Constructing Edge Disjoint Steiner Paths in Lexicographic Product Networks*

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Abstract

Dirac showed that in a $(k-1)$ -connected graph there is a path through each k vertices. The path k -connectivity $\pi_k(G)$ of a graph G , which is a generalization of Dirac's notion, was introduced by Hager in 1986. It is natural to introduce the concept of path k -edge-connectivity $\omega_k(G)$ of a graph G . Denote by $G \circ H$ the lexicographic product of two graphs G and H . In this paper, we prove that $\omega_3(G \circ H) \geq \omega_3(G) \lfloor \frac{3|V(H)|}{4} \rfloor$ for any two graphs G and H . Moreover, the bound is sharp. We also derive an upper bound of $\omega_3(G \circ H)$, that is, $\omega_3(G \circ H) \leq \min\{2\omega_3(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}$. We demonstrate the usefulness of the proposed constructions by applying them to some instances of lexicographic product networks.

Keywords: Edge-connectivity; Steiner tree; edge-disjoint Steiner paths; packing; path edge-connectivity; lexicographic product.

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1 Introduction

An interpersonal network is represented as a graph, where a node denotes a processor and an edge denotes a communication link. Broadcasting is the process of sending a message from the source node to all other nodes in a network. It can be accomplished by message dissemination in such a way that each node repeatedly receives and forwards messages. Some of the nodes and/or links may be faulty. However, multiple copies of messages can be disseminated through disjoint paths. We say that the broadcasting

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succeeds if all the healthy nodes in the network finally obtain the correct message from the source node within a certain limit of time. A lot of attention has been devoted to fault-tolerant broadcasting in networks [16, 25, 28, 46]. In order to measure the degree of fault-tolerance, the above disjoint path structure connecting two nodes is generalized into some tree structures connecting more than two nodes, see [30, 33, 37]. To show the properties of these generalizations clearly, we hope to state from the connectivity in graph theory. We divide our introduction into the following four subsections to state the motivations and our results of this paper.

1.1 Connectivity and k -connectivity

All graphs considered in this paper are undirected, finite and simple. We refer to [4] for graph theoretical notation and terminology not described here. For a graph G , let $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices, the set of edges and the minimum degree of G , respectively. For $S \subseteq V(G)$, we denote by $G - S$ the subgraph obtained by deleting from G the vertices of S together with the edges incident with them. Connectivity is one of the most basic concepts in graph theory, both in combinatorial sense and in algorithmic sense. As we know, the classical connectivity has two equivalent definitions. The *connectivity* of G , written $\kappa(G)$, is the minimum size of a vertex set $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. We call this definition the ‘cut’ version definition of connectivity. A well-known theorem of Menger [4] provides an equivalent definition of connectivity, which can be called the ‘path’ version definition of connectivity. For any two distinct vertices x and y in G , the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y . Then $\kappa(G) = \min\{\kappa_G(x, y) \mid x, y \in V(G), x \neq y\}$ is defined to be the *connectivity* of G . Similarly, the classical edge-connectivity also has two equivalent definitions. The *edge-connectivity* of G , written $\lambda(G)$, is the minimum size of an edge set $M \subseteq E(G)$ such that $G - M$ is disconnected or has only one vertex. We call this definition the ‘cut’ version definition of edge-connectivity. Menger’s theorem also provides an equivalent definition of edge-connectivity, which can be called the ‘path’ version definition. For any two distinct vertices x and y in G , the *local edge-connectivity* $\lambda_G(x, y)$ is the maximum number of edge-disjoint paths connecting x and y . Then $\lambda(G) = \min\{\lambda_G(x, y) \mid x, y \in V(G), x \neq y\}$ is defined to be the *edge-connectivity* of G . For connectivity and edge-connectivity, Oellermann published a survey paper; see [41].

Although there are many elegant and powerful results on connectivity in graph theory, the classical connectivity and edge-connectivity cannot be satisfied considerably in practical uses. So people want some generalizations of both connectivity and edge-connectivity. For the ‘cut’ version definition of connectivity, we are looking for a minimum vertex-cut with no consideration about the number of components of $G - S$. Two graphs with the

same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n}$ and the path P_{n+1} ($n \geq 3$) are both trees of order $n + 1$ and therefore connectivity 1, but the deletion of a cut-vertex from $K_{1,n}$ produces a graph with n components while the deletion of a cut-vertex from P_{n+1} produces only two components. Chartrand et al. [7] generalized the ‘cut’ version definition of connectivity. For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the k -connectivity $\kappa'_k(G)$ is the smallest number of vertices whose removal from G produces a graph with at least k components or a graph with fewer than k vertices. Thus, for $k = 2$, $\kappa'_2(G) = \kappa(G)$. For more details about k -connectivity, we refer to [7, 11, 41, 42]. The k -edge-connectivity, which is a generalization of the ‘cut’ version definition of classical edge-connectivity was initially introduced by Boesch and Chen [5] and subsequently studied by Goldsmith in [19, 20] and Goldsmith et al. [21]. For more details, we refer to [2, 40].

1.2 Generalized connectivity and generalized edge-connectivity

The generalized connectivity of a graph G , introduced by Hager [22], is a natural and nice generalization of the ‘path’ version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an S -Steiner tree or a Steiner tree connecting S (or simply, an S -tree) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two S -Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint S -Steiner trees in G . Note that when $|S| = 2$ a minimal S -Steiner tree is just a path connecting the two vertices of S . For an integer k with $2 \leq k \leq n$, *generalized k -connectivity* (or *k -tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa(S) \mid S \subseteq V(G), |S| = k\}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G , that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G . By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. This concept appears to have been introduced by Hager in [22]. It is also studied in [8] for example, where the exact value of the generalized k -connectivity of complete graphs are obtained. Note that the generalized k -connectivity and the k -connectivity of a graph are indeed different. Take for example, the graph H_1 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. Then $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$. There are many results on the generalized connectivity or tree-connectivity, we refer to the book [35] and the papers [8, 32, 33, 36, 38, 34, 43].

The following Table 1 shows how the generalization proceeds.

	Classical connectivity	Generalized connectivity
Vertex subset	$S = \{x, y\} \subseteq V(G)$ ($ S = 2$)	$S \subseteq V(G)$ ($ S \geq 2$)
Set of Steiner trees	$\left\{ \begin{array}{l} \mathcal{P}_{x,y} = \{P_1, P_2, \dots, P_\ell\} \\ \{x, y\} \subseteq V(P_i) \\ E(P_i) \cap E(P_j) = \emptyset \\ V(P_i) \cap V(P_j) = \{x, y\} \end{array} \right.$	$\left\{ \begin{array}{l} \mathcal{T}_S = \{T_1, T_2, \dots, T_\ell\} \\ S \subseteq V(T_i) \\ E(T_i) \cap E(T_j) = \emptyset \\ V(T_i) \cap V(T_j) = S \end{array} \right.$
Local parameter	$\kappa(x, y) = \max \mathcal{P}_{x,y} $	$\kappa(S) = \max \mathcal{T}_S $
Global parameter	$\kappa(G) = \min_{x,y \in V(G)} \kappa(x, y)$	$\kappa_k(G) = \min_{S \subseteq V(G), S =k} \kappa(S)$

Table 1. Classical connectivity and generalized connectivity

As a natural counterpart of the generalized connectivity, we introduced in [38] the concept of generalized edge-connectivity, which is a generalization of the ‘path’ version definition of edge-connectivity. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint S -Steiner trees in G . For an integer k with $2 \leq k \leq n$, the *generalized k -edge-connectivity* $\lambda_k(G)$ of G is then defined as $\lambda_k(G) = \min\{\lambda(S) \mid S \subseteq V(G) \text{ and } |S| = k\}$. It is also clear that when $|S| = 2$, $\lambda_2(G)$ is nothing new but the standard edge-connectivity $\lambda(G)$ of G , that is, $\lambda_2(G) = \lambda(G)$, which is the reason why we address $\lambda_k(G)$ as the generalized edge-connectivity of G . Also set $\lambda_k(G) = 0$ when G is disconnected. Results on the generalized edge-connectivity can be found in [35, 37, 38].

1.3 Path connectivity and path edge-connectivity

Dirac [13] showed that in a $(k - 1)$ -connected graph there is a path through each k vertices. Related problems were inquired in [51]. In [23], Hager revised this statement to the question of how many internally disjoint paths P_i with the exception of a given set S of k vertices exist such that $S \subseteq V(P_i)$. The path connectivity of a graph G , introduced by Hager [23], is a natural specialization of the generalized connectivity and is also a natural generalization of the ‘path’ version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, a *Steiner path connecting S* or an *S -Steiner path* (or simply, an *S -path*) is a subgraph $P = (V', E')$ of G that is a path with $S \subseteq V'$. Note that an S -Steiner path is also a Steiner tree connecting S . Two S -Steiner paths P and P' are said to be *internally disjoint* if $E(P) \cap E(P') = \emptyset$ and $V(P) \cap V(P') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local path connectivity* $\pi_G(S)$ is the maximum number of internally disjoint S -Steiner paths in G , that is, we search for the maximum cardinality of edge-disjoint paths which contain S and are vertex-disjoint with the exception of the vertices in S . For an integer k with $2 \leq k \leq n$, the *path k -connectivity* is defined as $\pi_k(G) = \min\{\pi_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\pi_k(G)$ is the minimum value of $\pi_G(S)$

when S runs over all k -subsets of $V(G)$. Clearly, $\pi_1(G) = \delta(G)$ and $\pi_2(G) = \kappa(G)$. For $k \geq 3$, $\pi_k(G) \leq \kappa_k(G)$ holds because each path is also a tree.

The relation between generalized connectivity and path-connectivity are shown in the following Table 2.

	Generalized connectivity	Path-connectivity
Vertex subset	$S \subseteq V(G) \ (S \geq 2)$	$S \subseteq V(G) \ (S \geq 2)$
Set of Steiner trees	$\begin{cases} \mathcal{T}_S = \{T_1, T_2, \dots, T_\ell\} \\ S \subseteq V(T_i) \\ E(T_i) \cap E(T_j) = \emptyset \end{cases}$	$\begin{cases} \mathcal{P}_S = \{P_1, P_2, \dots, P_\ell\} \\ S \subseteq V(P_i) \\ E(P_i) \cap E(P_j) = \emptyset \end{cases}$
Local parameter	$\kappa(S) = \max \mathcal{T}_S $	$\pi(S) = \max \mathcal{P}_S $
Global parameter	$\kappa_k(G) = \min_{S \subseteq V(G), S =k} \kappa(S)$	$\pi_k(G) = \min_{S \subseteq V(G), S =k} \pi(S)$

Table 2 Two kinds of tree-connectivities

As a natural counterpart of path k -connectivity, we recently introduced the concept of path k -edge-connectivity. Two S -Steiner paths P and P' are said to be *edge-disjoint* if $E(P) \cap E(P') = \emptyset$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local path edge-connectivity* $\omega_G(S)$ is the maximum number of edge-disjoint S -Steiner paths in G . For an integer k with $2 \leq k \leq n$, the *path k -edge-connectivity* is defined as $\omega_k(G) = \min\{\omega_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\omega_k(G)$ is the minimum value of $\omega_G(S)$ when S runs over all k -subsets of $V(G)$. Clearly, we have

$$\begin{cases} \omega_k(G) = \delta(G), & \text{for } k = 1; \\ \omega_k(G) = \lambda(G), & \text{for } k = 2; \\ \omega_k(G) \leq \lambda_k(G), & \text{for } k \geq 3. \end{cases} \quad (1)$$

1.4 Application background of these parameters

In addition to being a natural combinatorial measure, the path k -(edge-)connectivity and generalized k -(edge-)connectivity can be motivated by their interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G , then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree for connecting a set of vertices is usually called a *Steiner tree*, and popularly used in the physical design of VLSI circuits (see [17, 18, 47]). In this application, a Steiner tree is needed to share an electric signal by a set of terminal nodes. Steiner tree is also used in computer communication networks (see [14]) and optical wireless communication networks (see [9]). Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The path k -connectivity and generalized

k -connectivity can serve for measuring the capability of a network G to connect any k vertices in G .

Product networks were proposed based upon the idea of using the cross product as a tool for “combining” two known graphs with established properties to obtain a new one that inherits properties from both [12]. Recently, there has been an increasing interest in a class of interconnection networks called Cartesian product networks; see [1, 12, 30, 33]. Lexicographic product is also studied extensively; see [24]. Some applications in networks of the lexicographic product were studied; see [3, 15, 31, 37].

1.5 Main results of this paper

The *lexicographic product* of two graphs G and H , written as $G \circ H$, is defined as follows: $V(G \circ H) = V(G) \times V(H)$, and two distinct vertices (u, v) and (u', v') of $G \circ H$ are adjacent if and only if either $(u, u') \in E(G)$ or $u = u'$ and $(v, v') \in E(H)$. Note that unlike the Cartesian product, the lexicographic product is a non-commutative product since $G \circ H$ is usually not isomorphic to $H \circ G$.

Observation 1.1 (1) *Let G be a connected graph. Then $\pi_3(G) \leq \omega_3(G) \leq \delta(G)$.*

(2) *Let G be a connected graph with minimum degree δ . If G has two adjacent vertices of degree δ , then $\omega_k(G) \leq \delta - 1$.*

In [27], Jain et al. obtained the following result.

Lemma 1.1 [27] *Let G be a graph and $S = \{v_1, v_2, v_3\}$ be a subset of $V(G)$. Assume that v_1 and v_2 are ℓ -edge-connected, and v_1 and v_3 are $\frac{\ell}{2}$ -edge-connected in G . Then G has $\frac{\ell}{2}$ edge-disjoint S -Steiner trees.*

In their proof, the $\frac{\ell}{2}$ edge-disjoint S -Steiner trees are actually edge-disjoint S -Steiner paths. So the following result is immediate.

Corollary 1.2 *For any connected graph G ,*

$$\omega_3(G) \geq \frac{1}{2}\lambda(G).$$

Moreover, the lower bound is sharp.

Li et al. [38] obtained the following result.

Lemma 1.2 [38] *For any connected graph G ,*

$$\lambda_k(G) \leq \lambda(G).$$

Moreover, the upper bound is sharp.

Yang and Xu [52] investigated the classical edge-connectivity of the lexicographic product of two graphs.

Lemma 1.3 [52] *Let G and H be two non-trivial graphs. If G is connected, then*

$$\lambda(G \circ H) = \min\{\lambda(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}.$$

We now derive an upper bound of $\omega_3(G \circ H)$ by the above lemmas and corollary.

Theorem 1.3 *Let G and H be two non-trivial graphs. If G is connected, then*

$$\omega_3(G \circ H) \leq \min\{2\omega_3(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}.$$

Proof. From Corollary 1.2, we have $\omega_3(G) \geq \frac{1}{2}\lambda(G)$, hence $\lambda(G) \leq 2\omega_3(G)$. By Lemma 1.2 and (1), $\omega_3(G \circ H) \leq \lambda_3(G \circ H) \leq \lambda(G \circ H)$. Furthermore, by Lemma 1.3, we have

$$\begin{aligned} \omega_3(G \circ H) &\leq \lambda(G \circ H) \\ &= \min\{\lambda(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\} \\ &\leq \min\{2\omega_3(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}. \end{aligned}$$

In Section 2, we will show the following lower bound of $\omega_3(G \circ H)$.

Theorem 1.4 *Let G and H be two graphs. Then*

$$\omega_3(G \circ H) \geq \omega_3(G) \left\lfloor \frac{3|V(H)|}{4} \right\rfloor.$$

Moreover, the bound is sharp.

The following observation is immediate.

Observation 1.5 *For any connected graph G , if $\omega_3(G) \geq \ell$, then $\delta(G) \geq \ell$ and there are at most two vertices with degree ℓ .*

Example 1: Set $G = P_n$ and $H = 3K_1$. Clearly, $\omega_3(G) = 1$ and $|V(H)| = 3$. From Theorem 1.4, we obtain that $\omega_3(P_n \circ 3K_1) \geq 2$. Note that there are at least 6 vertices with minimum degree 3. From Observation 1.5, we have $\omega_3(P_n \circ 3K_1) \leq 2$. So $\omega_3(P_n \circ 3K_1) = 2$. So the bound in Theorem 1.4 is sharp.

2 Proof of Theorem 1.4

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$, respectively. Then $V(G \circ H) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u_i, v) \mid 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq m\}$. In the sequel, let K_n and P_n denote the complete graph of order n and path of order n , respectively. If G is a connected graph and $x, y \in V(G)$, then the *distance* $d_G(x, y)$ between x and y is the length of a shortest path connecting x and y in G . The degree of a vertex v in G is denoted by $d_G(v)$.

We now introduce the general idea of the proof of Theorem 1.4. In Section 2.1, we first study the path 3-edge-connectivity of the lexicographic product of a path P and a graph H and show $\omega_3(P \circ H) \geq \lfloor \frac{3|V(H)|}{4} \rfloor$. After this preparation, we consider the graph $G \circ H$ and prove $\omega_3(G \circ H) \geq \omega_3(G) \lfloor \frac{3|V(H)|}{4} \rfloor$ in Subsection 2.2.

Before realizing the above two steps, we introduce the following two well-known lemmas, which will be used later.

Given a vertex x and a set U of vertices, an (x, U) -*fan* is a set of paths from x to U such that any two of them share only the vertex x . The size of a (x, U) -fan is the number of internally disjoint paths from x to U .

Lemma 2.1 (Fan Lemma, [50], p-170) *A graph is k -connected if and only if it has at least $k + 1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an (x, U) -fan of size k .*

Lemma 2.2 (Expansion Lemma, [50], p-162) *If G is a k -connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G , then G' is k -connected.*

Let G be a k -connected graph. Choose $U \subseteq V(G)$ with $|U| = k$. Then the graph G' is obtained from G by adding a new vertex y and joining each vertex of U and the vertex y . We call this operation *an expansion operation at y and U* . Denote the resulting graph G' by $G' = G \vee \{y, U\}$.

2.1 Lexicographic product of a path and a connected graph

To start with, we show the following proposition, which is a preparation of the next subsection.

Proposition 2.1 *Let H be a connected graph and P_n be a path with n vertices. Then $\omega_3(P_n \circ H) \geq \lfloor \frac{3|V(H)|}{4} \rfloor$. Moreover, the bound is sharp.*

Set $V(H) = \{v_1, v_2, \dots, v_m\}$ and $V(P_n) = \{u_1, u_2, \dots, u_n\}$. Without loss of generality, let u_i and u_j be adjacent if and only if $|i - j| = 1$, where $1 \leq i \neq j \leq n$. It suffices to show that $\omega_{P_n \circ H}(S) \geq \lfloor \frac{3m}{4} \rfloor$ for any $S = \{x, y, z\} \subseteq V(P_n \circ H)$, that is, there exist

$\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths in $P_n \circ H$. We proceed our proof by the following three lemmas.

Lemma 2.3 *If x, y, z belongs to the same $V(H(u_i))$ ($1 \leq i \leq n$), then there exist $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths.*

Proof. Without loss of generality, we assume $x, y, z \in V(H(u_1))$. For any four vertices in $H(u_2)$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4})$, where $j_i \in \{1, 2, \dots, m\}$ and $1 \leq i \leq 4$. For any vertex in $H(u_2) - \{x, y, z\}$, we say (u_1, v_{i_1}) . Then the path induced by the edges in $\{x(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_4}), (u_2, v_{j_4})z\}$, the path induced by the edges in $\{y(u_2, v_{j_1}), (u_2, v_{j_1})(u_1, v_{i_1}), (u_1, v_{i_1})(u_2, v_{j_4}), (u_2, v_{j_4})x, x(u_2, v_{j_3}), (u_2, v_{j_3})z\}$ and the path induced by the edges in

$$\{x(u_2, v_{j_1}), (u_2, v_{j_1})z, z(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_1}), (u_1, v_{i_1})(u_2, v_{j_3}), (u_2, v_{j_3})y\}$$

are 3 edge-disjoint S -Steiner paths. For the arbitrariness of the four vertices in $H(u_2)$ and the vertex in $H(u_1)$, we can obtain $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths, as desired. ■

Lemma 2.4 *If only two vertices of $\{x, y, z\}$ belong to some copy $H(u_i)$ ($1 \leq i \leq n$), then there exist $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths.*

Proof. We may assume $x, y \in V(H(u_1))$ and $z \in V(H(u_i))$ ($2 \leq i \leq n$). In the following argument, we can see that this assumption has no impact on the correctness of our proof. We distinguish the following two cases to show this lemma.

Case 1. $i = 2$.

We may assume $x, y \in V(H(u_1))$ and $z \in V(H(u_2))$. Let x', y' be the vertices corresponding to x, y in $H(u_2)$, and let z' be the vertex corresponding to z in $H(u_1)$.

Figure 2.1 Graphs for Lemma 2.4.

Suppose $z' \notin \{x, y\}$. Without loss of generality, let $\{x, y, z'\} = \{(u_1, v_j) \mid 1 \leq j \leq 3\}$ and $\{x', y', z\} = \{(u_2, v_j) \mid 1 \leq j \leq 3\}$. Then the path Q_1 induced by the edges in $\{xz, yz\}$, the path Q_2 induced by the edges in $\{yy', y'x, xx', x'z', z'z\}$ and the path Q_3 induced by the edges in $\{x(u_2, v_4), (u_2, v_4)y, yx', x'(u_1, v_4), (u_1, v_4)z\}$ are 3 edge-disjoint S -Steiner paths; see Figure 2.1 (a). For any four vertices in $H(u_1) - \{x, y, z', (u_1, v_4)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3}), (u_1, v_{i_4})$, where $i_r \in \{5, 6, \dots, m\}$ and $1 \leq r \leq 4$. For any four vertices in $H(u_2) - \{x', y', z, (u_2, v_4)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4})$, where $j_r \in \{5, 6, \dots, m\}$ and $1 \leq r \leq 4$. Then the path induced by the edges in $\{y(u_2, v_{j_1}), (u_2, v_{j_1})x, x(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_1}), (u_1, v_{i_1})z\}$, the path induced by the edges in $\{x(u_2, v_{j_3}), (u_2, v_{j_3})y, y(u_2, v_{j_4}), (u_2, v_{j_4})(u_1, v_{i_2}), (u_1, v_{i_2})z\}$ and the path induced by the edges in $\{x(u_2, v_{j_4}), (u_2, v_{j_4})(u_1, v_{i_3}), (u_1, v_{i_3})z, z(u_1, v_{i_4}), (u_1, v_{i_4})(u_2, v_{j_2}), (u_2, v_{j_2})y\}$ are 3 edge-disjoint S -Steiner paths; see Figure 2.1 (a). For the arbitrariness of the four vertices in $H(u_1) - \{x, y, z', (u_1, v_4)\}$ and four vertices in $H(u_2) - \{x', y', z, (u_2, v_4)\}$, we can obtain $\lfloor \frac{3(m-4)}{4} \rfloor$ edge-disjoint S -Steiner paths. These paths together with Q_1, Q_2, Q_3 are $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths, as desired.

Suppose $z' \in \{x, y\}$. Without loss of generality, let $z' = y$, $\{x, y\} = \{(u_1, v_1), (u_1, v_2)\}$ and $\{x', z\} = \{(u_2, v_1), (u_2, v_2)\}$. Then the path Q_1 induced by the edges in $\{xz, yz\}$, the path Q_2 induced by the edges in $\{xx', x'y, y(u_2, v_3), (u_2, v_3)(u_1, v_3), (u_1, v_3)z\}$ and the path Q_3 induced by the edges in $\{y(u_2, v_4), (u_2, v_4)x, x(u_2, v_3), (u_2, v_3)(u_1, v_4), (u_1, v_4)z\}$ are 3 edge-disjoint S -Steiner paths; see Figure 2.1 (b). For any four vertices in $H(u_1) - \{x, y, (u_1, v_3), (u_1, v_4)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3}), (u_1, v_{i_4})$, where $i_r \in \{5, 6, \dots, m\}$ and $1 \leq r \leq 4$. For any four vertices in $H(u_2) - \{x', z, (u_2, v_3), (u_2, v_4)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4})$, where $j_r \in \{5, 6, \dots, m\}$ and $1 \leq r \leq 4$. Similarly to the proof of the above case, we can get 3 edge-disjoint S -Steiner paths; see Figure 2.1 (b). For the arbitrariness of the four vertices in $H(u_1) - \{x, y, (u_1, v_3), (u_1, v_4)\}$ and four vertices in $H(u_2) - \{x', z, (u_2, v_3), (u_2, v_4)\}$, we can obtain $\lfloor \frac{3(m-4)}{4} \rfloor$ edge-disjoint S -Steiner paths. These path together with Q_1, Q_2, Q_3 are $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths, as desired.

Case 2. $i \geq 3$.

Let $P' = u_2 u_3 \dots u_i$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 2.1, there is a (z, U) -fan in $P' \circ H$, where $U = V(H(u_2)) = \{(u_2, v_j) \mid 1 \leq j \leq m\}$. Thus, there exist m internally disjoint paths $P_1, P_2, \dots, P_m$ such that P_j ($1 \leq j \leq m$) is a path connecting z and (u_2, v_j) . Without loss of generality, let $\{x, y\} = \{(u_1, v_j) \mid 1 \leq j \leq 2\}$. For any four vertices in $H(u_1) - \{x, y\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2})$, where $i_r \in \{3, 4, \dots, m\}$ and $1 \leq r \leq 2$. For any four vertices in $H(u_2)$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4})$, where $j_r \in \{1, 2, \dots, m\}$ and $1 \leq r \leq 4$. Then the path induced by the edges in $\{x(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_1})\} \cup E(P_{j_1})$, the path induced by the edges in $\{y(u_2, v_{j_4}), (u_2, v_{j_4})x, x(u_2, v_{j_3})\} \cup E(P_{j_3})$ and the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})(u_1, v_{i_1}), (u_1, v_{i_1})(u_2, v_{j_2}), (u_1, v_{i_2})(u_2, v_{j_4}), (u_1, v_{i_2})(u_2, v_{j_3}), (u_2, v_{j_3})y\} \cup E(P_{j_2}) \cup E(P_{j_4})$ are 3 edge-disjoint S -Steiner paths;

Figure 2.2 Graphs for Case 2 of Lemma 2.4 and Case 1 of Lemma 2.5.

see Figure 2.2 (a). For the arbitrariness of the two vertices $(u_1, v_{i_1}), (u_1, v_{i_2})$ in $H(u_1) - \{x, y\}$ and four vertices $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4})$ in $H(u_2)$, we can obtain $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths, as desired. $\blacksquare$

Lemma 2.5 *If x, y, z are contained in distinct $H(u_i)$ s, then there exist $m-1$ edge-disjoint S -Steiner paths.*

Proof. We distinguish the following three cases to prove this lemma.

Case 1. $d_{P_n \circ H}(x, y) = d_{P_n \circ H}(y, z) = 1$.

Without loss of generality, let $x \in V(H(u_1)), y \in V(H(u_2)), z \in V(H(u_3))$. From the structure of lexicographic product graph, we may assume that $x = (u_1, v_1)$ and $y = (u_2, v_1)$ and $z = (u_3, v_1)$. In the following argument, we can see that this assumption has no influence on the correctness of our proof. Then the path P'_1 induced by the edges in $\{xy, yz\}$, the paths P_j induced by the edges in $\{x(u_2, v_{2j}), (u_1, v_{2j})(u_2, v_{2j}), y(u_1, v_{2j}), y(u_3, v_{2j+1}), (u_2, v_{2j+1})(u_3, v_{2j+1}), z(u_2, v_{2j+1})\}$ ($1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor$) and the paths Q_j induced by the edges in $\{x(u_2, v_{2j+1}), (u_1, v_{2j+1})(u_2, v_{2j+1}), y(u_1, v_{2j+1}), y(u_3, v_{2j}), (u_2, v_{2j})(u_3, v_{2j}), z(u_2, v_{2j})\}$ ($1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor$) are $m-1$ or m edge-disjoint S -Steiner paths; see Figure 2.2 (b).

Case 2. $d_{P_n \circ H}(x, y) = 1$ and $d_{P_n \circ H}(y, z) \geq 2$.

We may assume that $x \in V(H(u_1)), y \in V(H(u_2)), z \in V(H(u_i))$ ($4 \leq i \leq n$). In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let $P' = u_3 u_4 \cdots u_i$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 2.1, there is a (z, U) -fan in $P' \circ H$, where $U = V(H(u_3)) = \{(u_3, v_j) \mid 1 \leq j \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1, P_2, \dots, P_m$ such that each P_j ($1 \leq j \leq m$) is a path connecting z and (u_3, v_j) . Without loss of generality, let $x = (u_1, v_1)$ and $y = (u_2, v_1)$. In the following argument, we can see that this assumption has no influence on the correctness of our proof.

Then the paths Q_1 induced by the edges in $\{xy, y(u_3, v_1)\} \cup E(P_1)$ and the paths Q_j induced by the edges in $\{x(u_2, v_j), (u_1, v_j)(u_2, v_j), y(u_1, v_j), y(u_3, v_j)\} \cup E(P_j)$ ($2 \leq j \leq m$) are m edge-disjoint S -Steiner paths, as desired.

Case 3. $d_{P_n \circ H}(x, y) \geq 2$ and $d_{P_n \circ H}(y, z) \geq 2$.

We may assume that $x \in V(H(u_i))$, $y \in V(H(u_j))$, $z \in V(H(u_k))$, where $i < j < k$, $|j - i| \geq 2$, $|k - j| \geq 2$, $1 \leq i \leq n - 5$, $3 \leq j \leq n - 2$ and $5 \leq k \leq n$. Let $P' = u_i, u_{i+1}, \dots, u_{j-1}$ and $P'' = u_{j+1}, u_{j+2}, \dots, u_k$. Then P' and P'' are two paths of order at least 2. Since $\kappa(P' \circ H) \geq m$, it follows from Lemma 2.2 that, if we add the vertex y to $P' \circ H$ and join an edge from y to each (u_{j-1}, v_r) ($1 \leq r \leq m$), then $\kappa((P' \circ H) \vee \{y, V(H(u_{j-1}))\}) \geq m$. By the same reason, $\kappa((P'' \circ H) \vee \{y, V(H(u_{j+1}))\}) \geq m$. From Menger's Theorem, there exist m internally disjoint paths connecting x and y in $(P' \circ H) \vee \{y, V(H(u_{j-1}))\}$, say $P'_1, P'_2, \dots, P'_m$. Note that these paths are also edge-disjoint. Also, there exist m edge-disjoint paths connecting y and z in $(P'' \circ H) \vee \{y, V(H(u_{j+1}))\}$, say $P''_1, P''_2, \dots, P''_m$. Note that the union of any path in $\{P'_i \mid 1 \leq i \leq m\}$ with any path in $\{P''_j \mid 1 \leq j \leq m\}$ is an S -Steiner path. Then the paths Q_i induced by the edges in $E(P'_i) \cup E(P''_i)$ ($1 \leq i \leq m$) are m edge-disjoint S -Steiner paths, as desired. $\blacksquare$

From Lemmas 2.3, 2.4 and 2.5, we conclude that, for any $S \subseteq V(P_n \circ H)$, there exist $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths which use no edge in each $H(u_i)$ ($1 \leq i \leq n$), and hence $\omega_{P_n \circ H}(S) \geq \lfloor \frac{3m}{4} \rfloor$. From the arbitrariness of S , we have $\omega_3(P_n \circ H) \geq \lfloor \frac{3m}{4} \rfloor$. The proof of Proposition 2.1 is complete.

2.2 Lexicographic product of two general graphs

After the above preparations, we are ready to prove Theorem 1.4 in this subsection.

Proof of Theorem 1.4: Set $\omega_3(G) = \ell$. Recall that $V(G) = \{u_1, u_2, \dots, u_n\}$, $V(H) = \{v_1, v_2, \dots, v_m\}$. From the definition of $\omega_3(G \circ H)$, we need to prove that $\omega_{G \circ H}(S) \geq \ell \lfloor \frac{3m}{4} \rfloor$ for any $S = \{x, y, z\} \subseteq V(G \circ H)$. Furthermore, it suffices to show that there exist $\ell \lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths in $G \circ H$. Clearly, $V(G \circ H) = \bigcup_{i=1}^n V(H(u_i))$. Without loss of generality, let $x \in V(H(u_i))$, $y \in V(H(u_j))$ and $z \in V(H(u_k))$.

Suppose that x, y, z belongs to the same $V(H(u_i))$ ($1 \leq i \leq n$). Without loss of generality, let $x, y, z \in V(H(u_1))$. Since $\delta(G) \geq \omega_3(G) = \ell$, it follows that the vertex u_1 has ℓ neighbors in G , say $u_2, u_3, \dots, u_{\ell+1}$. From Proposition 2.1, there exist $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths in $P_i \circ H$ where $P_i = u_1 u_i$ ($2 \leq i \leq \ell + 1$). So there are $\ell \lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths in $G \circ H$, as desired.

Suppose that only two vertices of $\{x, y, z\}$ belong to some copy $H(u_i)$ ($1 \leq i \leq n$). Without loss of generality, let $x, y \in H(u_1)$ and $z \in H(u_2)$. Observe that $\lambda(G) \geq \omega_3(G) =$

ℓ . Therefore, there exist ℓ edge-disjoint paths connecting u_1 and u_2 in G , say $P_1, P_2, \dots, P_\ell$. From Proposition 2.1, there exist $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths in $P_i \circ H$ by Proposition 2.1. Observe that $\bigcup_{i=1}^\ell P_i$ is a subgraph of G and $(\bigcup_{i=1}^\ell P_i) \circ H$ is a subgraph of $G \circ H$. So the total number of the edge-disjoint S -Steiner paths is $\ell \lfloor \frac{3m}{4} \rfloor$, as desired.

Suppose that x, y, z are contained in distinct $H(u_i)$ s. Without loss of generality, let $x \in H(u_1)$, $y \in H(u_2)$ and $z \in H(u_3)$. Since $\omega_3(G) = \ell$, it follows that there exist ℓ edge-disjoint Steiner paths connecting $\{u_1, u_2, u_3\}$ in G , say $P_1, P_2, \dots, P_\ell$. From Proposition 2.1, there exist $\lfloor \frac{3m}{4} \rfloor$ edge-disjoint S -Steiner paths in $P_i \circ H$ by Proposition 2.1. Observe that $\bigcup_{i=1}^\ell P_i$ is a subgraph of G and $(\bigcup_{i=1}^\ell P_i) \circ H$ is a subgraph of $G \circ H$. Therefore, the total number of the edge-disjoint S -Steiner paths is $\ell \lfloor \frac{3m}{4} \rfloor$, as desired.

From the above argument, we conclude that, for any $S \subseteq V(G \circ H)$, $\omega_{G \circ H}(S) \geq \omega_{(\bigcup_{i=1}^\ell P_i) \circ H}(S) \geq \ell \lfloor \frac{3m}{4} \rfloor$, which implies that $\omega_3(G \circ H) \geq \ell \lfloor \frac{3m}{4} \rfloor = \omega_3(G) \lfloor \frac{3|V(G)|}{4} \rfloor$. The proof is now complete. $\blacksquare$

3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of lexicographic product networks.

3.1 Two-dimensional grid graph and r -dimensional mesh

A *two-dimensional grid graph* is an $m \times n$ graph $G_{n,m}$ that is the Cartesian product $P_n \square P_m$ of path graphs on m and n vertices. See Figure 3.1 (a) for the case $m = 3$. For more details on grid graph, we refer to [6, 26]. The network $P_n \circ P_m$ is the lexicographic product $P_n \circ P_m$ of path graphs on m and n vertices. See Figure 3.1 (b) for the case $m = 3$. For more details on $P_n \circ P_m$, we refer to [39].

Figure 3.1 (a) Two-dimensional grid graph $G_{n,3}$; (b) The network $P_n \circ P_3$.

Proposition 3.1 For network $P_n \circ P_m$ ($n \geq 4, m \geq 3$), we have

$$\left\lfloor \frac{3m}{4} \right\rfloor \leq \omega_3(P_n \circ P_m) \leq m + 1.$$

Proof. From Theorem 1.4, we have $\omega_3(P_n \circ P_m) \geq \lfloor \frac{3m}{4} \rfloor$. From Theorem 1.3, $\omega_3(P_n \circ P_m) \leq \min\{2\omega_3(P_n)|V(P_m)|^2, \delta(P_m) + \delta(P_n)|V(P_m)|\} = \min\{2m^2, 1 + m\} = m + 1$. ■

An r -dimensional mesh is the Cartesian product of r paths. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An r -dimensional hypercube is a special case of a r -dimensional mesh, in which the r linear arrays are all of size 2; see [29]. The network $P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}$ is investigated in [39].

Proposition 3.2 For r -dimensional mesh $P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}$,

$$\prod_{i=2}^r \left\lfloor \frac{3m_i}{4} \right\rfloor \leq \omega_3(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}) \leq 2^{r-1} \prod_{i=2}^r m_i^2.$$

Proof. From Theorem 1.4, we have $\omega_3(G \circ H) \geq \omega_3(G) \lfloor \frac{3|V(H)|}{4} \rfloor$ for any two graphs G and H , and hence

$$\begin{aligned} \omega_3(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}) &\geq \omega_3((P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}}) \circ P_{m_r}) \\ &\geq \omega_3(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}}) \left\lfloor \frac{3m_r}{4} \right\rfloor \\ &\geq \omega_3(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-2}}) \left\lfloor \frac{3m_{r-1}}{4} \right\rfloor \left\lfloor \frac{3m_r}{4} \right\rfloor \\ &\geq \dots \\ &\geq \omega_3(P_{m_1}) \prod_{i=2}^r \left\lfloor \frac{3m_i}{4} \right\rfloor \\ &= \prod_{i=2}^r \left\lfloor \frac{3m_i}{4} \right\rfloor. \end{aligned}$$

From Theorem 1.3, we have $\omega_3(G \circ H) \leq 2\omega_3(G)|V(H)|^2$ for any two graphs G and H , and hence

$$\begin{aligned} \omega_3(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}) &= \omega_3((P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}}) \circ P_{m_r}) \\ &\leq 2\omega_3(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}})m_r^2 \\ &\leq 2^2\omega_3(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-2}})m_{r-1}^2m_r^2 \\ &\leq \dots \\ &\leq 2^{r-1}\omega_3(P_{m_1})m_2^2 \cdots m_{r-1}^2m_r^2 \\ &= 2^{r-1} \prod_{i=2}^r m_i^2. \end{aligned}$$

3.2 r -dimensional torus

An r -dimensional torus is the Cartesian product of r cycles $C_{m_1}, C_{m_2}, \dots, C_{m_r}$ of size at least three. The cycles C_{m_i} are not necessary to have the same size. Ku et al. [30] showed that there are r edge-disjoint spanning trees in an r -dimensional torus. The network $C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_r}$ is investigated in [39]. Here, we consider the network constructed by $C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_r}$.

Proposition 3.3 For network $C_n \circ C_m$ ($n \geq 3, m \geq 3$),

$$\left\lfloor \frac{3m}{4} \right\rfloor \leq \omega_3(C_n \circ C_m) \leq 2m + 2.$$

Proof. From Theorem 1.4, we have $\omega_3(C_n \circ C_m) \geq \lfloor \frac{3m}{4} \rfloor$. From Theorem 1.3, $\omega_3(C_n \circ C_m) \leq \min\{2\omega_3(C_n)|V(C_m)|^2, \delta(C_m) + \delta(C_n)|V(C_m)|\} = \min\{2m^2, 2 + 2m\} = 2m + 2$. ■

Similarly to the proof of Proposition 3.2, we can derive the following result.

Proposition 3.4 For r -dimensional mesh $C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_r}$,

$$\prod_{i=2}^r \left\lfloor \frac{3m_i}{4} \right\rfloor \leq \omega_3(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_r}) \leq 2^{r-1} \prod_{i=2}^r m_i^2.$$

3.3 r -dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \geq 2$. An r -dimensional generalized hypercube [12, 16] is the Cartesian product of r cliques. Hager [22] obtained the exact value of path k -connectivity of complete graphs.

Lemma 3.1 [22] Let k, n be two integers with $2 \leq k \leq n$, and K_n be a complete graph of order n . Then

$$\pi_k(K_n) = \left\lfloor \frac{2n + k^2 - 3k}{2(k-1)} \right\rfloor.$$

Observation 3.5 Let K_n be a complete graph of order n . Then

$$\omega_3(K_n) \leq 2 + \left\lfloor \frac{3n - 11}{4} \right\rfloor.$$

Proof. For any $S \subseteq V(G)$ and $|S| = 3$, there are at most one S -Steiner path of size 2 and at most one S -Steiner path of size 3. Then each of the other S -Steiner paths must occupy four edges in $E_G[S, \bar{S}]$, where $\bar{S} = V(G) - S$. So $\omega_3(K_n) \leq 2 + \lfloor \frac{3(n-3)+3-5}{4} \rfloor = 2 + \lfloor \frac{3n-11}{4} \rfloor$. ■

We have the following:

Proposition 3.6 For network $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}$,

$$\left\lfloor \frac{\prod_{i=1}^r m_i}{2} \right\rfloor \leq \omega_3(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}) \leq 2 + \left\lfloor \frac{3 \prod_{i=1}^r m_i - 1}{4} \right\rfloor$$

Proof. From the definition of lexicographic product, $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}$ is a complete graph. From Lemma 3.1 and Observation 1.1, we have

$$\omega_3(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}) \geq \pi_3(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}) = \left\lfloor \frac{\prod_{i=1}^r m_i}{2} \right\rfloor.$$

From Observation 3.5, we have $\omega_3(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}) \leq 2 + \left\lfloor \frac{3 \prod_{i=1}^r m_i - 1}{4} \right\rfloor$, as desired. ■

3.4 r -dimensional hyper Petersen network

An r -dimensional hyper Petersen network HP_r is the product of the well-known Petersen graph and Q_{r-3} [10], where $r \geq 3$ and Q_{r-3} denotes an $(r-3)$ -dimensional hypercube. The cases $r = 3$ and 4 of hyper Petersen networks are depicted in Figure 3.2. Note that HP_3 is just the Petersen graph; see Figure 3.2 (a). We now consider the network HL_r , which is the lexicographic product of the Petersen graph and Q_{r-3} , where $n \geq 3$ and Q_{r-3} denotes an $(r-3)$ -dimensional hypercube. Note that HL_3 is just the Petersen graph, and HL_4 is a graph obtained from two copies of the Petersen graph by add one edge between one vertex in a copy of the Petersen graph and one vertex in another copy. See Figure 3.2 (c) for an example (We only show the edges $v_1 u_i$ ($1 \leq i \leq 10$)).

Figure 3.2 (a) Petersen graph; (b) The network HP_4 ; (c) The structure of HL_4 .

Mao [39] obtained the following result for path 3-connectivity.

Lemma 3.2 (1) For network HL_3 , $\pi_3(HL_3) = 2$.

(2) For network HL_4 , $\pi_3(HL_4) \geq 6$.

Proposition 3.7 (1) For network HL_3 , $\omega_3(HL_3) = 2$.

(2) For network HL_4 , $6 \leq \omega_3(HL_4) \leq 13$.

Proof. (1) Note that HL_3 is just the Petersen graph. Set $G = HL_3$. Since $\delta(G) = 3$ and there are two adjacent vertices of degree 3, it follows that $\omega_3(G) \leq 2$ by (2) of Observation 1.1. From (1) of Observation 1.1 and Lemma 3.2, we have $\omega_3(G) \geq \pi_3(G) = 2$.

(2) From Lemma 3.2 and (1) of Observation 1.1, we have $\omega_3(G) \geq \pi_3(G) \geq 6$. From Theorem 1.3, we have $\omega_3(HL_4) = \omega_3(P_2 \circ HL_3) \leq \min\{2\omega_3(P_2)|V(HL_3)|^2, \delta(HL_3) + \delta(P_2)|V(HL_3)|\} = \min\{2 \cdot 10^2, 3 + 10\} = 13$, as desired.

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