 3. This graphically depicts how the force balance equations are generated, as shown in Eq. (6). The intermediate stage is the first equation, while the final stage is the second equation. The moment balance is ignored for simplicity as the parallelogram axial stiffness derivation is dominated by large axial forces.

$$\begin{aligned}
& -R_m^T K_m R_m \begin{bmatrix} \delta x_i \\ \delta y_i \end{bmatrix} - R_m K_m R_m^T \begin{bmatrix} \delta x_i - \delta x_f \\ \delta y_i - \delta y_f \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2k_{lt} \delta y_i - \begin{bmatrix} 1 \\ 0 \end{bmatrix} k_{ue} \left(\delta x_i - \frac{\delta x_f}{2} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
& -R_m K_m R_m^T \begin{bmatrix} \delta x_f - \delta x_i \\ \delta y_f - \delta y_i \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} k_{fue} \delta x_f - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{k_{ue}}{2} \left(\frac{\delta x_f}{2} - \delta x_i \right) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2k_{lt} \delta y_f + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6) \\
& R_m = \begin{bmatrix} \cos(\theta_{ld}) & -\sin(\theta_{ld}) \\ \sin(\theta_{ld}) & \cos(\theta_{ld}) \end{bmatrix} \quad K_m = \begin{bmatrix} 2k_{mt} & 0 \\ 0 & 2k_{ma} \end{bmatrix}
\end{aligned}$$

The type m forces are calculated through Eq. (5), where a rotation matrix, R_m , is used to angle the type m flexure sets, which have a stiffness matrix K_m . The intermediate stage x-axis stiffness due to the UE linkage, k_{ue} , is calculated in Eq. (3), the final stage x-axis stiffness due to the UE linkage, k_{fue} , is calculated in Eq. (4). A y-axis force, F_y , is applied to the final stage. The UE linkage is included as a guiding spring on the intermediate stage. The UE linkage does not apply any force on the intermediate stage when x-axis motion occurs with the coupled 2:1 ratio, however, when this 2:1 scale is violated the high stiffness UE spring resists it. The forces of the UE on the intermediate and final stage are related by a torque balance on the UE. This balance is calculated in [1] via a moment summation around the type 2 flexure instant center, and simplifies to a 2:1 ratio of forces on the intermediate vs final stage due to the 2:1 lever arm ratio. The stiffness of the UE linkage with regards to 2:1 coupled motion is captured via k_{fue} in the final stage equation. The equations are set up for constant force conditions, but can be changed to constant displacement with two adjustments: set $\delta x_f = 0$ and add an F_x reaction force to the x-axis force balance on the final stage.

3.2.4 Displacements

The displacements can now be solved in closed form, as shown in Eq. (7). k_{lt} is included in Eq. (6) for completeness, however it is removed prior to solving for displacements, as typically the type 1 transverse stiffness is far lower than the type m axial stiffness $k_{lt}/k_{ma} \ll 1$. The force-controlled solution produces $\delta x_f = 0$, so the solution turns out to be identical regardless of the boundary conditions. force or displacement [28]. A small angle-dependent portion of the axial load is transferred to transverse loading on the angled beams, driving them to undergo arcuate motion which has a y-axis component at large displacements. The arcuate trajectories of the intermediate and final stages cancel out one another's x-axis motion, resulting in $\delta x_f = 0$ while $\delta x_i \neq 0$, which is an activation of the underconstrained DOF. The y-axis motions, however, are reinforced, resulting in $\delta y_f = 2\delta y_i$.

$$\begin{aligned}
\delta x_i &= -\frac{F_y \left(\frac{k_{ma} - k_{mt}}{k_{ma}} \right) \frac{\sin(2\theta_{ld})}{2}}{2k_{mt} + \frac{k_{iue}}{2} \left(\cos^2(\theta_{ld}) + \frac{k_{mt}}{k_{ma}} \sin^2(\theta_{ld}) \right)} \approx -\frac{F_y \left(\frac{x_f}{2L_m \gamma} \right)}{2k_{mt} + \frac{k_{iue}}{2}} \\
\delta y_i &= \delta y_f / 2 \\
\delta x_f &= 0 \\
\delta y_f &= \frac{F_y}{k_{ma}} \frac{(4k_{mt} + k_{iue}) + 4(k_{ma} - k_{mt}) \sin^2(\theta_{ld})}{4k_{mt} + k_{iue} \left(\cos^2(\theta_{ld}) + \frac{k_{mt}}{k_{ma}} \sin^2(\theta_{ld}) \right)} \approx \frac{F_y}{k_{ma}} \left[1 + \frac{4k_{ma}}{4k_{mt} + k_{iue}} \left(\frac{x_f}{2L_m \gamma} \right)^2 \right]
\end{aligned} \tag{7}$$

The full accuracy terms are shown, with approximations generated by making two assumptions. First, it is assumed that the type m transverse stiffness is far lower than the axial stiffness, $k_{mt}/k_{ma} \ll 1$, as is typically found in bearing flexures. Second, $\theta_{ld} < 0.1$ small angle assumptions are applied, as the flexure motion is typically still small relative to the length of the flexure even in large displacement analysis [2]. The small angle assumption is confirmed to be accurate within 5% up to about $x_f/h_m \approx 18$ ($\theta_{ld} \approx 0.23$) in this system.

3.3 Profile-based Stiffness via the Large Displacement Effect Scale

The beam profile-based axial stiffness variation is finally included into the model. This is analyzed via the Large Displacement Effect Scale (LDES) term, which captures the displacement at which the axial stiffness is halved due to large-displacement effects. The axial stiffness of clamped-guided flexures is a direct function of their end displacement. This functional dependence has been calculated for a simple single parallelogram flexure bearing [2], [5]. This is half the DP, without the UE linkage mechanism. The axial stiffness of a single beam in the parallelogram, $k_{ba}(x)$, can be extracted from such analysis [5], as shown in Eq. (8).

$$\begin{aligned}
k_{ba}(x) &= k_{bae} \frac{1}{1 + \left(\tau_b(a_0) \right)^{-2} \left(\frac{x_b}{h_b} \right)^2} \\
\tau_b(a_0) &= \sqrt{\frac{175(4a_0^2 - 6a_0 + 3)^3}{12a_0^2(576a_0^4 - 1480a_0^3 + 1440a_0^2 - 630a_0 + 105)}} \\
\tau_b(0.5) &= \sqrt{\frac{175}{3}} \approx 7.64
\end{aligned} \tag{8}$$

The beam's Eulerian axial stiffness, k_{bae} , is attenuated by the profile-based effect via a function of the beam tip displacement, x_b , divided by the beam bending thickness, h_b . The attenuation is governed by the characteristic LDES $\tau_b(a_0)$, which is the value of x_b/h_b at which the stiffness has fallen by half. The LDES is a function of the compliance distribution ratio a_0 . The compliance distribution ratio assumes a symmetric flexure with a rigid center section and flexible sections at either end. It is defined as the length of one of these end sections divided by the total length of the flexure [5]. The boundary conditions in Eq. (8) are that the guided end of the deformed beam is constrained to only deflect axially (relative to the undeformed shape) under axial load, with no transverse motion [5]. The single beam LDES can be improved from 7.64 at $a_0 = 0.5$ by reshaping the beam towards point compliance at the ends, $a_0 \rightarrow 0$, where the value rises without limit.

When two single parallelograms are stacked on one another to generate a DP bearing, as studied in this work, the effective axial stiffness reduction can be broken into two effects: the profile-driven axial stiffness reduction and an elastokinematic effect which drives the underconstrained DOF motion, δx_i . The axial (y-axis) stiffness, k_{DP} , of the DP bearing (without a UE mechanism) shown in Fig. 1 can be extracted from previous analysis [5], and is shown in Eq. (9) [5].

$$\begin{aligned}
 k_{DP}(X) &= k_{ma} \frac{1}{1 + (\tau_{DP}(a_0))^{-2} X^2} \\
 \tau_{DP}(a_0) &= \sqrt{\frac{140(4a_0^2 - 6a_0 + 3)^2}{3(720a_0^4 - 2240a_0^3 + 2688a_0^2 - 1470a_0 + 315)}} \\
 \tau_{DP}(0.5) &= \sqrt{\frac{140}{51}} \approx 1.66
 \end{aligned} \tag{9}$$

$X = x_f/h_m$ is the nondimensional displacement of the beam end as seen at the final stage. The underconstrained DOF elastokinematic LDES for a DP, $\tau_{DP}(0.5)$, is about 10x lower than the profile-driven LDES for a DP, $2\tau_b(0.5)$ when considering fully distributed compliance flexures. The factor of 2 in the profile-driven LDES accounts for the adjustment of the x-axis displacement of a single beam, x_b , to that of the final stage $x_f = 2x_b$. The elastokinematic LDES is reduced from 1.66 at $a_0 = 0.5$ by reshaping the beam towards point compliance at the ends, $a_0 \rightarrow 0$, where the value asymptotes to 1.16. The elastokinematic LDES forms an intrinsic limit to the DP bearing performance that cannot be improved by geometry adjustment, instead a UE is required.

The profile-based axial stiffness reduction occurs in all of the flexures, both in the UE and DP. All of these flexures including the type 11, 12, 2, and m are assumed to undergo effectively fixed guided motion, so that the analysis underlying Eq. (8) may be applied. The flexure's axial stiffnesses dominate the UE effect, so the profile-based reduction plays a significant role in the structure's performance. The effect is captured by $\tau_b(a_0)$ in Eq. (8), which can be used to replace the constant axial stiffness terms, k_{pa} , k_{ma} , in Eq. (2) with functions $k_{pa}(X)$, $k_{ma}(X)$, as shown in Eq. (10). The subscript p , as defined earlier, has a value of 11, 12, and 2 for the three kinds of flexures in the UE. The bending thickness of the type 11 flexures, h_{11} , and the type 12 flexures, h_{12} , are equal and set to h_1 , and the bending thickness of the type 2 flexures is h_2 .

$$\begin{aligned}
k_{pa}(X) &= \frac{A_p E_p}{L_p} \frac{1}{1 + \tau_p^{-2} X^2} \square k_{pb} \\
\tau_{11} = \tau_{12} &= \frac{2dh_1}{r_1 h_m} \tau_b(a_0) \quad \tau_2 = \frac{2dh_2}{r_2 h_m} \tau_b(a_0) \\
k_{ma}(X) &= k_{ma} \frac{1}{1 + \tau_m^{-2} X^2} \\
\tau_m &= 2\tau_b(a_0)
\end{aligned} \tag{10}$$

The LDESs τ_{11} , τ_{12} , τ_2 , and τ_m describe the scale at which the respective beams start to show a loss in their axial stiffness. Each beam's end displacement is related back to x_f and the beam width is normalized to the type m flexure's width, h_m , in order to define each function via X . The normalization term that scales $\tau_b(a_0)$ to produce τ_{11} , τ_{12} , τ_2 , and τ_m is calculated via Eq. 7 for each particular flexure. The calculation transforms the flexures' transverse displacement to bending thickness ratio with functions of x_f and h_m , respectively. A x_f/h_m ratio can then be extracted and replaced with X , leaving the excess terms to be gathered into the flexures LDES as shown in Eq. (10). This standardizes all functions to the same nondimensional ratio so that the LDESs can be directly compared. The rigid body relaxation stiffnesses are not subject to the profile-effect as they do not undergo large displacement. The modified axial stiffness expressions can now be used to define $k_{iue}(X)$ from Eq. (3) as shown in Eq. (11).

$$k_{iue}(X) = 2 \left[\frac{1}{4\sin(\theta_2)^2 k_{2a}(X)} + \frac{1}{k_{11a}(X)} + \frac{1}{4k_{12a}(X)} \right]^{-1} \tag{11}$$

3.4 Intermediate Stage Deformation Stiffness

An extra term must be included to account for the deformation of the intermediate stage, which sees moment-like loading at both ends. Only one side of the stage needs to be analyzed due to symmetry along the plane identified in Fig. 2a. The slope of the intermediate stage must be zero at the symmetry plane. The separation between the inner and outer type m flexures is captured by d_3 , as shown in Fig. 2a. The inner and outer sets of type m flexures apply equal but opposite axial loads, generating a moment approximately on the end of the intermediate stage. The deflection can then be adequately modeled via the moment-to-rotation stiffness of a simple fixed-free beam comprising half the end stage, as shown in Eq. (12), where E_i , I_i , and L_i are the Young's modulus, average second moment of area, and length respectively of the intermediate stage.

$$k_{yi} = \frac{2E_i I_i}{d_3^2 L_i} \quad (12)$$

The d_3 term is used once to change axial forces to a moment, and once to change the resulting rotation into a y-axis translation. This maps the intermediate stage deformation to a y-axis axial force-to-displacement stiffness, k_{yi} .

3.5 Net Large Displacement Axial Stiffness

The overall axial (y-axis) stiffness of the bearing, k_{yf} , is calculated in Eq. (13) by the inclusion of the intermediate stage stiffness as a spring in series, using Eq.'s (7) and (12). The LDES for the underconstrained DOF elastokinematic effect on the bearing is captured by $\tau_{yf}(X)$, which is a function of X due to the profile-based stiffness reduction occurring in both the UE and the type m flexures. This is the full expression for the bearing axial stiffness, and extends beyond previous models [2], [5] to include the discrete UE linkage.

$$k_{yf}(X) = k_{ma}(X) \frac{1}{1 + \left(\tau_{yf}(X)\right)^{-2} X^2} \square k_{yi} \quad (13)$$

$$\tau_{yf}(X) = \sqrt{\frac{4k_{mt} + k_{iue}(X)}{4k_{ma}(X)}} \frac{2L_m \gamma}{h_m}$$

The bearing elastokinematic LDES, τ_{yf} , can be simplified assuming beams with distributed compliance. The Euler expressions for the stiffnesses are used to expand the expression, which enables significant cancellation as shown in Eq. (14). The remaining terms can be broken into three segments. A 2γ term forms the baseline which is scaled by the other two terms. A $1+k_{iue}/(4k_{mt})$ term captures the effect of the UE on

scaling up the LDES. This term falls to unity as the UE stiffness is lowered. Finally, a profile sensitivity term remains from the $k_{ma}(X)$ function, which actually serves to boost the LDES by lowering the DP/UE stiffness ratio. This profile term only starts to grow significantly past unity when $X > \tau_m \approx 15$.

$$\tau_{yf}(X) = \sqrt{\left(1 + \tau_m^{-2} X^2\right) \left(1 + \frac{k_{iue}(X)}{4k_{mt}}\right)} 2\gamma \quad (14)$$

3.6 Net LDES

The net stiffness dropoff is a contribution from both the profile-based τ_m in $k_{ma}(X)$ and $\tau_{yf}(X)$, as well as the elastokinematic underconstrained DOF effect. The interaction between these terms must be considered when calculating the net LDES. The interaction is made apparent through analyzing the denominator of the DP bearing stiffness term from Eq. (13), as shown in Eq. (15). The k_{ma} and k_{yi} terms are removed for simplicity, as these have no X dependence.

$$\begin{aligned} \left(1 + \tau_m^{-2} X^2\right) \left[1 + \left[\tau_{yf}(X)\right]^{-2} X^2\right] &= 1 + \left[\tau_{d2}(X)\right]^{-2} X^2 + \left[\tau_{d4}(X)\right]^{-4} X^4 \\ \tau_{d2}(X) &= \left[\tau_m^{-2} + \left[\tau_{yf}(X)\right]^{-2}\right]^{-1/2} \quad \tau_{d4}(X) = \sqrt{\tau_m \tau_{yf}(X)} \end{aligned} \quad (15)$$

The expansion generates two terms: the effective τ 's for X^2 and for X^4 , as shown in Eq. (15). The $\tau_{d2}(X)$ term can be shown to always be lower, and thus will be the dominant term determining the LDES. The $\tau_{d2}(X)$ term is composed of parallel addition, so will always be lower than either τ_m or $\tau_{yf}(X)$. The parallel addition occurs to the 2nd power so the summation rapidly converges to the lower term from below. The $\tau_{d4}(X)$ term is composed of the geometric average of the two LDES, which will always lie between τ_m and $\tau_{yf}(X)$. The $\tau_{d2}(X)$ term is thus the primary expression required to interpret the actual axial stiffness dropoff, and can be used to define the net LDES of the DP UE bearing, τ_{nUE} .

The net LDES calculation requires a displacement value for X in $\tau_{yf}(X)$. The most accurate estimation is generated by calculating $\tau_{yf}(X)$ at the approximate net LDES scale, so $X \approx \tau_{nUE}$. Here the UE and non-UE cases must be separated in order to make accurate assumptions. It is expected that designers will attempt to drive τ_{yf} as high as possible when the UE is included, likely greater than τ_m . This would ensure the largest possible LDES. The parallel addition within the $\tau_{d2}(X)$ term means that τ_{nUE} can only asymptotically approach, but never exceed, τ_m . This establishes an upper bound and expected range of values for τ_{nUE} , so that

$X \approx \tau_m$ may be used as an approximation for design purposes, as shown in Eq. (16). This provides a reasonably accurate, simple expression for determining the LDES for the whole bearing.

$$\begin{aligned}\tau_{nUE} &\approx \left[\tau_m^{-2} + \left[\tau_{yf}(\tau_m) \right]^{-2} \right]^{-1/2} \\ \tau_{nDP} &\approx \left[\tau_m^{-2} + \left[\tau_{yf}(0) \right]^{-2} \right]^{-1/2} \approx 2\gamma \quad \text{where} \quad k_{iue}(X) = 0\end{aligned}\tag{16}$$

The LDES of the bearing without the UE, τ_{nDP} , is much simpler to calculate. Eq. (14) may be used for fully distributed compliance flexures, but flexures with $a_0 < 0.5$ must be analyzed with Eq. (13). The UE stiffness $k_{iue} = 0$, so the middle term in Eq. (14) falls to unity for fully distributed compliance flexures. The small- X value for τ_{yf} is now $2\gamma \approx 1.7$ for $a_0 = 0.5$, meaning $\tau_{yf}(0)$ may be used as an approximation in Eq. (16). The whole expression may be approximated as 2γ for $a_0 = 0.5$, as the error introduced by ignoring τ_m in the 2nd order parallel addition is $< 0.6\%$.

3.7 LDES Verification of PE Analysis

The non-UE model provides a means to test the validity of the perturbative Euler analysis model through comparison to literature models [2], [5], [18], shown in Eq. (9). Literature suggests for the common case of distributed compliance flexures, $\tau_{DP} \approx 1.657$ [2], [5], [18], while perturbative Euler analysis predicts $\tau_{nDP} \approx 1.693$, an overprediction of about 2%. In the extreme cases of $a_0 \rightarrow 0$, the literature model drops to 1.16, while perturbative Euler analysis drops to 0.98, capturing the trend with an error of only 16%. The new method of analysis matches existing work in the non-UE case while providing a framework into which the UE mechanism can be fit. The LDES figure-of-merit provides a representative means for visualizing the scale of large displacement effects. Insights can now be drawn from the simple expressions generated for the LDES.

4 Design Insights via LDES

The relative scale of effects can be examined via the LDES of the various flexures and topologies. This analysis is generalized in order to be applicable to all UE designs that can be specified via a k_{iue} . The overall LDES of a DP bearing is driven by the smallest LDES, which can come from either of the two modes contributing to stiffness loss in a DP: the profile-based effect and the underconstrained DOF elastokinematic effect. The LDES of a DP bearing alone is driven by the underconstrained DOF effect to $\tau_{DP}(a_0)$, and represents a low bound to DP performance. The LDES of a DP bearing whose underconstrained DOF mode

is inactivated (e.g. via a perfect UE) rises to the profile-based stiffness LDES, $\tau_m = 2\tau_b(a_0)$. This profile term represents an upper limit to DP UE bearing performance and can only be altered via a_0 . The degree of underconstraint removal is captured by $\tau_{yf}(X)$, which moves the net LDES within the bounds of τ_{DP} and τ_m . Better UE performance ($\tau_{yf}(\tau_m) \rightarrow \infty$), will asymptotically drive the net LDES towards τ_m from below, as shown in Fig. 4.

Fig. 4. Logarithmic plot of net LDES of the DP UE bearing vs the underconstrained DOF elastokinematics LDES, τ_{yf} , calculated for fully distributed compliance flexures. Any improvement to the elastokinematic LDES beyond the baseline term directly improves the net bearing performance. Design best practices for the tradeoff between the UE stiffness and LDES effect are shown.

The 2nd order nature of the parallel addition in Eq. (16) is such that little gains are made by making $\tau_{yf}(\tau_m) \gg \tau_m$. A detailed study of the variation of τ_{nDP} with respect to changing the UE stiffness can be carried out by combining Eq.'s (14) and (15), and varying the underconstraint DOF stiffness ratio $r_k = k_{iue}/(4k_{mt})$, as shown in Fig. 5. Several regimes are visible in the figure, including a low stiffness region where there is effectively no UE ($r_k < 1$), the region of major sensitivity to the UE ($1 < r_k < 160$) and a region of asymptotic behavior towards ideal performance ($r_k > 160$). Design best practices suggest a target ratio of upwards of $\tau_{yf}(\tau_m)/\tau_m \approx 2$, corresponding to a ratio of $k_{iue}(\tau_m)/(4k_{mt}) = 160$, which drives the net LDES, τ_{nDP} , to within 18%, respectively, of the asymptotic value τ_m .

Fig. 5. Logarithmic plot of net LDES of the DP UE bearing vs UE underconstraint DOF stiffness ratio, calculated for fully distributed compliance flexures. Bearing LDES performance gains are observed when r_k is $\gg 1$. Three regimes are highlighted, as well as design best practices for the tradeoff between the UE stiffness and LDES effect.

The UE LDES $\tau_{yf}(X)$ can be increased by balancing the LDESs for the flexures within it as calculated in Eq. (10). The lowest flexure LDES dominates the UE LDES. The flexures' bending thicknesses, h , and their compliance distribution ratios, a_0 , are some of the few adjustable parameters that can be used to make $(\tau_{11}, \tau_{12}, \tau_2) \geq \tau_m$. This will ensure the UE retains its stiffness well up to τ_m . In practice, τ_2 is generally the lowest in the UE by several orders of magnitude. The ratio $\tau_2/\tau_m = d/r_2 \cdot h_2/h_m$ can be brought to near unity by matching the type 2 and type m flexure bending thicknesses, assuming equal a_0 values for the flexure types. Otherwise the compliance distribution can be altered for the type 2 flexures. Either should have the effect of significantly raising τ_{yf} , and thus raising the net bearing LDES, at the expense of stiffening the main DOF.

5 Model Confirmation

FEA was used to confirm the large displacement model. Theory and literature suggest there should be no difference between force and displacement control over the final stage. The bearing stiffness is commonly parameterized to a controlled displacement [2], [5], however controlled force can be a more accurate model in certain conditions. The high-bandwidth pull-in effects on comb drives will likely occur before the main DOF controller has time to react and hold to a certain displacement, assuming closed loop control. Both constraints were used in the FEA for comparative purposes, as shown in Fig. 6.

Fig. 6. Logarithmic plot of axial stiffness vs nondimensional displacement, showing large displacement effects. Stage displacements were simulated using large displacement FEA with both force (solid points) and displacement (hollow points) control on the final stage, and compared to models presented in literature (LD DP Theory) [5], as well as the perturbative Euler analysis presented in this paper. The PE analysis was carried out for both the standard DP bearing (PE DP Theory) and the DP bearing with a nested UE linkage (PE DP-UE Theory).

The FEA was carried out to the small angle limit of $X \approx 20$ -30 as the simulations had difficulty in converging past this scale. This pushes the design past its hard stops in order to fully examine the bearing behavior at the LDES. The controlled force FEA results are shown as the solid marks, while the controlled displacement FEA results are shown as the hollow marks, in order to enable easy comparison of the values. The force and displacement control diverge slightly at the large displacement, likely due to stage warping under large loads. The DP bearing results are compared to both the literature (LD DP) [5] and the perturbative Euler (PE DP) analyses. The literature model is generated from Eq. (9) placed as springs in series with Eq. (12). These models accurately match the FEA results as well as one another, as can be predicted from their close ($\approx 2\%$ error) LDES values.

Fig. 7. Logarithmic plot of difference in stiffness predictions vs measurements, showing variations between the FEA results and theoretical predictions. The PE analysis terms have been subtracted from all values, which were then normalized to the small displacement axial stiffness.

The error was extracted from Fig. 6, and plotted in Fig. 7 by subtracting the PE analysis predicted value from both the FEA measurements and from the literature predicted theory. The error was normalized by dividing the absolute value of the difference by the original FEA or literature stiffness. The average error between the PE model and FEA is 10%, which can be separated into two main regimes, as seen in Fig. 7. Generally before the LDES dropoff, the average error magnitude is <1%. The FEA results cross the 50% stiffness dropoff LDES at $X = 8.36$, while the model crosses at 9.72 (16% error), and the simplified τ_{nUE} expression predicts 8.67 (4% error). The simplified LDES expression ignores two effects, the X^4 term in the denominator and the intermediate stage stiffness. The X^4 term raises the LDES, while k_{yi} lowers it. At large displacements, the X^4 term will dominate, which is what occurs in this case and causes the simplified expression to underpredict the full model. The simplified LDES expression is intended for general design scaling and comparison. After the dropoff, the error becomes more significant, with a maximum value of 39% and an average of 22%.

Several mechanisms are suggested to explain the deviation for interests lying past the LDES. The theory predicts a less sharp transition curve, followed by a -3 slope. The measurements show a sharper transition curve, followed by what appears to be a -3 slope that scales back to a -2 slope at around $X = 15$. The error is likely in the UE component of the model, as the DP-only perturbative Euler analysis model accurately predicts the DP performance. The slope change is consistent with a leveling off of the UE

stiffness, rather than the expected continual drop with larger displacement. This is most likely due to a failure in the assumptions underlying the profile-effect stiffness drop in the UE model at the large displacement. Strain stiffening in the beams would cause the transverse stiffness to start to rise. This was observed to be significant in the FEA at scale $X > 10$ or $\theta_{ld} > 0.13$, as shown in Fig. 8. The transverse stiffness was not included in k_{iue} as at small displacement the axial stiffness dominates. However, with the axial stiffness dropping, and strain stiffening causing the transverse stiffness to rise, a crossover point may be reached at this scale. This effect is deserving of further analysis.

Fig. 8. FEA results for large displacement simulation with $X = 21.2$, showing large displacement effects on the UE flexures. The type 1 and 2 flexures show a significant axial loading on the flexures, resulting in a variation from the typical sinusoidal profile. The displacement scale is 0.2 for this run.

6 Conclusion

This work presents a new method for describing the effect of an underconstraint eliminator linkage on the large displacement y-axis stiffness of a flexural double parallelogram linear bearing. Models are generated that are based on a new method for closed-form analysis which can easily incorporate the complex elastomechanics of a UE linkage, unlike existing models. Design rules and new figures-of-merit are developed to further simplify the design process. This work is generalizable to any UE and enables the rapid, direct design of the UE for desired large displacement behavior. Performance limitations are derived for both the DP bearing in general and also for the UE structure. These limits have not previously been identified.

This work will accelerate the adoption of UE linkages into DP bearings as designers can now quickly analyze the tradeoffs associated with their use in large-displacement applications. The large displacement

models are shown to provide predictive capability with an average 10% error. The models and LDES figure-of-merit provides a new tool for designing large displacement bearings, and can be applied to other UE designs. This figure-of-merit is a single term which captures the complexity of the large displacement response, can be rapidly calculated in closed form, and is provided in the context of performance bounds which inform the capabilities and limitations of bearing designs. These tools enable designers to bypass tedious iterative computational analysis of bearing designs. The form of this analysis lends itself to rapid parameter analysis and integration into other design methods including inversion for design synthesis, and optimization. The closed form analysis and LDES provide a means by which the design can be mapped to a constraint based optimization framework, which offers the potential for rapid application-specific performance maximization as has been done for other precision designs [40].

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