Scalable Modeling and Solution of Stochastic Multiobjective Optimization Problems

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Abstract

We present a scalable computing framework for the solution stochastic multiobjective optimization problems. The proposed framework uses a nested conditional value-at-risk (nCVR) objective to find compromise solutions among conflicting random objectives. We prove that the associated nCVR minimization problem can be cast as a standard stochastic programming problem with expected value (linking) constraints. We also show that these problems can be implemented in a modular and compact manner using PLASMO (a Julia-based structured modeling framework) and can be solved efficiently using PIPS-NLP (a parallel nonlinear solver). We apply the framework to a CHP design study in which we seek to find compromise solutions that trade-off cost, water, and emissions in the face of uncertainty in electricity and water demands.

Keywords: large scale; optimization; stochastic; multiobjective; CVaR

1 Motivation

Stochastic multiobjective optimization (SMOO) problems trade-off multiple random objectives, multiple statistics (metrics) of a single random objective, or combinations of both (e.g., expected cost vs. expected environmental impact, expected cost vs. worst-case cost, expected cost vs. worst-case environmental impact) [1, 19]. These types of problems also arise in multi-stakeholder decision-making in which a population of stakeholders seek to balance their conflicting objectives and priorities (e.g., attitudes and perceptions towards risk). **One special case of SMOO is deterministic multiobjective optimization (MOO), for which general formulations and computational frameworks are well-established [35, 41, 22].** SMOO formulations have a wide range of applications that include finance [2], energy systems [9], **chemical processes [43, 37, 31]**, transportation logistics [21], facility location [16, 14], manufacturing and production planning [42], supply chain management [40], telecommunication, health care management [3, 5], budget allocation [23], and project management [20].

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SMOO problems present several interesting technical challenges. In particular, SMOO problems are inherently ill-posed, in the sense that no unique solution exists. Consequently, a decision-maker needs to select appropriate metrics to identify efficient solutions (i.e., in the sense of Pareto or stochastic dominance) [19, 24]. SMOO problems are also inherently high-dimensional because they capture multiple realizations of random data and potentially many objectives. Consequently, in many applications it is impractical to compute the entire Pareto set. The most popular approach compute solutions for SMOO problems is scalarization. Here, the idea is to create a weighted combination of the random objectives and then minimize a summarizing statistic (e.g., expected value or a coherent risk metric) of the weighted objective function. Another approach is to explicitly enforce stochastic dominance over a predefined target distribution of the weighted objective [12] and for all the objectives [13, 4]. The stochastic dominance approach, while more flexible, is also computationally more demanding as it requires specialized mixed-integer reformulations. Because of this, the scalarization approach is often preferred but this requires experimentation with different metrics to shape the objective distributions.

In this work, we present a computational framework to solve high-dimensional continuous and nonconvex SMOO problems using a specialized scalarization technique. Our framework uses a general metric, that we call the nested conditional value at risk (nCVaR), to compute compromise solutions for such problems. This metric extracts subsets of the largest objectives and realizations that we seek to minimize. We show that the framework allows us to capture a wide range of scalarization metrics of interest and has expected value and worst-case metrics as extreme cases. We also show that the nCVaR minimization problem can be cast as standard stochastic programming problem with expected value constraints. This allows us to use linear algebra decomposition strategies to accelerate solutions [10]. We also illustrate how to model these problems in a compact manner using PLASMO [25], which is a Julia-based graph modeling framework that facilitates the expression and manipulation of structured problems.

The paper is structured as follows. In Section 2 we present the SMOO setting and introduce the nested CVaR metric and reformulations. In Section 3 we present a scalable modeling a solution framework for SMOO. In Section 4 we illustrate the developments using a design case study for a combined heat and power plant that seeks to trade-off cost, emissions, and water in the face of uncertainty in electricity and water demands. The paper closes with final remarks and directions of future work.

2 Stochastic Multiobjective Optimization Problems

In this section, we state the SMOO problem under study and present the nested CVaR metric to compute compromise solutions for this problem.

2.1 Setting and Basic Definitions

We are interested in finding solutions to the SMOO problem:

$$\min_{x \in \mathcal{X}} \{\mathbf{F}_{0,0}(x), \mathbf{F}_{0,1}(x), \dots, \mathbf{F}_{0,m-1}(x), \mathbf{F}_{1,0}(x), \mathbf{F}_{1,1}(x), \dots, \mathbf{F}_{1,m-1}(x), \dots, \mathbf{F}_{n-1,m-1}(x)\}. \quad (2.1)$$

Where $x \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ are the decision variables, $\mathbf{F}_{i,j} : \mathcal{X} \rightarrow \mathbb{R}$ are nonconvex and continuously differentiable objective functions, $\mathcal{X} := \{x \mid h(x) \geq 0\}$ is the feasible set (with continuously differentiable but possibly non-convex constraints $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_\nu}$ and with associated dual variables $\nu \in \mathbb{R}^{n_\nu}$)¹. The index $i \in \mathcal{O} := \{0, \dots, n-1\}$ is the objective index and $j \in \mathcal{S} := \{0, \dots, m-1\}$ is the scenario index. Each scenarios $j \in \mathcal{S}$ can be used to represent realizations of the objectives of the form $\mathbf{F}_{i,j}(x) := p_j \mathbf{F}_i(x, \xi_j)$ where $\xi_j \in \mathbb{R}^{n_\xi}$ is a realization of a random variable with probability $p_j \in \mathbb{R}_+$. Objectives can also be of the form $\mathbf{F}_{i,j}(x) := w_i \mathbf{F}_i(x)$ where $w_i \in \mathbb{R}_+$ are weights used to prioritize different objectives and scenarios. We thus note that probabilities and weights can be interpreted in a similar way to create the more general form $\mathbf{F}_{i,j}(x) := w_{i,j} \mathbf{F}_i(x)$. This setting can thus be used to capture multiple objectives (e.g., economic, environmental, social) and/or priorities from different stakeholders (e.g., in a competitive setting) [24].

For convenience, we define the objective matrix $\mathbf{F}(x) \in \mathbb{R}^{n \times m}$ with entries $\mathbf{F}_{i,j}(x)$. We use the notation $\mathbf{F}_{i,:} \in \mathbb{R}^m$ to denote the i -th row of matrix $\mathbf{F}$ and $\mathbf{F}_{:,j} \in \mathbb{R}^n$ to denote the j -th column of matrix $\mathbf{F}$. To highlight differences between scenario and objective spaces, we will refer to $\frac{1}{n} \sum_{i \in \mathcal{O}} \mathbf{F}_{i,j}$ (for a given $j \in \mathcal{S}$) as the *average* objective [in the objective space](#). In addition, we will refer to $\frac{1}{m} \sum_{j \in \mathcal{S}} \mathbf{F}_{i,j}$ (for a fixed $i \in \mathcal{O}$) as the *mean* objective [in the scenario space](#). Similarly, we refer to $\max_{i \in \mathcal{O}} \mathbf{F}_{i,j}$ as the *largest* objective and to $\max_{j \in \mathcal{S}} \mathbf{F}_{i,j}$ as the *worst-case* objective.

In a multiobjective setting it is often desired to scale the objectives. This can be done by using the so-called utopia solutions, which are computed from the individual problems:

$$\mathbf{F}_{i,j}^* := \min_{x \in \mathcal{X}} \mathbf{F}_{i,j}(x), \quad (i, j) \in \mathcal{O} \times \mathcal{S}. \quad (2.2)$$

With this, we can scale the objectives as $\mathbf{F}_{i,j}(x) \leftarrow \mathbf{F}_{i,j}(x) / \mathbf{F}_{i,j}^*$ (provided that $\mathbf{F}_{i,j}^* > 0$). Many other scaling alternatives as possible (e.g., linear scaling using utopia and nadir points) [34, 14].

The objectives $\mathbf{F}_{i,j}(x)$ are usually conflicting and thus SMOO does not have a well-defined solution. We thus seek to find *compromise solutions* that balance all objectives and in all scenarios. This

¹We generalize this formulation to consider recourse variables in Section 3.

can also be done in multiple ways; consider, for instance, the following problems:

$$\min_{x \in \mathcal{X}} \frac{1}{n \cdot m} \sum_{(i,j) \in \mathcal{O} \times \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.3a)$$

$$\min_{x \in \mathcal{X}} \frac{1}{n} \sum_{i \in \mathcal{O}} \max_{j \in \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.3b)$$

$$\min_{x \in \mathcal{X}} \frac{1}{m} \sum_{j \in \mathcal{S}} \max_{i \in \mathcal{O}} \mathbf{F}_{i,j}(x) \quad (2.3c)$$

$$\min_{x \in \mathcal{X}} \max_{j \in \mathcal{S}} \frac{1}{n} \sum_{i \in \mathcal{O}} \mathbf{F}_{i,j}(x) \quad (2.3d)$$

$$\min_{x \in \mathcal{X}} \max_{i \in \mathcal{O}} \frac{1}{m} \sum_{j \in \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.3e)$$

$$\min_{x \in \mathcal{X}} \max_{(i,j) \in \mathcal{O} \times \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.3f)$$

Problem (2.3a) minimizes the mean of the average objectives; (2.3f) minimizes the largest worst-case objective; (2.3b) minimizes the mean of the worst-case objectives; (2.3c) minimizes the average of the largest objectives, and so on. The solutions of these problems have *different interpretations* that might be desirable in different decision-making contexts. To highlight how different interpretations arise, we consider the following problem:

$$\mathbf{F}_{:,j} := \min_{x \in \mathcal{X}} \frac{1}{n} \sum_{i \in \mathcal{O}} \mathbf{F}_{i,j}(x), \quad j \in \mathcal{S}. \quad (2.4)$$

The solution of this problem minimizes the average objective for a given scenario $j \in \mathcal{S}$. This solution can also be interpreted as minimum distance solution (under an ℓ_1 norm) to the utopia point $\mathbf{F}_{:,j}^*$ (the vector $\mathbf{F}_{:,j}$ contains the coordinates of the minimum-distance point) [48]. One can prove that the minimum distance solution is a Pareto solution associated to the scenario $j \in \mathcal{S}$ (i.e., each scenario has an associated Pareto set). Consequently, by solving (2.3a), we find a solution that minimizes the mean of the average-distance solutions to the utopia point and one can prove that this solution is Pareto optimal in the combined space of the objectives and scenarios. This solution can also be interpreted as the minimization of an entry-wise ℓ_1 matrix norm $\|\mathbf{F}(x)\|_1$. Following this line of thought, the solution of the problem

$$\mathbf{F}_{:,j} := \min_{x \in \mathcal{X}} \max_{i \in \mathcal{O}} \mathbf{F}_{i,j}(x), \quad j \in \mathcal{S} \quad (2.5)$$

gives the minimum distance solution (under the ℓ_∞ norm) to the utopia point and the vector $\mathbf{F}_{:,j}$ contains the coordinates of the minimum-distance point [48]. This solution can also be interpreted as that of finding the minimum largest objective for a given scenario $j \in \mathcal{S}$. Consequently, by solving (2.3c) we find a solution that minimizes the average of the ℓ_∞ -distance solutions to the utopia point, while by solving (2.3f) we find a solution that minimizes the worst-case of the ℓ_∞ -distance solutions to the utopia point. This can also be interpreted as the minimization of an entry-wise ℓ_∞ matrix norm $\|\mathbf{F}(x)\|_\infty$.

2.2 Nested CVaR Metric

We use the conditional value at risk (CVaR) to create a general framework to compute a wide range of compromise solutions for SMOO. For clarity in the presentation, we define CVaR using finite-dimensional vector spaces (which also have geometric interpretations). For a given vector $\mathbf{v} \in \mathbb{R}^{n_v}$ with entries given by realizations v_ℓ , $\ell \in \{1, \dots, n_v\}$, we denote its CVaR with probability level $\alpha \in [0, 1]$ as $\langle\langle \mathbf{v} \rangle\rangle_\alpha$. Assume now (without loss of generality) that the entries of $\mathbf{v}$ are arranged as $v_1 \leq v_2 \leq \dots \leq v_{n_v}$. For a fixed $k \in \{0, \dots, n_v - 1\}$ with the probability level $\alpha(k) := k/n_v$, CVaR is computed as:

$$\langle\langle \mathbf{v} \rangle\rangle_{\alpha(k)} = \frac{1}{n_v - k} \sum_{\ell=k+1}^{n_v} v_\ell. \quad (2.6)$$

In other words, CVaR with probability level $\alpha(k)$ is the average of the $n_v - k$ largest entries of $\mathbf{v}$. Note also that $\langle\langle \mathbf{v} \rangle\rangle_{\alpha(0)} = \frac{1}{n_v} \sum_{\ell=1}^{n_v} v_\ell$ and $\langle\langle \mathbf{v} \rangle\rangle_{\alpha(n_v-1)} = \max_{\ell \in \{1, \dots, n_v\}} \{v_\ell\}$. It is also well-known that CVaR can be computed by solving the optimization problem:

$$\langle\langle \mathbf{v} \rangle\rangle_{\alpha(k)} = \min_t t + \frac{1}{(1 - \alpha(k)) \cdot n_v} \sum_{\ell=1}^{n_v} [v_\ell - t]_+, \quad (2.7)$$

where $[v_\ell]_+ = \max\{0, v_\ell\}$ and $k \in \{0, \dots, n_v - 1\}$ and the value of t that minimizes the right-hand side of (2.7) is the value-at-risk [38]. We highlight that we can compute CVaR for a non-integer k by using a weighted sum of the nearest integer values. The interested reader can refer to [38, 14] for more details. In the following, we drop the dependence of $\alpha(k)$ on k and simply use α . Moreover, we set $\langle\langle \mathbf{v} \rangle\rangle_{\alpha=1} = \langle\langle \mathbf{v} \rangle\rangle_{\alpha(n_v-1)}$ and use the notation $\alpha = 0$ to imply that $k = 0$ and $\alpha = 1$ to imply that $k = n_v - 1$. Moreover, the notation $\alpha = 1$ indicates that (2.7) is solved with $\alpha(n_v - 1)$.

We now define the following quantities:

$$\hat{\mathbf{F}}_{\alpha_{\mathcal{O}}}^j(x) := \langle\langle \mathbf{F}_{:,j} \rangle\rangle_{\alpha_{\mathcal{O}}}, \quad j \in \mathcal{S} \quad (2.8a)$$

$$\hat{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i(x) := \langle\langle \mathbf{F}_{i,:} \rangle\rangle_{\alpha_{\mathcal{S}}}, \quad i \in \mathcal{O} \quad (2.8b)$$

where $\alpha_{\mathcal{O}}$ and $\alpha_{\mathcal{S}}$ are probability levels for vectors with cardinality $n = |\mathcal{O}|$ and $m = |\mathcal{S}|$, respectively. We also define vectors $\hat{\mathbf{F}}_{\alpha_{\mathcal{O}}}(x) \in \mathbb{R}^m$ with entries $\hat{\mathbf{F}}_{\alpha_{\mathcal{O}}}^j(x)$ and $\hat{\mathbf{F}}_{\alpha_{\mathcal{S}}}(x) \in \mathbb{R}^n$ with entries $\hat{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i(x)$.

From these definitions and the properties of CVaR we can immediately establish that:

$$\hat{\mathbf{F}}_{\alpha_{\mathcal{O}}=0}^j(x) = \frac{1}{n} \sum_{i \in \mathcal{O}} \mathbf{F}_{i,j}(x), \quad j \in \mathcal{S} \quad (2.9a)$$

$$\hat{\mathbf{F}}_{\alpha_{\mathcal{S}}=0}^i(x) = \frac{1}{m} \sum_{j \in \mathcal{S}} \mathbf{F}_{i,j}(x), \quad i \in \mathcal{O} \quad (2.9b)$$

$$\hat{\mathbf{F}}_{\alpha_{\mathcal{O}}=1}^j(x) = \max_{i \in \mathcal{O}} \{\mathbf{F}_{i,j}(x)\}, \quad j \in \mathcal{S} \quad (2.9c)$$

$$\hat{\mathbf{F}}_{\alpha_{\mathcal{S}}=1}^i(x) = \max_{j \in \mathcal{S}} \{\mathbf{F}_{i,j}(x)\}, \quad i \in \mathcal{O}. \quad (2.9d)$$

We use these quantities to define the *nested CVaR* (nCVaR) metrics:

$$\langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_{\mathcal{O}}, \alpha_{\mathcal{S}}} := \langle\langle \hat{\mathbf{F}}_{\alpha_{\mathcal{S}}}(x) \rangle\rangle_{\alpha_{\mathcal{O}}} \quad (2.10a)$$

$$\langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_{\mathcal{S}}, \alpha_{\mathcal{O}}} := \langle\langle \hat{\mathbf{F}}_{\alpha_{\mathcal{O}}}(x) \rangle\rangle_{\alpha_{\mathcal{S}}}. \quad (2.10b)$$

The nCVaR metrics are used to state the optimization problems:

$$\min_{x \in \mathcal{X}} \langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{O}}, \alpha_{\mathcal{S}}} \quad (2.11a)$$

$$\min_{x \in \mathcal{X}} \langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{S}}, \alpha_{\mathcal{O}}} \quad (2.11b)$$

which we will use to compute compromise solutions for SMOO. Nested CVaR is a generalized metric that [can be used to compute](#) families of compromise solutions with different interpretations and has problems (2.3) as limiting cases. For instance, we have that:

$$\langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{O}}=0, \alpha_{\mathcal{S}}=0} = \langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{S}}=0, \alpha_{\mathcal{O}}=0} = \frac{1}{n \cdot m} \sum_{(i,j) \in \mathcal{O} \times \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.12a)$$

$$\langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{O}}=1, \alpha_{\mathcal{S}}=1} = \langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{S}}=1, \alpha_{\mathcal{O}}=1} = \max_{(i,j) \in \mathcal{O} \times \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.12b)$$

$$\langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{O}}=0, \alpha_{\mathcal{S}}=1} = \frac{1}{n} \sum_{i \in \mathcal{O}} \max_{j \in \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.12c)$$

$$\langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{O}}=1, \alpha_{\mathcal{S}}=0} = \max_{i \in \mathcal{O}} \frac{1}{m} \sum_{j \in \mathcal{S}} \mathbf{F}_{i,j}(x) \quad (2.12d)$$

$$\langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{S}}=0, \alpha_{\mathcal{O}}=1} = \frac{1}{m} \sum_{j \in \mathcal{S}} \max_{i \in \mathcal{O}} \mathbf{F}_{i,j}(x) \quad (2.12e)$$

$$\langle \mathbf{F}(x) \rangle_{\alpha_{\mathcal{S}}=1, \alpha_{\mathcal{O}}=0} = \max_{j \in \mathcal{S}} \frac{1}{n} \sum_{i \in \mathcal{O}} \mathbf{F}_{i,j}(x). \quad (2.12f)$$

The nCVaR metric can also be used to trade-off CVaR metrics for different objectives. For instance, we can trade-off CVaR for an economic metric, CVaR for environmental metric, and CVaR for a social metric. In particular, consider the case in which we want to minimize the average of the 90%-probability CVaR metrics for the objectives. This problem can be formulated as (2.11a) with $\alpha_{\mathcal{O}} = 0$ and $\alpha_{\mathcal{S}} = 0.9$ and is equivalent to:

$$\min_{x \in \mathcal{X}} \frac{1}{n} \sum_{i \in \mathcal{O}} \langle \mathbf{F}_{i,:}(x) \rangle_{\alpha_{\mathcal{S}}} \quad (2.13)$$

or, equivalently,

$$\min_{x \in \mathcal{X}} \frac{1}{n} \sum_{i \in \mathcal{O}} \left(\min_{t_i} \left\{ t_i + \frac{1}{(1 - \alpha_{\mathcal{S}}) \cdot m} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x) - t_i]_+ \right\} \right). \quad (2.14)$$

Similarly, we can minimize the largest of the 90%-probability CVaR metrics for the objectives by solving (2.11a) with $\alpha_{\mathcal{O}} = 1$ and $\alpha_{\mathcal{S}} = 0.9$. As another example, by minimizing (2.11a) with $\alpha_{\mathcal{O}} = 0.9, \alpha_{\mathcal{S}} = 0$ would minimize the largest 10% of the mean objectives and has the form:

$$\min_{x \in \mathcal{X}} \left\langle \frac{1}{m} \sum_{j \in \mathcal{S}} \mathbf{F}_{:,j}(x) \right\rangle_{\alpha_{\mathcal{O}}}. \quad (2.15)$$

This situation arises when we seek to find a solution that finds a compromise between the objectives of multiple stakeholders (in this case $i \in \mathcal{O}$ represents a stakeholder). [We also note that since CvaR is a vector norm \[38\], it can be verified that the nCVaR metric can be interpreted as a matrix norm \(provided that \$\mathbf{F}_{i,j}\(x\) \geq 0\$ for all \$x \in \mathcal{X}\$ \).](#)

2.3 Nested CVaR Reformulations

By combining (2.7), (2.8a), and (2.10a) we can formulate (2.11a) as:

$$\min_{x \in \mathcal{X}, z} z + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \left[\min_{t_i} \left\{ t_i + \frac{1}{m(1 - \alpha_S)} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x) - t_i]_+ \right\} - z \right]_+. \quad (2.16)$$

This problem can also be reformulated as:

$$\begin{aligned} \min_{x \in \mathcal{X}, z, \hat{\mathbf{F}}_{\alpha_S}^i} \quad & \varphi(z, \hat{\mathbf{F}}_{\alpha_S}^i) \\ \text{s.t.} \quad & \hat{\mathbf{F}}_{\alpha_S}^i = \min_{t_i} \left\{ t_i + \frac{1}{m(1 - \alpha_S)} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x) - t_i]_+ \right\}, \quad i \in \mathcal{O} \end{aligned} \quad (\text{A})$$

where $\varphi(z, \hat{\mathbf{F}}_{\alpha_S}^i) := z + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} [\hat{\mathbf{F}}_{\alpha_S}^i - z]_+$. Problems (2.16) and (A) cannot be solved directly using off-the-shelf NLP solvers because they include inner min and max operators. We now proceed to show that we reformulate (A) as a standard NLP. We first prove that (A) can be reformulated as:

$$\begin{aligned} \min_{x \in \mathcal{X}, z, \bar{\mathbf{F}}_{\alpha_S}^i, t_i} \quad & \varphi(z, \bar{\mathbf{F}}_{\alpha_S}^i) \\ \text{s.t.} \quad & \bar{\mathbf{F}}_{\alpha_S}^i = t_i + \frac{1}{m(1 - \alpha_S)} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x) - t_i]_+, \quad i \in \mathcal{O}. \end{aligned} \quad (\text{B})$$

We define an optimal solution of (A) as $(x^A, z^A, \hat{\mathbf{F}}_{\alpha_S}^A, t^A)$ where $t^A \in \mathbb{R}^n$ is a vector with entries t_i^A . We define the associated objective value of (A) as $v^A := \varphi(z^A, \hat{\mathbf{F}}_{\alpha_S}^A)$. We also define an optimal solution of (B) as $(x^B, z^B, \bar{\mathbf{F}}_{\alpha_S}^B, t^B)$ and the corresponding objective value as $v^B := \varphi(z^B, \bar{\mathbf{F}}_{\alpha_S}^B)$.

Theorem 1 *An optimal solution of (B) is an optimal solution of (A).*

Proof: It suffices to show that $v^A = v^B$. It is straightforward to see that $(x^A, z^A, \hat{\mathbf{F}}_{\alpha_S}^A, t^A)$ is a feasible solution to (B) and the objective function of (B) evaluated at $(x^A, z^A, \hat{\mathbf{F}}_{\alpha_S}^A, t^A)$ is $\varphi(z^A, \hat{\mathbf{F}}_{\alpha_S}^A) = v^A$. We thus have that $v^B \leq v^A$. If we now define $\hat{\mathbf{F}}_{\alpha_S}^{i,B} := \min_{t_i} \left\{ t_i + \frac{1}{m(1 - \alpha_S)} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x^B) - t_i]_+ \right\}$ we note that $(x^B, z^B, \hat{\mathbf{F}}_{\alpha_S}^B)$ is a feasible solution to (A). Moreover, because $\hat{\mathbf{F}}_{\alpha_S}^{i,B} \leq \bar{\mathbf{F}}_{\alpha_S}^{i,B}$, the objective value of (A) evaluated at the feasible solution $(x^B, z^B, \hat{\mathbf{F}}_{\alpha_S}^B)$ satisfies

$$\begin{aligned} v^A & \leq \varphi(z^B, \hat{\mathbf{F}}_{\alpha_S}^B) \\ & = z^B + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} [\hat{\mathbf{F}}_{\alpha_S}^{i,B} - z^B]_+ \\ & \leq z^B + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} [\bar{\mathbf{F}}_{\alpha_S}^{i,B} - z^B]_+ \\ & = \varphi(z^B, \bar{\mathbf{F}}_{\alpha_S}^B) \\ & = v^B. \end{aligned}$$

Therefore, we also have that $v^A \leq v^B$ and the proof is complete. $\square$

Theorem 1 implies that problem (2.16) is equivalent to:

$$\min_{x \in \mathcal{X}, z, t_i} z + \frac{1}{n(1-\alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \left[t_i + \frac{1}{m(1-\alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x) - t_i]_+ - z \right]_+. \quad (2.17)$$

In other words, the inner min operator can be factored out. To avoid the use of the non-smooth max functions, we now show that problem (B) can be reformulated as the standard nonlinear programming (NLP) problem:

$$\begin{aligned} \min_{x \in \mathcal{X}, z, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i, t_i, \eta_i, \mu_{i,j}} \quad & z + \frac{1}{n(1-\alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \eta_i \\ \text{s.t.} \quad & \eta_i \geq 0, \quad i \in \mathcal{O} \\ & \eta_i \geq \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i - z, \quad i \in \mathcal{O} \\ & \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i = t_i + \frac{1}{m(1-\alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} \mu_{i,j}, \quad i \in \mathcal{O} \\ & \mu_{i,j} \geq 0, \quad i \in \mathcal{O}, j \in \mathcal{S} \\ & \mu_{i,j} \geq \mathbf{F}_{i,j}(x) - t_i, \quad i \in \mathcal{O}, j \in \mathcal{S}. \end{aligned} \quad (\text{C})$$

We use $(x^C, z^C, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^C, t^C, \eta^C, \mu^C)$ and v^C to denote an optimal solution and corresponding objective value of (C), respectively.

Theorem 2 *An optimal solution of (C) is an optimal solution of (B).*

Proof: It suffices to show that $v^B = v^C$. If we define $\eta_i^B := [\tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^{i,B} - z^B]_+$ and $\mu_{i,j}^B := [\mathbf{F}_{i,j}(x^B) - t_i]_+$ we have that $(x^B, z^B, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^B, t^B, \eta^B, \mu^B)$ is feasible to (C) and the objective function of (C) evaluated at $(x^B, z^B, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^B, t^B, \eta^B, \mu^B)$ is $z^B + \frac{1}{n(1-\alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \eta_i^B = \varphi(z^B, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^B) = v^B$. We thus have that $v^C \leq v^B$.

If we now define $\tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^{i,C} := t_i^C + \frac{1}{m(1-\alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x^C) - t_i^C]_+$ we have that $(x^C, z^C, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^C, t^C)$ is feasible to (B). This follows because $\mu_{i,j}^C \geq [\mathbf{F}_{i,j}(x^C) - t_i^C]_+$ and

$$\begin{aligned} \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^{i,C} &= t_i^C + \frac{1}{m(1-\alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x^C) - t_i^C]_+ \\ &\leq t_i^C + \frac{1}{m(1-\alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} \mu_{i,j}^{i,C} \\ &= \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^{i,C}. \end{aligned}$$

The objective function of (B) evaluated at $(x^C, z^C, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^C, t^C)$ is

$$\begin{aligned} z^C + \frac{1}{n(1-\alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} [\tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^{i,C} - z^C]_+ &\leq z^C + \frac{1}{n(1-\alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} [\tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^{i,C} - z^C]_+ \\ &\leq z^C + \frac{1}{n(1-\alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \eta_i^C \\ &= v^C. \end{aligned}$$

Therefore, $v^B \leq v^C$. $\square$

Theorem 2 implies that problems (2.16) and (2.17) are equivalent to:

$$\begin{aligned} \min_{x \in \mathcal{X}, z, t_i, \mu_{i,j}} \quad & z + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \left[t_i + \frac{1}{m(1 - \alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} \mu_{i,j} - z \right]_+ \\ \text{s.t.} \quad & \mu_{i,j} \geq 0, \quad i \in \mathcal{O}, j \in \mathcal{S} \\ & \mu_{i,j} \geq \mathbf{F}_{i,j}(x) - t_i, \quad i \in \mathcal{O}, j \in \mathcal{S}. \end{aligned} \quad (2.19)$$

It is also easy to see that this problem is equivalent to:

$$\begin{aligned} \min_{x \in \mathcal{X}, z, t_i, \mu_{i,j}, \eta_i} \quad & z + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \eta_i \\ \text{s.t.} \quad & \eta_i \geq 0, \quad i \in \mathcal{O} \\ & \eta_i \geq t_i + \frac{1}{m(1 - \alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} \mu_{i,j} - z, \quad i \in \mathcal{O} \\ & \mu_{i,j} \geq 0, \quad i \in \mathcal{O}, j \in \mathcal{S} \\ & \mu_{i,j} \geq \mathbf{F}_{i,j}(x) - t_i, \quad i \in \mathcal{O}, j \in \mathcal{S} \end{aligned} \quad (2.20)$$

and to problem (C). In other words, Theorem 2 indicates that the inner max operators can also be factored out to achieve the standard NLP representation (C).

Similar reformulations and properties can be derived for (2.11b). In particular, we can reformulate (2.11b) as:

$$\begin{aligned} \min_{x \in \mathcal{X}, z, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^j, t_j, \eta_j, \mu_{i,j}} \quad & z + \frac{1}{m(1 - \alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} \eta_j \\ \text{s.t.} \quad & \eta_j \geq 0, \quad j \in \mathcal{S} \\ & \eta_j \geq \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^j - z, \quad j \in \mathcal{S} \\ & \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^j = t_j + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \mu_{i,j}, \quad j \in \mathcal{S} \\ & \mu_{i,j} \geq 0, \quad i \in \mathcal{O}, j \in \mathcal{S} \\ & \mu_{i,j} \geq \mathbf{F}_{i,j}(x) - t_j, \quad i \in \mathcal{O}, j \in \mathcal{S}. \end{aligned} \quad (\text{D})$$

2.4 Extensions

The previous definitions and properties hold for problems with recourse variables and constraints. In particular, consider the problem:

$$\min_{x \in \mathcal{X}, y_{i,j} \in \mathcal{Y}_{i,j}(x)} \quad \langle \mathbf{F}(x, y) \rangle_{\alpha_{\mathcal{O}}, \alpha_{\mathcal{S}}}. \quad (2.21)$$

Here, we note that we have generalized the concept of recourse in the sense that recourse variables can be used in the scenario and in the objective space. This can help us model multi-stakeholder

settings, for instance, in which stakeholders have different sets of recourse variables. In a traditional SMOO setting, recourse variables are only introduced in the scenario space, to obtain a restricted problem of the form:

$$\min_{x \in \mathcal{X}, y_j \in \mathcal{Y}_j(x)} \langle \mathbf{F}(x, y) \rangle_{\alpha_{\mathcal{O}}, \alpha_{\mathcal{S}}} \quad (2.22)$$

If we consider feasible sets of the form $\mathcal{X} := \{h(x) \geq 0\}$ and $\mathcal{Y}_{i,j}(x, y_{i,j}) := \{h_{i,j}(x, y_{i,j}) \geq 0\}$, problem (2.21) becomes an NLP of the form:

$$\min_{x, y_{i,j}, z, \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i, t_i, \eta_i, \mu_{i,j}} z + \frac{1}{n(1 - \alpha_{\mathcal{O}})} \sum_{i \in \mathcal{O}} \eta_i \quad (2.23a)$$

$$\text{s.t. } \eta_i \geq 0, \quad i \in \mathcal{O} \quad (2.23b)$$

$$\eta_i \geq \tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i - z, \quad i \in \mathcal{O} \quad (2.23c)$$

$$\tilde{\mathbf{F}}_{\alpha_{\mathcal{S}}}^i = t_i + \frac{1}{m(1 - \alpha_{\mathcal{S}})} \sum_{j \in \mathcal{S}} \mu_{i,j}, \quad i \in \mathcal{O} \quad (2.23d)$$

$$\mu_{i,j} \geq 0, \quad i \in \mathcal{O}, j \in \mathcal{S} \quad (2.23e)$$

$$\mu_{i,j} \geq \mathbf{F}_{i,j}(x, y_{i,j}) - t_i, \quad i \in \mathcal{O}, j \in \mathcal{S} \quad (2.23f)$$

$$h(x) \geq 0 \quad (2.23g)$$

$$h_{i,j}(x, y_{i,j}) \geq 0, \quad i \in \mathcal{O}, j \in \mathcal{S}. \quad (2.23h)$$

We also note that the nested CVaR metric can be easily adapted to handle the more general case in which we use different probability levels for each CVaR metric used in the scenario space (i.e., we use $\alpha_{\mathcal{S}}^i$ for $i \in \mathcal{O}$). For the example presented in (2.14), for instance, this would give a more general problem of the form:

$$\min_{x \in \mathcal{X}} \frac{1}{n} \sum_{i \in \mathcal{O}} \left(\min_{t_i} t_i + \frac{1}{(1 - \alpha_{\mathcal{S}}^i) \cdot m} \sum_{j \in \mathcal{S}} [\mathbf{F}_{i,j}(x) - t_i]_+ \right). \quad (2.24)$$

This solution would thus minimize the average of the CVaR metrics but each CVaR metric would have its own probability level (e.g., $\alpha_{\mathcal{S}}^0 = 0, \alpha_{\mathcal{S}}^1 = 0.9, \dots, \alpha_{\mathcal{S}}^{m-1} = 1.0$). The same observations hold for the reverse case in which we can pick a different a different probability level for CVaR in the objective space ($\alpha_{\mathcal{O}}^j$ for $j \in \mathcal{S}$). As can be seen, the proposed SMOO framework allows us to consider a wide range of problem classes.

3 Scalable Modeling and Solution of SMOO Problems

In this section we present a modeling and solution framework to address large-scale SMOO problems. In particular, we illustrate how to exploit linear algebra structures and how to create highly modular model implementations.

3.1 Problem Decomposition

The SMOO problems can be cast as structured NLPs of the form:

$$\min_{w_0, w_p} \phi_0(w_0) + \sum_{p \in \mathcal{P}} \phi_p(w_0, w_p) \quad (3.25a)$$

$$\text{s.t. } h_0(w_0) \geq 0 \quad (\nu_0) \quad (3.25b)$$

$$h_p(w_0, w_p) \geq 0, \quad p \in \mathcal{P} \quad (\nu_p) \quad (3.25c)$$

$$\sum_{p \in \mathcal{P}} (\bar{\Pi}_p w_0 + \underline{\Pi}_p w_p) = 0 \quad (\lambda) \quad (3.25d)$$

Here, w_p are the variables of the partition or scenario $p \in \mathcal{P} := \{1, 2, \dots, P\}$ (recourse or second-stage variables) and w_0 are the coupling variables (first-stage variables). Functions $\phi_p(\cdot)$ and $h_p(\cdot)$ are the objective and constraints of partition $p \in \mathcal{P}$ and $\phi_0(\cdot)$ and $h_0(\cdot)$ are the counterparts of the first-stage. The dual variables of the first-stage and second-stage constraints are ν_0 and ν_p , respectively. We highlight that problem (3.25) is not a standard two-stage stochastic NLP due to the presence of the linking constraints (3.25d). These constraints have associated dual variables λ .

The linking constraints couple variables across scenarios. In our context, these constraints arise from the nested CVaR reformulations. For instance, note that constraint (2.23d) induces coupling of variables across scenarios. Under appropriate modifications, the NLP (3.25) can still be decomposed to enable parallel solutions. To explain this and without loss of generality, we ignore the inequality constraints in (3.25) and only highlight the linking constraints. Under such simplifications, the Lagrange function of (3.25) is given by:

$$\mathcal{L} = \phi_0(w_0) + \sum_{p \in \mathcal{P}} (\phi_p(w_0, w_p) + \lambda^T (\bar{\Pi}_p w_0 + \underline{\Pi}_p w_p)). \quad (3.26)$$

The first-order optimality conditions are given by:

$$\nabla_{w_0} \mathcal{L} = 0 = \nabla_{w_0} \phi_0 + \sum_{p \in \mathcal{P}} (\nabla_{w_0} \phi_p + \bar{\Pi}_p^T \lambda) \quad (3.27a)$$

$$\nabla_{w_p} \mathcal{L} = 0 = \nabla_{w_p} \phi_p + \underline{\Pi}_p^T \lambda, \quad p \in \mathcal{P} \quad (3.27b)$$

$$\nabla_{\lambda} \mathcal{L} = 0 = \sum_{p \in \mathcal{P}} (\bar{\Pi}_p w_0 + \underline{\Pi}_p w_p). \quad (3.27c)$$

The linearization of these equations is simply,

$$W_{00} \Delta w_0 + \sum_{p \in \mathcal{P}} (W_{0p} \Delta w_p + \bar{\Pi}_p^T \Delta \lambda) = -\nabla_{w_0} \mathcal{L} \quad (3.28a)$$

$$W_{p0} \Delta w_0 + W_{pp} \Delta w_p + \underline{\Pi}_p^T \Delta \lambda = -\nabla_{w_p} \mathcal{L}, \quad p \in \mathcal{P} \quad (3.28b)$$

$$\sum_{p \in \mathcal{P}} (\bar{\Pi}_p \Delta w_0 + \underline{\Pi}_p \Delta w_p) = -\nabla_{\lambda} \mathcal{L}. \quad (3.28c)$$

where $W_{00} := \nabla_{w_0, w_0} \mathcal{L}$, $W_{pp} := \nabla_{w_p, w_p} \mathcal{L}$, $W_{0p} := \nabla_{w_0, w_p} \mathcal{L}$, and $W_{p0} := \nabla_{w_p, w_0} \mathcal{L}$. This linear system

can be expressed in matrix form as:

$$\begin{bmatrix} W_{11} & & & & W_{10} & \underline{\Pi}_1^T \\ & W_{22} & & & W_{20} & \underline{\Pi}_2^T \\ & & \ddots & & \vdots & \vdots \\ & & & W_{PP} & W_{P0} & \underline{\Pi}_P^T \\ \hline W_{01} & W_{02} & \dots & W_{0P} & W_{00} & \sum_{p \in \mathcal{P}} \bar{\Pi}_p^T \\ \underline{\Pi}_1 & \underline{\Pi}_2 & \dots & \underline{\Pi}_P & \sum_{p \in \mathcal{P}} \bar{\Pi}_p & \end{bmatrix} \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \\ \vdots \\ \Delta w_P \\ \Delta w_0 \\ \Delta \lambda \end{bmatrix} = - \begin{bmatrix} \nabla_{w_1} \mathcal{L} \\ \nabla_{w_2} \mathcal{L} \\ \vdots \\ \nabla_{w_P} \mathcal{L} \\ \nabla_{w_0} \mathcal{L} \\ \nabla_{\lambda} \mathcal{L} \end{bmatrix}. \quad (3.29)$$

We thus have that the linear system is block-bordered diagonal and thus can be solved using a Schur complement decomposition approach [27, 29]. The Schur complement is of the same size as the dimension of w_0 and λ and is often a dense matrix. [This approach exhibits linear scaling in the number of scenarios if the number of coupling variables and constraints is of a few thousand variables and constraints \[49, 27\].](#) In particular, the dense factorization of large Schur complement matrices dominates scaling in the scenario space. [More sophisticated approaches exist to deal with large Schur matrices but these do not exhibit linear scaling \[33, 8\].](#) We have implemented the proposed parallel decomposition approach in the interior-point solver PIPS-NLP [10].

3.2 Modeling SMOOs in PLASMO

We use PLASMO to implement the SMOO problems. PLASMO (Platform for Scalable Modeling and Optimization) is a Julia-based modeling framework that facilitates the construction and analysis of structured optimization models. To do this, it leverages a hierarchical graph abstraction wherein nodes and edges can be associated with individual optimization models that are linked together [25]. Given a graph structure with models and connections, PLASMO can produce either a pure (flattened) optimization model to be solved using off-the-shelf optimization solvers such as IPOPT [44], or it can communicate graph structures to parallel solvers such as PIPS-NLP and thus enable decomposition. PLASMO leverages the basic algebraic modeling syntax and automatic differentiation capabilities of JuMP [32]. By using a graph representation, automatic differentiation can be performed separately on each model associated to a given node. Moreover, by using a graph abstraction we can create sophisticated operations on individual nodes. For instance, we can warm-start a large-scale problem by first finding solutions for individual nodes.

A standard two-stage stochastic NLP induces a two-level hierarchical graph where in the parent node is a model that contains global variables and constraints (first-stage) while the children nodes are connected to the parent and contain their own model with local variables and constraints (second-stage or recourse). In this case, each children node corresponds to the model of each scenario or partition. In the case of the stochastic NLP with linking constraints (3.25), we can still pose the problem as a two-level graph by adding the coupling variables and linking constraints to the parent node.

The code snippet shown in Figure 1 illustrates how to implement problem (C) in PLASMO. As can be seen, the individual scenario models are created and appended to the parent node on-the-fly to create a [two-level](#) graph structure. The structure is directly communicated to PIPS-NLP and thus

Figure 1: Snippet of a nested CVaR implementation (Formulation (C)) in PLASMO

```

1  #call libraries
2  using PlasmO
3  using JuMP
4  #initialize MPI
5  MPI.Init()
6
7  # create two-stage model
8  stom=graphModel()
9
10 # define first-stage variables in parent node
11 @variable(stom,x[1:nx])
12
13 # define nested cvar variables in parent node
14 @variable(stom,eta[1:n]>=0)
15 @variable(stom,t[1:n])
16 @variable(stom,tildeF[1:n])
17 @variable(stom,z)
18 @constraint(stom,etacons[i in 1:n],eta[i]>=F[i]-z)
19
20 # create array of scenario models
21 scenm=Array{JuMP.Model,n}
22 for j in 1:m
23     # get scenario model and append to parent node
24     scenm[j] = get_scenario_model(j)
25     @addNode(stom,scenm[j],"s$j")
26     # add second-stage variables and constraints to children scenario node
27     @variable(scenm[j],mu[1:n]>=0)
28     @constraint(scenm[j],ocons[i in 1:n] mu[i]>=(getvariable(scenm[j],:F)[i])-t[i])
29     # connect children and parent variables
30     @constraint(stom,xcons[i in 1:nx], getvariable(scenm[j],:x)[i]==x[i])
31 end
32 # add nested cvar objective to parent node
33 @constraint(stom,tildeFcons[i in 1:n],
34     tildeF[i] == t[i] + (1/(1-as)*m)*sum{getvariable(scenm[j],:mu)[i], j in 1:m})
35 @objective(stom,Min,z+(1/n*(1-ao))*sum{eta[j],j in 1:n})
36
37 # solve two-stage program with PIPS-NLP
38 ParPipsNlp_solve(stom)
39
40 # alternatively, solve two-stage program as a general NLP with IPOPT
41 Ipopt_solve(stom)

```

the solver can execute the parallel Schur decomposition approach previously discussed. Note also that, under this modeling framework, the user does not need to have any knowledge on parallel computing. From the snippet we also note that PLASMO can also create a general (unstructured) NLP to be solved by off-the-shelf solvers like IPOPT [44]. This allows the user to compare computational performance and evaluate, for instance, if parallelization actually pays off.

4 CHP Case Study

We consider the optimal design of a CHP system that provides electricity and thermal energy to a residential housing complex [17]. As with most building systems, electricity and hot water demands follow very different patterns and are out of phase, and this greatly complicates the design and operation of CHP systems [47]. Because of this, CHP systems often interact with the local utility company, which allows the system to sell electrical power surplus or to buy electricity when production of the

CHP system is insufficient to meet the building demands [7]. This interaction is also essential because hot water is usually the limiting resource demand. Another complication in the design and operation is that demands vary strongly throughout the year due to seasonal effects.

The CHP system under study includes an internal combustion engine (ICE) and a thermal storage tank (ST). The ICE system under consideration uses natural gas to fuel the generator [46]. The thermal storage tank allows synchronization of electricity and thermal energy demands. For instance, it is possible to store the surplus of hot water during periods when electricity demand is far larger than hot water demands. The stored thermal energy can then be used in periods of high thermal demand and/or when it is not convenient to sell electricity to the utility plant [6]. The water is usually stored at high temperature to prevent the onset of pathogenic agents. Consequently, fresh water is often used to regulate the temperature of the delivered water to the residential buildings. The CHP system also [requires a water](#) purging system that allows to drain the excess hot water that can not be stored or consumed [45]. Purge water temperature must also be regulated before disposing of it. Water temperature regulation and storage management can thus lead to high water demands and are often in conflict with system economics. For instance, at certain times the CHP system might aggressively seek to sell electricity to the utility company but the thermal storage might already be at full capacity and thus there is an incentive to purge excess water. Thermal storage is also essential to prevent the ICE system from operating at partial loads, which has a strong effect on fuel efficiency and emissions. In particular, a CHP unit is most efficient at full load operation but, depending on real-time conditions on storage, utility prices, and demands; the CHP operator might seek to sacrifice fuel efficiency and increase emissions in order to improve economics [18, 39]. The size of the storage tank and the capacity of the CHP system together with operating policies thus play a critical role on the performance and trade-offs of the system.

4.1 Optimal Design and Control Formulation

We construct a simultaneous design and control model that seeks to trade-off water, emissions, and operational costs in CHP systems. The model incorporates energy and material balances of the CHP unit and the dynamics of the storage system. We also model the interactions with the utility company and quantify emissions and water use. The design seeks to mitigate variability in daily electricity and hot water demand profiles observed throughout the year. To do so, we model the daily demand profiles as scenarios. We begin our discussion by providing the equations describing a deterministic multiobjective optimization (MOO) formulation (for a single demand profile scenario) and we then describe extensions to the SMOO formulation.

4.1.1 Electricity Demands and Grid Interaction

The total electricity demand of the multi-building system (denoted as W_t^D where t is the time instant) is met using a fraction (W_t^C) of the total electricity production of the CHP unit (W_t^{CHP}). If the CHP system decides not to allocate all of the CHP generation to meet the demand, the system can buy electricity from the local utility company (this amount is denoted as W_t^B). The system can also decide to sell some fraction of the electricity produced in the CHP unit to the utility company (we denote

this as W_t^S). This logic is modeled using the power balances

$$W_t^D = W_t^C + W_t^B \quad (4.30)$$

$$W_t^{CHP} = W_t^C + W_t^S \quad (4.31)$$

These two balances imply the global balance:

$$W_t^D = W_t^{CHP} - W_t^S + W_t^B. \quad (4.32)$$

4.1.2 Hot Water Demands and Purge

The demand of hot water (G_t^D) is met using hot water from the storage (G_t^{ST}) and cold water from the utility company (G_t^{CW}) at ambient temperature (T_t^{amb}). The cold water is used for regulating the temperature of the hot water delivered to the buildings and to control the temperature of the purging system. The purge and demand temperature is given by T^D , which is typically required to be 50 °C [11]. This logic modeled using the following mass and energy balances:

$$G_t^D + G_t^P = G_t^{ST} + G_t^{CW} \quad (4.33)$$

$$(G_t^D + G_t^P)T^D = (G_t^{ST} T_t^{ST}) + (G_t^{CW} T_t^{amb}) \quad (4.34)$$

4.1.3 Thermal Storage Dynamics

The storage temperature (T_t^{ST}) and volume level (V_t^{ST}) depend on the inlet stream given by the hot water generated by the CHP unit (G_t^{CHP}), and by the outlet stream from the storage (G_t^{ST}) used for meeting the hot water demand of the building system. We also consider convective losses in the system, which are a function of the heat transfer area (A_t^{ST}) and the ambient temperature (T_t^{amb}) [28]. The transfer area is dynamic, as it depends on the storage level. The storage system dynamics are captured using the following set of equations:

$$\rho (V_t^{ST} - V_{t-1}^{ST}) = G_t^{CHP} - G_t^{ST} \quad (4.35)$$

$$\rho c_p (V_t^{ST} T_t^{ST} - V_{t-1}^{ST} T_{t-1}^{ST}) = (G_t^{CHP} c_p T^{CHP}) - (G_t^{ST} c_p T_t^{ST}) - Q_t^L \quad (4.36)$$

$$Q_t^L = U A_t^{ST} (T_t^{ST} - T_t^{amb}) \quad (4.37)$$

$$A_t^{ST} = 6 (V_t^{ST})^{2/3} \quad (4.38)$$

The water stored in the tank needs to satisfy a minimum temperature, which is usually set to $\underline{T}^{ST} = 70$ °C:

$$T_t^{ST} \geq \underline{T}^{ST}. \quad (4.39)$$

This temperature prevents proliferation of Legionella [30]. The storage volume level is constrained by the design capacity V^{cap} as:

$$V_t^{ST} \leq V^{cap}. \quad (4.40)$$

4.1.4 Fuel Use and Partial Load Effects

The thermal and electrical efficiencies (denoted as η_t^{QCHP} and η_t^{WCHP}) of the CHP unit are related to the production of heat and power and to the fuel flow as:

$$\eta_t^{QCHP} = \frac{Q_t^{CHP}}{F_t^{CHP}} \quad (4.41)$$

$$\eta_t^{WCHP} = \frac{W_t^{CHP}}{F_t^{CHP}} \quad (4.42)$$

The partial load (PL_t) is defined as the ratio of total production of electrical energy (W_t^{CHP}) and the CHP design capacity (W^{cap}):

$$PL_t = \frac{W_t^{CHP}}{W^{cap}}. \quad (4.43)$$

The effect of the partial load on the efficiency of the CHP unit is modeled using the equation proposed in [36]:

$$\eta_t^{WCHP} = (0.324 PL_t^2 + 0.44 PL_t + 0.235) \hat{\eta}^{WCHP} \quad (4.44)$$

where $\hat{\eta}^{WCHP}$ is the maximum (nominal) system efficiency. Note that efficiency decreases in a highly nonlinear manner with the partial load and that consumption of fuel increases. The total fuel consumption (F_t^{CHP}) can be met by natural gas (NG_t):

$$F_t^{CHP} = NG_t \quad (4.45)$$

The partial load also has a minimum limit in order to prevent system starvation effects:

$$PL_t \geq \underline{PL}. \quad (4.46)$$

The total hot water produced (G_t^{CHP}) at temperature (T^{CHP}) is related to the heat production as:

$$Q_t^{CHP} = G_t^{CHP} c_p (T^{CHP} - T_t^{amb}). \quad (4.47)$$

4.1.5 Total Direct Emissions

The total direct greenhouse gas emissions (GH) are calculated for natural gas according to their corresponding emissions factor (f_{GH}):

$$GH = \sum_{t=1}^T f_{GH}^{NG} \cdot NG_t \quad (4.48)$$

where T is the final time index (we consider a 24 hour horizon with time increments of one hour).

4.1.6 Total Water Consumption

The total water consumption (SW) accounts for the water used by the CHP unit to run the ICE and for the water used to regulate its temperature:

$$SW = \sum_{t=1}^T (G_t^{CHP} + G_t^{CW}). \quad (4.49)$$

4.1.7 Total Cost

The total negative revenue (total cost) factors capital costs as well as income from sales of heat and power, the cost of operation and maintenance, cost of fuel, and purchase of power from the grid:

$$TC = Cost^F + Cost^{OM} + Cost^{Fuel} + Cost^P - I^W - I^H \quad (4.50)$$

The revenue of the CHP system is calculated considering the sales of electrical power and thermal energy. The power sales (I^W) include the sales to the building system (W_t^C) and to the utility company (W_t^S). The price is set by a time-dependent tariff (PES_t) provided by the utility company. The total power sales are given by:

$$I^W = \sum_{t=1}^T PES_t \cdot (W_t^C + W_t^S). \quad (4.51)$$

The heat sales (I^H) are given by the total production of heat in the CHP unit (Q_t^{CHP}) minus the convective losses (Q_t^L). The revenue of thermal energy production is determined by the heating price (PHS):

$$I^H = PHS \cdot \sum_{t=1}^T (Q_t^{CHP} - Q_t^L). \quad (4.52)$$

The cost of maintenance of the CHP unit is based on the unitary cost of operation and maintenance associated to the total power production:

$$Cost^{OM} = UCost^{OM} \sum_{t=1}^T W_t^{CHP}. \quad (4.53)$$

The cost of fuel ($Cost^{Fuel}$) is determined by prices of natural gas (NG):

$$Cost^{Fuel} = \sum_{t=1}^T UCost^{NG} NG_t \quad (4.54)$$

The total cost of power purchased from the grid at price ($UCost_t^P$) is given by:

$$Cost^P = \sum_{t=1}^T (UCost_t^P W_t^B). \quad (4.55)$$

The capital (fixed) cost associated to the installation of the CHP system is given by:

$$Cost^F = \alpha^{ST} V^{cap} + \alpha^W W^{cap}. \quad (4.56)$$

4.1.8 Final MOO Representation

The deterministic design and control problem is a MOO problem of the form:

$$\begin{aligned} &\min (TC, GH, SW) \\ &\text{s.t. (4.30) -- (4.56).} \end{aligned}$$

The design variables are the thermal storage design capacity V^{ST} and the CHP design capacity W^{cap} . The operational variables are flows, temperatures, storage volume levels, and so on. The objective functions are daily cost, emissions, and water usage.

Figure 2: Daily electricity demand scenarios.

Figure 3: Daily hot water demand scenarios.

4.2 SMOO Formulation and Compromise Solutions

The SMOO formulation is obtained by considering multiple daily scenarios for the electrical and hot water demands W_t^D and G_t^D and of ambient temperatures T_t^{amb} . To capture the variability of demands, we discretize the 24 hour time horizon using a 5 minute mesh. The design (first-stage) decision variables are the thermal storage design volume capacity V^{ST} and the CHP design capacity W^{cap} . The operation (resource) variables are flows, temperatures, storage volume levels, and so on. The random objective functions are TC , GH , SW (which vary in each scenario).

We consider a residential housing complex comprising of 1,440 households and 7,632 inhabitants. We use an ICE with a maximum possible capacity for W^{cap} of 200 kW, efficiencies $\hat{\eta}^{WCHP} = 37.25\%$ and $\eta^{QCHP} = 47.5\%$, and minimum partial load $\underline{PL} = 0.35$. The maximum possible storage tank capacity V^{cap} is limited to 50 m^3 . The multi-building system is located in the State of Michoacan in

Mexico (by the Pacific coast and near the mountains). The climatic conditions in this region are stable throughout the year. Real demand data for electricity and hot water for the buildings is presented in Figures 2 and 3. As can be seen, demands follow quasi-periodic patterns but are out of phase (hot water demands peak at the beginning of the day when the residents take showers while electricity demand peaks at night). Moreover, one can see that there is a high variance of the demands at peak times. The electricity price from the utility company doubles during the time period of 5 p.m. and 10 p.m. The emission factor for natural gas f_{GH}^{NG} is 380 g CO₂/kWh [26]. The thermal energy price PHS is 0.015 \$/kWh.

We scale all the objective functions using the utopia solutions. We use the nested CVaR formulations to identify different compromise solutions. The models were implemented on PLASMO [15] and solved with PIPS-NLP [44]. All scripts are available at <http://zavalab.engr.wisc.edu/data>.

Table 1: Compromise solutions for Formulations C and D

Formulation C									
α_O	α_S	V^{cap} (m ³)	W^{cap} (kW)	$\overline{GH}$ (Ton)	$\overline{SW}$ (Ton)	$\overline{TC}$ (\$)	GH^{max} (Ton)	SW^{max} (Ton)	TC^{max} (\$)
0	0	22.55	113.68	2.41	77.011	671.40	2.79	88.555	861.72
1	1	22.55	113.68	2.74	89.161	672.87	2.79	99.677	705.23
1	0	49.94	113.68	2.62	85.014	596.21	2.79	88.528	769.05
0	1	22.55	113.68	2.69	87.065	690.01	2.79	87.407	728.39
Formulation D									
α_S	α_O	V^{cap} (m ³)	W^{cap} (kW)	$\overline{GH}$ (Ton)	$\overline{SW}$ (Ton)	$\overline{TC}$ (\$)	GH^{max} (Ton)	SW^{max} (Ton)	TC^{max} (\$)
1	0	22.55	113.68	2.62	84.892	707.86	2.79	89.301	861.67
0	1	22.55	113.68	2.62	85.522	602.24	2.79	99.426	705.65

The compromise solutions obtained with Formulation C and D are shown in Table 1. Here, we summarize the mean and worst-case values for the different objectives. The entire set of costs, water, and emissions along with their corresponding utopia points are shown in Figures 4-9. As can be seen, the average compromise solution (obtained by minimizing $\langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_O=0, \alpha_S=0}$) shows a large spread in the objective space. This spread is manifested in a large difference between the mean and worst-case objectives. For the cost we have \$671.40 and \$861.72 (a difference of 28%), for the emissions we have 2.41 and 2.79 Ton (a difference of 15%), and for water usage we have 77.01 Ton and 88.55 Ton (a difference of 15%). The worst-case compromise solution (obtained by minimizing $\langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_O=1, \alpha_S=1}$) reduces the relative differences to 4.9%, 2%, and 11.8%. The reduction in the variability can be clearly visualized by comparing Figures 4 and 5. Interestingly, the optimal size for the storage and the CHP system is the same for both compromise solutions. This indicates that the operational policy places a critical role in shaping the distribution of the objectives. We can also observe clear changes in the distribution of the objectives by comparing the compromise solutions obtained by minimizing the worst-case of the average scenario objectives (obtained by minimizing $\langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_O=1, \alpha_S=0}$) and obtained by minimizing the average of the worst-case scenario objectives (obtained by minimizing $\langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_O=0, \alpha_S=1}$). In particular, for the first case we see that the objectives form three clusters with a large variance in cost while in the second case we only observe two clusters with a smaller variance in cost.

Figure 4: Utopia and compromise solution for $\min_{x \in \mathcal{X}} \langle \mathbf{F}(x) \rangle_{\alpha_G=0, \alpha_S=0}$.

Interestingly, the only solution that has different design capacities is the one that minimizes the worst-case of the average objectives. This solution increases the capacity of the storage tank by 100% compared to the other designs. This allows the system to significantly reduce average cost (by 11%) compared to the closest design. This can be explained by the fact that a larger storage system allows the system to produce and sell more electricity to the utility company during periods of high prices. On the other hand, this design has a larger worst-case cost than that obtained by other designs. In other words, this design can be seen as a central solution between the average and worst-case compromise solutions. We also note that the optimal capacities for the rest of the compromise solutions do not hit the upper bounds V^{cap} and W^{cap} , which indicates that the design seeks to optimally balance sales and purchases of electricity from the utility. In other words, it is not optimal to build a large CHP system to fully satisfy the electricity demands of the residential complex. Such a large CHP design would force the system to operate at partial loads most of the time and will thus be inefficient from a fuel usage standpoint. Moreover, operating such a large CHP system will also waste significant amounts of water as the peak water demands do not match the peak demands of electricity.

In Table 1 and Figures 8 and 9 we compare the compromise solutions obtained by minimizing the worst-case over the average of the three objectives (obtained with $\langle \mathbf{F}(x) \rangle_{\alpha_S=0, \alpha_G=1}$) and by minimizing the average over the worst-case of the three objectives (obtained with $\langle \mathbf{F}(x) \rangle_{\alpha_S=1, \alpha_G=0}$). These solutions also obtain the same design as the solutions obtained with Formulation C. The most interesting behavior is obtained by minimizing $\langle \mathbf{F}(x) \rangle_{\alpha_S=0, \alpha_G=1}$. Under this solution, the objectives cluster around a plane with large water usage variance but narrow cost, which can be confirmed by verifying average and worst-case values in Table 1. This formulation has the interesting property that it first obtains the worst-case objective for each scenario and then it averages them. This solution can thus be interpreted as an average of the worst-case objectives. We again highlight that the proposed framework allows practitioners to obtain solutions with different interpretations. This can

Figure 5: Utopia and compromise solution for $\min_{x \in \mathcal{X}} \langle \mathbf{F}(x) \rangle_{\alpha_G=1, \alpha_S=1}$.

Figure 6: Utopia and compromise solution for $\min_{x \in \mathcal{X}} \langle \mathbf{F}(x) \rangle_{\alpha_G=1, \alpha_S=0}$.

Figure 7: Utopia and compromise solution for $\min_{x \in \mathcal{X}} \langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_G=0, \alpha_S=1}$.

allow decision-makers to identify designs and operational policies that satisfy their individual priorities. Of course, there is some ambiguity associated to the selection of the nested CVaR parameters but we note that the decision space has been significantly narrowed down from the full scenario and objective space to a handful of solutions. Moreover, we highlight that the minimization of the nested CVaR metric allows us to compute compromise solutions *without having to compute entire Pareto sets*.

Figure 8: Utopia and compromise solution for $\min_{x \in \mathcal{X}} \langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_S=0, \alpha_G=1}$.

4.3 Computational Performance

We now discuss computational performance of the framework. Formulation (C) of the CHP problem results in a two-level graph model with 13 variables, 4 equality constraints, and 3 inequality constraints in the parent node and with 5776 variables, 4909 equality constraints, and 868 inequality

Figure 9: Utopia and compromise solution for $\min_{x \in \mathcal{X}} \langle\langle \mathbf{F}(x) \rangle\rangle_{\alpha_S=1, \alpha_O=0}$.

constraints per scenario (children) node. The problem under study has 188 scenarios which gives rise a total of 1,085,901 variables, 922,896 equality constraints, and 163,187 inequality constraints. We solve this problem using `PIPS-NLP` using a Schur decomposition approach on a multi-core shared memory environment. We compare performance against the full-space solution with `IPOPT` performed on a single core. Schur decomposition is not used in `IPOPT` (a direct sparse factorization is used). We perform this comparison to evaluate how well does a full-space sparse factorization performs against parallel Schur decomposition.

The computational results are summarized in Table 2. As can be seen, for the $\alpha_O = 0, \alpha_S = 0$ case, `PIPS-NLP` is able to reduce the computational times from 3,894 seconds (over an hour) to 378 seconds (6.3 minutes) using 20 cores. The solution time of `IPOPT` is 1,167 seconds (19.4 minutes). We thus can see that the direct factorization used in `IPOPT` is fairly competitive but `PIPS-NLP` is still significantly faster and can in principle be accelerated further by using more cores. We also observe that the number of iterations is comparable for both solvers (around 300 iterations). In the $\alpha_O = 1, \alpha_S = 1$ case we see a significant increase in the number of iterations (to over 800). We can thus see that nested CVaR settings can have a significant effect on performance. We attribute this to the fact that the worst-case solutions lead to more degeneracy in the solution in which the different scenarios are nearly indistinguishable (this can be seen from the tight scenario clusters shown in Figure 5). In this case, however, we still perceive a significant acceleration using `PIPS-NLP`. The solution times are reduced from 12,627 seconds (3.5 hours) to 1,439 seconds (23.98 minutes). The solution time obtained with `IPOPT` is 3,959 seconds (66 minutes). We thus see that the proposed decomposition framework provides important savings in computing time and can handle large problems.

Table 2: Computational performance of PIPS-NLP and IPOPT on Formulation C.

		PIPS (20 Cores)		PIPS (10 Cores)		PIPS (5 Cores)		PIPS (1 Core)		IPOPT (1 Core)	
α_O	α_S	Time(s)	Iter	Time(s)	Iter	Time(s)	Iter	Time(s)	Iter	Time (s)	Iter
0	0	378	313	547	313	1,022	313	3,894	313	1,167	325
1	1	1,439	871	1,957	848	3,065	843	12,627	838	3,959	853

5 Conclusions

We presented a scalable computing framework for the solution stochastic multiobjective optimization problems. Our framework uses a nested conditional value-at-risk (nCVaR) objective to find compromise solutions among conflicting random objectives. We prove that the associated nCVaR minimization problems can be cast as standard stochastic programming problems with expected value constraints. We also show that these problems can be implemented in a highly modular fashion using PLASMO (a Julia-based structured modeling framework) and can be solved efficiently using the parallel solver PIPS-NLP. We show that the framework can be used to compute a rich set of compromise solutions that seek to balance objectives and performance in different scenarios. As part of future work, we will consider extensions to model a wider range of problems that involve multiple decision layers. Such problems arise, [for instance](#), when stakeholders disagree on which compromise solution should be used (i.e., which combination of probability levels α_O and α_S should be used).

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