New *a priori* and *a posteriori* probabilistic bounds for robust counterpart optimization: I. Unknown probability distributions

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Abstract

Optimization problems often have a subset of parameters whose values are not known exactly or have yet to be realized. Nominal solutions to models under uncertainty can be infeasible or yield overly optimistic objective function values given the actual parameter realizations. Worst-case robust optimization guarantees feasibility but yields overly conservative objective function values. The use of probabilistic guarantees greatly improves the performance of robust counterpart optimization. We present new *a priori* and *a posteriori* probabilistic bounds which improve upon existing methods applied to models with uncertain parameters whose possible realizations are bounded and subject to unspecified probability distributions. We also provide new *a priori* and *a posteriori* bounds which, for the first time, permit robust counterpart optimization of models with parameters whose means are only known to lie within some range of values. The utility of the bounds is demonstrated through computational case studies involving a mixed-integer linear optimization problem and a linear multiperiod planning problem. These bounds reduce the conservatism, improve the performance, and augment the applicability of robust counterpart optimization.

1. Introduction

The parameters of applied mathematical models often have multiple possible values which they can achieve due to limited information or measurement error. The solutions and objective function values produced from optimizing models with uncertain parameters can vary greatly based on which values the uncertain parameters realize. One method of handling uncertain parameters is to guarantee feasibility of the constraints for all possible parameter points

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contained within deterministically defined parameter spaces, called uncertainty sets; the corresponding optimization model is known as a robust counterpart. For bounded uncertainty, that is, for the case where upper and lower bounds on each uncertain parameter are known, Soyster (1973) formulated the so-called “worst-case” robust counterpart, where each constraint’s uncertainty set is defined to include all possible parameter points; the probability that parameters would realize values rendering an optimal solution infeasible is zero. El Ghaoui and Lebret (1997) applied robust optimization methods to uncertain least-squares optimization problems, and El Ghaoui et al. (1998) solved uncertain semidefinite optimization problems.

Less conservative solutions can also be obtained for models with bounded or unbounded uncertain parameters, where the uncertainty sets can be defined with a corresponding probability of constraint violation greater than zero. For bounded uncertainty in linear optimization models (LPs), Ben-Tal and Nemirovski (2000) proposed the interval + ellipsoidal uncertainty set and derived a method to provide an upper bound on constraint violation given the size of the uncertainty set. Bertsimas and Sim (2004) proposed and characterized the interval + polyhedral uncertainty set, which yielded a linear robust counterpart. The robust counterpart optimization framework was extended to mixed-integer linear optimization models (MILPs) by Floudas and coworkers (Janak et al., 2007; Lin et al., 2004; Verderame and Floudas, 2009a,b) with parameters subject to uncertainty with known or unknown distributions, including unbounded probability distributions.

In a traditional robust counterpart optimization framework, the quality of an optimal solution to the robust counterpart relies heavily on *a priori* methods that define uncertainty sets which are guaranteed to satisfy a particular upper bound on the probability of constraint violation; a tighter probabilistic bound would allow the imposition of a smaller uncertainty set with the same guarantee of feasibility and can drastically improve the objective function values. In other words, a tighter probabilistic upper bound on constraint violation leads to less conservative solutions. Both Ben-Tal and Nemirovski (2000) and Bertsimas and Sim (2004) derived *a priori* methods that provided an upper bound on the probability of constraint violation based on uncertainty set size for interval + ellipsoidal and interval + polyhedral sets, respectively. Kang et al. (2013) utilized distribution-dependent bounds with the interval + polyhedral uncertainty set. Li et al. (2011) extended these bounds to apply to other uncertainty sets so that a variety of *a priori* bounds were available for box, ellipsoidal, interval + ellipsoidal, polyhedral, and interval + polyhedral uncertainty sets. An alternative approach is to characterize the probability of constraint violation of a particular solution, that is, to obtain a probabilistic guarantee *a posteriori* (Kang et al., 2013; Li and Floudas, 2014; Li et al., 2012; Paschalidis et al., 2008), which would yield a tighter bound than an *a priori* bound of equivalent structure. An *a posteriori* bound can be incorporated into the robust counterpart, permitting feasibility only when the bound is met, as an alternative to an uncertainty set, though this yields a highly nonconvex optimization problem (Li and Floudas, 2014). Li and Floudas (2014) proposed an iterative

method which yields better solutions than those obtained from the traditional robust counterpart optimization framework (i.e., defining uncertainty sets via *a priori* bounds), by converging them towards the tighter results given by *a posteriori* bounds. Utilizing tight *a priori* and *a posteriori* bounds with this iterative method can provide drastically improved solutions when compared to the worst-case approach, as well as the traditional one-pass robust counterpart optimization framework.

We present fundamental theoretical results on new *a priori* and *a posteriori* bounds on the probability of constraint violation which improve upon existing methods. Situations for which one or more of the new bounds are applicable include (i) bounded, symmetric or asymmetric uncertain parameters with known means and unknown probability distributions, and (ii) bounded uncertain parameters with limited information on their means. With the proposed *a priori* and *a posteriori* bounds, robust counterpart optimization can be applied to models with parameters matching the latter case for the first time. The new *a priori* bounds yield smaller uncertainty sets while guaranteeing the same probability of constraint violation when compared with existing methods. A new *a posteriori* bound for symmetric, unspecified probability distributions can yield lower probabilities of constraint violation for the same robust solution. An *a posteriori* bound given uncertain parameters with limited information on their means is also provided. These methods can provide greatly improved objective function values, both when used within a traditional robust counterpart framework and when applied to the *a priori* – *a posteriori* iterative algorithm of Li and Floudas (2014).

2. Background

The scope of this work includes uncertain parameters which participate linearly or are coefficients of variables participating linearly in a constraint or objective function. Given continuous variables x and integer variables y , consider the nonlinear function $f_i(x, y)$ participating in constraint i :

$$f_i(x, y) + \sum_k a_{ik}x_k + \sum_\ell b_{i\ell}y_{i\ell} + \sum_m p_{im} \leq 0. \quad (1)$$

Relevant to this work is the case where the exact value of some or all of the parameters a_{ik} , $b_{i\ell}$, and p_{im} are unknown. Without loss of generality, constraint (1) can be reformulated as:

$$\begin{aligned} f_i(x, y) + t_i &\leq 0 \\ -t_i + \sum_k a_{ik}x_k + \sum_\ell b_{i\ell}y_{i\ell} + \sum_m p_{im} &\leq 0. \end{aligned} \quad (2)$$

The same type of reformulation can be applied to an objective function under uncertainty. Thus, we will assume that all uncertain constraints only involve linearly participating variables which are continuous or integer. The typical presentation of robust optimization goes further and assumes that the entire model

is an LP or MILP. We present the background section under this assumption so as to maintain consistency. The study of uncertainty in parameters that participate nonlinearly in the objective function or constraints is beyond the scope of this paper and will be the subject of future work.

The general form of a LP or MILP under uncertainty is as follows:

$$\begin{aligned}
& \max_{x,y} \quad \sum_k \tilde{c}_k x_k + \sum_\ell \tilde{d}_\ell y_\ell \\
& \text{s.t.} \quad \sum_k \tilde{a}_{ik} x_k + \sum_\ell \tilde{b}_{i\ell} y_\ell \leq \tilde{p}_i \quad \forall i \\
& \quad \quad y_\ell \in \{0, 1\} \quad \forall \ell
\end{aligned} \tag{3}$$

where any parameter denoted with a tilde, say $\tilde{p}$, denotes a parameter subject to uncertainty. Different realizations of the uncertain parameters can change the solution and objective function value of (3). Model (3) can be equivalently written (Li et al., 2012) as:

$$\begin{aligned}
& \max_{x,y,z} \quad z \\
& \text{s.t.} \quad z - \sum_k \tilde{c}_k x_k - \sum_\ell \tilde{d}_\ell y_\ell \leq 0 \\
& \quad \quad \tilde{p}_i x_0 + \sum_k \tilde{a}_{ik} x_k + \sum_\ell \tilde{b}_{i\ell} y_\ell \leq 0 \quad \forall i \\
& \quad \quad x_0 = -1 \\
& \quad \quad y_\ell \in \{0, 1\} \quad \forall \ell.
\end{aligned} \tag{4}$$

An uncertain parameter $\tilde{a}$ can be represented by a constant nominal value a , which is generally set to $\mathbf{E}[\tilde{a}]$, and a random variable ξ which subjects a to perturbations weighted by a known positive constant $\hat{a}$:

$$\tilde{a} = a + \xi \hat{a}. \tag{5}$$

Without loss of generality, models (3) and (4) can be represented as:

$$\begin{aligned}
& \max_x \quad \sum_j c_j x_j \\
& \text{s.t.} \quad \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \leq b_i \quad \forall i
\end{aligned} \tag{6}$$

where x_j is a continuous or integer variable, and $J_i := \{j : \hat{a}_{ij} > 0\}$ is the set of indices where the coefficients of x in constraint i are subject to uncertainty.

In general, the nominal solution to a model under uncertainty, where each uncertain parameter is held at its nominal value, can only assure feasibility of the model if every uncertain parameter realizes its nominal value. A robust solution to a model under uncertainty will permit uncertain parameters to realize multiple values while remaining feasible. If the nominal value of each uncertain

parameter is chosen as its expected value, then an optimal solution using the nominal parameter point in parameter space can be considered to be a type of base or average case. One can then extend outwards from the nominal parameter point and require a robust solution to guarantee feasibility for more parameter points in parameter space, thus increasing the robustness of the solution as more points are included. A set of points in parameter space that a robust solution must account for is called an uncertainty set, and is defined separately for each constraint under uncertainty. We denote the uncertainty set of constraint i as U_i . Given the notation of (6), U_i will contain some or all realizations of the random vector ξ_i with elements ξ_{ij} , $j \in J_i$, including the nominal parameter point, $\xi_i = \mathbf{0}$, which corresponds to the nominal solution. The robust counterpart of model (6) can thus be written as

$$\begin{aligned} \max_x \quad & \sum_j c_j x_j \\ \text{s.t.} \quad & \sum_j a_{ij} x_j + \max_{\xi_i \in U_i} \left\{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \right\} \leq b_i \quad \forall i. \end{aligned} \quad (7)$$

Uncertainty sets are generally constructed geometrically by including parameter points within some norm distance around the nominal point. The worst-case interval set proposed by Soyster (1973) can be generalized as an ∞ -norm distance about the nominal point, and is called a box uncertainty set (Li et al., 2011):

$$U_i^\infty = \{\xi_i : \|\xi_i\|_\infty = \max_{j \in J_i} |\xi_{ij}| \leq \Psi_i\}. \quad (\text{B})$$

Similarly, the 1-norm or 2-norm can be used, yielding the polyhedral and ellipsoidal uncertainty set, respectively (Ben-Tal and Nemirovski, 1998, 1999, 2000; Bertsimas and Sim, 2004; El Ghaoui and Lebret, 1997; El Ghaoui et al., 1998; Li et al., 2011):

$$U_i^1 = \{\xi_i : \|\xi_i\|_1 = \sum_{j \in J_i} |\xi_{ij}| \leq \Gamma_i\} \quad (\text{P})$$

$$U_i^2 = \{\xi_i : \|\xi_i\|_2 = \sqrt{\sum_{j \in J_i} \xi_{ij}^2} \leq \Omega_i\}. \quad (\text{E})$$

Note that increasing parameters Ψ_i , Γ_i , and Ω_i from zero increases the size of U_i . We will use the notation Δ_i to denote the size parameter of arbitrary uncertainty set U_i . If it is assumed that $\xi_{ij} \in [-1, 1]$, $\forall j \in J_i$, then when $\Psi_i = 1$, all possible parameter realizations are included in the box uncertainty set (thus becoming the worst-case or interval set). For polyhedral and ellipsoidal sets, this occurs when $\Gamma_i = |J_i|$ and $\Omega_i = \sqrt{|J_i|}$, respectively (Li et al., 2011). However, with these values of Γ_i and Ω_i , more parameter points than the worst-case set are included due to the geometry of the underlying norm. Ben-Tal and Nemirovski (2000) proposed the interval + ellipsoidal uncertainty set, which is

the intersection of the interval and ellipsoidal sets and is never larger than the worst-case set:

$$U_i^{2 \cap \infty} = \{\xi_i : \|\xi_i\|_2 \leq \Omega_i, \|\xi_i\|_\infty \leq 1\}. \quad (\text{IE})$$

Similarly, Bertsimas and Sim (2004) proposed the interval + polyhedral uncertainty set:

$$U_i^{1 \cap \infty} = \{\xi_i : \|\xi_i\|_1 \leq \Gamma_i, \|\xi_i\|_\infty \leq 1\}. \quad (\text{IP})$$

The five uncertainty sets considered in this work are summarized in Table 4. The introduction of uncertainty sets into a robust counterpart formulation is summarized in Table 5 and discussed in detail in (Li et al., 2011; Li and Floudas, 2014; Li et al., 2012).

If the solution to a model under uncertainty must be feasible for any possible realization of its uncertainty parameters, all of which are bounded, then the robust counterpart should include only interval uncertainty sets ($\Psi_i = 1$), which typically yields very conservative solutions. However, if one or more parameters are unbounded, or if the situation can tolerate some small, nonzero probability that constraint i is infeasible, the objective function value can be drastically improved when compared to the worst-case solution. One way this is accomplished is by reducing the size of U_i via selecting Δ_i values that are less than the worst-case value prior to solving the robust counterpart. The size of U_i is selected such that any subsequent solution is guaranteed to be feasible in constraint i using any parameter realization with a user-specified probability of ϵ_i^{prio} or less:

$$\Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \leq \epsilon_i^{prio}. \quad (8)$$

Methods which can relate the size of U_i to ϵ_i^{prio} are thus called *a priori* probabilistic bounds.

Alternatively, a specific solution x^* to the robust counterpart can be characterized by determining an upper bound ϵ_i^{post} on the probability that the solution will be feasible in constraint i with any parameter realization:

$$\Pr \left\{ \sum_j a_{ij} x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j^* > b_i \right\} \leq \epsilon_i^{post}. \quad (9)$$

Methods which can relate a specific solution of the robust counterpart to the probability of constraint violation are called *a posteriori* probabilistic bounds.

It is desirable to utilize the tightest *a priori* or *a posteriori* bounding methods available; at the same value of ϵ_i^{prio} , a tighter *a priori* method can yield a better objective function value than a looser method, while a tighter *a posteriori* method can give a lower estimate of ϵ_i^{post} for a given solution than a looser method.

3. Preliminaries

Given that, for all $j \in J_i$, random variable ξ_{ij} is independent, the following relations for the *a priori* probability of constraint violation hold for the i th constraint, any uncertainty set U_i outlined above, and any $\theta > 0$ (Ben-Tal and Nemirovski, 2000; Bertsimas and Sim, 2004; Li et al., 2012):

$$\Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \leq \Pr \left\{ \sum_{j \in J_i} \delta_{ij} \xi_{ij} > \Delta_i \right\} \quad (10)$$

$$\leq e^{-\theta \Delta_i} \prod_{j \in J_i} \mathbf{E}[e^{\theta \delta_{ij} \xi_{ij}}], \quad (11)$$

where the definitions of δ_{ij} are dependent on the classification of uncertainty set U_i and are given in Table 6. Each $\delta_{ij} \in [-1, 1]$ and is essentially $\hat{a}_{ij} x_j$ scaled by a factor related to U_i ; for more details, see (Li et al., 2012). The *a priori* value of δ_{ij} is not known; the full range and combination of values of elements δ_{ij} , $\forall j \in J_i$, of vector δ_i must be taken into account. Based on the type of uncertainty set U_i , the various norm distances of the vector δ_i are constrained to be less than or equal to different values. Parameter Δ_i represents size parameter Ψ_i , Ω_i , or Γ_i for box, ellipsoidal/interval + ellipsoidal, or polyhedral/interval + polyhedral uncertainty sets, respectively (see Tables 4 and 5). Relation (11) follows from Markov's inequality and independence of ξ_{ij} . Relations (10) and (11) will be used as starting points for some of the theorems in Section 4.

We also require three lemmas. The first was noted by Hoeffding (1963) but is not the well-known Hoeffding's lemma, though it is tighter. It can be seen as a specific case of Bennett's Lemma where the standard deviation of the probability distribution is not known (Bennett, 1962; Dembo and Zeitouni, 2009).

Lemma 1. *Given the random variable ξ is subject to a probability distribution that is supported on the bounded interval $[a, b]$, and given that its moment generating function $M_\xi(\theta) = \mathbf{E}[e^{\theta \xi}]$ exists, then for any real number θ ,*

$$\mathbf{E}[e^{\theta \xi}] \leq \frac{b - \mathbf{E}[\xi]}{b - a} e^{\theta a} + \frac{\mathbf{E}[\xi] - a}{b - a} e^{\theta b}.$$

Proof. The function $\exp(\theta \xi)$ is convex in ξ . Thus,

$$\exp(\theta((1 - \lambda)\xi_1 + \lambda\xi_2)) \leq (1 - \lambda)e^{\theta \xi_1} + \lambda e^{\theta \xi_2} \quad \forall \xi_1, \xi_2 \in [a, b], \forall \lambda \in [0, 1].$$

Taking $\xi_1 = a$, $\xi_2 = b$, and $(1 - \lambda)\xi_1 + \lambda\xi_2 = \xi$, the relation becomes

$$e^{\theta \xi} \leq \frac{b - \xi}{b - a} e^{\theta a} + \frac{\xi - a}{b - a} e^{\theta b} \quad \forall \xi \in [a, b].$$

Therefore,

$$\mathbf{E}[e^{\theta\xi}] \leq \mathbf{E}\left[\frac{b-\xi}{b-a}e^{\theta a} + \frac{\xi-a}{b-a}e^{\theta b}\right] = \frac{b-\mathbf{E}[\xi]}{b-a}e^{\theta a} + \frac{\mathbf{E}[\xi]-a}{b-a}e^{\theta b}.$$

□

Solely by applying Lemma 1, better *a priori* bounds can be derived. However, bounds can be further tightened by considering the natural upper bound on $\|\delta_i\|_1$, denoted as $\delta_i^{(1)}$, which depends on the uncertainty set type (see Table 6).

Lemma 2. *Let $f_j : X \rightarrow Y$, where $X = \{x_j \in \mathbb{R} : 0 \leq x^L \leq x_j \leq x^U\}$ and $Y = \{y \in \mathbb{R} : 0 \leq y < \infty\}$, be nondecreasing and log-convex over X , $j = 1, 2, \dots, n$, with*

$$\frac{f_1(x_i)}{f_1(x_j)} \geq \frac{f_2(x_i)}{f_2(x_j)} \geq \dots \geq \frac{f_n(x_i)}{f_n(x_j)} \quad \forall (i, j) : x_i, x_j \in X, x_i \geq x_j. \quad (12)$$

Given that the nonconvex optimization problem

$$\max_x \left\{ \prod_{j=1}^n f_j(x_j) : \sum_{j=1}^n x_j \leq x^\sigma; x_j \in X, \forall j \right\}, \quad (\text{P1})$$

is feasible (i.e., $x^L \leq x^\sigma/n$), then the point $x^* = (x_1^*, x_2^*, \dots, x_j^*, \dots, x_n^*)^\top$, where

$$\begin{aligned} x_j^* &= \begin{cases} x^U & j \leq \bar{j} \\ x^m & j = \bar{j} + 1 \\ x^L & j \geq \bar{j} + 2, \end{cases} \\ \bar{j} &:= \left\lfloor \frac{x^\sigma - nx^L}{x^U - x^L} \right\rfloor, \\ x^m &:= x^\sigma - \bar{j}(x^U - x^L) - (n-1)x^L, \end{aligned}$$

is a global maximizer to problem (P1).

Proof. Note that the codomain of $f_j(x_j)$ over X is nonnegative. Then a global minimizer of the following optimization problem:

$$\min_x \left\{ -\sum_{j=1}^n \ln f_j(x_j) : \sum_{j=1}^n x_j \leq x^\sigma; x_j \in X, \forall j \right\}, \quad (13)$$

which is a concave minimization problem over a polytope, will be a global maximizer of problem (P1). Furthermore, at least one of the vertices of the polytope will be a global minimizer of problem (13) (Falk and Hoffman, 1976; Tardella, 2003).

If $x^L = x^\sigma/n$, the only feasible solution is $x'_j = x^L, \forall j$. Then $\bar{j} = 0$, $x^m = x^L$, and $x'_j = x_j^*, \forall j$.

If $x^U \leq x^\sigma/n$, then the constraint $\sum_{j=1}^n x_j \leq x^\sigma$ is redundant. As $f_j(x)$ is nondecreasing, a global minimizer is $x'_j = x^U, \forall j$. Then $\bar{j} \geq n$ and $x'_j = x_j^*, \forall j$.

If $x^L < x^\sigma/n < x^U$, the feasible region has a nonempty interior and the constraint $\sum_{j=1}^n x_j \leq x^\sigma$ is not redundant. If this constraint is inactive ($\sum_{j=1}^n x_j < x^\sigma$), then there exists some $x_j < x^U$ which, if increased, would not increase the value of the objective function of (13) given that every $f_j(x_j)$ is nondecreasing over X . Thus, the set of vertex points where the equality $\sum_{j=1}^n x_j = x^\sigma$ holds contains a global minimizer.

Define the set of indices denoting variables which are on the interior of their interval bounds as

$$I := \{j : x^L < x'_j < x^U\}.$$

If a given feasible point x' involves two or more variables which are on the interior of their interval bounds, i.e., $|I| > 1$, then x' is not a vertex point, as it can be written as a convex combination of two other feasible points x^a and x^b where an increase in $x_i^a, x_j^b, i \neq j, i \in I, j \in I$ can correspond to a decrease in x_j^a, x_i^b . However, with $|I| \leq 1$, point x' cannot be written as a convex combination of two other feasible points and thus is vertex point. Depending on x^L, x^U , and x^σ , vertices where $\sum_{j=1}^n x_j = x^\sigma$ holds will have either one index in I or no indices in I .

Consider the weighted sum of two values x^L and x^U , where the sum of the weights equals n and the weighted sum equals x^σ :

$$\alpha x_U + (n - \alpha)x^L = x^\sigma.$$

Solving for α yields

$$\alpha = \frac{x^\sigma - nx^L}{x^U - x^L}.$$

Note that α is not necessarily an integer amount. Relating this situation to the proof, the floor of α represents the number of variables at x^U . If $\alpha - \bar{j} = 0$, then $|I| = 0$ and $x^m = x^L$. Otherwise, $\alpha - \bar{j} > 0, |I| = 1$, and it must hold that

$$x_{\bar{j}+1}^\sigma = x^\sigma - \bar{j}x^U - (n - \bar{j} - 1)x^L = x^m$$

for $\sum_{j=1}^n x_j = x^\sigma$ to hold. In both cases, $x_j = x_j^*, \forall j$.

By the definition of x^* , the ordering by value of x_j^* in x^* is $x_1^* \geq x_2^* \geq \dots \geq x_n^*$. Assume that switching some x_i^* with $x_k^*, i < k, x_i > x_k$, would improve the

objective function of (13). This means

$$\begin{aligned}
& - \sum_{j \notin \{i,k\}} \ln f_j(x_j) - \ln f_i(x_k) - \ln f_k(x_i) \\
& \qquad \qquad \qquad < - \sum_{j \notin \{i,k\}} \ln f_j(x_j) - \ln f_i(x_i) - \ln f_k(x_k) \quad x_i > x_k \\
\rightarrow & -\ln f_i(x_k) - \ln f_k(x_i) < -\ln f_i(x_i) - \ln f_k(x_k) \quad x_i > x_k \\
\rightarrow & \ln(f_i(x_k)f_k(x_i)) > \ln(f_i(x_i)f_k(x_k)) \quad x_i > x_k \\
\rightarrow & f_i(x_k)f_k(x_i) > f_i(x_i)f_k(x_k) \quad x_i > x_k \\
\rightarrow & \frac{f_k(x_i)}{f_k(x_k)} > \frac{f_i(x_i)}{f_i(x_k)} \quad x_i > x_k
\end{aligned}$$

which contradicts conditions (12). $\square$

Remark 1. If functions $\ln f_j(x_j)$ are differentiable over X , conditions (12) are equivalent to

$$\begin{aligned}
& \frac{f_1(x_i)}{f_1(x_j)} \geq \dots \geq \frac{f_n(x_i)}{f_n(x_j)} \quad \forall (i,j) : x_i, x_j \in X, x_i \geq x_j \\
\rightarrow & \ln f_1(x_i) - \ln f_1(x_j) \geq \dots \\
& \geq \ln f_n(x_i) - \ln f_n(x_j) \quad \forall (i,j) : x_i, x_j \in X, x_i \geq x_j \\
\rightarrow & \int_{x_j}^{x_i} \frac{d \ln f_1(x)}{dx} dx \geq \dots \\
& \geq \int_{x_j}^{x_i} \frac{d \ln f_n(x)}{dx} dx \quad \forall (i,j) : x_i, x_j \in X, x_i \geq x_j,
\end{aligned}$$

which implies

$$\frac{d \ln f_1(x)}{dx} \geq \dots \geq \frac{d \ln f_n(x)}{dx} \quad \forall x \in X \quad (14)$$

as alternate conditions to conditions (12) which can be used to qualify Lemma 2.

Remark 2. If $x^L = 0$ and $x^U = 1$, $\bar{j} = \lfloor x^\sigma \rfloor$. Using $|\delta_{ij}|$ for x_j ($|\delta_{ij}| \in [0, 1]$), $\bar{j} = \lfloor \delta_i^{(1)} \rfloor$.

For box, polyhedral, and interval + polyhedral sets, any vector δ_i which satisfies the constraint $\|\delta_i\|_1 \leq \delta_i^{(1)}$ will also satisfy the constraint $\|\delta_i\|_2 \leq \delta_i^{(2)}$, where $\delta_i^{(2)}$ is the natural upper bound on $\|\delta_i\|_2$. However, for ellipsoidal and interval + ellipsoidal sets, the latter constraint is both an equality and more restrictive than the former; vectors δ_i which satisfy the 1-norm constraint can violate the 2-norm constraint. The following holds for any vector $x \in \mathbb{R}^n$ due to the Cauchy-Schwarz inequality:

$$\|x\|_1 \leq n^{1/2} \|x\|_2.$$

Thus, because $\delta_i^{(1)} = |J_i|^{1/2} \delta_i^{(2)}$ for ellipsoidal and interval + ellipsoidal sets, we only need to consider the constraint on $\|\delta_i\|_2$.

Lemma 3. *Let $f : X \rightarrow Y$, where $X = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and $Y = \{y \in \mathbb{R} : 0 < y < \infty\}$, be nondecreasing over X , and let the composite function $(f \circ g)(x)$, where $g(x) = \sqrt{x}$, be concave over X . Then the optimization problem*

$$\max_x \left\{ \sum_j f(x_j) : \sum_j x_j^2 \leq 1; x_j \in [0, 1], \forall j \right\} \quad (\text{P2})$$

with $j = 1, 2, \dots, n$, has an optimal solution $x_j^* = 1/\sqrt{n}$

Proof. With a change of variables $w_j = x_j^2$, problem (P2) is equivalent to

$$\max_w \left\{ \sum_j f(\sqrt{w_j}) : \sum_j w_j \leq 1; w_j \in [0, 1], \forall j \right\}, \quad (15)$$

which is a concave maximization over a convex set. Because the function $f(x_j)$ is nondecreasing over the domain, the function $f(\sqrt{w_j})$ is also nondecreasing over the domain. Thus, the region where the equality $\sum_j w_j = 1$ holds will contain an optimal solution. At any feasible point w^* where the equality holds, if there exists some $w_i^* < w_k^*$, $i \neq k$, an increase in w_i^* (and a corresponding decrease in w_k^* to satisfy $\sum_j w_j = 1$) will not decrease the objective function value due to the concavity of $f(\sqrt{w_j})$. Thus, an optimal solution to (15) is $w_j^* = 1/n$, $\forall j$, corresponding to $x_j^* = 1/\sqrt{n}$, $\forall j$. $\square$

4. A Priori Bounds

We organize the *a priori* bounds by the amount of information that is known about the uncertain parameters in constraint i , that is, assumptions on their means and whether they are subject to a symmetric distribution. Existing *a priori* bounds and new *a priori* bounds proposed in this work are summarized in Tables 7 and 8, and their comparisons are summarized in Table 9.

4.1. Uncertain parameters are bounded and have bounded means

In this case, the uncertain parameters have known lower and upper bounds, and their means are known only to within some range of values, that is, a single expected value cannot be confidently imposed. Instead of the nominal value of $\tilde{a}_{ij}$ representing the mean, yielding $\mathbf{E}[\xi_{ij}] = 0$, the nominal value is chosen such that $|\mathbf{E}[\xi_{ij}]| \leq \mu_{ij}$. To our knowledge, this is the first *a priori* bound that applies to this case.

Theorem 1. *Given that, for all $j \in J_i$, random variable ξ_{ij} is independent and subject to a probability distribution that is supported on the bounded interval*

$[-1, 1]$, and given that $|\mathbf{E}[\xi_{ij}]| \leq \mu_{ij}$, the following probability bound of constraint violation holds for the i th constraint with $\Delta_i > \sum_{k \in K_i} \mu_{ik}$ and any $\theta > 0$:

$$\Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \leq \exp \left(\min_{\theta > 0} \left\{ -\theta \Delta_i + \sum_{k \in K_i} \ln G_{ik}(\theta) \right\} \right) \quad (\text{GMF1})$$

where $G_{ij}(\theta) := \mu_{ij} \sinh(\theta) + \cosh(\theta)$ and K_i contains the indices of the $\lceil \delta_i^{(1)} \rceil$ largest μ_{ij} . For $0 \leq \Delta_i \leq \sum_{k \in K_i} \mu_{ik}$, the bound approaches a probability of 1 as θ approaches 0. The parameter $\delta_i^{(1)}$ depends on U_i and is given in Table 6.

Proof.

$$\begin{aligned} \Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} &\leq e^{-\theta \Delta_i} \prod_{j \in J_i} \mathbf{E}[e^{\theta \delta_{ij} \xi_{ij}}] \\ &\stackrel{1}{\leq} e^{-\theta \Delta_i} \prod_{j \in J_i} \left(\frac{1 - \mathbf{E}[\xi_{ij}]}{2} e^{-\theta \delta_{ij}} + \frac{\mathbf{E}[\xi_{ij}] + 1}{2} e^{\theta \delta_{ij}} \right) \end{aligned}$$

where relation 1 is due to Lemma 1. Note that $\mathbf{E}[\xi_{ij}] \in [-1, 1]$; the term within the product operator is nonnegative. If $\delta_{ij} \geq 0$, then $\exp(\theta \delta_{ij}) \geq \exp(-\theta \delta_{ij})$, $\forall x \geq 0$, and the expression is maximized when $\mathbf{E}[\xi_{ij}] = \mu_{ij}$. If $\delta_{ij} \leq 0$, then $\exp(\theta \delta_{ij}) \leq \exp(-\theta \delta_{ij})$, $\forall x \geq 0$, and the expression is maximized when $\mathbf{E}[\xi_{ij}] = -\mu_{ij}$. Thus,

$$\frac{1 - \mathbf{E}[\xi_{ij}]}{2} e^{-\theta \delta_{ij}} + \frac{\mathbf{E}[\xi_{ij}] + 1}{2} e^{\theta \delta_{ij}} \leq \frac{1 - \mu_{ij}}{2} e^{-\theta |\delta_{ij}|} + \frac{\mu_{ij} + 1}{2} e^{\theta |\delta_{ij}|}. \quad (16)$$

Then,

$$\begin{aligned} e^{-\theta \Delta_i} \prod_{j \in J_i} \left(\frac{1 - \mathbf{E}[\xi_{ij}]}{2} e^{-\theta \delta_{ij}} + \frac{\mathbf{E}[\xi_{ij}] + 1}{2} e^{\theta \delta_{ij}} \right) &\stackrel{2}{\leq} e^{-\theta \Delta_i} \prod_{j \in J_i} \left(\frac{1 - \mu_{ij}}{2} e^{-\theta |\delta_{ij}|} + \frac{\mu_{ij} + 1}{2} e^{\theta |\delta_{ij}|} \right) \\ &= e^{-\theta \Delta_i} \prod_{j \in J_i} (\mu_{ij} \sinh(\theta |\delta_{ij}|) + \cosh(\theta |\delta_{ij}|)) \\ &= e^{-\theta \Delta_i} \prod_{j \in J_i} G_{ij}(\theta |\delta_{ij}|). \end{aligned}$$

where relation 2 is due to (16).

The optimization problem

$$\max_{|\delta_{ij}|} \left\{ \prod_{j \in J_i} G_{ij}(\theta|\delta_{ij}|) : \|\delta_{ij}\|_1 \leq \delta_i^{(1)}; |\delta_{ij}| \in [0, 1], \forall j \in J_i \right\}$$

matches (P1) of Lemma 2. Ordering $G_{ij}(\theta|\delta_{ij}|)$ by decreasing μ_{ij} assures conditions (14) are satisfied:

$$\begin{aligned} & \frac{d \ln G_{i1}(\theta|\delta_{ij}|)}{d|\delta_{ij}|} \geq \frac{d \ln G_{i2}(\theta|\delta_{ij}|)}{d|\delta_{ij}|} \quad \forall |\delta_{ij}| \in [0, 1], \forall \theta \geq 0 \\ \rightarrow & \theta \frac{\mu_{i1} \cosh(\theta\delta_{ij}) + \sinh(\theta\delta_{ij})}{\mu_{i1} \sinh(\theta\delta_{ij}) + \cosh(\theta\delta_{ij})} \\ & \geq \theta \frac{\mu_{i2} \cosh(\theta\delta_{ij}) + \sinh(\theta\delta_{ij})}{\mu_{i2} \sinh(\theta\delta_{ij}) + \cosh(\theta\delta_{ij})} \quad \forall |\delta_{ij}| \in [0, 1], \forall \theta \geq 0 \\ \rightarrow & \mu_{i1} (\cosh^2(\theta\delta_{ij}) - \sinh^2(\theta\delta_{ij})) \\ & \geq \mu_{i2} (\cosh^2(\theta\delta_{ij}) - \sinh^2(\theta\delta_{ij})) \quad \forall |\delta_{ij}| \in [0, 1], \forall \theta \geq 0 \\ \rightarrow & \mu_{i1} \geq \mu_{i2}. \end{aligned}$$

Then,

$$\begin{aligned} & e^{-\theta\Delta_i} \prod_{j \in J_i} G_{ij}(\theta|\delta_{ij}|) \\ & \stackrel{3}{\leq} e^{-\theta\Delta_i} \left(\prod_{k \in K_i \setminus \{\bar{j}\}} G_{ik}(\theta) \right) G_{i\bar{j}} \left(\theta[\delta_i^{(1)} - \lfloor \delta_i^{(1)} \rfloor] \right) \\ & = \exp \left(-\theta\Delta_i + \sum_{k \in K_i \setminus \{\bar{j}\}} \ln G_{ik}(\theta) + \ln G_{i\bar{j}} \left(\theta[\delta_i^{(1)} - \lfloor \delta_i^{(1)} \rfloor] \right) \right) \\ & \stackrel{4}{\leq} \exp \left(-\theta\Delta_i + \sum_{k \in K_i \setminus \{\bar{j}\}} \ln G_{ik}(\theta) + \left(\delta_i^{(1)} - \lfloor \delta_i^{(1)} \rfloor \right) \ln G_{i\bar{j}}(\theta) \right) \\ & \leq \exp \left(-\theta\Delta_i + \sum_{k \in K_i} \ln G_{ik}(\theta) \right) \end{aligned}$$

where relation 3 is due to Lemma 2, and relation 4 is due to convexity of $\ln G_{ij}(\theta)$ and the fact that $(\delta_i^{(1)} - \lfloor \delta_i^{(1)} \rfloor) \in [0, 1]$. The tightest bound is obtained by minimizing the expression over $\theta > 0$.

We can solve for the allowable range of Δ_i by determining when $\theta^* = 0$ after

imposing first-order optimality conditions:

$$\begin{aligned}
\frac{d}{d\theta} \left(-\theta \Delta_i + \sum_{k \in K_i} \ln G_{ik}(\theta) \right) \Big|_{\theta=0} &= \left[-\Delta_i + \sum_{k \in K_i} \frac{\mu_{ik} \cosh(\theta) + \sinh(\theta)}{\mu_{ik} \sinh(\theta) + \cosh(\theta)} \right]_{\theta=0} \\
&= -\Delta_i + \sum_{k \in K_i} \mu_{ik} = 0|_{\theta=\theta^*} \\
\rightarrow \Delta_i &= \sum_{k \in K_i} \mu_{ik}
\end{aligned}$$

□

Example 1. Consider the constraint

$$\widetilde{1}x_1 + \widetilde{1}x_2 \leq \widetilde{20} \quad (17)$$

where parameters $\widetilde{1}$, $\widetilde{1}$, and $\widetilde{20}$ are subject to bounded uncertainty. Each parameter can realize values within 10% of their nominal values 1, 1, and 20, respectively, and can be written as:

$$\begin{aligned}
\widetilde{20} &= 20 + 2\xi_0 \\
\widetilde{1} &= 1 + 0.1\xi_1 \\
\widetilde{1} &= 1 + 0.1\xi_2,
\end{aligned}$$

where random variable $\xi_j \in [-1, 1]$, $j = 0, 1, 2$. Constraint (17) can be reformulated as:

$$\begin{aligned}
x_1 + x_2 + 2\xi_0x_0 + 0.1\xi_1x_1 + 0.1\xi_2x_2 &\leq 20 \\
x_0 &= -1.
\end{aligned} \quad (18)$$

The expected values of the parameters are only known to be within 1% of their nominal values:

$$\begin{aligned}
\mathbf{E}[\widetilde{20}] &\in [19.8, 20.2] \rightarrow \mathbf{E}[\xi_0] \in [-0.1, 0.1] \rightarrow |\mathbf{E}[\xi_0]| \leq 0.1 = \mu_0 \\
\mathbf{E}[\widetilde{1}] &\in [0.99, 1.01] \rightarrow \mathbf{E}[\xi_1] \in [-0.1, 0.1] \rightarrow |\mathbf{E}[\xi_1]| \leq 0.1 = \mu_1 \\
\mathbf{E}[\widetilde{1}] &\in [0.99, 1.01] \rightarrow \mathbf{E}[\xi_2] \in [-0.1, 0.1] \rightarrow |\mathbf{E}[\xi_2]| \leq 0.1 = \mu_2.
\end{aligned}$$

A probability of constraint violation of 0.2 or less will be met using bound (GMF1) and an interval + polyhedral set. Thus, $|K_i| = \lceil \delta_i^{(1)} \rceil = 3$, and $K_i = \{0, 1, 2\}$. The bisection method with bound (GMF1) can be performed to determine the value of Δ_i (that is, Γ) which will guarantee the desired probability of constraint violation (see Section 6 for details). Selecting $\Delta_i = \Gamma = 2.930$

and using bound (GMF1):

$$\exp \left(\min_{\theta > 0} \{-2.930\theta + 3 \ln(0.1 \sinh(\theta) + \cosh(\theta))\} \right) = \exp(-1.61) = 0.20,$$

where $\theta^* = 2.119$. As per Table 5, the robust counterpart of the constraint with an interval + polyhedral set is:

$$\begin{aligned} x_1 + x_2 + p_0 + p_1 + p_2 + 2.930z &\leq 20 \\ z + p_0 &\geq 2 \\ z + p_1 &\geq 0.1|x_1| \\ z + p_2 &\geq 0.1|x_2| \\ z, p_0, p_1, p_2 &\geq 0. \end{aligned} \tag{19}$$

Corollary 1.1. *For ellipsoidal and interval + ellipsoidal uncertainty sets, the following probability bound of constraint violation holds for the i th constraint with $\Delta_i > \bar{\mu}_i \sqrt{|J_i|}$ and any $\theta > 0$:*

$$\begin{aligned} \Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \\ \leq \exp \left(\min_{\theta > 0} \left\{ -\theta \Delta_i + |J_i| \ln \bar{G}_i \left(\theta / \sqrt{|J_i|} \right) \right\} \right) \quad (\text{GMF1}') \end{aligned}$$

where $\bar{\mu}_i = \max_{j \in J_i} \mu_{ij}$ and $\bar{G}_i(\theta) := \bar{\mu}_i \sinh(\theta) + \cosh(\theta)$. For $0 \leq \Delta_i \leq \bar{\mu}_i \sqrt{|J_i|}$, the bound approaches a probability of 1 as θ approaches 0.

Proof.

$$\begin{aligned} \Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \\ \leq e^{-\theta \Delta_i} \prod_{j \in J_i} \mathbf{E}[e^{\theta \delta_{ij} \xi_{ij}}] \\ \stackrel{1}{\leq} e^{-\theta \Delta_i} \prod_{j \in J_i} \left(\frac{1 - \mathbf{E}[\xi_{ij}]}{2} e^{-\theta \delta_{ij}} + \frac{\mathbf{E}[\xi_{ij}] + 1}{2} e^{\theta \delta_{ij}} \right) \\ \stackrel{2}{\leq} e^{-\theta \Delta_i} \prod_{j \in J_i} \left(\frac{1 - \bar{\mu}_i}{2} e^{-\theta |\delta_{ij}|} + \frac{\bar{\mu}_i + 1}{2} e^{\theta |\delta_{ij}|} \right) \\ = e^{-\theta \Delta_i} \prod_{j \in J_i} \bar{G}_i(\theta |\delta_{ij}|) \\ = \exp \left(-\theta \Delta_i + \sum_{j \in J_i} \ln \bar{G}_i(\theta |\delta_{ij}|) \right) \end{aligned}$$

where relation 1 is due to Lemma 1, and relation 2 is to (16).

The optimization prob

$$\max_{|\delta_{ij}|} \left\{ \sum_{j \in J_i} \ln \bar{G}_i(\theta |\delta_{ij}|) : \|\delta_{ij}\|_2 \leq \delta_i^{(2)}; |\delta_{ij}| \in [0, 1], \forall j \in J_i \right\}$$

matches problem (P2) of Lemma 3. The conditions of Lemma 3 require concavity of $\bar{G}_i(\theta \sqrt{|\delta_{ij}|})$. With $z_{ij} := 2\theta \sqrt{|\delta_{ij}|}$, it must hold that

$$\begin{aligned} & \frac{d^2 \bar{G}_i(\theta \sqrt{|\delta_{ij}|})}{d|\delta_{ij}|} \leq 0 \quad \forall |\delta_{ij}| \in [0, 1], \forall \theta \geq 0 \\ \rightarrow & -\theta \left[(2 + \bar{\mu}_i^2) \sinh(z_{ij}) + 2\bar{\mu}_i \cosh(z_{ij}) \right. \\ & \left. - (1 - \bar{\mu}_i^2) z_{ij} \right] / \left[8|\delta_{ij}|^{3/2} \left(\bar{G}_i(\theta \sqrt{|\delta_{ij}|}) \right)^2 \right] \leq 0 \quad \forall |\delta_{ij}| \in [0, 1], \forall \theta \geq 0 \\ \rightarrow & (2 + \bar{\mu}_i^2) \sinh(z_{ij}) + 2\bar{\mu}_i \cosh(z_{ij}) - (1 - \bar{\mu}_i^2) z_{ij} \geq 0 \quad \forall z_{ij} \geq 0 \\ \rightarrow & \frac{(1 + \bar{\mu}_i)^2}{2} e^{z_{ij}} - \frac{(1 - \bar{\mu}_i)^2}{2} e^{-z_{ij}} - (1 - \bar{\mu}_i^2) z_{ij} \geq 0 \quad \forall z_{ij} \geq 0. \end{aligned}$$

The final relation is nondecreasing as $\bar{\mu}_i$ increases for any z_{ij} , and is thus minimized at $\bar{\mu}_i = 0$:

$$\sinh(z_{ij}) - z_{ij} \geq 0 \quad \bar{\mu}_i = 0, \forall z_{ij} \geq 0$$

which holds true.

Thus,

$$\exp \left(-\theta \Delta_i + \sum_{j \in J_i} \ln \bar{G}_i(\theta |\delta_{ij}|) \right) \stackrel{3}{\leq} \exp \left(-\theta \Delta_i + |J_i| \bar{G}_i(\theta / \sqrt{|J_i|}) \right)$$

where relation 3 is due to Lemma 3. The tightest bound is obtained by minimizing the expression over $\theta > 0$. As in Theorem 1, first-order optimality conditions yield the ranges of Δ_i . $\square$

Example 2. Consider the constraint

$$\tilde{1}x_1 + \tilde{1}x_2 \leq \widetilde{20} \quad (20)$$

where parameters $\tilde{1}$, $\tilde{1}$, and $\widetilde{20}$ are subject to bounded uncertainty. Each parameter can realize values within 10% of their nominal values 1, 1, and 20, respectively, and can be written as:

$$\begin{aligned} \widetilde{20} &= 20 + 2\xi_0 \\ \tilde{1} &= 1 + 0.1\xi_1 \\ \tilde{1} &= 1 + 0.1\xi_2, \end{aligned}$$

where random variable $\xi_j \in [-1, 1]$, $j = 0, 1, 2$. Constraint (20) can be reformulated as:

$$\begin{aligned} x_1 + x_2 + 2\xi_0 x_0 + 0.1\xi_1 x_1 + 0.1\xi_2 x_2 &\leq 20 \\ x_0 &= -1. \end{aligned} \quad (21)$$

The expected values of the parameters are only known to be within 1% of their nominal values:

$$\begin{aligned} \mathbf{E}[\widetilde{20}] &\in [19.8, 20.2] \rightarrow \mathbf{E}[\xi_0] \in [-0.1, 0.1] \rightarrow |\mathbf{E}[\xi_0]| \leq 0.1 = \mu_0 \\ \mathbf{E}[\widetilde{1}] &\in [0.99, 1.01] \rightarrow \mathbf{E}[\xi_1] \in [-0.1, 0.1] \rightarrow |\mathbf{E}[\xi_1]| \leq 0.1 = \mu_1 \\ \mathbf{E}[\widetilde{1}] &\in [0.99, 1.01] \rightarrow \mathbf{E}[\xi_2] \in [-0.1, 0.1] \rightarrow |\mathbf{E}[\xi_2]| \leq 0.1 = \mu_2. \end{aligned}$$

A probability of constraint violation of 0.2 or less will be met using bound (GMF1') and an interval + ellipsoidal set. The bisection method with bound (GMF1') can be performed to determine the value of Δ_i (that is, Ω) which will guarantee the desired probability of constraint violation (see Section 6 for details). Selecting $\Delta_i = \Omega = 1.692$ and using bound (GMF1'):

$$\begin{aligned} \exp\left(\min_{\theta > 0} \left\{ -1.692\theta + 3 \ln \left(0.1 \sinh(\theta/\sqrt{3}) + \cosh(\theta/\sqrt{3}) \right) \right\}\right) \\ = \exp(-1.61) = 0.20, \end{aligned}$$

where $\theta^* = 3.679$. As per Table 5, the robust counterpart of the constraint with an interval + ellipsoidal set is:

$$\begin{aligned} x_1 + x_2 + 2|-1 - z_0| + 0.1|x_1 - z_1| \\ + 0.1|x_2 - z_2| + 1.692\sqrt{4z_0^2 + 0.01z_1^2 + 0.01z_2^2} \leq 20. \end{aligned} \quad (22)$$

4.2. Uncertain parameters are bounded and have known means

In this case, the uncertain parameters have known lower and upper bounds, and their means are known. The bounds in this section are included in Table 7. Without loss of generality, we assume uncertain parameter $\tilde{a}_{ij}$ is represented as in (5) with $\xi_{ij} \in [-1, 1]$ and $\mathbf{E}[\xi_{ij}] = 0$. Existing bounds for this case are provided by Ben-Tal and Nemirovski (2000) and Bertsimas and Sim (2004), respectively:

$$\Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \leq \exp \left(\frac{-\Delta_i^2}{2} \right), \quad (\text{BN})$$

$$\Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \leq \exp \left(\frac{-\Delta_i^2}{2|J_i|} \right). \quad (\text{BS1})$$

Bound (BN) was derived for interval + ellipsoidal uncertainty sets and can also be applied to ellipsoidal and box uncertainty sets but not polyhedral and interval

+ polyhedral uncertainty sets (Li et al., 2012). Bound (BS1) was derived for interval + polyhedral uncertainty sets and can be applied to the other four sets as well (Li et al., 2012), though it is looser than (BN) for box, ellipsoidal, and interval + ellipsoidal sets. While the derivations of these two bounds utilized the conditions that the underlying probability distributions must be symmetric, we note that they can both be derived using Hoeffding's inequality (Hoeffding, 1963) and thus do not require symmetry as a condition.

Theorem 2. *Given that, for all $j \in J_i$, random variable ξ_{ij} is independent and subject to a probability distribution that is supported on the bounded interval $[-1, 1]$ with $\mathbf{E}[\xi_{ij}] = 0$, the following probability bound of constraint violation holds for the i th constraint:*

$$\Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \leq (\alpha^\alpha \beta^\beta)^{-\delta_i^{(1)}/2} \quad (\text{GMF2})$$

where $\alpha := 1 - \Delta_i/\delta_i^{(1)}$ and $\beta := 1 + \Delta_i/\delta_i^{(1)}$. The parameter $\delta_i^{(1)}$ depends on U_i and is given in Table 6.

Proof. This is a special case of Theorem 1 with $\mu_{ij} = 0, \forall j \in J_i$. Starting from the penultimate relation in Theorem 1,

$$\begin{aligned} & \Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \\ & \leq \exp \left(-\theta \Delta_i + \sum_{k \in K_i \setminus \{\bar{j}\}} \ln G_{ik}(\theta) + \left(\delta_i^{(1)} - \lfloor \delta_i^{(1)} \rfloor \right) \ln G_{i\bar{j}}(\theta) \right) \\ & = \exp \left(-\theta \Delta_i + \lfloor \delta_i^{(1)} \rfloor \ln \cosh(\theta) + \left(\delta_i^{(1)} - \lfloor \delta_i^{(1)} \rfloor \right) \ln \cosh(\theta) \right) \\ & = \exp \left(-\theta \Delta_i + \delta_i^{(1)} \ln \cosh(\theta) \right). \end{aligned}$$

The tightest bound is obtained by minimizing the expression over $\theta > 0$. By first-order optimality conditions, the following must hold:

$$\theta^* = \text{artanh}(\Delta_i/\delta_i^{(1)}) = \frac{1}{2} \ln(1 + \Delta_i/\delta_i^{(1)}) - \frac{1}{2} \ln(1 - \Delta_i/\delta_i^{(1)}).$$

Thus,

$$\begin{aligned}
& \exp \left(\min_{\theta > 0} \left\{ -\theta \Delta_i + \delta_i^{(1)} \ln \cosh(\theta) \right\} \right) \\
&= \exp \left(-\frac{1}{2} \Delta_i \ln \left(1 + \Delta_i / \delta_i^{(1)} \right) + \frac{1}{2} \Delta_i \ln \left(1 - \Delta_i / \delta_i^{(1)} \right) \right. \\
&\quad \left. - \frac{1}{2} \delta_i^{(1)} \ln \left(1 + \Delta_i / \delta_i^{(1)} \right) - \frac{1}{2} \delta_i^{(1)} \ln \left(1 - \Delta_i / \delta_i^{(1)} \right) \right) \\
&= \exp \left(\ln \left(1 + \Delta_i / \delta_i^{(1)} \right)^{-\frac{1}{2} \delta_i^{(1)} (1 + \Delta_i / \delta_i^{(1)})} \right. \\
&\quad \left. + \ln \left(1 - \Delta_i / \delta_i^{(1)} \right)^{-\frac{1}{2} \delta_i^{(1)} (1 - \Delta_i / \delta_i^{(1)})} \right) \\
&= \left(\left(1 + \Delta_i / \delta_i^{(1)} \right)^{(1 + \Delta_i / \delta_i^{(1)})} \left(1 - \Delta_i / \delta_i^{(1)} \right)^{(1 - \Delta_i / \delta_i^{(1)})} \right)^{-\delta_i^{(1)} / 2}.
\end{aligned}$$

□

Example 3. Consider the constraint

$$\widetilde{1}x_1 + \widetilde{2}x_2 \leq \widetilde{12} \quad (23)$$

where parameters $\widetilde{1}$, $\widetilde{2}$, and $\widetilde{12}$ are subject to bounded uncertainty. Each parameter can realize values within 10% of their nominal values 1, 2, and 12, respectively, and can be written as:

$$\begin{aligned}
\widetilde{12} &= 12 + 1.2\xi_0 \\
\widetilde{1} &= 1 + 0.1\xi_1 \\
\widetilde{2} &= 2 + 0.2\xi_2,
\end{aligned}$$

where random variable $\xi_j \in [-1, 1]$, $j = 0, 1, 2$. Constraint (23) can be reformulated as:

$$\begin{aligned}
x_1 + 2x_2 + 1.2\xi_0x_0 + 0.1\xi_1x_1 + 0.2\xi_2x_2 &\leq 12 \\
x_0 &= -1.
\end{aligned} \quad (24)$$

The nominal value of each parameter was chosen as its expected value; therefore, $\mathbf{E}[\xi_j] = 0$, $j = 0, 1, 2$. A probability of constraint violation of 0.2 or less will be met using bound (GMF2) and an interval + polyhedral set. Thus, $\delta_i^{(1)} = 3$. The bisection method with bound (GMF2) can be performed to determine the value of Δ_i (that is, Γ), which will guarantee the desired probability of constraint violation (see Section 6 for details). Selecting $\Delta_i = \Gamma = 2.781$ and

using bound (GMF2):

$$\begin{aligned}\alpha &= 1 - \frac{2.781}{3} = 0.073 \\ \beta &= 1 + \frac{2.781}{3} = 1.927 \\ \left[(0.073)^{(0.073)} (1.927)^{(1.927)} \right]^{-3/2} &= 0.20.\end{aligned}$$

As per Table 5, the robust counterpart of the constraint with an interval + polyhedral set is:

$$\begin{aligned}x_1 + 2x_2 + p_0 + p_1 + p_2 + 2.781z &\leq 12 \\ z + p_0 &\geq 1.2 \\ z + p_1 &\geq 0.1|x_1| \\ z + p_2 &\geq 0.2|x_2| \\ z, p_0, p_1, p_2 &\geq 0.\end{aligned}\tag{25}$$

Corollary 2.1. *For ellipsoidal and interval + ellipsoidal uncertainty sets, the following probability bound of constraint violation holds for the i th constraint:*

$$\Pr \left\{ \sum_j a_{ij}x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j > b_i \right\} \leq (\alpha^\alpha \beta^\beta)^{-|J_i|/2} \tag{GMF2'}$$

where $\alpha := 1 - \Delta_i/\delta_i^{(1)}$ and $\beta := 1 + \Delta_i/\delta_i^{(1)}$. The parameter $\delta_i^{(1)}$ depends on U_i and is given in Table 6.

Proof. This is a special case of Corollary 1.1 with $\bar{\mu}_i = 0$. Starting from (GMF1'),

$$\begin{aligned}\Pr \left\{ \sum_j a_{ij}x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j > b_i \right\} \\ \leq \exp \left(\min_{\theta > 0} \left\{ -\theta \Delta_i + |J_i| \ln \bar{G}_i \left(\theta / \sqrt{|J_i|} \right) \right\} \right) \\ = \exp \left(\min_{\theta > 0} \left\{ -\theta \Delta_i + |J_i| \ln \cosh \left(\theta / \sqrt{|J_i|} \right) \right\} \right).\end{aligned}$$

By first-order optimality conditions, the following must hold:

$$\begin{aligned}\theta^* &= \sqrt{|J_i|} \operatorname{artanh}(\Delta_i / \sqrt{|J_i|}) = \sqrt{|J_i|} \operatorname{artanh}(\Delta_i / \delta_i^{(1)}) \\ &= \frac{1}{2} \sqrt{|J_i|} \ln(1 + \Delta_i / \delta_i^{(1)}) - \frac{1}{2} \sqrt{|J_i|} \ln(1 - \Delta_i / \delta_i^{(1)}).\end{aligned}$$

Thus,

$$\begin{aligned}
& \exp \left(\min_{\theta > 0} \left\{ -\theta \Delta_i + |J_i| \ln \cosh(\theta / \sqrt{|J_i|}) \right\} \right) \\
&= \exp \left(-\frac{1}{2} \sqrt{|J_i|} \Delta_i \ln \left(1 + \Delta_i / \delta_i^{(1)} \right) + \frac{1}{2} \sqrt{|J_i|} \Delta_i \ln \left(1 - \Delta_i / \delta_i^{(1)} \right) \right. \\
&\quad \left. - \frac{1}{2} |J_i| \ln \left(1 + \Delta_i / \delta_i^{(1)} \right) - \frac{1}{2} |J_i| \ln \left(1 - \Delta_i / \delta_i^{(1)} \right) \right) \\
&= \exp \left(\ln \left(1 + \Delta_i / \delta_i^{(1)} \right)^{-\frac{1}{2} |J_i| (1 + \Delta_i / \delta_i^{(1)})} \right. \\
&\quad \left. + \ln \left(1 - \Delta_i / \delta_i^{(1)} \right)^{-\frac{1}{2} |J_i| (1 - \Delta_i / \delta_i^{(1)})} \right) \\
&= \left(\left(1 + \Delta_i / \delta_i^{(1)} \right)^{(1 + \Delta_i / \delta_i^{(1)})} \left(1 - \Delta_i / \delta_i^{(1)} \right)^{(1 - \Delta_i / \delta_i^{(1)})} \right)^{-|J_i|/2}.
\end{aligned}$$

□

Example 4. Consider the constraint

$$\widetilde{1}x_1 + \widetilde{2}x_2 \leq \widetilde{12} \quad (26)$$

where parameters $\widetilde{1}$, $\widetilde{2}$, and $\widetilde{12}$ are subject to bounded uncertainty. Each parameter can realize values within 10% of their nominal values 1, 2, and 12, respectively, and can be written as:

$$\begin{aligned}
\widetilde{12} &= 12 + 1.2\xi_0 \\
\widetilde{1} &= 1 + 0.1\xi_1 \\
\widetilde{2} &= 2 + 0.2\xi_2,
\end{aligned}$$

where random variable $\xi_j \in [-1, 1]$, $j = 0, 1, 2$. Constraint (26) can be reformulated as:

$$\begin{aligned}
x_1 + 2x_2 + 1.2\xi_0x_0 + 0.1\xi_1x_1 + 0.2\xi_2x_2 &\leq 12 \\
x_0 &= -1.
\end{aligned} \quad (27)$$

where random variable $\xi_j \in [-1, 1]$, $j = 0, 1, 2$. The nominal value of each parameter was chosen as its expected value; therefore, $\mathbf{E}[\xi_j] = 0$, $j = 0, 1, 2$. A probability of constraint violation of 0.2 or less will be met using bound (GMF2') and an interval + ellipsoidal set. Thus, $\delta_i^{(1)} = \sqrt{3}$. The bisection method with bound (GMF2') can be performed to determine the value of Δ_i (that is, Ω) which will guarantee the desired probability of constraint violation

(see Section 6 for details). Selecting $\Delta_i = \Omega = 1.606$ and using bound (GMF2'):

$$\begin{aligned}\alpha &= 1 - \frac{1.606}{\sqrt{3}} = 0.073 \\ \beta &= 1 + \frac{1.606}{\sqrt{3}} = 1.927 \\ \left[(0.073)^{(0.073)} (1.927)^{(1.927)} \right]^{-3/2} &= 0.20.\end{aligned}$$

As per Table 5, the robust counterpart of the constraint with an interval + ellipsoidal set is:

$$\begin{aligned}x_1 + 2x_2 + 1.2|-1 - z_0| + 0.1|x_1 - z_1| \\ + 0.2|x_2 - z_2| + 1.606\sqrt{1.44z_0^2 + 0.01z_1^2 + 0.04z_2^2} \leq 12.\end{aligned}\quad (28)$$

Remark 1. For box uncertainty sets (where (BN) is as tight as or tighter than (BS1)), bound (GMF2) is as tight as or tighter than (BN). As $\Delta_i \rightarrow 1$, bound (GMF2) approaches 2^{-1} as opposed to $\sqrt{e}^{-1}$ ($\sqrt{e} \approx 1.65$).

Remark 2. For ellipsoidal uncertainty sets (where (BN) is as tight as or tighter than (BS1)), bound (GMF2') is tighter than (BN). As $\Delta_i \rightarrow \sqrt{|J_i|}$, bound (GMF2) approaches $2^{-|J_i|}$ as opposed to $\sqrt{e}^{-|J_i|}$.

Remark 3. For polyhedral uncertainty sets (where only (BS1) applies), bound (GMF2) is as tight as or tighter than (BS1). As $\Delta_i \rightarrow |J_i|$, bound (GMF2) approaches $2^{-|J_i|}$ as opposed to $\sqrt{e}^{-|J_i|}$.

4.3. Uncertain parameters are bounded and symmetric

In this case, the uncertain parameters have known lower and upper bounds and are assumed to be subject to symmetric probability distributions. The bounds of this section are included in Table 8. The bounds of the previous section (where symmetry was not assumed) apply to this case, as does the following bound provided by Bertsimas and Sim (2004) derived for interval + polyhedral uncertainty sets with $\Gamma_i \geq 1$:

$$\begin{aligned}\Pr \left\{ \sum_j a_{ij}x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j \geq b_i \right\} \\ \leq \frac{1}{2^n} \left[(1 - (\nu - \lfloor \nu \rfloor)) \binom{n}{\lfloor \nu \rfloor} + \sum_{k=\lfloor \nu \rfloor+1}^n \binom{n}{k} \right]\end{aligned}\quad (\text{BS2})$$

where $\nu := (\Gamma_i + n)/2$ and $n := |J_i|$. Li et al. (2012) noted that (BS2) can also apply to polyhedral, ellipsoidal, interval + ellipsoidal, and box uncertainty sets, though it has no utility with box uncertainty sets due to it providing a

probability bound of constraint violation only at $\Psi_i = 1$ with a value of 0.5, which is the worst-case model for bounded uncertainty where the probability of constraint violation is 0.

None of the aforementioned *a priori* bounds successfully incorporate the fact that an upper bound on the probability of constraint violation in the nominal case ($\Delta_i = 0$) with bounded and symmetric uncertain parameters should be 0.5. The sum of random variables $\sum_{j \in J_i} \delta_{ij} \xi_{ij}$ can be considered a single, new random variable Ξ_i with an expected value of 0 and a symmetric probability distribution. Then, $\Pr\{\Xi_i > 0\}$ is at most 0.5, corresponding to the case where $\Pr\{\Xi_i = 0\} = 0$. A higher upper bound is only possible if Ξ_i (that is, any ξ_{ij}) is not symmetric about 0. Presented below, bound (GMF3) is as tight as or tighter than (BS2) for $\Delta_i \geq 1$, applies to all $\Delta_i \geq 0$ while (BS2) only applies to $\Delta_i \geq 1$, and returns 0.5 as the probability bound of constraint violation at $\Delta_i = 0$.

Theorem 3. *Given that, for all $j \in J_i$, random variable ξ_{ij} is independent and subject to a symmetric probability distribution that is supported on the bounded interval $[-1, 1]$ with $\mathbf{E}[\xi_{ij}] = 0$, the following probability bound of constraint violation holds for the i th constraint:*

$$\Pr \left\{ \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j > b_i \right\} \leq \frac{1}{2^{\tilde{n}}} \sum_{k=\lceil \tilde{\nu} \rceil}^{\tilde{n}} \binom{\tilde{n}}{k} \quad (\text{GMF3})$$

where $\tilde{\nu} := (\Delta_i + \tilde{n})/2$ and

$$\tilde{n} := \begin{cases} |J_i| + 1 - \text{sgn } \Delta_i & \text{if } \lceil \Delta_i \rceil + |J_i| \text{ is even} \\ |J_i| - \text{sgn } \Delta_i & \text{if } \lceil \Delta_i \rceil + |J_i| \text{ is odd.} \end{cases}$$

Proof. Denote a Rademacher random variable as $\xi^{(\text{Ra})}$ (that is, $\Pr\{\xi^{(\text{Ra})} = -1\} = 0.5$ and $\Pr\{\xi^{(\text{Ra})} = 1\} = 0.5$), a Bernoulli random variable with $p = 0.5$ as $\xi^{(\text{Be})}$, and a binomial random variable with $p = 0.5$ as $\xi^{(\text{Bi})}$. Each product $\delta_{ij} \xi_{ij}$ can be considered a new symmetric random variable ξ'_{ij} which is supported on $[-1, 1]$ and has an expected value of 0. First, for $\Delta_i > 0$,

$$\begin{aligned}
\Pr \left\{ \sum_{j \in J_i} \delta_{ij} \xi_{ij} > \Delta_i \right\} &= \Pr \left\{ \sum_{j \in J_i} \xi'_{ij} > \Delta_i \right\} \\
&\leq \Pr \left\{ \sum_{j \in J_i} \xi'_{ij} \geq \Delta_i \right\} \\
&\stackrel{1}{\leq} \begin{cases} \Pr \left\{ \sum_{j=1}^{|J_i|} \xi_{ij}^{(\text{Ra})} \geq \Delta_i \right\} & \text{if } \lceil \Delta_i \rceil + |J_i| \text{ is even} \\ \Pr \left\{ \sum_{j=1}^{|J_i|-1} \xi_{ij}^{(\text{Ra})} \geq \Delta_i \right\} & \text{if } \lceil \Delta_i \rceil + |J_i| \text{ is odd} \end{cases} \\
&\stackrel{2}{=} \Pr \left\{ \sum_{j=1}^{\tilde{n}} \xi_{ij}^{(\text{Ra})} \geq \Delta_i \right\} \\
&\stackrel{3}{=} \Pr \left\{ \sum_{j=1}^{\tilde{n}} \left(2\xi_{ij}^{(\text{Be})} - 1 \right) \geq \Delta_i \right\} \\
&= \Pr \left\{ \sum_{j=1}^{\tilde{n}} \xi_{ij}^{(\text{Be})} \geq \tilde{\nu} \right\} \\
&\stackrel{4}{=} \Pr \left\{ \xi_i^{(\text{Bi})} \geq \tilde{\nu} \right\} \\
&= \Pr \left\{ \xi_i^{(\text{Bi})} = \tilde{\nu} \right\} + \Pr \left\{ \xi_i^{(\text{Bi})} > \tilde{\nu} \right\} \\
&= \Pr \left\{ \xi_i^{(\text{Bi})} = \tilde{\nu} \right\} + 1 - \Pr \left\{ \xi_i^{(\text{Bi})} \leq \tilde{\nu} \right\} \\
&= \Pr \left\{ \xi_i^{(\text{Bi})} = \tilde{\nu} \right\} + 1 - \frac{1}{2^{\tilde{n}}} \sum_{k=0}^{\lfloor \tilde{\nu} \rfloor} \binom{\tilde{n}}{k} \\
&= \Pr \left\{ \xi_i^{(\text{Bi})} = \tilde{\nu} \right\} + \frac{1}{2^{\tilde{n}}} \sum_{k=\lfloor \tilde{\nu} \rfloor + 1}^{\tilde{n}} \binom{\tilde{n}}{k}
\end{aligned}$$

where relation 1 is due to Theorem 1.1 in (Dzindzalieta et al., 2012), relation 2 is by definition of $\tilde{n}$ with $\text{sgn } \Delta_i = 1$, relation 3 is due to the relationship between Bernoulli and Rademacher random variables, and relation 4 is due to the relationship between a sum of i.i.d. Bernoulli random variables and the binomial distribution. Note that, in the last expression, the first term, $\Pr \left\{ \xi_i^{(\text{Bi})} = \tilde{\nu} \right\}$, is only defined for integer $\tilde{\nu}$. If $\tilde{\nu}$ is not integer, then the first

term is zero and

$$\frac{1}{2^{\tilde{n}}} \sum_{k=\lfloor \tilde{\nu} \rfloor + 1}^{\tilde{n}} \binom{\tilde{n}}{k} = \frac{1}{2^{\tilde{n}}} \sum_{k=\lceil \tilde{\nu} \rceil}^{\tilde{n}} \binom{\tilde{n}}{k}.$$

Otherwise, if $\tilde{\nu}$ is integer,

$$\begin{aligned} \Pr \left\{ \xi_i^{(\text{Bi})} = \tilde{\nu} \right\} + \frac{1}{2^{\tilde{n}}} \sum_{k=\lfloor \tilde{\nu} \rfloor + 1}^{\tilde{n}} \binom{\tilde{n}}{k} &= \frac{1}{2^{\tilde{n}}} \binom{\tilde{n}}{\tilde{\nu}} + \frac{1}{2^{\tilde{n}}} \sum_{k=\tilde{\nu}+1}^{\tilde{n}} \binom{\tilde{n}}{k} \\ &= \frac{1}{2^{\tilde{n}}} \sum_{k=\tilde{\nu}}^{\tilde{n}} \binom{\tilde{n}}{k} \\ &= \frac{1}{2^{\tilde{n}}} \sum_{k=\lceil \tilde{\nu} \rceil}^{\tilde{n}} \binom{\tilde{n}}{k}. \end{aligned}$$

Finally, we will show that the bound equals 0.5 for $\Delta_i \in [0, 1]$. First note that, regardless of Δ_i , $\tilde{n} \geq 1$ for nonempty J_i , and recall the identity

$$\sum_{k=0}^n \binom{n}{k} = 2^n. \quad (29)$$

For odd n , (29) contains an even number of terms which have a 1-to-1 correspondence for $k < n/2$ and $k > n/2$, i.e.,

$$\binom{n}{k_1} = \binom{n}{k_2} \quad k_1 = n - k_2, \quad k_2 = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

Say $\Delta_i \in (0, 1]$ ($\text{sgn } \Delta_i = 1$). Then $\tilde{n}$ is always odd:

$$\begin{aligned} |J_i| \text{ even} &\rightarrow 1 + |J_i| \text{ odd} \rightarrow \tilde{n} = |J_i| - 1 \text{ odd} \\ |J_i| \text{ odd} &\rightarrow 1 + |J_i| \text{ even} \rightarrow \tilde{n} = |J_i| \text{ odd}, \end{aligned}$$

and

$$\lceil \tilde{\nu} \rceil = \left\lceil \frac{\Delta_i}{2} + \frac{\tilde{n}}{2} \right\rceil = \left\lceil \frac{\tilde{n}}{2} \right\rceil.$$

Thus,

$$\frac{1}{2^{\tilde{n}}} \sum_{k=\lceil \frac{\tilde{n}}{2} \rceil}^{\tilde{n}} \binom{\tilde{n}}{k} = \frac{1}{2^{\tilde{n}}} \left(\frac{1}{2} 2^{\tilde{n}} \right) = \frac{1}{2}.$$

Say $\Delta_i = 0$ ($\text{sgn } \Delta_i = 0$). Then $\tilde{n}$ is always odd:

$$\begin{aligned} |J_i| \text{ even} &\rightarrow 0 + |J_i| \text{ even} \rightarrow \tilde{n} = |J_i| + 1 \text{ odd} \\ |J_i| \text{ odd} &\rightarrow 0 + |J_i| \text{ odd} \rightarrow \tilde{n} = |J_i| \text{ odd}, \end{aligned}$$

and

$$\lceil \tilde{\nu} \rceil = \left\lceil \frac{\tilde{n}}{2} \right\rceil.$$

Once again,

$$\frac{1}{2^{\tilde{n}}} \sum_{k=\lceil \frac{\tilde{n}}{2} \rceil}^{\tilde{n}} \binom{\tilde{n}}{k} = \frac{1}{2}.$$

□

Example 5. Consider the constraint

$$\tilde{1}x_1 - \tilde{1}x_2 \leq -\tilde{4} \quad (30)$$

where parameters $\tilde{1}$, $-\tilde{1}$, and $-\tilde{4}$ are subject to bounded and symmetric uncertainty. Each parameter can realize values within 10% of their nominal values 1, -1 , and -4 , respectively, and can be written as:

$$\begin{aligned} -\tilde{4} &= -4 + 0.4\xi_0 \\ \tilde{1} &= 1 + 0.1\xi_1 \\ -\tilde{1} &= -1 + 0.1\xi_2, \end{aligned}$$

where random variable $\xi_j \in [-1, 1]$, $j = 0, 1, 2$. Constraint (30) can be reformulated as:

$$\begin{aligned} x_1 - x_2 + 0.4\xi_0x_0 + 0.1\xi_1x_1 + 0.1\xi_2x_2 &\leq -4 \\ x_0 &= -1. \end{aligned} \quad (31)$$

where random variable $\xi_j \in [-1, 1]$, $j = 0, 1, 2$. The nominal value of each parameter was chosen as its expected value; therefore, ξ_j , $j = 0, 1, 2$, is symmetric about 0. A probability of constraint violation of 0.2 or less will be met using bound (GMF3) and an interval + polyhedral set. The bisection method with bound (GMF3) can be performed to determine the value of Δ_i (that is, Γ) which will guarantee the desired probability of constraint violation (see Section 6 for details). Selecting $\Delta_i = \Gamma = 2.001$ and using bound (GMF3):

$$\begin{aligned} \lceil 2.001 \rceil + 3 &= 6 \text{ (even)} \\ \tilde{n} &= 3 + 1 - 1 = 3 \\ \tilde{\nu} &= \frac{2.001 + 3}{2} = 2.5005 \\ \frac{1}{2^{\tilde{n}}} \sum_{k=\lceil 2.5005 \rceil}^3 \binom{3}{k} &= \frac{1}{8} \binom{3}{3} = 0.125. \end{aligned}$$

Note that with $\Gamma = 2.000$, the probability bound of constraint violation is 0.25, which is larger than ϵ_i^{prio} . As per Table 5, the robust counterpart of the

constraint with an interval + polyhedral set is:

$$\begin{aligned}
x_1 - x_2 + p_0 + p_1 + p_2 + 2.001z &\leq -4 \\
z + p_0 &\geq 0.4 \\
z + p_1 &\geq 0.1|x_1| \\
z + p_2 &\geq 0.1|x_2| \\
z, p_0, p_1, p_2 &\geq 0.
\end{aligned} \tag{32}$$

As proposed by Bertsimas and Sim (2004), the term $1/2^n$ and the binomial coefficient pose computational difficulties at very large $|J_i|$, but can be replaced by the function $C(n, k)$, defined below, with a slight loss of tightness:

$$\begin{aligned}
\Pr \left\{ \sum_j a_{ij}x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j > b_i \right\} &\leq \frac{1}{2^n} \sum_{k=\lceil \tilde{\nu} \rceil}^{\tilde{n}} \binom{\tilde{n}}{k} \\
&\leq \sum_{k=\lceil \tilde{\nu} \rceil}^{\tilde{n}} C(\tilde{n}, k)
\end{aligned} \tag{GMF3'}$$

where

$$C(n, k) = \begin{cases} \frac{1}{2^n} & \text{if } k = 0 \text{ or } k = n \\ \sqrt{\frac{n}{2\pi(n-k)k}} \exp \left(n \ln \left(\frac{n}{2(n-k)} \right) + k \ln \left(\frac{n-k}{k} \right) \right) & \text{otherwise.} \end{cases} \tag{33}$$

The discussion in Section 4 included situations where, based on the mathematical structure of the bounds, it can be seen that certain bounds will always be better than other bounds. A more comprehensive overview is given in Table 9.

5. *A Posteriori* Bounds

Deriving an upper bound on the probability of constraint violation for a given solution (i.e., an *a posteriori* bound) is easier than deriving one for *all* possible solutions (i.e., an *a priori* bound). In principle, an *a posteriori* bound can follow from the derivation of an *a priori* bound with specific solution values as opposed to handling all possible values. The *a posteriori* bound for solution x^* in constraint i ,

$$\Pr \left\{ \sum_j a_{ij}x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j^* > b_i \right\} \leq \epsilon_i^{post},$$

is equivalent to

$$\Pr \left\{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j^* > h_i(x^*) \right\} \leq \epsilon_i^{post} \tag{34}$$

where $h_i(x) := b_i - \sum_j a_{ij}x_j$.

We first present the *a posteriori* analogue to bound (GMF1) for when $\mathbf{E}[\xi_{ij}]$ is not known exactly.

Theorem 4. *Given that, for all $j \in J_i$, random variable ξ_{ij} is independent and subject to a probability distribution that is supported on the bounded interval $[-1, 1]$, and given that $|\mathbf{E}[\xi_{ij}]| \leq \mu_{ij}$, the following probability bound of constraint violation holds for the i th constraint and any $\theta > 0$:*

$$\Pr \left\{ \sum_j a_{ij}x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j^* > b_i \right\} \leq \exp \left(\min_{\theta > 0} \left\{ -\theta h_i(x^*) + \sum_{j \in J_i} \ln G_{ij}(|\hat{a}_{ij}x_j^*|\theta) \right\} \right). \quad (\text{GMF4})$$

where $G_{ij}(\theta) := \mu_{ij} \sinh(\theta) + \cosh(\theta)$.

Proof. See Appendix A.

As an analogue to the *a priori* bounds (BN) and (BS1), Li et al. (2012) derived the following *a posteriori* bound for bounded uncertainty using Hoeffding's inequality (Hoeffding, 1963):

$$\Pr \left\{ \sum_j a_{ij}x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j^* > b_i \right\} \leq \exp \left(\frac{-h_i(x^*)^2}{2 \sum_{j \in J_i} \hat{a}_{ij}^2 x_j^{*2}} \right). \quad (\text{LTF})$$

Note that the presentation of (LTF) slightly differs from that of Li et al. (2012) due to the assumption without loss of generality that $\mathbf{E}[\xi_{ij}] = 0, \forall j \in J_i$.

Just as bounds (GMF2) and (GMF2') improve upon (BN) and (BS1), the following bound by Paschalidis et al. (2008) for bounded uncertainty improves upon (LTF) using an inequality similar to Lemma 1:

$$\Pr \left\{ \sum_j a_{ij}x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij}x_j^* > b_i \right\} \leq \exp \left(\min_{\theta > 0} \left\{ -\theta h_i(x^*) + \sum_{j \in J_i} \ln \cosh(\hat{a}_{ij}x_j^*\theta) \right\} \right). \quad (\text{PKL})$$

Note that the presentation of (PKL) differs from that of Paschalidis et al. (2008) due to the assumptions that $\mathbf{E}[\xi_{ij}] = 0$ and $\xi_{ij} \in [-1, 1]$.

There are two ways to extend the principles of (GMF3) to an *a posteriori* setting. The proof of the bound requires a sum of bounded and symmetric random variables with expected values of zero and supported within $[-1, 1]$. It is applicable to a sum of weighted, symmetric random variables $\xi_{ij} \in [-1, 1]$ with $\mathbf{E}[\xi_{ij}] = 0$, so long as their coefficients are also within $[-1, 1]$.

Theorem 5. *Given that, for all $j \in J_i$, random variable ξ_{ij} is independent and subject to a symmetric probability distribution that is supported on the bounded interval $[-1, 1]$ with $\mathbf{E}[\xi_{ij}] = 0$, the following probability bound of constraint violation holds for the i th constraint:*

$$\Pr \left\{ \sum_j a_{ij} x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j^* > b_i \right\} \leq \frac{1}{2^{n_a}} \sum_{k=\lceil \nu_a \rceil}^{n_a} \binom{n_a}{k} \quad (35)$$

where

$$\begin{aligned} \nu_a &:= \frac{h_a + n_a}{2} \\ n_a &:= \begin{cases} |J_i^{\delta, *}| + 1 - \operatorname{sgn} h_a & \text{if } \lceil h_a \rceil + |J_i^{\delta, *}| \text{ is even} \\ |J_i^{\delta, *}| - \operatorname{sgn} h_a & \text{if } \lceil h_a \rceil + |J_i^{\delta, *}| \text{ is odd} \end{cases} \\ h_a &:= \frac{\Delta_i}{\max_{j \in J_i^{\delta, *}} |\delta_{ij}^*|} \\ J_i^{\delta, *} &:= \{j : |\delta_{ij}^*| > 0\}, \end{aligned}$$

and constants δ_{ij}^* are given by their definitions and the solution of the robust counterpart.

Proof. See Appendix A.

Remark 1. Note that (35) would only yield different (and lower) values from (GMF3) if $\max_{j \in J_i^{\delta, *}} |\delta_{ij}^*| < 1$; other than the trivial case where $|J_i^{\delta, *}| = 0$, this does not occur with polyhedral and interval + polyhedral sets.

Bound (35) does not yield the expected reduction in the *a posteriori* probability when constraint i was inactive in the solution. An alternative is to assure that the coefficients $\hat{a}_{ij} x_j^*$ on ξ_{ij} are within $[-1, 1]$.

Theorem 6. *Given that, for all $j \in J_i$, random variable ξ_{ij} is independent and subject to a symmetric probability distribution that is supported on the bounded interval $[-1, 1]$ with $\mathbf{E}[\xi_{ij}] = 0$, the following probability bound of constraint violation holds for the i th constraint:*

$$\Pr \left\{ \sum_j a_{ij} x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j^* > b_i \right\} \leq \frac{1}{2^{n_b}} \sum_{k=\lceil \nu_b \rceil}^{n_b} \binom{n_b}{k} \quad (36)$$

where

$$\begin{aligned}\nu_b &:= \frac{h_b + n_b}{2} \\ n_b &:= \begin{cases} |J_i^*| + 1 - \text{sgn } h_b & \text{if } \lceil h_b \rceil + |J_i^*| \text{ is even} \\ |J_i^*| - \text{sgn } h_b & \text{if } \lceil h_b \rceil + |J_i^*| \text{ is odd} \end{cases} \\ h_b &:= \frac{h_i(x^*)}{\max_{j \in J_i^*} \hat{a}_{ij} |x_j^*|} \\ J_i^* &:= \{j : \hat{a}_{ij} |x_j^*| > 0\}.\end{aligned}$$

Proof. See Appendix A.

Though (36) would on average yield better probabilities than (GMF3) and (35), it is not exactly the analogue of (GMF3), and thus might yield higher *a posteriori* probabilities. The minimum of the two variants is preferred:

$$\begin{aligned}\Pr \left\{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j^* > h_i(x^*) \right\} \\ \leq \min \left\{ \frac{1}{2^{n_a}} \sum_{k=\lceil \nu_a \rceil}^{n_a} \binom{n_a}{k}, \frac{1}{2^{n_b}} \sum_{k=\lceil \nu_b \rceil}^{n_b} \binom{n_b}{k} \right\}. \quad (\text{GMF5})\end{aligned}$$

Note that, for all uncertainty sets but interval + ellipsoidal, the set of indices $J_i^{\delta,*}$ is equal to J_i^* , simplifying bound (GMF5) to

$$\Pr \left\{ \sum_j a_{ij} x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j^* > b_i \right\} \leq \frac{1}{2^{n_c}} \sum_{k=\lceil \nu_c \rceil}^{n_c} \binom{n_c}{k} \quad (37)$$

where c denotes a if $h_a > h_b$, and b otherwise.

As with (GMF3), the approximation for very high $|J_i^{\delta,*}|$ or $|J_i^*|$ is

$$\begin{aligned}\Pr \left\{ \sum_j a_{ij} x_j^* + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j^* > b_i \right\} \\ \leq \min \left\{ \sum_{k=\lceil \nu_a \rceil}^{n_a} C(n_a, k), \sum_{k=\lceil \nu_b \rceil}^{n_b} C(n_b, k) \right\} \quad (\text{GMF5'})\end{aligned}$$

where $C(n, k)$ is given in (33).

6. Implementation

The *a priori* bounds presented here yield an upper bound on the probability of constraint violation, ϵ_i^{prio} , for a given Δ_i . However, formulating the robust

counterpart typically seeks the value of Δ_i which will guarantee a particular ϵ_i^{prio} . A few bounds are able to be analytically rearranged to yield Δ_i as a function of ϵ_i^{prio} . Those which cannot can be easily solved via the bisection method over Δ_i by noting the monotonicity of ϵ_i^{prio} as a function of Δ_i . Complicating matters is the fact that two bounds contain an inner (convex) minimization over $\theta > 0$. Performing the bisection method over Δ_i would require solving the minimization at every iteration. However, this is not necessary; the inner minimization can be avoided if the following proposition is applicable.

Proposition 1. *Given the NLP*

$$\min_x \{-cx + f(x) : x \geq x^L\}$$

where $c > 0$, $f(x)$ is a $\mathcal{C}^1$ -continuous convex function over $x \geq x^L$ with $f(x^L) = cx^L$ and $0 \leq df(x)/dx|_{x^L} < c$, and x^* is an optimal solution, then

- (i) $x^* > x^L$,
- (ii) x^* is nondecreasing as c increases.

Proof. The conclusions follow from the properties of $f(x)$.

- (i) At $x = x^L$, the objective function value is 0. At $x = x^L + \epsilon$, $\epsilon \rightarrow 0^+$, the objective function value is decreasing because $cx^L + c\epsilon > f(x^L + \epsilon)$. Thus $x^* > x^L$.
- (ii) First order optimality conditions indicate that $c = df(x)/dx|_{x^*}$. Due to convexity,

$$df(x)/dx|_{x_1} \geq df(x)/dx|_{x_2} \quad \forall x_1, x_2 : x_1 \geq x_2.$$

□

Proposition 1 applies to all the bounds presented here with inner minimization problems. The implications of Proposition 1 are that ϵ_i^{prio} is monotonic in θ^* , and that if $\Delta_i > 0$, constraint $\theta > 0$ can be dropped. Then, the inner minimization could be performed using simple algorithms such as a golden section search. Simplifying the procedure even further, one can derive Δ_i as a function of θ^* using first-order optimality conditions, that is, the value of Δ_i which, if the inner minimization were solved, would yield solution θ^* . Then, the bisection method over θ^* would select θ^* , calculate the corresponding Δ_i , and then calculate ϵ_i^{prio} . For bound (GMF1),

$$\Delta_i(\theta^*) = \sum_{k \in K_i} \frac{\mu_{ik} \cosh(\theta^*) + \sinh(\theta^*)}{\mu_{ik} \sinh(\theta^*) + \cosh(\theta^*)}. \quad (38)$$

For bound (GMF1'),

$$\Delta_i(\theta^*) = \sqrt{|J_i|} \frac{\bar{\mu}_i \cosh\left(\theta^*/\sqrt{|J_i|}\right) + \sinh\left(\theta^*/\sqrt{|J_i|}\right)}{\mu_{ik} \sinh\left(\theta^*/\sqrt{|J_i|}\right) + \cosh\left(\theta^*/\sqrt{|J_i|}\right)}. \quad (39)$$

While there is no upper bound on θ^* , it can be observed that relatively low values of θ^* (magnitude of 10 to 100) will yield Δ_i values very close to their maximum attainable values due to rapid convergence of $\Delta_i(\theta^*)$ to a constant ($|K_i|$ and $\sqrt{|J_i|}$ for bounds (GMF1) and (GMF1'), respectively). For the *a posteriori* bound (PKL), which also contains an inner minimization, Proposition 1 can convert solving the inner minimization with optimization to simply meeting first-order optimality conditions with a root-finding method.

7. Comparisons of *a priori* bounds

This section compares the existing and new *a priori* bounds computationally with two example problems. A summary of their comparisons that is not tied to computational results but is based on their mathematical structure is given in Table 9.

7.1. Example MILP

We first demonstrate the tightness of the proposed *a priori* bounds through case studies that evaluate Δ_i and the resulting objective function values of an uncertain MILP. Tightness refers to the ability of an *a priori* bound to produce a smaller Δ_i parameter at the same probability of constraint violation, which can result in a better objective function value. To investigate this, a simple MILP is adapted from Example 7.1 in Li et al. (2011). The nominal MILP is

$$\begin{aligned}
\max_{x,y} \quad & \widetilde{3}x_1 + \widetilde{2}x_2 - \widetilde{10}y_1 - 5y_2 \\
\text{s.t.} \quad & \widetilde{1}x_1 + \widetilde{1}x_2 \leq \widetilde{20} \\
& \widetilde{1}x_1 + \widetilde{2}x_2 \leq \widetilde{12} \\
& x_1 - 20y_1 \leq 0 \\
& x_2 - 20y_2 \leq 0 \\
& \widetilde{1}x_1 - \widetilde{1}x_2 \leq -\widetilde{4} \\
& 0 \leq x_1 \leq 10 \\
& 0 \leq x_2 \leq 10 \\
& y_1, y_2 \in \{0, 1\}.
\end{aligned} \tag{40}$$

For these case studies, uncertainty will be considered in selected right-hand side, left-hand side, and objective function parameters. Specifically, all parameters denoted with a tilde (12 in total) are subject to bounded uncertainty. Each uncertain constraint i has three uncertain parameters ($|J_i| = 3$). The values of all $\hat{a}_{ij}$ parameters will be 10% of the nominal parameter values. With equivalent assumptions for each parameter and equivalent uncertainty set types for each constraint, applying the same bound to each uncertain constraint i will yield the same value of Δ_i . For example, the resulting robust counterpart when

applying a box uncertainty set with each uncertain constraint is

$$\begin{aligned}
& \max_{x,y,z} \quad z \\
& \text{s.t.} \quad z - 3x_1 - 2x_2 + 10y_1 + 5y_2 + \Psi(0.3x_1 + 0.2x_2 + y_1) \leq 0 \\
& \quad x_1 + x_2 + \Psi(0.1x_1 + 0.1x_2 + 2) \leq 20 \\
& \quad x_1 + 2x_2 + \Psi(0.1x_1 + 0.2x_2 + 1.2) \leq 12 \\
& \quad x_1 - 20y_1 \leq 0 \\
& \quad x_2 - 20y_3 \leq 0 \\
& \quad x_1 - x_2 + \Psi(0.1x_1 + 0.1x_2 + 0.4) \leq -4 \\
& \quad 0 \leq x_1 \leq 10 \\
& \quad 0 \leq x_2 \leq 10 \\
& \quad y_1, y_2 \in \{0, 1\}.
\end{aligned} \tag{41}$$

7.1.1. Bounded probability distributions with known means

If the assumption can be made that the expected values of all random variables are zero, then four *a priori* bounds (and two ellipsoidal-specific variants) are immediately relevant, although bounds (GMF1) and (GMF1') simplify to bounds (GMF2) and (GMF2'), respectively. Going beyond this to an assumption of symmetry includes two more bounds, leaving six that are relevant for the case of bounded, symmetric probability distributions. The utility of these bounds are shown in Figures 1 to 6; (GMF1) and (GMF1') are not plotted due to their redundancy with (GMF2) and (GMF2'), respectively.

Note that the figures are constructed from the perspective of selecting a particular Δ_i that will guarantee a desired ϵ_i^{prio} . For example, bounds (BS2) and (GMF3) cannot return probabilities greater than 0.5; if the probability of constraint violation should be less than or equal to 0.6, it is optimal to utilize the smallest Δ_i (1 or 0, respectively) which does not return a probability exceeding 0.6. Bound (GMF3) is upper semi-continuous at positive integer values; it returns the same probability at positive integer Δ_i as well as just below Δ_i at Δ_i^- , and a different (lower) probability just above Δ_i at Δ_i^+ . For any ϵ_i^{prio} falling between the probabilities around Δ_i , selecting the larger value, Δ_i^+ , validly constrains the probability of constraint violation, while selecting Δ_i does not. Thus, bound (GMF3) is plotted with open circles at Δ_i^+ and the higher probability.

Figures 1 to 6 display the strong performance of (GMF3) for each uncertainty set; despite the discontinuous nature of the bound, it will provide the tightest Δ_i values and best objective function values for any probability of constraint violation in the box, polyhedral, or interval + polyhedral uncertainty sets. This is clearly seen in with the example MILP ($|J_i| = 3$). With ellipsoidal or interval + ellipsoidal sets, at higher $|J_i|$ values and low probabilities of constraint violation, there are cases in which (GMF2') provides better results than (GMF3). This is seen in Figure 3 at low probabilities of constraint violation for the example MILP. Thus, this case study demonstrates that when the only assumption regarding the probability distributions is that they are bounded and

Figure 1: Probability of constraint violation as a function of parameter Ψ for the box uncertainty set using *a priori* probabilities of constraint violation for unspecified but bounded and symmetric probability distributions, $|J_i| = 3$.

Figure 2: Objective value from the example MILP at varying probabilities of constraint violation for the box uncertainty set, using *a priori* bounds for unspecified but bounded and symmetric probability distributions, $|J_i| = 3$.

Figure 3: Probability of constraint violation as a function of parameter Ω for the interval + ellipsoidal uncertainty set using *a priori* probabilities of constraint violation for unspecified but bounded and symmetric probability distributions, $|J_i| = 3$.

Figure 4: Objective value from the example MILP at varying probabilities of constraint violation for the interval + ellipsoidal uncertainty set, using *a priori* bounds for unspecified but bounded and symmetric probability distributions, $|J_i| = 3$.

Figure 5: Probability of constraint violation as a function of parameter Γ for the interval + polyhedral uncertainty set using *a priori* probabilities of constraint violation for unspecified but bounded and symmetric probability distributions, $|J_i| = 3$.

Figure 6: Objective value from the example MILP at varying probabilities of constraint violation for the interval + polyhedral uncertainty set, using *a priori* bounds for unspecified but bounded and symmetric probability distributions, $|J_i| = 3$.

Figure 7: Probability of constraint violation as a function of Ω for (GMF1') on an interval + ellipsoidal uncertainty set with increasingly loose bounds on $E[\xi_{ij}]$, $|J_i| = 3$.

symmetric, bounds (GMF3) and (GMF2') are the only two which should be considered.

7.1.2. Bounded probability distributions with bounded means

Bounds (GMF1) and (GMF1') permit robust counterpart optimization even in the case where the mean of the distribution of an uncertain parameter is not known exactly, but is assumed to lie within a range of values. The performances of (GMF1) and (GMF1') with the interval + polyhedral and interval + ellipsoidal sets, respectively, were evaluated with $|J_i| = 3$ and $|J_i| = 12$. The expected value of every uncertain parameter is not known exactly; a range of values is specified instead. The bounds on $|\mathbf{E}[\xi_{ij}]|$, denoted as μ_{ij} , were set at equal values, $\forall j \in J_i$, ranging from 0.1 to 0.5 with increments of 0.1. The results can be seen in Figures 7 through 10. Despite the fact that the only given information on the uncertain parameters is that (i) they are bounded, and (ii) there is an interval that their expected values lie within, bounds (GMF1) and (GMF1') provide tight results which are comparable to the bounds that assume $\mathbf{E}[\xi_{ij}] = 0$. At $|J_i| = 12$ and $\mu_{ij} = 0.5$, that is, with the true expected value of a parameter existing anywhere across 50% of the total range of the parameter support, better than worst-case values of Δ_i are able to be obtained at, say, $\epsilon_i^{prio} = 0.05$ with the interval + ellipsoidal or interval + polyhedral set.

Figure 8: Probability of constraint violation as a function of Γ for (GMF1) on an interval + polyhedral uncertainty set with increasingly loose bounds on $E[\xi_{ij}]$, $|J_i| = 3$.

Figure 9: Probability of constraint violation as a function of Ω for (GMF1') on an interval + ellipsoidal uncertainty set with increasingly loose bounds on $E[\xi_{ij}]$, $|J_i| = 12$.

Figure 10: Probability of constraint violation as a function of Γ for (GMF1) on an interval + polyhedral uncertainty set with increasingly loose bounds on $E[\xi_{ij}]$, $|J_i| = 12$.

7.2. Application: Multiperiod Production Planning LP

A linear programming application of robust optimization exists in multiperiod production planning. An example LP model is expanded from (Ravindran, 2008) and (Li et al., 2012) in which a company is planning for an upcoming year. The profit for the company is to be maximized over the year, which is broken into 12 months each with unique parameters relating to production. The company also must operate within a given annual budget, in this case, \$800,000. The model is written to decide how much product should be produced each month, given fluctuating production costs, selling prices, storage costs, production capacity, and product demand. Operations begin in period one, with the company beginning with 500 tons of product in storage; at the end of the year, the company seeks to have this same amount of product in storage again. The product produced in each period can be sold immediately, or stored until being sold in a future period. This problem is formulated as the following linear optimization problem:

$$\begin{aligned}
\max_{x,y,z} \quad & \sum_j (P_j z_j - C_j x_j - V_j y_j) \\
\text{s.t.} \quad & \sum_j (C_j x_j + V_j y_j) \leq B \\
& I + x_1 - (y_1 + z_1) = 0 \\
& y_{j-1} + x_j - (y_j + z_j) = 0 \quad \forall j = 2, \dots, 12 \\
& y_{12} = F \\
& x_j \leq U_j \quad \forall j \\
& z_j \leq D_j \quad \forall j \\
& x_j, y_j, z_j \geq 0 \quad \forall j.
\end{aligned} \tag{42}$$

In this formulation, x_j , y_j , and z_j are variables representing the production amount, amount of product in storage, and amount of product sold (tons) in each period, respectively. Parameters P_j , C_j , and V_j are financial parameters representing the selling price, production cost, and storage cost per ton of product. The amount sold in each period is limited by demand D_j , and the amount produced is limited by the production capacity U_j . Finally, the overall budget $B = \$800,000$ limits the total cost of production, while $I = F = 500$ tons, representing the initial (I) and final (F) amount of product in storage. The nominal values for each parameter in each month are shown in Table 1.

To analyze the new *a priori* bounds on the probability of constraint violation, uncertainty is considered in the selling price of the product. Thus, uncertainty is seen in the objective function alone and $|J_i| = 12$. The objective function can be transformed in the following manner such that variable ζ is maximized with the added constraint

$$\zeta - \sum_j \tilde{P}_j z_j + \sum_j (C_j x_j + V_j y_j) \leq 0. \tag{43}$$

Figure 11: Probability of constraint violation from Ψ values for the box uncertainty set considering uncertainty in the objective function of the multiperiod planning problem, $|J_i| = 12$. A uniform distribution is assumed for all uncertain parameters.

Then, the robust counterpart can be formulated, where $\tilde{P}_j$ represents the actual selling price and is described by $\tilde{P}_j = P_j + \xi_j \hat{P}_j$. It is assumed that the perturbations $\hat{P}$ are at most 20% of the nominal values P_j . Bounds presented in this paper are then applied to the box, interval + ellipsoidal, and interval + polyhedral uncertainty sets. The results are shown in Figures 11 through 16.

The new *a priori* probabilistic bounds are quantified for six different probabilities of constraint violation ranging from 5% to 30%. Traditionally, a “price of robustness” (Bertsimas and Sim, 2004) is used as a metric for conservatism by evaluating the change in the objective function relative to the nominal value when solving the robust counterpart formulation. If ζ_N and ζ_R represent the nominal and robust objective function values, respectively, then the price of robustness (PoR) is calculated as

$$\text{PoR} = \left| \frac{\zeta_N - \zeta_R}{\zeta_N} \right|. \quad (44)$$

A new quantification is proposed for bounded uncertainty to represent conservatism in a way that is not as dependent on the nominal objective function value. This “normalized price of robustness” (NPoR) is defined using the gap

Figure 12: Profit from the multiperiod planning problem as a function of probability of constraint violation for the box uncertainty set with parameters described by a uniform distribution, $|J_i| = 12$.

Figure 13: Probability of constraint violation from Ω values for the interval + ellipsoidal uncertainty set considering uncertainty in the objective function of the multiperiod planning problem, $|J_i| = 12$. A uniform distribution is assumed for all uncertain parameters.

Figure 14: Profit from the multiperiod planning problem as a function of probability of constraint violation for the interval + ellipsoidal uncertainty set with parameters described by a uniform distribution, $|J_i| = 12$.

Figure 15: Probability of constraint violation from Γ values for the interval + polyhedral uncertainty set considering uncertainty in the objective function of the multiperiod planning problem, $|J_i| = 12$. A uniform distribution is assumed for all uncertain parameters.

Figure 16: Profit from the multiperiod planning problem as a function of probability of constraint violation for the interval + polyhedral uncertainty set with parameters described by a uniform distribution, $|J_i| = 12$.

between the nominal and worst case (ζ_{WC}) objective function values such that

$$\text{NPoR} = \left| \frac{\zeta_N - \zeta_R}{\zeta_N - \zeta_{WC}} \right|. \quad (45)$$

The worst case value can simply be calculated using a box uncertainty set with $\Psi = 1$, and NPoR will vary from values of 0 at the nominal case to 1 at the worst case.

The inclusion of uncertainty is most notable in the reduction of profit from \$4,207,705.88 at the nominal case to \$3,206,164.71 in the worst case scenario, as seen in all of the objective function figures. Yet, the conservatism of the worst case values can be drastically reduced, as seen Figures 14 and 16, where even a 1% probability of constraint violation using (GMF3) leads to a \$159,000 increase (5% increase) in profit compared to the worst case for the interval + polyhedral set. Bound (GMF3) performs the strongest for all probabilities of constraint violation for the interval + polyhedral set, and is dominant at high probabilities of constraint violation for the interval + ellipsoidal set. For the interval + ellipsoidal case, the new (GMF2') outperforms the existing (BN).

These results are reiterated through Tables 2 and 3 for the interval + ellipsoidal and interval + polyhedral sets, respectively. Bounds (BN) and (GMF2') are the only bounds to reach an NPoR of 50% or less at $\epsilon_i^{prio} = 0.3$ and 0.35. Bound (GMF2') displays the best performance of all bounds at low probabilities of constraint violation, with the biggest improvements over (BN) coming at the lowest probabilities of constraint violation. For the interval + polyhedral case, (GMF3) is the strongest performer; at only 20% probability of constraint violation, this bound leads to profits closer to the nominal than the worst case. At 30% probability of constraint violation, the profit has been reduced by only 7% due to the inclusion of uncertainty. These results demonstrate that for unknown distributions with these $|J_i|$ values, the *a priori* bounds are tighter for the interval + polyhedral set than the interval + ellipsoidal set.

Table 1: Monthly Planning Data for the LP Production Planning Problem

Period	Selling Price (\$/ton)	Production Cost (\$/ton)	Storage Cost (\$/ton)	Production Capacity (tons)	Demand (tons)
j	P_j	C_j	V_j	U_j	D_j
1	180	20	2	1500	1100
2	180	25	2	2000	1500
3	250	30	2	2200	1800
4	270	40	2	3000	1600
5	300	50	2	2700	2300
6	320	60	2	2500	2500
7	300	80	2	1900	2300
8	270	55	2	1500	1600
9	250	45	2	1300	1800
10	180	30	2	1100	1500
11	180	30	2	1200	1100
12	160	25	2	1400	1200

Table 2: Comparison of existing and new *a priori* probabilistic bounds for the multiperiod production planning problem with unknown parameter distributions and an interval + ellipsoidal uncertainty set.

ϵ_i^{prio}	Existing Bounds								
	(BN)			(BS1)			(BS2)		
	Profit (MM\$)	PoR (%)	NPoR (%)	Profit (MM\$)	PoR (%)	NPoR (%)	Profit (MM\$)	PoR (%)	NPoR (%)
0.05	3.424	18.63	78.27	3.206	23.80	100.00	3.206	23.80	100.00
0.10	3.516	16.45	69.11	3.206	23.80	100.00	3.206	23.80	100.00
0.15	3.579	14.94	62.77	3.206	23.80	100.00	3.206	23.80	100.00
0.20	3.629	13.76	57.81	3.206	23.80	100.00	3.206	23.80	100.00
0.25	3.670	12.77	53.66	3.206	23.80	100.00	3.212	23.67	99.46
0.30	3.707	11.90	50.00	3.206	23.80	100.00	3.304	21.48	90.26
0.35	3.740	11.11	46.69	3.206	23.80	100.00	3.442	18.20	76.46
ϵ_i^{prio}	New Bounds								
	(GMF2')			(GMF3)					
	Profit (MM\$)	PoR (%)	NPoR (%)	Profit (MM\$)	PoR (%)	NPoR (%)			
0.05	3.455	17.88	75.12	3.206	23.80	100.00			
0.10	3.538	15.91	66.83	3.206	23.80	100.00			
0.15	3.596	14.53	61.05	3.206	23.80	100.00			
0.20	3.642	13.44	56.48	3.282	22.00	92.42			
0.25	3.681	12.52	52.59	3.282	22.00	92.42			
0.30	3.715	11.70	49.15	3.562	15.35	64.48			
0.35	3.747	10.95	46.00	3.562	15.35	64.48			

8. Comparisons of *a posteriori* bounds

We can compare the strength of the *a posteriori* bounds by applying them to specific solutions x^* . As in Section 7.2, we solved the multiperiod planning problem at different probabilities of constraint violation using the interval + ellipsoidal and interval + polyhedral uncertainty sets with Δ specified by *a priori* bounds (GMF2) and (GMF2'), respectively. A comparison of the probabilities obtained by each *a posteriori* bound versus the *a priori* probabilities can be seen in Figures 17 and 19. Any *a posteriori* bound will provide probabilities that are as tight as or tighter than those provided by its *a priori* analogue. Note that the *a posteriori* analogue of bound (GMF2) and (GMF2') is bound (PKL); the probabilities provided by bound (GMF2') were remarkably close and often equivalent to those provided by bound (PKL), despite the fact that it accounts for all solutions rather than a particular solution. Bound (GMF5) displayed good performance, particularly for the interval + polyhedral set, but could provide worse probabilities than other bounds at times due the fact that it does not similarly account for particular elements of the robust solution. The objective function values corresponding to the *a posteriori* probabilities can be seen in Figures 18 and 20, with the *a priori* probabilities and objective function values of bound (GMF2') and (GMF2), respectively, included for comparison. In this example, *a priori* bound (GMF2') essentially provided *a posteriori* levels of profit.

9. Conclusions

The performance of robust counterpart optimization is reliant upon the relationship between the probabilistic guarantees required by the user and the uncertain parameters included in a robust counterpart. We derived new *a priori* bounds on the probability of constraint violation which improve upon existing methods and are applicable to models under uncertainty with bounded uncertain parameters and known means, but unspecified probability distributions. We also provided the first *a priori* and *a posteriori* bounds for uncertain models with limited information on the means of its uncertain parameters. Given uncertain parameters subject to bounded uncertainty, the new *a priori* bounds will provide the best objective function values to a robust counterpart at an equivalent probability of constraint violation when compared to existing bounds. If, however, any further knowledge on the distributions of uncertain parameters exists, it can be leveraged to provide even starker improvements to the performance of robust counterpart optimization. We will explore this concept in subsequent work.

Figure 17: *A posteriori* probability of constraint violation of each solution obtained from the multiperiod planning problem with the interval + ellipsoidal uncertainty set versus *a priori* probability distribution; ϵ_i^{prio} was set by selecting Δ_i with bound (GMF2').

Figure 18: Profit versus *a posteriori* probability of constraint violation of each solution obtained from the multiperiod planning problem with the interval + ellipsoidal uncertainty set and bound (GMF2'); profit versus ϵ_i^{prio} as obtained by bound (GMF2') included for comparison.

Figure 19: *A posteriori* probability of constraint violation of each solution obtained from the multiperiod planning problem with the interval + polyhedral uncertainty set versus *a priori* probability distribution; ϵ_i^{prio} was set by selecting Δ_i with bound (GMF2).

Figure 20: Profit versus *a posteriori* probability of constraint violation of each solution obtained from the multiperiod planning problem with the interval + polyhedral uncertainty set and bound (GMF2'); profit versus ϵ_i^{PTIO} as obtained by bound (GMF2) included for comparison.

Table 3: Comparison of existing and new *a priori* probabilistic bounds for the multiperiod production planning problem with unknown parameter distributions and an interval + polyhedral uncertainty set.

ϵ_i^{prio}	Existing Bounds					
	(BS1)			(BS2)		
	Profit (MM\$)	PoR (%)	NPoR (%)	Profit (MM\$)	PoR (%)	NPoR (%)
0.05	3.342	20.57	86.43	3.431	18.45	77.52
0.10	3.395	19.30	81.10	3.544	15.78	66.28
0.15	3.441	18.23	76.58	3.616	14.05	59.04
0.20	3.487	17.13	71.99	3.687	12.36	51.95
0.25	3.525	16.22	68.13	3.734	11.26	47.30
0.30	3.559	15.41	64.74	3.785	10.04	42.18
0.35	3.590	14.68	61.67	3.857	8.34	35.06

ϵ_i^{prio}	New Bounds					
	(GMF2)			(GMF3)		
	Profit (MM\$)	PoR (%)	NPoR (%)	Profit (MM\$)	PoR (%)	NPoR (%)
0.05	3.360	20.15	84.65	3.505	16.70	70.14
0.10	3.409	18.98	79.76	3.592	14.64	61.51
0.15	3.457	17.85	74.99	3.682	12.50	52.53
0.20	3.499	16.84	70.75	3.772	10.36	43.54
0.25	3.535	15.98	67.15	3.772	10.36	43.54
0.30	3.567	15.22	63.95	3.910	7.09	29.77
0.35	3.597	14.52	61.01	3.910	7.09	29.77

Table 4: Definition of uncertainty sets.

Uncertainty set U_i	Parameter Δ_i^a	Definition	$\max \Delta_i^b$
Box	Ψ_i	$U_i^\infty = \{\xi_i : \ \xi_i\ _\infty \leq \Psi_i\}$	1
Ellipsoidal	Ω_i	$U_i^2 = \{\xi_i : \ \xi_i\ _2 \leq \Omega_i\}$	$\sqrt{ J_i }$
Interval + Ellipsoidal	Ω_i	$U_i^{2 \cap \infty} = \{\xi_i : \ \xi_i\ _2 \leq \Omega_i, \ \xi_i\ _\infty \leq 1\}$	$\sqrt{ J_i }$
Polyhedral	Γ_i	$U_i^1 = \{\xi_i : \ \xi_i\ _1 \leq \Gamma_i\}$	$ J_i $
Interval + Polyhedral	Γ_i	$U_i^{1 \cap \infty} = \{\xi_i : \ \xi_i\ _1 \leq \Gamma_i, \ \xi_i\ _\infty \leq 1\}$	$ J_i $

^a Original, set-specific notation of parameter denoted here; Δ_i used for the general case.

^b Applicable when $\xi_{ij} \in [-1, 1]$, $\forall j \in J_i$.

Table 5: Robust counterparts of constraint i by uncertainty set type.

U_i	$\sum_j a_{ij}x_j + \max_{\xi_i \in U_i} \left\{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \right\} \leq b_i$
B	$\sum_j a_{ij}x_j + \Psi_i \sum_{j \in J_i} \hat{a}_{ij} x_j \leq b_i$
E	$\sum_j a_{ij}x_j + \Omega_i \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2} \leq b_i$
IE	$\sum_j a_{ij}x_j + \sum_{j \in J_i} \hat{a}_{ij} x_j - z_{ij} + \Omega_i \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 z_j^2} \leq b_i$
P	(1) $\sum_j a_{ij}x_j + \Gamma_i z_i \leq b_i$ (2) $z_i \geq \hat{a}_{ij} x_j , \forall j \in J_i$
IP	(1) $\sum_j a_{ij}x_j + \sum_{j \in J_i} p_{ij} + \Gamma_i z_i \leq b_i$ (2) $z_i \geq 0$ (3) $z_i + p_{ij} \geq \hat{a}_{ij} x_j , \forall j \in J_i$ (4) $p_{ij} \geq 0, \forall j \in J_i$

 Table 6: Associated parameters of uncertainty sets.^a

U_i	Definition of δ_i	$\ \delta_i\ _1 \leq \delta_i^{(1)}$	$\ \delta_i\ _2 \leq \delta_i^{(2)}$
B	$\delta_{ij} := \frac{\hat{a}_{ij}x_j}{\sum_j \hat{a}_{ij} x_j }$	$\sum_j \delta_{ij} \leq 1$	$\sqrt{\sum_j \delta_{ij}^2} \leq 1$
E	$\delta_{ij} := \frac{\hat{a}_{ij}x_j}{\sqrt{\sum_j \hat{a}_{ij}^2 x_j^2}}$	$\sum_j \delta_{ij} \leq \sqrt{ J_i }^c$	$\sqrt{\sum_j \delta_{ij}^2} = 1$
IE ^b	$\delta_{ij} := \frac{\hat{a}_{ij}z_j}{\sqrt{\sum_j \hat{a}_{ij}^2 z_j^2}}$	$\sum_j \delta_{ij} \leq \sqrt{ J_i }^c$	$\sqrt{\sum_j \delta_{ij}^2} = 1$
P	$\delta_{ij} := \frac{\hat{a}_{ij}x_j}{\max_j \hat{a}_{ij} x_j }$	$\sum_j \delta_{ij} \leq J_i $	$\sqrt{\sum_j \delta_{ij}^2} \leq \sqrt{ J_i }$
IP ^{b,d}	$\delta_{ij} := \min \left\{ 1, \frac{\hat{a}_{ij} x_j }{\hat{a}_{ir} x_r } \right\}$	$\sum_j \delta_{ij} \leq J_i $	$\sqrt{\sum_j \delta_{ij}^2} \leq \sqrt{ J_i }$

^a Note that all $j \in J_i$, and that for all uncertainty sets, $\delta_{ij} \in [-1, 1]$. Notation uses vector δ_i with elements δ_{ij} .

^b Assumes that probability distributions are bounded.

^c Follows from the Cauchy-Schwarz inequality.

^d Index r is the index of the $(\max \{1, \lceil \Gamma_i \rceil\})$ th largest $\hat{a}_{ij}|x_j|$.

Table 7: Existing and new *a priori* probabilistic bounds on constraint violation: bounded distributions.

Source	Label	Bound	Applicable ^a U_i	Assumptions ^b
(^c)	(BN)	$\exp(-\Delta_i^2/2)$	B, E, IE	$\xi_{ij} \in [-1, 1], \mathbf{E}[\xi_{ij}] = 0$
(^d)	(BS1)	$\exp(-\Delta_i^2/2 J_i)$	B, E, IE, P, IP	$\xi_{ij} \in [-1, 1], \mathbf{E}[\xi_{ij}] = 0$
this work	(GMF1) ^e	$\exp\left(\min_{\theta>0}\left\{-\theta\Delta_i + \sum_{k \in K_i} \ln G_{ik}(\theta)\right\}\right)$	B, E, IE, P, IP	$\xi_{ij} \in [-1, 1]$, known bounds on $\mathbf{E}[\xi_{ij}]$
this work	(GMF1') ^f	$\exp\left(\min_{\theta>0}\left\{-\theta\Delta_i + J_i \ln \overline{G}_i\left(\theta/\sqrt{ J_i }\right)\right\}\right)$	E, IE	$\xi_{ij} \in [-1, 1]$, known bounds on $\mathbf{E}[\xi_{ij}]$
this work	(GMF2) ^g	$(\alpha^\alpha \beta^\beta)^{-\delta_i^{(1)}/2}$	B, E, IE, P, IP	$\xi_{ij} \in [-1, 1], \mathbf{E}[\xi_{ij}] = 0$
this work	(GMF2') ^g	$(\alpha^\alpha \beta^\beta)^{- J_i /2}$	E, IE	$\xi_{ij} \in [-1, 1], \mathbf{E}[\xi_{ij}] = 0$

^a Bound (BN) was formulated for IE uncertainty sets, while bound (BS1) was formulated for IP uncertainty sets. Their usage was extended other uncertainty sets in (Li et al., 2012).

^b Bounds (BN) and (BS1) were derived under assumptions of symmetry; we note that they apply to cases where all $\mathbf{E}[\xi_{ij}] = 0$.

^c Ben-Tal and Nemirovski, 2000.

^d Bertsimas and Sim, 2004.

^e $G_{ij}(\theta) := \mu_{ij} \sinh(\theta) + \cosh(\theta)$.

^f $\overline{G}_i(\theta) := (\max_{j \in J_i} \mu_{ij}) \sinh(\theta) + \cosh(\theta)$

^g $\alpha := 1 - \Delta_i/\delta_i^{(1)}, \beta := 1 + \Delta_i/\delta_i^{(1)}$.

Table 8: Existing and new *a priori* probabilistic bounds on constraint violation: bounded, symmetric distributions.^a

Source	Label	Bound	Applicable ^b U_i	Assumptions
(c)	(BS2) ^d	for $\Delta_i \geq 1$, $\frac{1}{2^n} \left[(1 - (\nu - \lfloor \nu \rfloor)) \binom{n}{\lfloor \nu \rfloor} i + \sum_{k=\lfloor \nu \rfloor+1}^n \binom{n}{k} \right]$	B, E, IE, P, IP	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$, symmetric
(c)	(BS2') ^{d,e}	for $\Delta_i \geq 1$, $(1 - (\nu - \lfloor \nu \rfloor)) C(n, \lfloor \nu \rfloor) + \sum_{k=\lfloor \nu \rfloor+1}^n C(n, k)$	B, E, IE, P, IP	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$, symmetric
this work	(GMF3) ^f	$\frac{1}{2^{\tilde{n}}} \sum_{k=\lceil \tilde{\nu} \rceil}^{\tilde{n}} \binom{\tilde{n}}{k}$	B, E, IE, P, IP	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$, symmetric
this work	(GMF3') ^{e,f}	$\sum_{k=\lceil \tilde{\nu} \rceil}^{\tilde{n}} C(\tilde{n}, k)$	B, E, IE, P, IP	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$, symmetric

^a The bounds of Table 7 also apply.

^b Bounds (BS2) and (BS2') were formulated for IP uncertainty sets. Their usage was extended other uncertainty sets in (Li et al., 2012).

^c Bertsimas and Sim, 2004.

^d $\nu := (\Delta_i + n)/2$, $n := |J_i|$.

^e Less computationally expensive for large $|J_i|$ with slight loss of tightness.

^f $\tilde{\nu} := (\Delta_i + \tilde{n})/2$, $\tilde{n} := |J_i| + 1 - \text{sgn } \Delta_i$ or $|J_i| - \text{sgn } \Delta_i$.

Table 9: Tightest *a priori* bounds by case.^a

Available Information	Uncertainty Set		
	B	E, IE	P, IP
ξ_{ij} bounded, known bounds on $\mathbf{E}[\xi_{ij}]$	(GMF1)	(GMF1) or (GMF1')	(GMF1)
ξ_{ij} bounded, $\mathbf{E}[\xi_{ij}]$ known	(GMF2)	(GMF2')	(GMF2)
ξ_{ij} bounded and symmetric	(GMF3)	(GMF2') or (GMF3)	(GMF3)

^a New bounds are in boldface.

Table 10: Existing and new *a posteriori* probabilistic bounds on constraint violation.

Source	Label	Bound	Assumptions
this work	(GMF4) ^{a,b}	$\exp \left(\min_{\theta > 0} \left\{ -\theta h_i(x^*) + \sum_{j \in J_i} \ln G_{ij}(\hat{a}_{ij} x_j^* \theta) \right\} \right)$	$\xi_{ij} \in [-1, 1]$, known bounds on $\mathbf{E}[\xi_{ij}]$
(c)	(LTF) ^a	$\exp \left(\frac{-h_i(x^*)^2}{2 \sum_{j \in J_i} \hat{a}_{ij}^2 x_j^{*2}} \right)$	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$
(d)	(PKL) ^a	$\exp \left(\min_{\theta > 0} \left\{ -\theta h_i(x^*) + \sum_{j \in J_i} \ln \cosh(\hat{a}_{ij} x_j^* \theta) \right\} \right)$	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$
this work	(GMF5) ^{e,f}	$\min \left\{ \frac{1}{2^{n_a}} \sum_{k=\lceil \nu_a \rceil}^{n_a} \binom{n_a}{k}, \frac{1}{2^{n_b}} \sum_{k=\lceil \nu_b \rceil}^{n_b} \binom{n_b}{k} \right\}$	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$, symmetric
this work	(GMF5') ^{e,f,g}	$\min \left\{ \sum_{k=\lceil \nu_a \rceil}^{n_a} C(n_a, k), \sum_{k=\lceil \nu_b \rceil}^{n_b} C(n_b, k) \right\}$	$\xi_{ij} \in [-1, 1]$, $\mathbf{E}[\xi_{ij}] = 0$, symmetric

^a $h_i(x) := b_i - \sum_j a_{ij} x_j$.^b $G_{ij}(\theta) := \mu_{ij} \sinh(\theta) + \cosh(\theta)$.^c Li et al., 2012.^d Paschalidis et al., 2008.^e $\nu_a := (h_a + n_a)/2$, $n_a := |J_i^{\delta,*}| + 1 - \text{sgn } h_a$ or $|J_i^{\delta,*}| - \text{sgn } h_a$, $h_a := \Delta_i / (\max_{j \in J_i^{\delta,*}} |\delta_{ij}^*|)$, $J_i^{\delta,*} := \{j : |\delta_{ij}^*| > 0\}$.^f $\nu_b := (h_b + n_b)/2$, $n_b := |J_i^*| + 1 - \text{sgn } h_b$ or $|J_i^*| - \text{sgn } h_b$, $h_b := h_i(x^*) / (\max_{j \in J_i^*} \hat{a}_{ij} |x_j^*|)$, $J_i^* := \{j : \hat{a}_{ij} |x_j^*| > 0\}$.^g Less computationally expensive for large $|J_i^{\delta,*}|$ or $|J_i^*|$ with slight loss of tightness.

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