Binary Bell Polynomials, Hirota Bilinear Approach to Levi Equation

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Abstract

Combining the binary Bell polynomials and Hirota method, we obtained two kinds of equivalent bilinear equations for the Levi equation. Then, we got the double Wronskian solutions of the Levi equation by virtue of one of the bilinear equations. Furthermore, we constructed the bilinear Bäcklund transformation and the Lax pair. Finally, we also derived the Darboux transformation and the infinite conservation laws of the Levi equation.

Key words: Binary Bell polynomials, Double Wronskian solutions, Bäcklund transformation and Lax pair, Darboux transformation, Infinite conservation laws

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1 Introduction

Nonlinear equations are widely used in the fields of physics, chemistry, atmospheric dynamics, life sciences and engineering sciences etc. There have been many great researches on the nonlinear equations in recent years. Particularly in soliton theory, many different approaches can be used to deal with the nonlinear equations. Inverse scattering transformation [1], Bäcklund transformation [2], Darboux transformation [3], Hirota bilinear method [4, 5] and Wronskian technique [6] are all very cogent methods. Among those methods, the bilinear method developed by Hirota is powerful for analysing soliton equations. The first step of this method is to transform the given soliton equations into bilinear forms by proper dependent variables transformation. By means of the bilinear representations for soliton equations, not only the multi-soliton solutions but also the main integrability features of these equations such as bilinear Bäcklund transformations,

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Lax pairs and Darboux transformation can be derived. However, the well chosen dependent variables transformation for deriving the bilinear forms can not be obtained easily, because it relies on clever guesswork and tedious calculation. Recently, Lember, Gilson *et al.* [7-10] found there exists a deep connection between the Bell polynomials and Hirota's bilinear expressions and then proposed a lucid and systematic method to overcome these difficulties. So both the bilinear forms and bilinear Bäcklund transformations can be obtained directly. Lots of studies have devoted to the application and extension of this method [11-21].

In this paper, we will study the Levi equation

$$u_t - 2uu_x - 2v_x + u_{xx} = 0, \quad (1.1a)$$

$$v_t - 2(uv)_x - v_{xx} = 0, \quad (1.1b)$$

Eqs.(1.1) has been investigated in [22] for solving the the double Wronskian solutions. However, the bilinear Bäcklund transformations, Lax pair, Darboux transformation as well as the infinite conservation Laws for Eqs.(1.1) have not been studied based on the binary Bell polynomials and Hirota method.

On account of the above analysis, this paper is organized as follows. In Section 2, we will briefly introduce concepts and formulae about the binary Bell polynomials. In Section 3, we will derive two kinds of equivalent bilinear equations for Eqs.(1.1), then we will get the double Wronskian solutions of Eqs.(1.1). In Section 4, we will give the bilinear Bäcklund transformation and the corresponding Lax pair. In Section 5, we will obtain the Darboux transformation by introducing a gauge transformation. In Section 6, we also will get the infinite conservation laws of Eqs.(1.1). Finally, we will draw some conclusions.

2 The binary Bell polynomials

Let $f = f(x_1, \dots, x_l)$ is a C^∞ function of multi-variables, the definition of multi-dimensional Bell polynomials (see [7 – 10] for details) reads as

$$Y_{n_1x_1, \dots, n_lx_l}(f) \equiv Y_{n_1, \dots, n_l}(f_{r_1x_1, \dots, r_lx_l}) = e^{-f} \partial_{x_1}^{n_1} \dots \partial_{x_l}^{n_l} e^f, \quad (2.1)$$

where $f_{r_1x_1, \dots, r_lx_l} = \partial_{x_1}^{r_1} \dots \partial_{x_l}^{r_l} f$, $r_1 = 0, \dots, n_1; \dots; r_l = 0, \dots, n_l$.

Based on the above Bell polynomials, when $l = 2$ in (2.1), the binary Bell polynomials ($\mathcal{Y}$ -polynomials) can be defined by

$$\mathcal{Y}_{n_1x, n_2t}(v, w) = Y_{n_1x, n_2t}(f) \Big|_{f_{r_1x, r_2t} = \begin{cases} v_{r_1x, r_2t}, & r_1 + r_2 = \text{odd}, \\ w_{r_1x, r_2t}, & r_1 + r_2 = \text{even}. \end{cases}} \quad (2.2)$$

For example, the first few lowest order $\mathcal{Y}$ -polynomials are

$$\begin{aligned}\mathcal{Y}_{x,t}(v, w) &= w_{xt} + v_x v_t, \\ \mathcal{Y}_{2x,t}(v, w) &= v_{2x,t} + w_{2x} v_t + 2w_{xt} v_x + v_x^2 v_t, \\ \mathcal{Y}_{3x}(v, w) &= v_{3x} + 3w_{2x} v_x + v_x^3, \dots\end{aligned}$$

The link between $\mathcal{Y}$ -polynomials and the Hirota's standard bilinear expressions

$$D_x^{n_1} D_t^{n_2} F \cdot G = (\partial_x - \partial_{x'})^{n_1} (\partial_t - \partial_{t'})^{n_2} \times F(x, t) G(x', t') \big|_{x'=x, t'=t} \quad (2.3)$$

is given by the identity

$$\mathcal{Y}_{n_1 x, n_2 t}(v = \ln F/G, w = \ln FG) = (FG)^{-1} D_x^{n_1} D_t^{n_2} F \cdot G. \quad (2.4)$$

Particularly, in the case of $F = G$, we have

$$\begin{aligned}F^{-2} D_x^{n_1} D_t^{n_2} F \cdot F &= \mathcal{Y}_{n_1 x, n_2 t}(0, w = 2 \ln F) \\ &= \begin{cases} 0, & n_1 + n_2 = \text{odd}, \\ P_{n_1 x, n_2 t}(w), & n_1 + n_2 = \text{even}, \end{cases} \end{aligned} \quad (2.5)$$

which discloses that P -polynomials can be obtained by restricting the Bell Polynomials to even part partitions.

Moreover, there has a relationship between the binary Bell polynomials and Lax pair through the following expression

$$\begin{aligned}& \mathcal{Y}_{n_1 x, n_2 t}(v = \ln \Psi, w = v + Q) \\ &= \Psi^{-1} \sum_{a_1=0}^{n_1} \sum_{a_2=0}^{n_2} \binom{n_1}{a_1} \binom{n_2}{a_2} P_{a_1 x, a_2 t}(Q) Y_{(n_1-a_1)x, (n_2-a_2)t}(v = \ln \Psi), \end{aligned} \quad (2.6)$$

where Ψ and Q are functions of x and t .

3 Bilinear equations and double Wronskian solutions

In this section, with the help of the binary Bell polynomials, we will give two kinds of equivalent bilinear forms of Eqs.(1.1).

Let

$$u = p_x, \quad v = \frac{1}{2}(q_{xx} + p_{xx}), \quad (3.1)$$

Eqs.(1.1) are transformed to

$$p_t - (p_x^2 + q_{xx}) = 0, \quad (3.2a)$$

$$q_{xt} - p_{xxx} - 2p_x q_{xx} = 0, \quad (3.2b)$$

using Eq.(3.2a), Eqs.(3.2) are further transformed to

$$p_t - (p_x^2 + q_{xx}) = 0, \quad (3.3a)$$

$$(q_{xt} + p_x p_t) - (p_{xxx} + 3p_x q_{xx} + p_x^3) = 0. \quad (3.3b)$$

From the expressions (2.2), Eqs.(3.3) are equivalent to

$$\mathcal{Y}_t(p) - \mathcal{Y}_{2x}(p, q) = 0, \quad (3.4a)$$

$$\mathcal{Y}_{xt}(p, q) - \mathcal{Y}_{3x}(p, q) = 0. \quad (3.4b)$$

By setting

$$p = \ln(h/f), \quad q = \ln(hf), \quad (3.5)$$

then by using Eq.(2.3) and Eq.(2.4), the bilinear equations for Eqs.(1.1) are obtained as

$$(D_t - D_x^2)h \cdot f = 0, \quad (3.6a)$$

$$(D_x D_t - D_x^3)h \cdot f = 0, \quad (3.6b)$$

where h and f are the functions of x and t .

Based on the above bilinear equations, we can get their double Wronskian determinant solutions, but the process for proving the existing of Wronskian solutions will be very difficult because Eq.(3.6b) is third-order on h and f . In order to make the problem not too difficult, we may first reduce Eq.(3.6b) to two second-order bilinear equations on h and f by adding an auxiliary variable.

Theorem 3.1 Let

$$f^2 \ln(f)_{xx} = -gh, \quad (3.7)$$

where g is an auxiliary variable (which is the function of x and t), then Eqs.(3.6) are equivalent to the following three second-order bilinear equations

$$(D_t - D_x^2)h \cdot f = 0, \quad (3.8a)$$

$$D_x^2 f \cdot f = -2gh, \quad (3.8b)$$

$$(D_t + D_x^2)g \cdot f = 0, \quad (3.8c)$$

Proof: From Eq.(3.7) and Eq.(2.3), we can get Eq.(3.8b) directly, hence we only need to prove Eq.(3.8c).

Taking the derivative with respect to x on both sides of Eq.(3.2b), we have

$$q_{xxt} - p_{xxx} - 2(p_x q_{xx})_x = 0. \quad (3.9)$$

Based on the transformation (3.5) and Eq.(3.7), we can get $q_{xx} = p_{xx} - \frac{2g}{f} \cdot \frac{h}{f}$, substituting it into Eq.(3.9), we have

$$p_{xxt} - \left(\frac{2g}{f} \cdot \frac{h}{f}\right)_t - p_{xxxx} - 2(p_x p_{xx})_x + 2\left(p_x \frac{2g}{f} \cdot \frac{h}{f}\right)_x = 0, \quad (3.10)$$

using $p_{xxt} - 2(p_x p_{xx})_x - p_{xxxx} = -\left(\frac{2g}{f} \cdot \frac{h}{f}\right)_{xx}$ and $p_x = \left(\frac{h}{f}\right)_x \cdot \frac{f}{h}$, Eq.(3.10) is transformed to

$$-\left(\frac{g}{f}\right)_{xx} \cdot \frac{h}{f} + \frac{g}{f} \cdot \left(\frac{h}{f}\right)_{xx} - \left(\frac{h}{f}\right)_t \cdot \frac{g}{f} - \frac{h}{f} \cdot \left(\frac{g}{f}\right)_t = 0. \quad (3.11)$$

From Eq.(3.3a) and the transformation (3.5), we can get

$$\left(\frac{h}{f}\right)_t = \frac{h}{f} (\ln \left(\frac{h}{f}\right))_x^2 + \frac{h}{f} (\ln \left(\frac{h}{f}\right))_{xx} - \frac{h}{f} \cdot \frac{2gh}{f^2},$$

substituting it into Eq.(3.11), we obtain

$$\left(\frac{g}{f}\right)_{xx} + \frac{g}{f} \frac{2gh}{f^2} + \left(\frac{g}{f}\right)_t = 0. \quad (3.12)$$

That is just Eq.(3.8c) by using Eq.(2.3) and Eq.(3.8b).

Let we give the double Wronskian solutions based on Eqs.(3.8).

Theorem 3.2 Eqs.(3.8) have the double Wronskian solutions

$$h = -2W^{N,M+2}(\Phi; \Psi), \quad f = W^{N+1,M+1}(\Phi; \Psi), \quad g = 2W^{N+2,M}(\Phi; \Psi), \quad (3.13)$$

where the entries $\Phi, \Psi (j = 1, \dots, N)$ satisfy the following linear differential conditions :

$$\Phi_{i,x} = -k_i \Phi_i, \quad \Phi_{i,t} = -2\Phi_{i,xx}, \quad \Psi_{i,x} = k_i \Psi_i, \quad \Psi_{i,t} = 2\Psi_{i,xx}, \quad 1 \leq i \leq N, \quad (3.14)$$

and the double Wronskian is defined as [5, 6]

$$W^{N,M}(\Phi; \Psi) = \det(\Phi, \partial_x \Phi, \dots, \partial_x^{N-1} \Phi; \Psi, \partial_x \Psi, \dots, \partial_x^{N-1} \Psi),$$

with $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_{N+M+2})^T, \Psi = (\Psi_1, \Psi_2, \dots, \Psi_{N+M+2})^T$.

Proof: In the following, we use the abbreviated notation of Freeman and Nimmo for the Wronskian and its derivatives [23-26], then Eqs. (3.13) become

$$h = -2|\widehat{N-1}, \widehat{M+1}|, \quad f = |\widehat{N}, \widehat{M}|, \quad g = 2|\widehat{N+1}, \widehat{M-1}|,$$

we have

$$\begin{aligned}
f_x &= |\widehat{N-1}, N+1, \widehat{M}| + |\widehat{N}, \widehat{M-1}, M+1|, \\
f_{xx} &= |\widehat{N-2}, N, N+1, \widehat{M}| + |\widehat{N-1}, N+2, \widehat{M}| + 2|\widehat{N-1}, N+1, \widehat{M-1}, M+1| \\
&\quad + |\widehat{N}, \widehat{M-2}, M, M+1| + |\widehat{N}, \widehat{M-1}, M+2|, \\
f_t &= 2\{|\widehat{N-2}, N, N+1, \widehat{M}| - |\widehat{N-1}, N+2, \widehat{M}| - |\widehat{N}, \widehat{M-2}, M, M+1| \\
&\quad + |\widehat{N}, \widehat{M-1}, M+2|\}, \\
h_x &= -2\{|\widehat{N-2}, N, \widehat{M+1}| + |\widehat{N-1}, \widehat{M}, M+2|\}, \\
h_{xx} &= -2\{|\widehat{N-3}, N-1, N, \widehat{M+1}| + |\widehat{N-2}, N+1, \widehat{M+1}| + 2|\widehat{N-2}, N, \widehat{M}, M+2| \\
&\quad + |\widehat{N-1}, \widehat{M-1}, M+1, M+2| + |\widehat{N-1}, \widehat{M}, M+3|\}, \\
h_t &= -2\{2|\widehat{N-3}, N-1, N, \widehat{M+1}| - 2|\widehat{N-2}, N+1, \widehat{M+1}| \\
&\quad - 2|\widehat{N-1}, \widehat{M-1}, M+1, M+2| + 2|\widehat{N-1}, \widehat{M}, M+3|\},
\end{aligned}$$

so,

$$\begin{aligned}
&f(h_t - h_{xx}) \\
&= -2|\widehat{N}, \widehat{M}|\{|\widehat{N-3}, N-1, N, \widehat{M+1}| - 3|\widehat{N-2}, N+1, \widehat{M+1}| \\
&\quad - 2|\widehat{N-2}, N, \widehat{M}, M+2| - 3|\widehat{N-1}, \widehat{M-1}, M+1, M+2| + |\widehat{N-1}, \widehat{M}, M+3|\} \\
&= -2|\widehat{N}, \widehat{M}|\{|\widehat{N-3}, N-1, N, \widehat{M+1}| + |\widehat{N-2}, N+1, \widehat{M+1}| \\
&\quad - 2|\widehat{N-2}, N, \widehat{M}, M+2| + |\widehat{N-1}, \widehat{M-1}, M+1, M+2| + |\widehat{N-1}, \widehat{M}, M+3| \\
&\quad - 4|\widehat{N-2}, N+1, \widehat{M+1}| - 4|\widehat{N-1}, \widehat{M-1}, M+1, M+2|\}, \\
&-h(f_t + f_{xx}) \\
&= -2|\widehat{N-1}, \widehat{M+1}|\{-3|\widehat{N-2}, N, N+1, \widehat{M}| + |\widehat{N-1}, N+2, \widehat{M}| \\
&\quad - 2|\widehat{N-1}, N+1, \widehat{M-1}, M+1| + |\widehat{N}, \widehat{M-2}, M, M+1| - 3|\widehat{N}, \widehat{M-1}, M+2|\} \\
&= -2|\widehat{N-1}, \widehat{M+1}|\{|\widehat{N-2}, N, N+1, \widehat{M}| + |\widehat{N-1}, N+2, \widehat{M}| \\
&\quad - 2|\widehat{N-1}, N+1, \widehat{M-1}, M+1| + |\widehat{N}, \widehat{M-2}, M, M+1| + |\widehat{N}, \widehat{M-1}, M+2| \\
&\quad - 4|\widehat{N-2}, N, N+1, \widehat{M}| - 4|\widehat{N}, \widehat{M-1}, M+2|\}, \\
&2h_x f_x = -4\{|\widehat{N-1}, N+1, \widehat{M}| + |\widehat{N}, \widehat{M-1}, M+1|\} \cdot \\
&\quad \{|\widehat{N-2}, N, \widehat{M+1}| + |\widehat{N-1}, \widehat{M}, M+2|\}.
\end{aligned}$$

Noting

$$|\widehat{N}, \widehat{M}|[\sum_{j=1}^{N+M+2} k_j]^2 |\widehat{N}, \widehat{M}| = \{[\sum_{j=1}^{N+M+2} k_j] |\widehat{N}, \widehat{M}|\}^2,$$

and

$$|\widehat{N}, \widehat{M}|[\sum_{j=1}^{N+M+2} k_j]^2 |\widehat{N-1}, \widehat{M+1}| = \{[\sum_{j=1}^{N+M+2} k_j] |\widehat{N}, \widehat{M}|\} \{[\sum_{j=1}^{N+M+2} k_j] |\widehat{N-1}, \widehat{M+1}|\}$$

$$= |\widehat{N-1}, \widehat{M+1}| \left[\sum_{j=1}^{N+M+2} k_j \right]^2 |\widehat{N}, \widehat{M}|,$$

we get

$$\begin{aligned} & |\widehat{N}, \widehat{M}| (|\widehat{N-2}, N, N+1, \widehat{M}| + |\widehat{N-1}, N+2, \widehat{M}| - 2|\widehat{N-1}, N+1, \widehat{M-1}, M+1| \\ & + |\widehat{N}, \widehat{M-2}, M, M+1| + |\widehat{N}, \widehat{M-1}, M+2|) \\ & = (|\widehat{N-1}, N+1, \widehat{M}| - |\widehat{N}, \widehat{M-1}, M+1|)^2, \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} & |\widehat{N-1}, \widehat{M+1}| \{ |\widehat{N-2}, N, N+1, \widehat{M}| + |\widehat{N-1}, N+2, \widehat{M}| \\ & - 2|\widehat{N-1}, N+1, \widehat{M-1}, M+1| + |\widehat{N}, \widehat{M-2}, M, M+1| + |\widehat{N}, \widehat{M-1}, M+2| \} \\ & = |\widehat{N}, \widehat{M}| \{ |\widehat{N-3}, N-1, N, \widehat{M+1}| + |\widehat{N-2}, N+1, \widehat{M+1}| - 2|\widehat{N-2}, N, \widehat{M}, M+2| \\ & + |\widehat{N-1}, \widehat{M-1}, M+1, M+2| + |\widehat{N-1}, \widehat{M}, M+3| \} \\ & = \{ |\widehat{N-1}, N+1, \widehat{M}| - |\widehat{N}, \widehat{M-1}, M+1| \} \{ |\widehat{N-2}, N, \widehat{M+1}| - |\widehat{N-1}, \widehat{M}, M+2| \}, \end{aligned} \quad (3.16)$$

then, utilizing the identities (3.16), by tedious calculus,

$$\begin{aligned} & (D_t - D_x^2)h \cdot f \\ & = f(h_t - h_{xx}) - h(f_t + f_{xx}) + 2h_x f_x \\ & = 8|\widehat{N}, \widehat{M}| \{ |\widehat{N-2}, N+1, \widehat{M+1}| + |\widehat{N-1}, \widehat{M-1}, M+1, M+2| \} \\ & \quad + 8|\widehat{N-1}, \widehat{M+1}| \{ |\widehat{N-2}, N, N+1, \widehat{M}| + |\widehat{N}, \widehat{M-1}, M+2| \} \\ & \quad - 8\{ |\widehat{N-1}, N+1, \widehat{M}| |\widehat{N-2}, N, \widehat{M+1}| + |\widehat{N}, \widehat{M-1}, M+1| |\widehat{N-1}, \widehat{M}, M+2| \}. \end{aligned}$$

According to Lemma 1 in [25], it is readily checked the above expression is equal to zero, which shows that Eq.(3.8a) is satisfied. Similarly, we can verify that Eq.(3.8c) is also satisfied.

For Eq.(3.8b), using the identity (3.15) and Lemma 1 in [25], we can easily get

$$\begin{aligned} & D_x^2 f \cdot f + 2gh \\ & = 2(ff_{xx} - f_x^2 + gh) \\ & = 8|\widehat{N}, \widehat{M}| |\widehat{N-1}, N+1, \widehat{M-1}, M+1| - 8|\widehat{N-1}, N+1, \widehat{M}| |\widehat{N}, \widehat{M-1}, M+1| \\ & \quad - 8|\widehat{N+1}, \widehat{M-1}| |\widehat{N-1}, \widehat{M+1}| \end{aligned}$$

is also equal to zero, so the proof of Eq.(3.8b) is finished.

4 Bilinear Bäcklund transformation and Lax pair

4.1 Bilinear Bäcklund transformation

The Bäcklund transformation (BT) [27] provides a way of constructing new solutions from known ones for the soliton equations. In this section, based on the transformation (3.5), we will obtain the bilinear BT for Eqs. (1.1).

Theorem 4.1 The bilinear BT for Eqs. (1.1) is as follows:

$$(D_t - D_x^2)g' \cdot g = 0, \quad (4.1a)$$

$$(D_t - D_x^2)f' \cdot f = 0, \quad (4.1b)$$

$$D_x f' \cdot g = \mu f g', \quad (4.1c)$$

$$(D_x^2 - \gamma D_x)f' \cdot g + \mu D_x g' \cdot f = 0, \quad (4.1d)$$

where γ and μ are two arbitrary real constants.

Proof: Let $p' = \ln(h'/f')$, $q' = \ln(h'f')$, and introduce the following relations:

$$\begin{aligned} v_1 &= \ln \frac{f'}{f}, & v_2 &= \ln \frac{h'}{h}, & v_3 &= \ln \frac{h'}{f}, & v_4 &= \ln \frac{f'}{h}, \\ w_1 &= \ln f f', & w_2 &= \ln h h', & w_3 &= \ln f h', & w_4 &= \ln h f', \end{aligned} \quad (4.2)$$

combining the transformation (3.5), we have

$$\begin{aligned} p' - p &= v_2 - v_1 = w_3 - w_4, & p' + p &= v_3 - v_4 = w_2 - w_1, \\ q' - q &= v_1 + v_2 = v_3 + v_4, & q' + q &= w_1 + w_2 = w_3 + w_4. \end{aligned} \quad (4.3)$$

Considering

$$E_1 = (p'_t - p_t) - (p_x'^2 - p_x^2) - (q'_{xx} - q_{xx}) = 0, \quad (4.4)$$

$$E_2 = (q'_{xt} - q_{xt}) - (p'_{xxx} - p_{xxx}) - 2(p'_x q'_{xx} - p_x q_{xx}) = 0, \quad (4.5)$$

we have

$$E_1 = (v_2 - v_1)_t - 2v_{3,x}(v_2 - v_1)_x + (v_{2,x}^2 - v_{1,x}^2) - (w_1 - w_2)_{xx} - 2v_{3,xx} = 0, \quad (4.6)$$

where we have used the relations: $v_1 + v_2 = w_1 - w_2 + 2v_3$.

$$E_2 = (v_1 + v_2)_{xt} - (v_2 - v_1)_{xxx} - (v_2 - v_1)_x(w_1 + w_2)_{xx} - (2v_3 - v_1 - v_2)_x(v_1 + v_2)_{xx} = 0, \quad (4.7)$$

where we have used the relations: $v_3 - v_4 = 2v_3 - v_1 - v_2$.

Eq.(4.6) can be decoupled into the following bilinearizable constraints:

$$\mathcal{Y}_t(v_1, w_1) + \mathcal{Y}_{xx}(v_1, w_1) = 0, \quad (4.8a)$$

$$\mathcal{Y}_t(v_2, w_2) + \mathcal{Y}_{xx}(v_2, w_2) = 0, \quad (4.8b)$$

$$\mathcal{Y}_x(v_3, w_3) - \mu e^{v_1 - v_2} = 0. \quad (4.8c)$$

Differentiating the first two equations of Eqs.(4.8) with respect to x and using the relation $w_1 + w_2 + v_2 - v_1 = 2w_3$, Eq. (4.7) can be recasted in the form

$$w_{3,3x} + (v_2 - v_1)_x w_{3,2x} + v_{3,x}(v_3 + v_4)_{xx} = 0,$$

on account of Eq.(4.8c), the above expression amounts to the condition

$$v_{3,x} \left(\frac{w_{3,2x}}{v_{3,x}} \right)_x = -v_{3,x}(v_3 + v_4)_{xx},$$

which is satisfied if

$$\begin{aligned} \frac{w_{3,2x}}{v_{3,x}} + (v_3 + v_4)_x &= \gamma \Leftrightarrow \\ \mathcal{Y}_{xx}(v_3, w_3) - \gamma \mathcal{Y}_x(v_3, w_3) + \mu e^{v_1 - v_2} \mathcal{Y}_x(v_4, w_4) &= 0, \end{aligned} \quad (4.9)$$

so, according to Eq.(2.4) and the relations (4.2), Eqs.(4.8) and Eq.(4.9) can be transformed to Eqs.(4.1) for Eqs. (1.1).

4.2 Lax pair

Through the relations (4.2), we can get the following relations:

$$v_3 = v_2 + p, \quad v_4 = v_1 - p, \quad w_2 = w_3 + p, \quad w_4 = w_1 + p.$$

Setting that $w_i = v_i + Q_i$, $v_i = \ln \Psi_i$ ($i = 1, 3$), we can get

$$Q_1 = w_1 - v_1 = Q_3 = w_3 - v_3 = q - p. \quad (4.10)$$

From Eq.(4.8c), Eq.(4.9) and the relations (4.10), we can get

$$\Psi_{1,x} = \frac{\gamma}{2} \Psi_1 - \frac{(q-p)_{xx}}{2\mu e^p} \Psi_3, \quad (4.11)$$

from the BT Eq.(4.1c), we have

$$\Psi_{3,x} = \mu e^p \Psi_1, \quad (4.12)$$

so, in combination with Eq.(4.11) and Eq.(4.12), we have

$$\Psi_x = \begin{pmatrix} \frac{\gamma}{2} & -\frac{(q-p)_{xx}}{2\mu e^p} \\ \mu e^p & 0 \end{pmatrix} \Psi = M \Psi, \quad (4.13)$$

where $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_3 \end{pmatrix}$.

Again, from Eq.(4.8a) and Eq.(4.8b), we can get directly

$$\Psi_t = N\Psi,$$

with

$$\begin{aligned} N &= \begin{pmatrix} \frac{-\gamma^2 - 2(q-p)_{xx}}{4} & \frac{\gamma(q-p)_{xx} - 2p_x(q-p)_{xx} + 2(q-p)_{xxx}}{4\mu e^p} \\ \frac{\mu e^p(2p_x - \gamma)}{2} & \frac{2p_t - 2p_x^2 - p_{xx} - q_{xx}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\gamma^2 - 2(q-p)_{xx}}{4} & \frac{\gamma(q-p)_{xx} - 2p_x(q-p)_{xx} + 2(q-p)_{xxx}}{4\mu e^p} \\ \frac{\mu e^p(2p_x - \gamma)}{2} & \frac{(q-p)_{xx}}{2} \end{pmatrix}. \end{aligned} \quad (4.14)$$

M and N are just the Lax pair of Eqs. (1.1), but they are very complicated, we will simplify it by two transformations.

Let $\psi = e^p$, $\varphi = -\frac{(q-p)_{xx}}{2e^p}$, M and N are further transformed to

$$M = \begin{pmatrix} \frac{\gamma}{2} & \frac{\varphi}{\mu} \\ \mu\psi & 0 \end{pmatrix}, \quad N = \begin{pmatrix} \frac{-\gamma^2}{4} + \psi\varphi & \frac{1}{\mu}(-\varphi_x - \frac{\gamma}{2}\varphi) \\ \mu(\psi_x - \frac{\gamma}{2}\psi) & -\psi\varphi \end{pmatrix}. \quad (4.15)$$

In particular, if we let $\gamma = 4\lambda$, and

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\phi}{\mu} \\ \phi & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad (4.16)$$

where $\phi = e^{\lambda x - 2\lambda^2 t}$, and λ is an arbitrary real constant, then we can get

$$\chi_x = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda & \psi \\ \varphi & \lambda \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \overline{M} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad (4.17)$$

and

$$\chi_t = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_t = \begin{pmatrix} 2\lambda^2 - \psi\varphi & \psi_x - 2\lambda\psi \\ -\varphi_x - 2\lambda\varphi & -2\lambda^2 + \psi\varphi \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \overline{N} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \quad (4.18)$$

$\overline{M}$ and $\overline{N}$ are also the Lax pair of Eqs. (1.1). When $u = (\ln \psi)_x$, $v = -\psi\varphi + (\ln \psi)_{xx}$, we can verify that the Levi equation (1.1) can be recovered by the compatibility condition $\overline{M}_t - \overline{N}_x + [\overline{M}, \overline{N}] = 0$.

5 Darboux transformation

In what follows, we are going to discuss the Darboux transformation [28, 29] of the spectral problems (4.17) and (4.18). Now we introduce a gauge transformation

$$\overline{\chi} = T\chi, \quad (5.1)$$

where T is defined by

$$T_x + T\overline{M} = \overline{\overline{M}}T, \quad (5.2a)$$

$$T_t + T\overline{N} = \overline{\overline{N}}T. \quad (5.2b)$$

The Lax pair (4.17) and (4.18) can be transformed into

$$\overline{\chi}_x = \overline{\overline{M}}\overline{\chi}, \quad \overline{\chi}_t = \overline{\overline{N}}\overline{\chi}, \quad (5.3)$$

where $\overline{\overline{M}}$ and $\overline{\overline{N}}$ have the same forms as $\overline{M}$ and $\overline{N}$, respectively, except replacing ψ and φ with $\overline{\psi}$ and $\overline{\varphi}$.

Let the matrix T be in the form of

$$T = T(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (5.4)$$

where

$$\begin{aligned} A &= \lambda^N + \sum_{k=0}^{N-1} \lambda^k A_k, & B &= \sum_{k=0}^{N-1} \lambda^k B_k, \\ C &= \sum_{k=0}^{N-1} \lambda^k C_k, & D &= \lambda^N + \sum_{k=0}^{N-1} \lambda^k D_k, \end{aligned}$$

with A_k, B_k, C_k and D_k are the functions of x and t , they can be determined by the following linear algebraic system:

$$\sum_{k=0}^{N-1} \lambda_j^k (A_k + B_k \sigma_j) = -\lambda_j^N, \quad \sum_{k=0}^{N-1} \lambda_j^k (C_k + D_k \sigma_j) = -\sigma_j \lambda_j^N, \quad (5.5)$$

where

$$\sigma_j = \frac{\zeta_2(\lambda_j) - r_j \eta_2(\lambda_j)}{\zeta_1(\lambda_j) - r_j \eta_1(\lambda_j)}, \quad (1 \leq j \leq 2N), \quad (5.6)$$

here $\zeta = (\zeta_1, \zeta_2)^T$ and $\eta = (\eta_1, \eta_2)^T$ are two basic solutions of the spectral problems (4.17) and (4.18), λ_j and $r_j(\lambda_k \neq \lambda_j, r_k \neq r_j \text{ as } k \neq j)$ are some suitable parameters such that the determinant of the coefficients for Eq.(5.5) is nonzero. Therefore, A_k, B_k, C_k and $D_k(0 \leq k \leq N-1)$ are uniquely determined by Eq.(5.5).

Theorem 5.1 The transformation from the old potentials ψ and φ into the new ones $\overline{\psi}$ and $\overline{\varphi}$ are given by

$$\overline{\psi} = \psi + 2B_{N-1}, \quad \overline{\varphi} = \varphi - 2C_{N-1}, \quad (5.7)$$

then the matrices $\overline{\overline{M}}$ and $\overline{\overline{N}}$ have the same forms as $\overline{M}$ and $\overline{N}$, respectively.

Proof: Let $T^{-1} = T^*/\det T$ and

$$(T_x + T\overline{M})T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}, \quad (5.8)$$

where T^* denotes the adjoint matrix of T . it is easy to see that $f_{11}(\lambda), f_{22}(\lambda)$ are $2N+1$ th-order polynomials in λ , while $f_{12}(\lambda), f_{21}(\lambda)$ are $2N$ th-order polynomials in λ . From the spectral problems (4.17),(4.18) and the expression (5.6), we have a Riccati equation

$$\sigma_{jx} = \varphi + 2\lambda_j\sigma_j - \psi\sigma_j^2, \quad \sigma_{jt} = (-\varphi_x - 2\lambda_j\varphi) - 2(2\lambda_j^2 - \psi\varphi)\sigma_j - (\psi_x - 2\lambda_j\psi)\sigma_j^2. \quad (5.9)$$

Through direct calculations, all $\lambda_j (1 \leq j \leq 2N)$ are roots of $f_{sl}(s, l = 1, 2)$, therefore, Eq.(5.8) gives

$$(T_x + T\overline{M})T^* = (\det T)P(\lambda),$$

with

$$P(\lambda) = \begin{pmatrix} P_{11}^{(1)}\lambda + P_{11}^{(0)} & P_{12}^{(0)} \\ P_{21}^{(0)} & P_{22}^{(1)}\lambda + P_{22}^{(0)} \end{pmatrix},$$

where $P_{kj}^{(l)}(k, j = 1, 2, l = 0, 1)$ are independent of λ . Now Eq.(5.8) can be written as

$$(T_x + T\overline{M}) = P(\lambda)T, \quad (5.10)$$

By comparing the coefficients of $\lambda^{N+1}, \lambda^N, \lambda^{N-1}$ in (5.10), we find

$$\begin{aligned} P_{11}^{(1)} &= -1, & P_{11}^{(0)} &= 0, \\ P_{12}^{(0)} &= \psi + 2B_{N-1}, & P_{21}^{(0)} &= \varphi - 2C_{N-1}, \\ P_{22}^{(1)} &= 1, & P_{22}^{(0)} &= 0, \end{aligned}$$

noticing the transformation (5.7), we see that

$$P(\lambda) = \begin{pmatrix} -\lambda & \overline{\psi} \\ \overline{\varphi} & \lambda \end{pmatrix} = \overline{\overline{M}}.$$

Similarly,

$$(T_t + T\overline{N})T^* = (\det T)Q(\lambda), \quad (5.11)$$

with

$$Q(\lambda) = \begin{pmatrix} g_{11}^{(1)}\lambda^2 + g_{11}^{(0)} & g_{12}^{(1)}\lambda + g_{12}^{(0)} \\ g_{21}^{(1)}\lambda + g_{21}^{(0)} & g_{22}^{(1)}\lambda^2 + g_{22}^{(0)} \end{pmatrix},$$

where $g_{kj}^{(l)}(k, j = 1, 2, l = 0, 1)$ are independent of λ . Now Eq.(5.11) can be written as

$$(T_t + T\overline{N}) = Q(\lambda)T, \quad (5.12)$$

By comparing the coefficients of $\lambda^{N+2}, \lambda^{N+1}, \lambda^N$ in Eq.(5.12), we find

$$\begin{aligned} g_{11}^{(1)} &= 2, & g_{11}^{(0)} &= -(\psi + 2B_{N-1})(\varphi - 2C_{N-1}), \\ g_{12}^{(1)} &= -2(\psi + 2B_{N-1}), & g_{12}^{(0)} &= \psi_x - 4B_{N-2} - 2\psi(A_{N-1} - D_{N-1}) + 4B_{N-1}D_{N-1}, \\ g_{21}^{(1)} &= -2(\varphi - 2C_{N-1}), & g_{21}^{(0)} &= -\varphi_x + 4C_{N-2} + 2\varphi(A_{N-1} - D_{N-1}) - 4A_{N-1}C_{N-1}, \\ g_{22}^{(1)} &= -2, & g_{22}^{(0)} &= (\psi + 2B_{N-1})(\varphi - 2C_{N-1}). \end{aligned}$$

Comparing the coefficients of λ^{N-1} in Eq.(5.10), we have

$$\begin{aligned} B_{N-1,x} &= -2B_{N-2} - \psi(A_{N-1} - D_{N-1}) + 2B_{N-1}D_{N-1}, \\ C_{N-1,x} &= 2C_{N-2} + \varphi(A_{N-1} - D_{N-1}) - 2A_{N-1}C_{N-1}, \end{aligned}$$

noticing the transformation (5.7), we obtain

$$Q(\lambda) = \begin{pmatrix} 2\lambda^2 - \bar{\psi}\bar{\varphi} & -2\bar{\psi}\lambda + \bar{\psi}_x \\ -2\bar{\varphi}\lambda - \bar{\varphi}_x & -2\lambda^2 + \bar{\psi}\bar{\varphi} \end{pmatrix} = \bar{\bar{N}}.$$

The proof is completed.

6 Infinite conservation laws

In this section, we derive the infinite conservation laws [30] for Eqs.(1.1) through the Lax equations (4.17) and (4.18).

Let $w(x, \lambda) = \frac{\chi_2}{\chi_1}$, Eq.(4.17) is transformed to

$$w(x, \lambda)_x = -\psi w(x, \lambda)^2 + 2\lambda w(x, \lambda) + \varphi,$$

or

$$\psi w(x, \lambda)_x = -\psi^2 w(x, \lambda)^2 + 2\lambda \psi w(x, \lambda) + \psi \varphi, \quad (6.1)$$

inserting expansion

$$\psi w(x, \lambda) = \sum_{n=0}^{\infty} w_n(x) (2\lambda)^{-(n+1)} \quad (6.2)$$

into Eq.(6.1), and comparing the coefficients of $(2\lambda)^{-(n+1)} (n = 0, 1, 2, 3, \dots)$, we get

$$\begin{aligned} w_0(x) &= -\psi \varphi = v - u_x, \\ w_1(x) &= -\psi \varphi_x = v_x - u_{xx} - uv + uu_x, \\ w_{n+1}(x) &= \psi \left(\frac{w_n(x)}{\psi} \right)_x + \sum_{j=0}^{n-1} w_j(x) w_{n-j-1}(x) \quad (n = 1, 2, 3, \dots). \end{aligned}$$

This is the formular of the coefficients of Eq.(6.2). Dividing both sides of the first equation of the Lax equations (4.17) and (4.18) by χ_1 , we get

$$\begin{aligned}(\ln \chi_1)_x &= -\lambda + \psi w(x, \lambda), \\ (\ln \chi_1)_t &= 2\lambda^2 - \psi\varphi + (\psi_x - 2\lambda\psi)w(x, \lambda).\end{aligned}$$

From the compatibility condition $(\ln \chi_1)_{tx} = (\ln \chi_1)_{xt}$, we have

$$(-\lambda + \psi w(x, \lambda))_t = (2\lambda^2 - \psi\varphi + (\psi_x - 2\lambda\psi)w(x, \lambda))_x,$$

that conservation laws is just

$$(\psi w(x, \lambda))_t = (2\lambda^2 - \psi\varphi + (\psi_x - 2\lambda\psi)w(x, \lambda))_x. \quad (6.3)$$

The first conservation laws and the n th conservation laws are

$$\begin{aligned}(w_0(x))_t &= (\psi\varphi)_t = (\psi_x\varphi - \psi\varphi_x)_x, \\ (w_n(x))_t &= (-w_{n+1}(x) + \frac{\psi_x}{\psi}w_n(x))_x, \quad n = 1, 2, 3, \dots\end{aligned}$$

7 Conclusions

In this paper, we investigated the Levi equation (1.1) by using the binary Bell polynomials approach and Hirota method. Based on the bilinear equations for Eqs.(1.1), the double Wronskian solutions, the bilinear Bäcklund transformation, the corresponding Lax pair, the Darboux transformation and the infinite conservation laws of Eqs.(1.1) are all derived. These results show, combining the binary Bell polynomials approach and Hirota method not only can get more integrability properties of a nonlinear equation but also can avoid a lot of difficulties.

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