The 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the PMC model and MM* model *

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Abstract

Diagnosability of a multiprocessor system is one important study topic. In 2012, Peng et al. proposed a measure for fault tolerance of the system, which is called the g -good-neighbor diagnosability that restrains every fault-free node containing at least g fault-free neighbors. In 2016, Zhang et al. proposed a new measure for fault diagnosis of the system, namely, the g -extra diagnosability, which restrains that every fault-free component has at least $(g + 1)$ fault-free nodes. As a favorable topology structure of interconnection networks, the n -dimensional alternating group graph network AN_n has many good properties. In this paper, we obtain that (a) the 2-good-neighbor diagnosability of AN_n is $3n - 7$ for $n \geq 4$ under the PMC model and MM* model; (b) the 2-extra diagnosability of AN_n is $3n - 7$ for $n \geq 4$ under the PMC model, and the 2-extra diagnosability of AN_n is $3n - 7$ for $n \geq 5$ under the MM* model. These results are optimal with respect to 2-extra diagnosability of AN_n .

Key words: Interconnection network; Combinatorics; Diagnosability; Connectivity; Alternating group graph network

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1. Introduction

Many multiprocessor systems have interconnection networks (networks for short) as underlying topologies. A network is usually represented by a graph where nodes represent processors and links represent communication links between processors. So we use graphs and networks interchangeably. For the system, study of the topological properties of its network is important. Furthermore, some processors may fail in the system, so processor fault identification plays an important role for reliable computing. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. A system is said to be t -diagnosable if all faulty processors can be identified without replacement, provided that the number of faults presented does not exceed t . The diagnosability $t(G)$ of a system G is the maximum value of t such that G is t -diagnosable [6,7,11]. For a t -diagnosable system, Dahbura and Masson [6] proposed an algorithm with time complex $O(n^{2.5})$, which can effectively identify the set of faulty processors.

Several diagnosis models were proposed to identify the faulty processors. One major approach is the Preparata, Metze, and Chien's (PMC) diagnosis model introduced by Preparata et al. [15]. The diagnosis of the system is achieved through two linked processors testing each other. Another major approach is the Maeng and Malek's (MM) diagnosis model, namely the comparison diagnosis model, proposed by Maeng and Malek [13]. In the MM model, to diagnose a system, a node sends the same task to two of its neighbors, and then compares their responses. Sengupta and Dahbura [17] proposed a special case of the MM model, called the MM* model, in which each node must test its any pair of adjacent nodes. The PMC model and MM* model have been widely used in the pioneering work. In 2005, Lai et al. [11] introduced a restricted diagnosability of a multiprocessor system called conditional diagnosability. They considered the situation that no fault set can contain all the neighbors of any vertex in the system. In 2012, Peng et al. [14] proposed a measure for fault diagnosis of systems, namely, the g -good-neighbor diagnosability (which is also called the g -good-neighbor conditional diagnosability), which requires that every fault-free node has at least g fault-free neighbors. In [14], they studied the g -good-neighbor diagnosability of the n -dimensional hypercube

under the PMC model. In 2016, Wang and Han [18] studied the g -good-neighbor diagnosability of the n -dimensional hypercube under the MM^* model. In 2016, Xu et al. [23] studied the g -good-neighbor diagnosability of complete cubic networks under the PMC model and MM^* model. Yuan et al. [24,25] studied that the g -good-neighbor diagnosability of the k -ary n -cube ($k \geq 3$) under the PMC model and MM^* model. As a favorable topology structure of interconnection networks, the Cayley graph CT_n generated by the transposition tree Γ_n has many good properties. In [19,20], Wang et al. studied the g -good-neighbor diagnosability of CT_n under the PMC model and MM^* model for $g = 1, 2$. In 2016, Zhang and Yang [26] proposed a new measure for fault diagnosis of the system, namely, the g -extra diagnosability, which restrains that every fault-free component has at least $(g + 1)$ fault-free nodes. In [26], they studied the g -extra diagnosability of the n -dimensional hypercube under the PMC model and MM^* model. In 2016, Wang et al. [21] studied the 2-extra diagnosability of the bubble-sort star graph BS_n under the PMC model and MM^* model. As a favorable topology structure of interconnection networks, the n -dimensional alternating group graph network AN_n has many good properties. In this paper, it is proved that (a) the 2-extra connectivity of AN_n is $3n - 9$ and this result is optimal; (b) the 2-good-neighbor diagnosability of AN_n is $3n - 7$ for $n \geq 4$ under the PMC model and MM^* model; (c) the 2-extra diagnosability of AN_n is $3n - 7$ for $n \geq 4$ under the PMC model, and the 2-extra diagnosability of AN_n is $3n - 7$ for $n \geq 5$ under the MM^* model. These results are optimal with respect to 2-extra diagnosability of AN_n .

2. Preliminaries

In this section, some definitions and notations needed for our discussion, the alternating group graph network, the PMC model and the MM^* model are introduced.

2.1. Notations

A multiprocessor system is modeled as an undirected simple graph $G = (V, E)$, whose vertices (nodes) represent processors and edges (links) represent communication links. Given a nonempty vertex subset V' of V , the *induced subgraph* by V' in G , denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of

G with both endpoints in V' . The *degree* $d_G(v)$ of a vertex v is the number of edges incident with v . The *minimum degree* of a vertex in G is denoted by $\delta(G)$. For any vertex v , we define the *neighborhood* $N_G(v)$ of v in G to be the set of vertices adjacent to v . u is called a neighbor vertex or a neighbor of v for $u \in N_G(v)$. Let $S \subseteq V$. We use $N_G(S)$ to denote the set $\cup_{v \in S} N_G(v) \setminus S$. For neighborhoods and degrees, we will usually omit the subscript for the graph when no confusion arises. A graph G is said to be k -regular if for any vertex v , $d_G(v) = k$. The *connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left when G is complete. A complete graph of order n is denoted by K_n . A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y ; such a partition (X, Y) is called a bipartition of the graph, and X and Y its parts. We denote a bipartite graph G with bipartition (X, Y) by $G = (X, Y; E)$. If $G = (X, Y; E)$ is simple and every vertex in X is joined to every vertex in Y , then $G = (X, Y; E)$ is called a complete bipartite graph, denoted by $K_{n,m}$, where $|X| = n$ and $|Y| = m$. Let F_1 and F_2 be two distinct subsets of V , and let the *symmetric difference* $F_1 \Delta F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Let $B_1, \dots, B_k$ ($k \geq 2$) be the components of $G - F_1$. If $|V(B_1)| \leq \dots \leq |V(B_k)|$ ($k \geq 2$), then B_k is called the maximum component of $G - F_1$. For graph-theoretical terminology and notation not defined here we follow [2].

Let $G = (V, E)$ be a connected graph. A fault set $F \subseteq V$ is called a *g -good-neighbor faulty set* if $|N(v) \cap (V \setminus F)| \geq g$ for every vertex v in $V \setminus F$. A *g -good-neighbor cut* of a graph G is a g -good-neighbor faulty set F such that $G - F$ is disconnected. The minimum cardinality of g -good-neighbor cuts is said to be the *g -good-neighbor connectivity* of G , denoted by $\kappa^{(g)}(G)$. A fault set $F \subseteq V$ is called a *g -extra faulty set* if every component of $G - F$ has at least $(g + 1)$ vertices. A *g -extra cut* of G is a g -extra faulty set F such that $G - F$ is disconnected. The minimum cardinality of g -extra cuts is said to be the *g -extra connectivity* of G , denoted by $\tilde{\kappa}^{(g)}(G)$.

Proposition 2.1. ([16]) *Let G be a connected graph. Then $\tilde{\kappa}^{(g)}(G) \leq \kappa^{(g)}(G)$. In particular, $\kappa^{(1)}(G) = \tilde{\kappa}^{(1)}(G)$.*

2.2. The PMC model and the MM* model

Under the PMC model [15], to diagnose a system G , two adjacent nodes in G have the capability of performing tests on each other. For two adjacent nodes u and v in $V(G)$, the test performed by u on v is represented by the ordered pair (u, v) . The outcome of a test (u, v) is 1 (resp. 0) if u evaluates v as faulty (resp. fault-free). In the PMC model, we usually assume that the testing result is reliable (resp. unreliable) if the node u is fault-free (resp. faulty). A test assignment T for a system G is a collection of tests for every adjacent pair of vertices. It can be modeled as a directed testing graph $T = (V(G), L)$, where $(u, v) \in L$ implies that u and v are adjacent in G . The collection of all test results for a test assignment T is called a syndrome. Formally, a syndrome is a function $\sigma : L \mapsto \{0, 1\}$.

The set of all faulty processors in the system is called a faulty set. This can be any subset of $V(G)$. For a given syndrome σ , a subset of vertices $F \subseteq V(G)$ is said to be consistent with σ if the syndrome σ can be produced from the situation that, for any $(u, v) \in L$ such that $u \in V \setminus F$, $\sigma(u, v) = 1$ if and only if $v \in F$. This means that F is a possible set of faulty processors. Since a test outcome produced by a faulty processor is unreliable, a given set F of faulty vertices may produce a lot of different syndromes. On the other hand, different fault sets may produce the same syndrome. Let $\sigma(F)$ denote the set of all syndromes which F is consistent with.

Under the PMC model, two distinct sets F_1 and F_2 in $V(G)$ are said to be indistinguishable if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$, otherwise, F_1 and F_2 are said to be distinguishable. Besides, we say (F_1, F_2) is an indistinguishable pair if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$; else, (F_1, F_2) is a distinguishable pair.

Using the MM model [13], the diagnosis is carried out by sending the same testing task from one processor to a pair of processors and comparing their responses. Under the MM model, we always assume the output of a comparison performed by a faulty processor is unreliable. The comparison scheme of a system $G = (V, E)$ is modeled as a multigraph, denoted by $M = (V(G), L)$, where L is the labeled-edge set. A labeled edge $(u, v)_w \in L$ represents a comparison in which two vertices u and v are compared by a vertex w , which implies $uw, vw \in E(G)$. The collection of all comparison results in

$M = (V(G), L)$ is called the syndrome, denoted by σ^* , of the diagnosis. If the comparison $(u, v)_w$ disagrees, then $\sigma^*((u, v)_w) = 1$. Otherwise, $\sigma^*((u, v)_w) = 0$. Hence, a syndrome is a function from L to $\{0, 1\}$. The MM^* model is a special case of the MM model. In the MM^* model, all comparisons of G are in the comparison scheme of G , i.e., if $uw, vw \in E(G)$, then $(u, v)_w \in L$.

Similar to the PMC model, we can define a subset of vertices $F \subseteq V(G)$ is consistent with a given syndrome σ^* and two distinct sets F_1 and F_2 in $V(G)$ are indistinguishable (resp. distinguishable) under the MM^* model.

A system $G = (V, E)$ is *g -good-neighbor t -diagnosable* if F_1 and F_2 are distinguishable for each pair of distinct g -good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$. The g -good-neighbor diagnosability $t_g(G)$ of G is the maximum value of t such that G is g -good-neighbor t -diagnosable.

Proposition 2.2. ([14]) *For any given system G , $t_g(G) \leq t_{g'}(G)$ if $g \leq g'$.*

In a system $G = (V, E)$, a faulty set $F \subseteq V$ is called a *conditional faulty set* if it does not contain all the neighbor vertices of any vertex in G . G is *conditional t -diagnosable* if each pair of distinct conditional faulty subsets $F_1, F_2 \subseteq V$ with $|F_1| \leq t, |F_2| \leq t$, are distinguishable. The conditional diagnosability $t_c(G)$ of G is the maximum number of t such that G is conditional t -diagnosable. By [10], $t_c(G) \geq t(G)$.

Theorem 2.3. ([19]) *For a system G , $t(G) = t_0(G) \leq t_1(G) \leq t_c(G)$.*

In [19], Wang et al. proved that the 1-good-neighbor diagnosability of the bubble-sort graph B_n under the PMC model is $2n - 3$ for $n \geq 4$. In [27], Zhou et al. proved the conditional diagnosability of B_n is $4n - 11$ for $n \geq 4$ under the PMC model. Therefore, $t_1(B_n) < t_c(B_n)$ when $n \geq 5$ and $t_1(B_n) = t_c(B_n)$ when $n = 4$.

In a system $G = (V, E)$, a faulty set $F \subseteq V$ is called a *g -extra faulty set* if every component of $G - F$ has more than g nodes. G is *g -extra t -diagnosable* if and only if for each pair of distinct faulty g -extra vertex subsets $F_1, F_2 \subseteq V(G)$ such that $|F_i| \leq t$, F_1 and F_2 are distinguishable. The g -extra diagnosability of G , denoted by $\tilde{t}_g(G)$, is the maximum value of t such that G is g -extra t -diagnosable.

Proposition 2.4. ([21]) *For any given system G , $\tilde{t}_g(G) \leq \tilde{t}_{g'}(G)$ if $g \leq g'$.*

Theorem 2.5. ([21]) *For a system G , $t(G) = \tilde{t}_0(G) \leq \tilde{t}_g(G) \leq t_g(G)$. In particular,*

$$\tilde{t}_1(G) = t_1(G).$$

2.3. Alternating group graph networks

In this section, we give the definition and some properties of the alternating group graph network.

For simplicity, denote $[n] = \{1, 2, \dots, n\}$. An (n, k) -star graph [5], denoted by $S_{n,k}$, is specified by two integers n and k , where $1 \leq k < n$. The vertex set of $S_{n,k}$ is denoted by $\{p_1 p_2 \dots p_k \mid p_i \in [n] \text{ and } p_i \neq p_j \text{ for } i \neq j\}$. The adjacency is defined as follows: $p_1 p_2 \dots p_{i-1} p_i p_{i+1} \dots p_k$ is adjacent to (1) $p_i p_2 \dots p_{i-1} p_1 p_{i+1} \dots p_k$ through an edge of dimension i , where $2 \leq i \leq k$ (swap p_1 with p_i), and (2) $x p_2 \dots p_k$ through dimension 1, where $x \in [n] - \{p_i \mid 1 \leq i \leq k\}$.

In the permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$, $i \longrightarrow p_i$. For the convenience, we denote the permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$ by $p_1 p_2 \dots p_n$. Every permutation can be denoted by a product of cycles [9]. For example, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$. Specially, $\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} = (1)$. The product $\sigma\tau$ of two permutations is the composition function τ followed by σ , that is, $(12)(13) = (132)$. For terminology and notation not defined here we follow [9].

Let S_n be the symmetric group on $[n]$ containing all permutations $p = p_1 p_2 \dots p_n$ of $[n]$. The alternating group A_n is the subgroup of S_n containing all even permutations. Let $\Omega = \{(123), (132)\} \cup \{(12)(3i) : 4 \leq i \leq n\}$. It is well known that Ω is a generating set for A_n . The n -dimensional alternating group graph network AN_n [27] is the graph with vertex set $V(AN_n) = A_n$ in which two vertices u, v are adjacent if and only if $u = vs$ for $s \in \Omega$. Note that if $u = vs$, then $v = us^{-1}$. It is easy to see from the definition that AN_n is a $(n-1)$ -regular graph on $n!/2$ vertices. AN_n can be decomposed into n sub-alternating group graph network, $AN_n^1, AN_n^2, \dots, AN_n^n$, where each AN_n^i has a fixed i in the last position of the label strings which represents the vertices and is isomorphic to AN_{n-1} . We use $E_n^{i,j}$ to denote the set of edges between AN_n^i and AN_n^j . Fixing two distinct symbols i, j from $[n]$, there are $(n-2)$ distinct remaining symbols, and there are $(n-2)!$ permutations in total. Half of these are even permutations, which are the corresponding vertices in AN_n^i and AN_n^j . Thus, there are exactly $|E_n^{i,j}| = (n-2)!/2$ independent edges (cross-edges) between AN_n^i and AN_n^j for all $i \neq j$, which constitute a perfect matching. Ji [8] showed that AN_n is Hamiltonian and has diameter $\lceil 3(n-2)/2 \rceil$.

Chen et al. [3] presented an optimal routing algorithm for this class of networks. Zhou et al. [28] further expanded these results on AN_n by constructing containers of the maximum width, deriving the best containers, and then computed the wide diameter and fault diameter for AN_n . The graphs AN_3 and AN_4 are depicted in Fig.1.

Fig. 1. Alternating group graph networks: (a) AN_3 , (b) AN_4 .

Theorem 2.6. ([1]) AN_n is vertex transitive.

Proposition 2.7. ([27]) The connectivity $\kappa(AN_n) = n - 1$.

Theorem 2.8. ([4]) Let $n \geq 3$. Then AN_n is isomorphic to $(n, n - 2)$ -star.

Theorem 2.9. ([12]) $\kappa^{(h)}(S_{n,k}) = n + h(k - 2) - 1$ for $2 \leq k \leq n - 1$ and $0 \leq h \leq n - k$.

Theorem 2.10. ([9]) Every nonidentity permutation in the symmetric group is uniquely (up to the order of the factors) a product of disjoint cycles, each of which has length at least 2.

Lemma 2.11. ([27]) Let F be a faulty vertex subset of AN_n with $|F| \leq 3n - 10$ for $n \geq 4$. Then, the $AN_n - F$ satisfies one of the following conditions:

- (1) $AN_n - F$ is connected;
- (2) $AN_n - F$ has two components, one of which has is an isolated vertex;
- (3) $AN_n - F$ has two components, one of which is a K_2 ;
- (4) $AN_n - F$ has three components, two of which are isolated vertices.

Proposition 2.12. *Let $n \geq 4$. Then there is no 4-cycle in AN_n .*

Proof. Let $a \in A_n$ with $a \neq (1)$, and let (1) and a be not adjacent. Suppose, on the contrary, that $(1)xy(1)$ is a 4-cycle. Let $a = xg = yh$. We consider the following cases.

Case 1. $x = (12)(3i)$ and $y = (12)(3j)$ ($i, j \geq 4$).

If $g = (12)(3m)$, (123) or (132) , then $a = xg = (3mi)$, $(2i3)$ or $(1i3)$, where $(m \geq 4)$. If $h = (12)(3l)$, (123) or (132) , then $a = yh = (3lj)$, $(2j3)$ or $(1j3)$, where $(l \geq 4)$. By Theorem 2.10, $m = l$ and $i = j$, a contradiction to the fact that $x \neq y$.

Case 2. $x = (12)(3i)$ and $y = (123)$ ($i \geq 4$).

If $g = (12)(3m)$, (123) or (132) , then $a = xg = (3mi)$, $(2i3)$ or $(1i3)$, where $(m \geq 4)$. If $h = (12)(3l)$ or (123) , then $a = yh = (13l)$ or (132) , a contradiction to the fact that $xg = yh$, where $(l \geq 4)$.

Case 3. $x = (12)(3i)$ and $y = (132)$ ($i \geq 4$).

If $g = (12)(3m)$, (123) or (132) , then $a = xg = (3mi)$, $(2i3)$ or $(1i3)$, where $(m \geq 4)$. If $h = (12)(3l)$ or (132) , then $a = yh = (23l)$ or (123) , a contradiction to the fact that $xg = yh$, where $(l \geq 4)$.

Case 4. $x = (123)$ and $y = (132)$.

If $g = (12)(3m)$ or (123) , then $a = xg = (13m)$ or (132) , where $(m \geq 4)$. If $h = (12)(3l)$ or (132) , then $a = yh = (23l)$ or (123) , where $(l \geq 4)$. This is a contradiction to the fact that $xg = yh$.

Case 5. $x = (123)$ and $y = (12)(3j)$ ($j \geq 4$).

If $g = (12)(3m)$ or (123) , then $a = xg = (13m)$ or (132) , where $(m \geq 4)$. If $h = (12)(3l)$, (123) or (132) , then $a = yh = (3lj)$, $(2j3)$ or $(1j3)$, where $(l \geq 4)$. This is a contradiction to the fact that $xg = yh$.

Case 6. $x = (132)$ and $y = (123)$.

This case is considered by Case 4.

Case 7. $x = (132)$ and $y = (12)(3j)$ ($j \geq 4$).

If $g = (12)(3m)$ or (132) , then $a = xg = (23m)$ or (123) , where $(m \geq 4)$. If $h = (12)(3l)$, (123) or (132) , then $a = yh = (3lj)$, $(2j3)$ or $(1j3)$, where $(l \geq 4)$. This is a contradiction to the fact that $xg = yh$.

By Theorem 2.6, there is no 4-cycle in AN_n . □

By Proposition 2.12, we have the following lemma.

Lemma 2.13. *Let AN_n be the alternating group graph network. Then for two vertices u, v there is at most one common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| \leq 1$.*

Proposition 2.14. *Let $n \geq 4$. Then there is no 5-cycle in AN_n .*

Proof. It is easy to see that there is no 5-cycle in AN_4 (see Fig. 1). We prove this proposition by induction on n . The result holds for $n = 4$. Assume $n \geq 5$ and the result holds for AN_{n-1} . We decompose AN_n into n sub-alternating group graph network, $AN_n^1, AN_n^2, \dots, AN_n^n$, where each AN_n^i has a fixed i in the last position of the label strings which represents the vertices and is isomorphic to AN_{n-1} . Suppose, on the contrary, that AN_n has a 5-cycle C . By Theorem 2.6, we may assume, without loss of generality, that $(1) \in V(C)$ and $C = (1)v_1v_2v_3v_4v_5(1)$. Then $(1) \in V(AN_n^n)$. By the inductive hypothesis, AN_n^n has no 5-cycle. Therefore, we consider the following cases.

Case 1. $|V(C) \cap V(AN_n^n)| = 4$.

Without loss of generality, let $v_1 \in V(AN_n^3)$. Then $(1)v_1$ and v_1v_2 are two cross-edges, a contradiction to that all cross-edges are a perfect matching of AN_n .

Case 2. $|V(C) \cap V(AN_n^n)| = 3$.

Since all cross-edges are a perfect matching of AN_n , there are v_i, v_{i+1} in C such that $v_i, v_{i+1} \in V(AN_n^j)$. Without loss of generality, let $v_1, v_2 \in V(AN_n^3)$. Then $v_1 = (12)(3n)$. Note that v_2 is one of $\{(2n3), (1n3), (3in)\}$ and cross-edges $\{(2n3), (1n2)\}, \{(1n3), (12n)\}, \{(3in), (12)(in)\}$. Since $(1n2), (12n), (12)(in) \notin V(AN_n^n)$, this is a contradiction to $v_3 \in V(AN_n^n)$.

Case 3. $|V(C) \cap V(AN_n^n)| = 2$.

Similar to the discussion on Cases 1 and 2, we have that there is no 5-cycle in AN_n .

Case 4. $|V(C) \cap V(AN_n^n)| = 1$.

Similar to the discussion on Cases 1 and 2, we have that there is no 5-cycle in AN_n . $\square$

Theorem 2.15. *Let $n \geq 4$ and let AN_n be the alternating group graph network. Then the 2-extra connectivity of AN_n is equal to the 2-good-neighbor connectivity of AN_n , i.e., $\tilde{\kappa}^{(2)}(AN_n) = \kappa^{(2)}(AN_n) = 3n - 9$ and this result is optimal.*

Proof. By Proposition 2.1, $\tilde{\kappa}^{(2)}(AN_n) \leq \kappa^{(2)}(AN_n)$. By Theorems 2.8 and 2.9, $\tilde{\kappa}^{(2)}(AN_n) \leq 3n - 9$. We suppose that $F \subseteq A_n$ such that $|F| \leq 3n - 10$. By Lemma

2.11, the $AN_n - F$ satisfies one of the following conditions: (1) $AN_n - F$ is connected; (2) $AN_n - F$ has two components, one of which is an isolated vertex; (3) $AN_n - F$ has two components, one of which is a K_2 ; (4) $AN_n - F$ has three components, two of which are isolated vertices. Therefore, F is not a 2-extra cut of AN_n and $\tilde{\kappa}^{(2)}(AN_n) \geq 3n - 9$.

By Proposition 2.1, this result is optimal. $\square$

3. The 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the PMC model

In this section, we will give the 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the PMC model.

Theorem 3.1. ([24,26]) *A system $G = (V, E)$ is g -good-neighbor (g -extra) t -diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ for each pair of distinct g -good-neighbor (2-extra) faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$.*

Lemma 3.2. *Let $A = \{(1), (123), (132)\}$, $n \geq 4$ and let $F_1 = N_{AN_n}(A)$, $F_2 = A \cup N_{AN_n}(A)$. Then $|F_1| = 3n - 9$, $|F_2| = 3n - 6$, $\delta(AN_n - F_1) \geq 2$, $\delta(AN_n - F_2) \geq 2$, F_1 is a 2-extra cut of AN_n , and $AN_n - F_1$ has two components $AN_n - F_2$ and $AN_n[A]$.*

Proof. By $A = \{(1), (123), (132)\}$, we have $AN_n[A] = K_3$. Note that $N_{AN_n}((1)) \setminus A = \{(12)(3i) : 4 \leq i \leq n\}$, $N_{AN_n}((123)) \setminus A = \{(13i) : 4 \leq i \leq n\}$ and $N_{AN_n}((132)) \setminus A = \{(23i) : 4 \leq i \leq n\}$. Therefore, $(N_{AN_n}((1)) \setminus A) \cap (N_{AN_n}((123)) \setminus A) \cap (N_{AN_n}((132)) \setminus A) = \emptyset$. Thus, $|F_1| = |N_{AN_n}(A)| = 3(n - 3) = 3n - 9$ and $|F_2| = 3n - 6$.

Claim 1. $|N_{AN_n}(x) \cap F_1| \leq 1$ for any $x \in A_n \setminus F_2$.

Suppose, on the contrary, that $|N_{AN_n}(x) \cap F_1| \geq 2$ for same $x \in A_n \setminus F_2$. It is sufficient to suppose that $xu_1, xu_2 \in E(AN_n)$, where $u_1, u_2 \in F_1$ and $|\{u_1, u_2\}| = 2$. By Lemma 2.6, we may assume, without loss of generality, u_1 is adjacent (1). Then $u_1 = (12)(3i)$, ($i \geq 4$). We consider the following cases.

Case 1. u_2 is adjacent to (1).

This case is a contradiction to Lemma 2.12.

Case 2. u_2 is adjacent to (123).

Note that $u_1 = (12)(3i)$, where $4 \leq i \leq n$ and $u_2 = (123)(12)(3m)$, where $4 \leq m \leq n$. Let $x = u_1g = u_2h$. If $g = (12)(3j)$, (123) or (132) , then $x = u_1g = (3ji)$, $(2i3)$ or $(1i3)$, where $(j \geq 4)$. If $h = (12)(3l)$, (123) or (132) , then $x = u_2h = (123lm)$, $(12m)$ or $(23)(1m)$, where $(l \geq 4)$. This is a contradiction to the fact that $x = u_1g = u_2h$.

Case 3. u_2 is adjacent to (132) .

Note that $u_1 = (12)(3i)$, where $4 \leq i \leq n$ and $u_2 = (132)(12)(3m)$, where $4 \leq m \leq n$. Let $x = u_1g = u_2h$. If $g = (12)(3j)$, (123) or (132) , then $x = u_1g = (3ji)$, $(2i3)$ or $(1i3)$, where $(j \geq 4)$. If $h = (12)(3l)$, (123) or (132) , then $x = u_2h = (13lm2)$, $(1m2)$ or $(13)(2m)$, where $(l \geq 4)$. This is a contradiction to the fact that $x = u_1g = u_2h$. By Case 1-3, this completes the proof of this claim.

By Claim 1, $\delta(AN_n - F_2) \geq (n-1)-1 = n-2$. $AN_n - F_1$ has two parts (for convenience): $AN_n[A]$ and $AN_n - F_2$. Note that $\delta(AN_n[A]) = 2$. Since $n \geq 4$, $\delta(AN_n - F_2) \geq n-2 \geq 2$. Therefore, $\delta(AN_n - F_1) \geq 2$.

Claim 2. $AN_n - F_2$ is connected.

Note that $AN_4 - F_2$ is a 6-cycle (see Fig 1). Therefore, the result holds for $n = 4$. We proceed by induction on n . Assume $n \geq 5$ and the result holds for AN_{n-1} . We decompose AN_n along the last position, denoted by AN_n^i ($i = 1, \dots, n$). Then AN_n^i and AN_{n-1} are isomorphic. Let $F_2^i = F_2 \cap V(AN_n^i)$ for $i \in \{1, 2, \dots, n\}$ with $|F_2^1| \geq |F_2^2| \geq \dots \geq |F_2^n|$.

Note that $A \subseteq V(AN_n^n)$, $(1)(12)(3n) = (12)(3n)$, $(123)(12)(3n) = (13n)$ and $(132)(12)(3n) = (23n)$. Therefore, $|N(A) \cap V(AN_n^1)| = |N(A) \cap V(AN_n^2)| = |N(A) \cap V(AN_n^3)| = 1$, $|N(A) \cap V(AN_n^4)| = \dots = |N(A) \cap V(AN_n^{n-1})| = 0$. By Proposition 2.7, $AN_n - F_2^i$ is connected for $i \in \{1, 2, \dots, n-1\}$. Since $(n-2)!/2 > 2$, $AN_n[V(AN_n^1 - F_2^1) \cup V(AN_n^2 - F_2^2) \cup \dots \cup V(AN_n^{n-1} - F_2^{n-1})]$ is connected. Since $n \geq 5$, we have $(n-1)!/2 - [3(n-1) - 6] > 3$ and hence $AN_n[V(AN_n^1 - F_2^1) \cup V(AN_n^2 - F_2^2) \cup \dots \cup V(AN_n^{n-1} - F_2^{n-1}) \cup V(AN_n^n - F_2^n)]$ is connected. The proof of this claim is complete.

By Claim 2, $AN_n - F_2$ is connected. Note that $|V(AN_n - F_2)| = (n-1)!/2 - [3(n-1) - 6] > 3$ and $|A| = 3$. Therefore, F_1 is a 2-extra cut of AN_n . $\square$

Lemma 3.3. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the n -dimensional alternating group graph network AN_n under the PMC model is less than or equal to $3n - 7$, i.e., $t_2(AN_n) \leq 3n - 7$.*

Proof. Let A be defined in Lemma 3.2, and let $F_1 = N_{AN_n}(A)$, $F_2 = A \cup N_{AN_n}(A)$. By Lemma 3.2, $|F_1| = 3n - 9$, $|F_2| = 3n - 6$, $\delta(AN_n - F_1) \geq 2$ and $\delta(AN_n - F_2) \geq 2$. Therefore, F_1 and F_2 are both 2-good-neighbor faulty sets of AN_n with $|F_1| = 3n - 9$ and $|F_2| = 3n - 6$. Since $A = F_1 \triangle F_2$ and $N_{AN_n}(A) = F_1 \subset F_2$, there is no edge of AN_n between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 3.1, we can deduce that AN_n is not 2-good-neighbor $(3n - 6)$ -diagnosable under the PMC model. Hence, by the definition of 2-good-neighbor diagnosability, we conclude that the 2-good-neighbor diagnosability of AN_n is less than $3n - 6$, i.e., $t_2(AN_n) \leq 3n - 7$. $\square$

Lemma 3.4. *Let $n \geq 4$. Then the 2-extra diagnosability of the n -dimensional alternating group graph network AN_n under the PMC model is less than or equal to $3n - 7$, i.e., $\tilde{t}_2(AN_n) \leq 3n - 7$.*

Proof. Let A be defined in Lemma 3.2, and let $F_1 = N_{AN_n}(A)$, $F_2 = A \cup N_{AN_n}(A)$. By Lemma 3.2, F_1 and F_2 are both 2-extra faulty sets of AN_n with $|F_1| = 3n - 9$ and $|F_2| = 3n - 6$. Since $A = F_1 \triangle F_2$ and $N_{AN_n}(A) = F_1 \subset F_2$, there is no edge of AN_n between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 3.1, we can deduce that AN_n is not 2-extra $(3n - 6)$ -diagnosable under PMC model. Hence, by the definition of 2-extra diagnosability, we conclude that the 2-extra diagnosability of AN_n is less than $3n - 6$, i.e., $\tilde{t}_2(AN_n) \leq 3n - 7$. $\square$

Lemma 3.5. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the n -dimensional alternating group graph network AN_n under the PMC model is more than or equal to $3n - 7$, i.e., $t_2(AN_n) \geq 3n - 7$.*

Proof. By the definition of 2-good-neighbor diagnosability, it is sufficient to show that AN_n is 2-good-neighbor $(3n - 7)$ -diagnosable. By Theorem 3.1, to prove AN_n is 2-good-neighbor $(3n - 7)$ -diagnosable, it is equivalent to prove that there is an edge $uv \in E(AN_n)$ with $u \in V(AN_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of $V(AN_n)$ with $|F_1| \leq 3n - 7$ and $|F_2| \leq 3n - 7$.

We prove this statement by contradiction. Suppose that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of AN_n with $|F_1| \leq 3n - 7$ and $|F_2| \leq 3n - 7$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Theorem 3.1, i.e., there are no edges between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that

$F_2 \setminus F_1 \neq \emptyset$. Assume $V(AN_n) = F_1 \cup F_2$. By the definition of A_n , $|F_1 \cup F_2| = |A_n| = n!/2$. We claim that $n!/2 > 6n - 14$ for $n \geq 4$, i.e., $n! > 12n - 28$ for $n \geq 4$. When $n = 4$, $n! = 24, 12n - 28 = 20$. So $n! > 12n - 28$ for $n = 4$. Assume that $n! > 12n - 28$ for $n \geq 4$. $(n+1)! = n!(n+1) > (n+1)(12n - 28) = n(12n - 28) + (12n - 16) - 12 = [12(n+1) - 28] + n(12n - 28) - 12 = [12(n+1) - 28] + 4(3n^2 - 7n - 3)$. It is sufficient to show that $3n^2 - 7n - 3 \geq 0$ for $n \geq 4$. Let $y = 3x^2 - 7x - 3$. Then $y = 3x^2 - 7x - 3$ is a quadratic function. When $x \geq 3$, $y = 3x^2 - 7x - 3 \geq 0$.

Since $n \geq 4$, we have that $n!/2 = |V(AN_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(3n - 7) = 6n - 14$, a contradiction to $n!/2 > 6n - 14$. Therefore, let $V(AN_n) \neq F_1 \cup F_2$ as follows.

Since there are no edges between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a 2-good-neighbor faulty set, $AN_n - F_1$ has two parts $AN_n - F_1 - F_2$ and $AN_n[F_2 \setminus F_1]$ (for convenience). Thus, $\delta(AN_n - F_1 - F_2) \geq 2$ and $\delta(AN_n[F_2 \setminus F_1]) \geq 2$. Similarly, $\delta(AN_n[F_1 \setminus F_2]) \geq 2$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 2-good-neighbor faulty set. Since there are no edges between $V(AN_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut. Let $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subset F_2$. By the above discuss, $F_1 \cap F_2 = F_1$ is a 2-good-neighbor cut. If $F_1 \cap F_2 = \emptyset$, this is a contradiction to that AN_n is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. Since $n \geq 4$, by Theorems 2.8 and 2.9, $|F_1 \cap F_2| \geq 3n - 9$. Since $\delta(AN_n[F_2 \setminus F_1]) \geq 2$, we have $|F_2 \setminus F_1| \geq 3$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 3 + 3n - 9 = 3n - 6$, which contradicts with that $|F_2| \leq 3n - 7$. So AN_n is 2-good-neighbor $(3n - 7)$ -diagnosable. By the definition of $t_2(AN_n)$, $t_2(AN_n) \geq 3n - 7$. $\square$

Lemma 3.6. *Let $n \geq 4$. Then the 2-extra diagnosability of the n -dimensional alternating group graph network AN_n under the PMC model is more than or equal to $3n - 7$, i.e., $\tilde{t}_2(AN_n) \geq 3n - 7$.*

Proof. By the definition of 2-extra diagnosability, it is sufficient to show that AN_n is 2-extra $(3n - 7)$ -diagnosable. By Theorem 3.1, suppose, on the contrary, that there are two distinct 2-extra faulty subsets F_1 and F_2 of AN_n with $|F_1| \leq 3n - 7$ and $|F_2| \leq 3n - 7$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Theorem 3.1, i.e., there are no edges between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similar to the discussion on $V(AN_n) \neq F_1 \cup F_2$ in Lemma 3.5, we can deduce $V(AN_n) \neq F_1 \cup F_2$. Therefore, $V(AN_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a 2-extra faulty set, $AN_n - F_1$ has two parts $AN_n - F_1 - F_2$ and $AN_n[F_2 \setminus F_1]$ (for convenience). Thus, every component G_i of $AN_n - F_1 - F_2$ has $|V(G_i)| \geq 3$ and every component B'_i of $AN_n[F_2 \setminus F_1]$ has $|V(B'_i)| \geq 3$. Similarly, every component B''_i of $AN_n[F_1 \setminus F_2]$ has $|V(B''_i)| \geq 3$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 2-extra faulty set of AN_n . Note that $F_1 \cap F_2 = F_1$ is also a 2-extra faulty set when $F_1 \setminus F_2 = \emptyset$. Since there are no edges between $V(AN_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-extra cut of AN_n . If $F_1 \cap F_2 = \emptyset$, this is a contradiction to that AN_n is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. Since $n \geq 4$, by Theorem 2.15, $|F_1 \cap F_2| \geq 3n - 9$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 3 + 3n - 9 = 3n - 6$, which contradicts with that $|F_2| \leq 3n - 7$. So AN_n is 2-extra $(3n - 7)$ -diagnosable. By the definition of $\tilde{t}_2(AN_n)$, $\tilde{t}_2(AN_n) \geq 3n - 7$. $\square$

Combining Lemmas 3.3 and 3.5, we have the following theorem.

Theorem 3.7. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the n -dimensional alternating group graph network AN_n under PMC model is $3n - 7$.*

Combining Lemmas 3.4 and 3.6, we have the following theorem.

Theorem 3.8. *Let $n \geq 4$. Then the 2-extra diagnosability of the n -dimensional alternating group graph network AN_n under PMC model is $3n - 7$ and this result is optimal.*

By Theorem 2.5, $\tilde{t}_2(AN_n) \leq t_2(AN_n)$. By Theorems 3.7 and 3.8, $\tilde{t}_2(AN_n) = t_2(AN_n) = 3n - 7$. Therefore, this result is optimal with respect to $\tilde{t}_2(AN_n)$.

4. The 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the MM* model

Before discussing the 2-good-neighbor (2-extra) diagnosability of the n -dimensional alternating group graph network AN_n under the MM* model, we first give an existing result.

Theorem 4.1. *([6, 24, 26]) A system $G = (V, E)$ is g -good-neighbor (g -extra) t -diagnosable under the MM* model if and only if each pair of distinct g -good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions.*

(1) There are two vertices $u, w \in V \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E$ and $vw \in E$.

(2) There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

(3) There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

Lemma 4.2. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the n -dimensional alternating group graph network AN_n under the MM^* model is less than or equal to $3n - 7$, i.e., $t_2(AN_n) \leq 3n - 7$.*

Proof. Let A , F_1 and F_2 be defined in Lemma 3.2. By the Lemma 3.2, $|F_1| = 3n - 9$, $|F_2| = 3n - 6$, $\delta(AN_n - F_1) \geq 2$ and $\delta(AN_n - F_2) \geq 2$. So both F_1 and F_2 are 2-good-neighbor faulty sets. By the definitions of F_1 and F_2 , $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(AN_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any one condition in Theorem 4.1, and AN_n is not 2-good-neighbor $(3n - 6)$ -diagnosable. Hence, $t_2(AN_n) \leq 3n - 7$. Thus, the proof is complete. $\square$

Lemma 4.3. *Let $n \geq 4$. Then the 2-extra diagnosability of the n -dimensional alternating group graph network AN_n under the MM^* model is less than or equal to $3n - 7$, i.e., $\tilde{t}_2(AN_n) \leq 3n - 7$.*

Proof. Let A , F_1 and F_2 be defined in Lemma 3.2. By the Lemma 3.2, $|F_1| = 3n - 9$, $|F_2| = 3n - 6$, both F_1 and F_2 are 2-extra faulty sets of AN_n . By the definitions of F_1 and F_2 , $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(AN_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any one condition in Theorem 4.1, and AN_n is not 2-extra $(3n - 6)$ -diagnosable. Hence, $\tilde{t}_2(AN_n) \leq 3n - 7$. Thus, the proof is complete. $\square$

Lemma 4.4. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the n -dimensional alternating group graph network AN_n under the MM^* model is more than or equal to $3n - 7$, i.e., $t_2(AN_n) \geq 3n - 7$.*

Proof. By the definition of 2-good-neighbor diagnosability, it is sufficient to show that AN_n is 2-good-neighbor $(3n - 7)$ -diagnosable. By Theorem 4.1, suppose, on the contrary,

that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of AN_n with $|F_1| \leq 3n - 7$ and $|F_2| \leq 3n - 7$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 4.1. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Assume $V(AN_n) = F_1 \cup F_2$. By the definition of A_n , $|F_1 \cup F_2| = |A_n| = n!/2$. We claim that $n!/2 > 6n - 14$ for $n \geq 4$, i.e., $n! > 12n - 28$ for $n \geq 4$. When $n = 4$, $n! = 24, 12n - 28 = 20$. So $n! > 12n - 28$ for $n = 4$. Assume that $n! > 12n - 28$ for $n \geq 4$. $(n + 1)! = n!(n + 1) > (n + 1)(12n - 28) = n(12n - 28) + (12n - 16) - 12 = [12(n + 1) - 28] + n(12n - 28) - 12 = [12(n + 1) - 28] + 4(3n^2 - 7n - 3)$. It is sufficient to show that $3n^2 - 7n - 3 \geq 0$ for $n \geq 4$. Let $y = 3x^2 - 7x - 3$. Then $y = 3x^2 - 7x - 3$ is a quadratic function. When $x \geq 3$, $y = 3x^2 - 7x - 3 \geq 0$.

Since $n \geq 4$, we have that $n!/2 = |V(AN_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(3n - 7) = 6n - 14$, a contradiction to $n!/2 > 6n - 14$. Therefore, let $V(AN_n) \neq F_1 \cup F_2$ as follows.

Claim 1. $AN_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $AN_n - F_1 - F_2$ has at least one isolated vertex w . Since F_1 is a 2-good-neighbor faulty set, $d_{AN_n[w \cup (F_2 \setminus F_1)]}(w) \geq 2$, and hence there are two vertices $u, v \in F_2 \setminus F_1$ such that u and v are adjacent to w . Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 4.1, this is a contradiction. Therefore, $AN_n - F_1 - F_2$ has no isolated vertex. The proof of Claim 1 is complete.

Let $u \in V(AN_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $AN_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 4.1, by the condition (1) of Theorem 4.1, for any pair of adjacent vertices $u, w \in V(AN_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(AN_n)$ and $vw \in E(AN_n)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u , there is no edge between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 2-good-neighbor faulty set, $\delta(AN_n - F_1 - F_2) \geq 2$, $\delta(AN_n[F_2 \setminus F_1]) \geq 2$ and $|F_2 \setminus F_1| \geq 3$. Similarly, $\delta(AN_n[F_1 \setminus F_2]) \geq 2$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 2-good-neighbor faulty set. Since there are no edges between $V(AN_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut. Let $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subset F_2$. By the above discuss, $F_1 \cap F_2 = F_1$ is a 2-good-neighbor cut. If $F_1 \cap F_2 = \emptyset$, this is a contradiction to that AN_n is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. By Theorem 2.9, we

have $|F_1 \cap F_2| \geq 3n - 9$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 3 + (3n - 9) = 3n - 6$, which contradicts $|F_2| \leq 3n - 7$. Therefore, AN_n is 2-good-neighbor $(3n - 7)$ -diagnosable and $t_2(AN_n) \geq 3n - 7$. The proof is complete. $\square$

Lemma 4.5. *Let $n \geq 5$. Then the 2-extra diagnosability of the n -dimensional alternating group graph network AN_n under the MM^* model is more than or equal to $3n - 7$, i.e., $\tilde{t}_2(AN_n) \geq 3n - 7$.*

Proof. By the definition of 2-extra diagnosability, it is sufficient to show that AN_n is 2-extra $(3n - 7)$ -diagnosable.

By Theorem 4.1, suppose, on the contrary, that there are two distinct 2-extra faulty subsets F_1 and F_2 of AN_n with $|F_1| \leq 3n - 7$ and $|F_2| \leq 3n - 7$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 4.1. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similar to the discussion on $V(AN_n) \neq F_1 \cup F_2$ in Lemma 2.13, we can deduce $V(AN_n) \neq F_1 \cup F_2$. Therefore, $V(AN_n) \neq F_1 \cup F_2$.

Claim 1. $AN_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $AN_n - F_1 - F_2$ has at least one isolated vertex w_1 . Since F_1 is one 2-extra faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w_1 . Meanwhile, since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 4.1, by the condition (3) of Theorem 4.1, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w_1 . Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w_1 . If $F_1 \setminus F_2 = \emptyset$, then $F_1 \subseteq F_2$. Since F_2 is a 2-extra faulty set, every component G_i of $AN_n - F_1 - F_2 = AN_n - F_2$ has $|V(G_i)| \geq 3$. Therefore, $AN_n - F_1 - F_2$ has no isolated vertex. Thus, let $F_1 \setminus F_2 \neq \emptyset$. Similarly, we can deduce that there is just a vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w_1 . Let $W \subseteq A_n \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $AN_n[A_n \setminus (F_1 \cup F_2)]$, and let H be the induced subgraph by the vertex set $A_n \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(n - 3)$ neighbors in $F_1 \cap F_2$. Since $|F_2| \leq 3n - 7$, we have $\sum_{w \in W} |N_{NA_n[(F_1 \cap F_2) \cup W]}(w)| = |W|(n - 3) \leq \sum_{v \in F_1 \cap F_2} d_{NA_n}(v) \leq |F_1 \cap F_2|(n - 1) \leq (|F_2| - 1)(n - 1) \leq (3n - 8)(n - 1) = 3n^2 - 11n + 8$. It follows that $|W| \leq \frac{3n^2 - 11n + 8}{n - 3} \leq 3n - 1$ for $n \geq 5$. Note $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2(3n - 7) - (n - 3) = 5n - 11$. Suppose $V(H) = \emptyset$. Then $n!/2 = |V(NA_n)| = |F_1 \cup F_2| + |W| \leq 5n - 11 + 3n - 1 = 8n - 12$. This is a contradiction to $n \geq 5$. So $V(H) \neq \emptyset$. Since the vertex set pair (F_1, F_2) is not satisfied with the condition (1) of Theorem 4.1,

and any vertex of $V(H)$ is not isolated in H , we induce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Thus, F_1 is a vertex cut of NA_n . Since F_1 is a 2-extra faulty set of AN_n , we have that every component H_i of H has $|V(H_i)| \geq 3$ and every component B_i of $AN_n[W \cup (F_2 \setminus F_1)]$ has $|V(B_i)| \geq 3$. Therefore, F_1 is also a 2-extra cut of NA_n . If $F_1 \cap F_2 = \emptyset$, then this is a contradiction to that AN_n is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. By Theorem 2.15, $|F_1| \geq 3n - 9$. Since $|F_1| \leq 3n - 7$, we have $3n - 9 \leq |F_1| \leq 3n - 7$. Since every component B_i of $AN_n[W \cup (F_2 \setminus F_1)]$ has $|V(B_i)| \geq 3$, we have $|F_2 \setminus F_1| \geq 2$ and hence $|F_1| = 3n - 7$ and $|F_2 \setminus F_1| = 2$. Since F_2 is a 2-extra faulty set of AN_n , we have that $W = \emptyset$ when $F_1 \setminus F_2 = \emptyset$. Therefore, let $F_1 \setminus F_2 \neq \emptyset$. Similarly, we can deduce that F_2 is also a 2-extra cut of NA_n , $|F_2| = 3n - 7$ and $|F_1 \setminus F_2| = 2$. Let $F_2 \setminus F_1 = \{u, v\}$, $F_1 \setminus F_2 = \{a, b\}$, and let $vu w_1 ab$ be a path in AN_n (See Fig. 2).

Suppose that there are two isolated vertices w_1 and w_2 in $AN_n - F_1 - F_2$. Suppose that w_2 is adjacent to u . If w_2 is adjacent to a , then this is a contradiction to Proposition 2.12 (AN_n has no 4-cycle). If w_2 is adjacent to b , then this is a contradiction to Proposition 2.14 (AN_n has no 5-cycle). Therefore, w_2 is adjacent to v . If w_2 is adjacent to a , then this is a contradiction to Proposition 2.14. Therefore, w_2 is adjacent to v and b (See Fig. 3). Suppose that there are three isolated vertices w_1, w_2 and w_3 in $AN_n - F_1 - F_2$. Suppose that w_3 is adjacent to u . If w_3 is adjacent to a , then this is a contradiction to Proposition 2.12. If w_3 is adjacent to b , then this is a contradiction to Proposition 2.14. Suppose that w_3 is adjacent to v . If w_3 is adjacent to a , then this is a contradiction to Proposition 2.14. If w_3 is adjacent to b , then this is a contradiction to Proposition 2.12. Therefore, $|W| \leq 2$.

Suppose that there is exactly one isolated vertex w_1 in $AN_n - F_1 - F_2$ (See Fig. 2). Since there is no edge between $V(H)$ and $(F_1 \triangle F_2) \cup W$, we have $V(H) \cap N(\{v, u, w_1, a, b\}) = \emptyset$ and hence $N(\{v, u, w_1, a, b\}) \subseteq (F_1 \cap F_2)$. Note that $vu w_1 ab$ is a path in AN_n . By Propositions 2.12 and 2.14, $N(v) \cap N(w_1) = \emptyset$, $N(v) \cap N(a) = \emptyset$, $N(u) \cap N(a) = \emptyset$, $N(u) \cap N(b) = \emptyset$ and $N(w_1) \cap N(b) = \emptyset$. Therefore, $|F_1 \cap F_2| \geq |N(\{v, u, w_1, a, b\})| \geq 5(n-1) - (\binom{5}{2} - 5) = 5n - 10$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| = 2 + 5n - 10 = 5n - 8$. Note that $5n - 8 > 3n - 7$, a contradiction to $|F_2| \leq 3n - 7$.

Suppose that there are exactly two isolated vertices w_1 and w_2 in $AN_n - F_1 - F_2$ (See Fig. 3). Since there is no edge between $V(H)$ and $(F_1 \triangle F_2) \cup W$, we have $V(H) \cap$

$N(\{w_2, v, u, w_1, a, b\}) = \emptyset$ and hence $N(\{w_2, v, u, w_1, a, b\}) \subseteq (F_1 \cap F_2)$. Note that $w_2vwu_1abw_2$ is a 6-cycle in AN_n . By Propositions 2.12 and 2.14, $N(v) \cap N(w_1) = \emptyset$, $N(v) \cap N(a) = \emptyset$, $N(u) \cap N(a) = \emptyset$, $N(u) \cap N(b) = \emptyset$, $N(w_1) \cap N(b) = \emptyset$, $N(w_1) \cap N(w_2) = \emptyset$ and $N(a) \cap N(w_2) = \emptyset$. Therefore, $|F_1 \cap F_2| \geq |N(\{w_2, v, u, w_1, a, b\})| \geq 6(n-1) - \binom{6}{2} - 7 = 6n - 14$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| = 2 + 6n - 14 = 6n - 12$. Note that $6n - 12 > 3n - 7$, a contradiction to $|F_2| \leq 3n - 7$. This completes the proof of this claim.

Let $u \in V(AN_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $AN_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 4.1, by the condition (1) of Theorem 4.1, for any pair of adjacent vertices $u, w \in V(AN_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(AN_n)$ and $vw \in E(AN_n)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u , there is no edge between $V(AN_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$.

Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 2-extra faulty set, we have that every component H_i of $AN_n - F_1 - F_2$ has $|V(H_i)| \geq 3$, every component G_i of $AN_n[F_2 \setminus F_1]$ has $|V(G_i)| \geq 3$ and hence $|F_2 \setminus F_1| \geq 3$. Similarly, every component G'_i of $AN_n[F_1 \setminus F_2]$ has $|V(G'_i)| \geq 3$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 2-extra faulty set. Since there are no edges between $V(AN_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-extra cut. Let $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subset F_2$. By the above discuss, $F_1 \cap F_2 = F_1$ is a 2-extra cut. If $F_1 \cap F_2 = \emptyset$, this is a contradiction to that AN_n is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. By Theorem 2.9, we have $|F_1 \cap F_2| \geq 3n - 9$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 3 + (3n - 9) = 3n - 6$, which contradicts $|F_2| \leq 3n - 7$. Therefore, AN_n is 2-extra $(3n - 7)$ -diagnosable and $\tilde{t}_2(AN_n) \geq 3n - 7$. The proof is complete. $\square$

By Lemmas 4.2 and 4.4, we have the following theorem.

Theorem 4.6. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the the n -dimensional alternating group graph network AN_n under the MM^* model is $3n - 7$.*

Combining Lemmas 4.3 and 4.5, and Theorem 2.5, we have the following theorem.

Theorem 4.7. *Let $n \geq 5$. Then the 2-extra diagnosability of the the n -dimensional alternating group graph network AN_n under the MM^* model is $3n - 7$ and this result is optimal.*

Fig. 2. Illustration of one isolated vertex w_1 .

Fig. 3. Illustration of two isolated vertices w_1 and w_2 .

5. Conclusions

In this paper, we investigate the problem of the 2-good-neighbor (2-extra) diagnosability of the n -dimensional alternating group graph network AN_n under the PMC model and MM* model. It is proved that the 2-good-neighbor (2-extra) diagnosability of AN_n under the PMC model and MM* model is $3n - 7$ when $n \geq 5$. The above results show

that the 2-good-neighbor (2-extra) diagnosability is several times larger than the classical diagnosability of AN_n depending on the condition: 2-good-neighbors or 2-extras. The work will help engineers to develop more different measures of the 2-good-neighbor (2-extra) diagnosability based on application environment, network topology, network reliability, and statistics related to fault patterns.

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