Distributed Constraint Optimization on Networked Multi-Agent Systems

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Abstract

This paper deals with a distributed constraint optimization problem on networked multi-agent systems. First, we propose a distributed algorithm based on the Lagrangian method, where a new update law of the Lagrangian multiplier is designed. This update law enables each agent to estimate the value of the Lagrangian multiplier in a distributed manner. Next, we derive a necessary and sufficient condition that the optimization problem is solvable in a distributed manner over a graph. Finally, we apply the proposed method to power grid control via distributed pricing to maintain the supply-demand balance.

Keywords: distributed optimization, multi-agent system, Lagrangian method

1. Introduction

Recently, distributed optimization has been vigorously investigated to solve problems in large-scale systems. The most successful example is the Internet protocols [1, 2, 3, 4], where congestion control is implemented. Distributed control techniques are inevitable to reduce computation burdens in such an enormously large-scale system. In recent years, these techniques are becoming more and more important because the sizes of many public infrastructures are growing (e.g., power grids [5, 6, 7, 8] and traffic systems [9, 10, 11]) and the management of these systems is getting harder.

This paper deals with a distributed optimization problem in the terminology of networked multi-agent systems. Let $x_i \in \mathbb{R}^{d_i}$ be the variable updated by agent $i \in \{1, 2, \dots, n\}$. For $x = [x_1^\top \ x_2^\top \ \cdots \ x_n^\top]^\top$, let $F(x) \in \mathbb{R}$ and

1 $G_j(x) \in \mathbb{R}$, $j = 1, 2, \dots, m$ be the objective and constraint functions. Then,
2 the constraint optimization problem is formulated as follows.

$$\begin{cases} \text{minimize} & F(x) \\ \text{subject to} & G_j(x) \leq 0, \quad j = 1, 2, \dots, m \end{cases} \quad (1)$$

3 Note that equality constraints can be easily handled by the choice of $G_j(x)$.
4 Now, these agents can communicate with each other over a network $\mathcal{G}$ (which
5 is described by a graph).

6 Then, each agent can update the value of x_i by using only local informa-
7 tion over $\mathcal{G}$. Let $x_i[k] \in \mathbb{R}^{d_i}$ be the updated value of x_i at time k , and $\lambda_j \in \mathbb{R}$
8 be the Lagrangian multiplier (a kind of penalty) according to the constraint
9 $G_j(x)$. Then, the distributed update law based on the Lagrangian method
10 is of the form

$$x_i[k+1] = f_i(x_i[k], x_{i_1}[k], x_{i_2}[k], \dots, x_{i_{n_i}}[k], \lambda_1, \lambda_2, \dots, \lambda_m), \quad (2)$$

11 where $i_1, i_2, \dots, i_{n_i}$ are the neighbors of agent i over the network $\mathcal{G}$. (2) is
12 distributed in the sense that the agent requires only its own, its neighbors'
13 information and the Lagrangian multipliers.

14 Note that the Lagrangian multiplier λ_j is common among the agents,
15 hence it is global information. As long as the update law (2) requires the
16 information on λ_j , a supervisor is inevitable to compute and broadcast the
17 value of λ_j . In a large-scale system, a huge amount of information will be
18 concentrated on the supervisor. Thus, the update law (2) is not completely
19 distributed nor scalable. In order to make the update law (2) distributed,
20 the value of λ_j should be individually computed by each agent. However,
21 generally, global information, the value of $G_j(x)$, is required to update λ_j . Is
22 it possible to obtain the solution of (1) by distributed computations of not
23 only x_i but also λ_j ?

24 In this paper, we propose a new distributed algorithm to solve the op-
25 timization problem (1) over the network $\mathcal{G}$. Especially, we propose a dis-
26 tributed method to estimate the Lagrangian multiplier λ_j . By using this
27 method, agents can individually obtain the estimated values of λ_j through
28 information exchange between the agents. Then, whether the solution of (1)
29 is obtained or not depends on the structure of $\mathcal{G}$. So, we derive a necessary
30 and sufficient condition of $\mathcal{G}$ with which the problem (1) is solvable in the
31 distributed way. From this result, if an optimization problem is given, we

1 can answer a strict class of networks with which the problem is distribut-
2 edly solvable. Conversely, if a network is given, we can know what kind of
3 optimization problem is solvable over this network.

4 As an application of the proposed method, we consider power grid control
5 via pricing to maintain supply-demand balance. The Lagrangian multiplier
6 λ_j is corresponding to the electricity price. Our proposed method gives a ba-
7 sic rule to determine the price by negotiations between consumers, suppliers
8 and distributors over a communication network.

9 1.1. Contributions comparing to related researches

10 Distributed optimization methods are summarized in [12]. For example,
11 dual decomposition was developed in 1960s and has been modified by several
12 researchers [13, 14, 15, 16]. The main idea of the dual decomposition is to
13 deal with separable functions with respect to agents, namely,

$$\begin{cases} \text{minimize} & \sum_{i=1}^n F_i(x_i) \\ \text{subject to} & \sum_{i=1}^n G_{ij}(x_i) \leq 0, \quad j = 1, 2, \dots, m \end{cases} \quad (3)$$

14 where $F_i(x_i)$ and $G_{ij}(x_i)$ depend on one agent's variable x_i . Then, the update
15 law of each agent is given by

$$x_i[k+1] = x_i[k] - \alpha \left(\frac{\partial F_i}{\partial x_i}(x_i[k]) + \sum_{j=1}^m \lambda_j \frac{\partial G_{ij}}{\partial x_i}(x_i[k]) \right), \quad (4)$$

16 **where α is a step gain.** The update law (4) is of the form (2) because other
17 agents' information is not required. However, this update law requires the
18 global information λ_j . Moreover, the objective and constraint functions are
19 restricted to separable forms as (3). **The other primal-dual methods**
20 **including alternating direction method of multipliers (ADMM),**
21 **alternating direction augmented Lagrangian (ADAL) and so on**
22 **[17, 18] have the same structure.**

23 Recently, consensus-based distributed optimization has been investigated
24 [19, 20, 21, 22, 23]. **The consensus-based distributed optimization for**
25 **the objective function $F(x) = \sum_{i=1}^n F_i(x)$ with agent-wise functions**
26 **$F_i(x)$ is reduced to**

$$\begin{cases} \text{minimize} & \sum_{i=1}^n F_i(x_i) \\ \text{subject to} & x_i = x_j, \quad i, j = 1, 2, \dots, n \end{cases} \quad (5)$$

1 **by introducing the constraints** $x_i = x_j$. Then, the update law is given
 2 by

$$x_i[k+1] = x_i[k] - \alpha \left(\frac{\partial F_i}{\partial x_i}(x_i[k]) - \sum_{j \in \mathcal{N}_i} (x_i[k] - x_j[k]) \right), \quad (6)$$

3 where $\mathcal{N}_i$ is the set of neighbors of agent i over the network $\mathcal{G}$. The update
 4 law (6) is distributed and the solution is obtained only if $\mathcal{G}$ is connected.

5 **Note that the objective function in (5) is separable, and other**
 6 **types of objective functions cannot be treated in this form.**

7 Since these existing researches restrict optimization problems to the forms
 8 (3) and (5), the problems can be solved by the distributed update laws (4) and
 9 (6) (although (4) is not completely distributed because of λ_j). The question
 10 here is that *if we do not restrict optimization problems to these forms, what*
 11 *kind of problems can be solved in a distributed manner over the network $\mathcal{G}$?*
 12 *Are there more problems solvable in a distributed manner?* To answer these
 13 questions, we give a complete characterization of the class of optimization
 14 problems solvable in a distributed manner over $\mathcal{G}$. The derived class indicates
 15 the limit of the graph $\mathcal{G}$'s ability to solve distributed optimization problems.
 16 As far as the authors' knowledge, this is the first paper which specifies the
 17 graph's ability in this way. **As a result, this paper reveals that we**
 18 **can solve a wider class of optimization problems than (3) and (5)**
 19 **in a distributed manner. See Example 1 and Section 4 for such**
 20 **examples.**

21 1.2. Notations

22 The following notations are used in this paper. Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{Z}_+$ be
 23 the sets of the real numbers, nonnegative real numbers and nonnegative
 24 integers. The set of n -dimensional (non-negative) real numbers is represented
 25 as $\mathbb{R}^n$ ($\mathbb{R}_+^n$). The cardinal number of a set is described as $|\cdot|$. For a set
 26 $\mathcal{S} \subset \{1, 2, \dots, n\}$ with the cardinal number $|\mathcal{S}| = m$, there exists a strictly
 27 increasing sequence $s_1, s_2, \dots, s_m \leq n$ of natural numbers such that $\mathcal{S} =$
 28 $\{s_1, s_2, \dots, s_m\}$. For $\mathcal{S}$ and $x_i \in \mathbb{R}^{d_i}$, $i = 1, 2, \dots, n$, we use the notation
 29 $x_{\mathcal{S}} = [x_{s_1}^\top \ x_{s_2}^\top \ \cdots \ x_{s_m}^\top]^\top$. Let the function $[\cdot]^+ : \mathbb{R}^n \rightarrow \mathbb{R}_+^n$ be the operation
 30 of replacing negative components of a vector with zero such that

$$([x]^+)_i = \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 & \text{if } x_i < 0 \end{cases},$$

31 where $([x]^+)_i \in \mathbb{R}_+$ is the i -th component of the vector $[x]^+ \in \mathbb{R}_+^n$.

1 2. Problem setting

2 2.1. Lagrangian method

3 Consider the optimization problem (1). We assume that functions $F(x)$
 4 and $G_j(x) \in \mathbb{R}$, $j = 1, 2, \dots, m$ for $x \in \mathbb{R}^N$ are continuously differentiable.
 5 The Lagrangian method is introduced to solve (1) [24]. Let

$$L(x, \lambda) = F(x) + \lambda^\top G(x) \quad (7)$$

6 be the Lagrangian function with the Lagrangian multiplier $\lambda \in \mathbb{R}_+^m$ and the
 7 constraint function $G(x) = [G_1(x) \ G_2(x) \ \cdots \ G_m(x)]^\top$. Let

$$\Phi(\lambda) = \min_{x \in \mathbb{R}^N} L(x, \lambda) \quad (8)$$

8 be the Lagrangian dual function. Then,

$$\begin{cases} \text{maximize } \Phi(\lambda) \\ \text{subject to } \lambda \geq 0 \end{cases} \quad (9)$$

9 is called the dual problem of the **primal** problem (1).

10 If the solution λ of (9) is achieved, the solution x of (1) is obtained from
 11 (8) under the assumption of the strong duality. To obtain the solution based
 12 on the dual problem, we update x so as to **decrease** $L(x, \lambda)$ from (8), then
 13 update λ so as to increase $L(x, \lambda)$ according to (9), and repeat these updates
 14 alternatively. An algorithm based on this method is given as follows.

15 *Algorithm A:*

16 (A.1) Give the initial vectors $x^{[0]} \in \mathbb{R}^N$, $\lambda^{[0]} \in \mathbb{R}_+^m$ and constants $\alpha, \beta > 0$.

17 Set the step $s = 0$.

18 (A.2) Set $x[0] = x^{[s]} \in \mathbb{R}^N$ and execute the recursive computation

$$x[k+1] = x[k] - \alpha \left(\frac{\partial F}{\partial x}(x[k]) + (\lambda^{[s]})^\top \frac{\partial G}{\partial x}(x[k]) \right) \quad (10)$$

19 for $k \in \mathbb{Z}_+$. Then, update $x^{[s]}$ to $x^{[s+1]} = \lim_{k \rightarrow \infty} x[k]$.

20 (A.3) Update $\lambda^{[s]} \in \mathbb{R}_+^m$ to

$$\lambda^{[s+1]} = [\lambda^{[s]} + \beta G(x^{[s+1]})]^\top. \quad (11)$$

21 (A.4) Add 1 to the step s and return to (A.2).

1 The update variable $x^{[s]}$ converges to the solution of (1) under two as-
2 sumptions. The first assumption is a certain relaxed condition to guar-
3 antee the strong duality. Then, there is no gap between the solutions of
4 the primal and dual problems. The second assumption is the **twice con-**
5 **tinuous differentiability** and the convexity of the functions $F(x)$ and
6 $G_j(x)$, $j = 1, 2, \dots, m$. This is to achieve the global minimum by using
7 the gradient-based methods. We assume these conditions hereafter.

8 **Remark 1.** In the iteration of (A.2) of the algorithm, the limit
9 $\lim_{k \rightarrow \infty} x[k]$ exists under the assumptions of the functions $F(x)$ and
10 $G(x)$. On the other hand, in practice, the iteration has to be trun-
11 cated, so there is numerical error to the limit $\lim_{k \rightarrow \infty} x[k]$. In this
12 case, we obtain the update variable $x^{[s+1]} = \lim_{k \rightarrow \infty} x[k] + \varepsilon^{[s+1]}$ with
13 the error $\varepsilon^{[s+1]}$. Then, the optimization error remains in proportion
14 to the absolute value of $\varepsilon^{[s+1]}$ and its square [25]. Therefore, the
15 optimization error can be arbitrarily small as the number of the
16 iteration increases.

17 2.2. Distributed update

18 Consider a multi-agent system with n agents over the network represented
19 by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the node set $\mathcal{V} = \{1, 2, \dots, n\}$ and the edge set
20 $\mathcal{E}$ represent the group of agents and their network connections, respectively.
21 Assume that $\mathcal{G}$ is simple and undirected, and the edge between $i, j \in \mathcal{V}$ is
22 represented as $\{i, j\} \in \mathcal{E}$. Let $x_i \in \mathbb{R}^{d_i}$ be the variable updated by agent i .
23 Our goal is to solve the constraint optimization problem (1) in a distributed
24 manner over the graph $\mathcal{G}$, where $N = \sum_i^n d_i$.

25 Now, we try to employ Algorithm A for this purpose. The problem here is
26 that the Lagrangian multiplier λ is global information. Therefore, the value
27 $\lambda^{[s]} \in \mathbb{R}_+^m$ in Algorithm A should be updated and broadcast by a supervisor.
28 Our stance in this paper is to adopt a completely distributed manner without
29 supervisors. To attain this, we let each agent estimate the value of $\lambda^{[s]} \in \mathbb{R}_+^m$
30 individually. The estimate value of agent i is described as $\hat{\lambda}_i^{[s]} \in \mathbb{R}_+^m$. Now,
31 agent i has to update two variables $x_i^{[s]}$ and $\hat{\lambda}_i^{[s]}$ in a distributed manner over
32 $\mathcal{G}$.

33 Let $x_i[k] \in \mathbb{R}^{d_i}$ and $\hat{\lambda}_i[l] \in \mathbb{R}^m$ be the temporal update values for $x_i^{[s]}$ and
34 $\hat{\lambda}_i^{[s]}$. Let $\mathcal{N}_i$ be the set of neighbors of agent i over $\mathcal{G}$ as

$$\mathcal{N}_i = \{j \in \mathcal{V} | \{i, j\} \in \mathcal{E}\}. \quad (12)$$

1 Consider some functions $f_i : \mathbb{R}^{d_i} \times \mathbb{R}^{D_i} \times \mathbb{R}^{d_i} \times \mathbb{R}^{D_i} \times \mathbb{R}^m \times \mathbb{R}^{m|\mathcal{N}_i|} \rightarrow \mathbb{R}^{d_i}$,
 2 $g_i : \mathbb{R}^{d_i} \times \mathbb{R}^{D_i} \times \mathbb{R}^m \times \mathbb{R}^{m|\mathcal{N}_i|} \rightarrow \mathbb{R}^m$, and $h_i : \mathbb{R}^m \times \mathbb{R}^{m|\mathcal{N}_i|} \times \mathbb{R}^{d_i} \times \mathbb{R}^{D_i} \times \mathbb{R}^m \times$
 3 $\mathbb{R}^{m|\mathcal{N}_i|} \rightarrow \mathbb{R}^m$ where $D_i = \sum_{j \in \mathcal{N}_i} d_j$. Then, an algorithm to update $x_i^{[s]}$ and
 4 $\hat{\lambda}_i^{[s]}$ in a distributed manner is described as follows.

5 *Algorithm B:*

6 (B.1) Each agent gives initial vectors $x_i^{[0]} \in \mathbb{R}^{d_i}$ and $\hat{\lambda}_i^{[0]} \in \mathbb{R}_+^m$. Set the
 7 step $s = 0$.

8 (B.2) Each agent sets $x_i[0] = x_i^{[s]}$, executes

$$x_i[k+1] = f_i(x_i[k], x_{\mathcal{N}_i}[k], x_i^{[s]}, x_{\mathcal{N}_i}^{[s]}, \hat{\lambda}_i^{[s]}, \hat{\lambda}_{\mathcal{N}_i}^{[s]}) \quad (13)$$

9 for $k \in \mathbb{Z}_+$, and updates $x_i^{[s]}$ to $x_i^{[s+1]} = \lim_{k \rightarrow \infty} x_i[k]$.

10 (B.3) Each agent sets

$$\hat{\lambda}_i[0] = g_i(x_i^{[s+1]}, x_{\mathcal{N}_i}^{[s+1]}, \hat{\lambda}_i^{[s]}, \hat{\lambda}_{\mathcal{N}_i}^{[s]}) \quad (14)$$

11 and executes

$$\hat{\lambda}_i[l+1] = h_i(\hat{\lambda}_i[l], \hat{\lambda}_{\mathcal{N}_i}[l], x_i^{[s+1]}, x_{\mathcal{N}_i}^{[s+1]}, \hat{\lambda}_i^{[s]}, \hat{\lambda}_{\mathcal{N}_i}^{[s]}) \quad (15)$$

12 for $l \in \mathbb{Z}_+$, and updates $\hat{\lambda}_i^{[s]}$ to $\hat{\lambda}_i^{[s+1]} = [\lim_{l \rightarrow \infty} \hat{\lambda}_i[l]]^+$.

13 (B.4) Add 1 to the step s and return to (B.2).

14 **In this paper, we say an algorithm is distributed over the graph**
 15 **$\mathcal{G}$ if the computations of each agent in the algorithm is given by a**
 16 **function depending on its own information and the neighbors' one.**

17 In Algorithm B, agent i uses only its own variable $x_i[k]$, $\hat{\lambda}_i[l]$, $x_i^{[s]}$, $x_i^{[s+1]}$, $\hat{\lambda}_i^{[s]}$
 18 and the neighbors' variables $x_{\mathcal{N}_i}[k]$, $\hat{\lambda}_{\mathcal{N}_i}[l]$, $x_{\mathcal{N}_i}^{[s]}$, $x_{\mathcal{N}_i}^{[s+1]}$, $\hat{\lambda}_{\mathcal{N}_i}^{[s]}$. In this sense, this
 19 algorithm is distributed.

20 Now, we design functions f_i , g_i and h_i such that Algorithm B works as
 21 Algorithm A. Then, the solution of the constraint optimization problem (1)
 22 is obtained in a distributed manner. Do there really exist such functions f_i ,
 23 g_i and h_i ? Actually, the feasibility depends on the structure of the graph
 24 $\mathcal{G}$ because it determines the neighbors $\mathcal{N}_i$ in (13), (14) and (15). Now, the
 25 main problem is formulated as follows.

26 **Problem 1.** For continuously differentiable functions $F : \mathbb{R}^N \rightarrow \mathbb{R}$ and
 27 $G : \mathbb{R}^N \rightarrow \mathbb{R}^m$ and a graph $\mathcal{G}$, derive a necessary and sufficient condition such
 28 that there exist continuous functions f_i , g_i and h_i and positive constants α, β
 29 satisfying the following conditions.

1 (i) (13) is equivalent to

$$x_i[k+1] = x_i[k] - \alpha \left(\frac{\partial F}{\partial x_i}(x[k]) + (\hat{\lambda}_i^{[s]})^\top \frac{\partial G}{\partial x_i}(x[k]) \right). \quad (16)$$

2 (ii) (14) and (15) generate $\hat{\lambda}_i^{[s+1]} = [\lim_{l \rightarrow \infty} \hat{\lambda}_i[l]]^+$ such that

$$\lambda_i^{[s+1]} = \left[\hat{\lambda}_i^{[s]} + \beta G(x^{[s+1]}) \right]^+ \quad (17)$$

$$\hat{\lambda}_i^{[s+1]} = \hat{\lambda}_j^{[s+1]}, \quad \forall i, j \in \mathcal{V}. \quad (18)$$

3 Moreover, design these functions f_i , g_i and h_i .

4 **The big difference between Algorithms A and B is that the**
5 **latter is based on the agent-wise variables $\hat{\lambda}_i^{[s]}$ as a Lagrangian mul-**
6 **tiplier. However, if Problem 1 is solved, then $\hat{\lambda}_i^{[s]}$ are updated into**
7 **the same value among all agents as (18), and each agent can use it**
8 **as a Lagrangian multiplier because of (17). Then, Algorithm B works**
9 **as Algorithm A and we can obtain the solution of the constraint optimization**
10 **problem (1) in a distributed manner.**

11 **Remark 2.** It seems that the update variable $\hat{\lambda}_i^{[s+1]}$ is possibly obtained by
12 the direct computation of (17). However, to do so, all information on $x^{[s+1]}$
13 related to $G(x^{[s+1]})$ is necessary. Therefore, it is not distributed anymore.
14 Instead, we update $\hat{\lambda}_i^{[s]}$ in a distributed manner through the repetitive cal-
15 culation (14) and (15). We expect that $\hat{\lambda}_i^{[s]}$ is updated into the same value
16 satisfying (17) as a result.

17 **Remark 3.** The concept of the distributedness here is different
18 from the other primal-dual methods as ADMM and ADAL [17, 18].
19 These existing methods are said to be distributed if the variable x_i
20 of each agent can be updated by using its own information and the
21 information on the Lagrange multiplier λ . Note that λ is updated
22 by a supervisor and that the information on neighbors is not used.
23 On the other hand, in our paper, we assume that there is no super-
24 visor who updates λ , but instead the agents individually estimate
25 the value of λ through communication between neighbors.

1 3. Main result

2 First, consider condition (i) in Problem 1. (13) and (16) are equivalent
 3 if and only if there exists a continuous function $\hat{f}_i : \mathbb{R}^{d_i} \times \mathbb{R}^m \times \mathbb{R}^{D_i} \rightarrow \mathbb{R}^{d_i}$
 4 such that

$$\frac{\partial F}{\partial x_i}(x) + \hat{\lambda}_i^\top \frac{\partial G}{\partial x_i}(x) \equiv \hat{f}_i(x_i, \hat{\lambda}_i, x_{\mathcal{N}_i}). \quad (19)$$

5 Let $\mathcal{S}_{\text{clq}}(\mathcal{G})$ be the set of all cliques on the graph $\mathcal{G}$. A clique on $\mathcal{G}$ is a node
 6 subset $\mathcal{I} \subset \mathcal{V}$ such that the subgraph induced by $\mathcal{I}$ is complete. Therefore,
 7 $\mathcal{S}_{\text{clq}}(\mathcal{G})$ is described as follows.

$$\mathcal{S}_{\text{clq}}(\mathcal{G}) = \{\mathcal{I} \subset \mathcal{V} | \{i, j\} \in \mathcal{E} \ \forall i, j \in \mathcal{V}, i \neq j\} \quad (20)$$

8 Then, the necessary and sufficient condition for (19) is derived from the
 9 authors' paper [26].

10 **Lemma 1.** *For continuously differentiable functions $F : \mathbb{R}^N \rightarrow \mathbb{R}$ and $G :$
 11 $\mathbb{R}^N \rightarrow \mathbb{R}^m$, a vector $x = [x_1^\top \ x_2^\top \ \cdots \ x_n^\top]^\top \in \mathbb{R}^N$ with $x_i \in \mathbb{R}^{d_i}$ and a graph $\mathcal{G}$,
 12 there exist a continuous function $\hat{f}_i : \mathbb{R}^{d_i} \times \mathbb{R}^m \times \mathbb{R}^{D_i} \rightarrow \mathbb{R}^{d_i}$ for any $i \in \mathcal{V}$
 13 satisfying (19) if and only if there exist continuously differentiable functions
 14 $\bar{F}^\mathcal{I} : \mathbb{R}^{\sum_{j \in \mathcal{I}} d_j} \rightarrow \mathbb{R}$ and $\bar{G}^\mathcal{I} : \mathbb{R}^{\sum_{j \in \mathcal{I}} d_j} \rightarrow \mathbb{R}^m$ for $\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})$ such that*

$$F(x) = \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})} \bar{F}^\mathcal{I}(x_\mathcal{I}) \quad (21)$$

$$G(x) = \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})} \bar{G}^\mathcal{I}(x_\mathcal{I}). \quad (22)$$

15 □

16 Because $F(x)$ and $G(x)$ have to be of the forms (21) and (22), the update
 17 law (16) is reduced to

$$x_i[k+1] = x_i[k] - \alpha \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq},i}(\mathcal{G})} \left(\frac{\partial \bar{F}^\mathcal{I}}{\partial x_i}(x_\mathcal{I}[k]) + (\hat{\lambda}_i^{[s]})^\top \frac{\partial \bar{G}^\mathcal{I}}{\partial x_i}(x_\mathcal{I}[k]) \right), \quad (23)$$

18 where $\mathcal{S}_{\text{clq},i}(\mathcal{G})$ is the set of the cliques including i as follows.

$$\mathcal{S}_{\text{clq},i}(\mathcal{G}) = \{\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G}) | i \in \mathcal{I}\} \quad (24)$$

1 Note that (23) is of the distributed form (13) because $j \in \mathcal{N}_i$ holds for any
 2 $j \in \mathcal{I} \in \mathcal{S}_{\text{clq},i}(\mathcal{G})$.

3 Next, consider condition (ii) in Problem 1. Because $G(x)$ is of the form
 4 (22), (17) is reduced to

$$\hat{\lambda}_i^{[s+1]} = \left[\hat{\lambda}_i^{[s]} + \beta \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})} \bar{G}^{\mathcal{I}}(x_{\mathcal{I}}^{[s+1]}) \right]^+. \quad (25)$$

5 It is impossible to directly compute the right-hand side of (25) in a distributed
 6 manner because agent i needs the information on the variables $x_{\mathcal{I}}^{[s+1]}$ for all
 7 the cliques $\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})$. The information that agent i can obtain is at most
 8 the variables $x_{\mathcal{I}}^{[s+1]}$ for the cliques $\mathcal{I} \in \mathcal{S}_{\text{clq},i}(\mathcal{G})$ which he belongs to. From
 9 this viewpoint, we propose the following update law of $\hat{\lambda}_i^{[s]}$:

$$\hat{\lambda}_i[0] = \hat{\lambda}_i^{[s]} + \kappa \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq},i}(\mathcal{G})} \gamma^{\mathcal{I}} \bar{G}^{\mathcal{I}}(x_{\mathcal{I}}^{[s+1]}) \quad (26)$$

$$\hat{\lambda}_i[l+1] = \hat{\lambda}_i[l] - \eta \sum_{j \in \mathcal{N}_i} (\hat{\lambda}_i[l] - \hat{\lambda}_j[l]) \quad (27)$$

10 for some positive constants κ , $\gamma^{\mathcal{I}}$ and η . Then, (26) and (27) are of the
 11 distributed forms (14) and (15). Note that (27) is a consensus controller,
 12 hence all $\hat{\lambda}_i[l]$ converge to the same value if $\mathcal{G}$ is connected and $\eta < 1/|\mathcal{N}_i|$
 13 **holds for all** $i \in \mathcal{V}$ [27]. Thus, (18) is achieved. Moreover, (17) is obtained
 14 as proved in the following theorem. The main result of this paper is derived
 15 as follows.

16 **Theorem 1.** *Consider continuously differentiable functions $F : \mathbb{R}^N \rightarrow \mathbb{R}$,
 17 $G : \mathbb{R}^N \rightarrow \mathbb{R}^m$ and a graph $\mathcal{G}$. There exist functions f_i, g_i, h_i and positive
 18 constants α, β such that (i) (13) and (16) are equivalent, (ii) (14) and (15)
 19 generate $\hat{\lambda}_i^{[s+1]} = [\lim_{l \rightarrow \infty} \hat{\lambda}_i[l]]^+$ satisfying (17) and (18) if and only if $\mathcal{G}$ is
 20 connected and there exist functions $\bar{F}^{\mathcal{I}}(x_{\mathcal{I}})$ and $\bar{G}^{\mathcal{I}}(x_{\mathcal{I}})$ for $\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})$ such
 21 that $F(x)$ and $G(x)$ are of the forms (21) and (22). Under these conditions,
 22 such f_i, g_i, h_i are given by the right-hand sides of (23), (26) and (27) for
 23 $\gamma^{\mathcal{I}} = 1/|\mathcal{I}|$, $\kappa > 0$ and a positive constant $\eta < 1/|\mathcal{N}_i|$ for all $i \in \mathcal{V}$.*

24 **Proof.** To show the sufficiency, assume that $\mathcal{G}$ is connected and that $F(x)$
 25 and $G(x)$ are of the forms (21) and (22). Then, (i) is satisfied from Lemma 1.

1 As for (ii), (18) is achieved because (27) is a consensus controller. Moreover,
 2 $\hat{\lambda}_i[l]$ converges to the average of the initial states $\hat{\lambda}_j[0]$ for all agents $j \in \mathcal{V}$
 3 under the assumption that $\eta < 1/|\mathcal{N}_i|$ [27]. Thus, from (26) and $\gamma^{\mathcal{I}} = 1/|\mathcal{I}|$,
 4 the following is derived.

$$\begin{aligned}
 \lim_{l \rightarrow \infty} \hat{\lambda}_i[l] &= \frac{1}{n} \sum_{j=1}^n \hat{\lambda}_j[0] \\
 &= \frac{1}{n} \sum_{j=1}^n \left(\hat{\lambda}_j^{[s]} + \kappa \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq},j}(\mathcal{G})} \frac{1}{|\mathcal{I}|} \bar{G}^{\mathcal{I}}(x_{\mathcal{I}}^{[s+1]}) \right) \\
 &= \frac{1}{n} \sum_{j=1}^n \hat{\lambda}_j^{[s]} + \frac{\kappa}{n} \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})} \sum_{j \in \mathcal{I}} \frac{1}{|\mathcal{I}|} \bar{G}^{\mathcal{I}}(x_{\mathcal{I}}^{[s+1]}) \\
 &= \bar{\lambda}^{[s]} + \beta \sum_{\mathcal{I} \in \mathcal{S}_{\text{clq}}(\mathcal{G})} \bar{G}^{\mathcal{I}}(x_{\mathcal{I}}^{[s+1]}), \tag{28}
 \end{aligned}$$

5 where

$$\bar{\lambda}^{[s]} = \frac{1}{n} \sum_{j=1}^n \hat{\lambda}_j^{[s]}, \quad \beta = \frac{\kappa}{n}.$$

6 From the update of the previous step, $\hat{\lambda}_i^{[s]} = \hat{\lambda}_j^{[s]}$ hold for all $j \in \mathcal{V}$, hence
 7 $\bar{\lambda}^{[s]} = \hat{\lambda}_i^{[s]}$ holds. Therefore, from (22) and (28), we obtain (17) for $\hat{\lambda}_i^{[s+1]} =$
 8 $[\lim_{l \rightarrow \infty} \hat{\lambda}_i[l]]^+$.

9 The necessity is shown. If $\mathcal{G}$ is not connected, consensus is not achieved
 10 [27]. Thus, (18) is not realized. If either $F(x)$ or $G(x)$ is not of the form
 11 (21) or (22), then (16) is not of the form (13) from Lemma 1. The proof is
 12 completed. $\square$

13 **Example 1.** Consider the functions

$$\begin{aligned}
 F(x) &= (x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_3 + x_4 + x_5)^4 \\
 G(x) &= \begin{bmatrix} x_1 + x_3 \\ x_3 + x_5 \end{bmatrix}.
 \end{aligned}$$

14 Then, $F(x)$ is of the form (21) if and only if the $\mathcal{G}$ contains the cliques
 15 $\{1, 2\}, \{2, 3\}, \{3, 4, 5\}$. $G(x)$ is of the form (22) if and only if the $\mathcal{G}$ contains
 16 the cliques $\{1, 3\}$ and $\{3, 5\}$. Such a graph is depicted in Fig. 1. Therefore,

1 in order to update $x_i^{[s]}$ in a distributed manner, the graph $\mathcal{G}$ has to contain at
2 least the edges in Fig. 1. Moreover, such $\mathcal{G}$ is connected. Then, Theorem 1
3 guarantees that the optimization problem is solvable in the distributed way.
4 $\square$

5 Theorem 1 gives the strict class of graphs $\mathcal{G}$ with which the optimization
6 problem (1) is solvable in the distributed way. Thus, if $\mathcal{G}$ is not in this class,
7 we can say that $\mathcal{G}$ has to be reconstructed. Conversely, if a connected graph
8 $\mathcal{G}$ is given in advance, Theorem 1 gives the strict classes of functions $F(x)$
9 and $G(x)$ solvable in the distributed way. These classes are given by (21) and
10 (22), which indicate the limit of the graph $\mathcal{G}$'s ability to solve the optimization
11 problem. **Note that the problem (1) with (21) and (22) contains a**
12 **wider class of optimization problems than the conventional ones as**
13 **(3) and (5). Actually, Example 1 is not of the conventional forms.**

14 4. Application to control of power system

15 Consider the power grid model depicted in Fig. 2. The communication
16 network between the agents is given by Fig. 3. There are three types of agents
17 in this grid: consumers, suppliers and distributors. Agents $i = 1, 2, \dots, 17$
18 have the variables $x_i \in \mathbb{R}$ representing the electric consumption, generation
19 and transmission values. For example, agents 1 and 2 are the supplier and
20 consumer in Region 1, and their generation and consumption values are given
21 by x_1 and x_2 , respectively. Agent 3 is the distributor which transmits the
22 electricity from Region 1 to 2, and the transmission value is given by x_3 . The
23 behavior of each agent is described by the utility function $k_i(x_i - x_i^*)^2$ with
24 the desired value x_i^* and price sensitivity $k_i > 0$. Then, the objective function
25 of the whole system is given by the sum of the agents' utility functions as

$$F(x) = \sum_{i=1}^{17} k_i(x_i - x_i^*)^2. \quad (29)$$

26 This grid is divided into some regions, each of which is managed to main-
27 tain the supply-demand balance by pricing. Then, the constraint function
28 representing the supply-demand balance is given by

$$G(x) = [\hat{G}(x)^\top - \hat{G}(x)^\top]^\top \quad (30)$$

$$\hat{G}(x) = \begin{bmatrix} x_1 - x_2 - x_3 + x_4 \\ x_3 + x_5 - x_6 + x_7 + x_8 + x_9 \\ -x_7 + x_{10} - x_{11} + x_{12} \\ -x_8 - x_{12} + x_{13} \\ -x_4 - x_9 + x_{14} - x_{15} \\ x_{15} + x_{16} - x_{17} \end{bmatrix},$$

where each term in $\hat{G}(x)$ represents the surplus or shortage electricity in each region. To keep the supply-demand balance in all the regions, $\hat{G}(x) = 0$, namely $G(x) \geq 0$, should be satisfied. **The functions $F(x)$ and $G(x)$ in (29) and (30) are obviously convex. Moreover, because $G(x)$ is linear and the constraint condition $G(x) \leq 0$ is satisfied for $x = 0$, the strong duality holds [28].**

Now, we solve the optimization problem (1) with (29) and (30). Then, $F(x)$ and $G(x)$ are of the forms (21) and (22) with

$$\begin{aligned} \bar{F}^{\{i\}}(x_i) &= k_i(x_i - x_i^*)^2 \\ \bar{G}^{\{1\}}(x_1) &= [x_1 \ 0 \ 0 \ 0 \ 0 \ 0 \ -x_1 \ 0 \ 0 \ 0 \ 0 \ 0]^\top \\ \bar{G}^{\{2\}}(x_2) &= [-x_2 \ 0 \ 0 \ 0 \ 0 \ 0 \ x_2 \ 0 \ 0 \ 0 \ 0 \ 0]^\top \\ \bar{G}^{\{3\}}(x_3) &= [-x_3 \ x_3 \ 0 \ 0 \ 0 \ 0 \ x_3 \ -x_3 \ 0 \ 0 \ 0 \ 0]^\top \\ &\vdots \\ \bar{F}^{\mathcal{I}}(x_{\mathcal{I}}) &= 0, \ \bar{G}^{\mathcal{I}}(x_{\mathcal{I}}) = 0 \ \text{if } |\mathcal{I}| \geq 2. \end{aligned}$$

Moreover, the graph in Fig. 3 is connected. Thus, the conditions in Theorem 1 are fulfilled, and the distributed constraint optimization problem is solvable by using the proposed method. Note that (23) is corresponding to the agent behavior model, and that (26) and (27) present the pricing law of the electricity because the Lagrangian multiplier represents the price. Then, this update law gives a rule to determine an appropriate price via negotiations between agents. The negotiations are carried out through the information exchanged over the graph given by Fig. 3. **This pricing system cannot be established by the conventional distributed optimization methods, e. g. [13, 14], which require supervisors to determine the Lagrangian multiplier.**

The simulation results are depicted in Figs. 4, 5 and 6, which show the values of $x_i^{[s]}$, $F(x^{[s]})$ and $G(x^{[s]})$ for step s . It is observed that $F(x^{[s]})$ is monotonically decreasing while $G(x^{[s]})$ converges to 0. Fig. 7 shows the Lagrangian multiplier $\lambda^{[s]}$, which is the result of the individual computation of

1 each agent according to (26) and (27). Fig. 8 shows the transition of the
2 computation, where the first components of $\hat{\lambda}_i[k]$ for $l \in \mathbb{Z}_+$ at a certain
3 step s are depicted. The dotted line represents the appropriate value of $\lambda^{[s+1]}$
4 given by (11). This figure shows that the agents successfully estimate the La-
5 grangian multiplier in a distributed manner. As a result, the supply-demand
6 balance is maintained by the distributed pricing. This result illustrates the
7 effectiveness of the proposed method.

8 5. Conclusion

9 In this paper, we considered the distributed constraint optimization prob-
10 lem on networked multi-agent systems. We proposed a distributed method
11 to estimate the Lagrangian multiplier, which is executable on each agent
12 by using only local information over the network. Then, a necessary and
13 sufficient condition was derived for the distributed constraint optimization
14 problem to be solvable. Finally, we applied this method to power grid control
15 to maintain the supply-demand balance by pricing. The proposed distributed
16 update law of the Lagrangian multiplier gives a pricing rule for the agents to
17 determine an appropriate price by negotiations. **As for Remark 1, the**
18 **number of the iteration in the proposed algorithm should be deter-**
19 **mined according to the tolerance optimization error. To determine**
20 **an appropriate number is important future work.**

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Figure 1: Example of a graph

Figure 2: Power grid

Figure 3: Communication graph for pricing

Figure 4: Variable $x_i^{[s]}$ of each agent $i \in \mathcal{V}$

Figure 5: Objective function $F(x^{[s]})$

Figure 6: All components of constraint function $G(x^{[s]})$

Figure 7: All components of Lagrange multiplier $\lambda^{[s]}$

Figure 8: First component of $\hat{\lambda}_i[l]$ for each $i \in \mathcal{V}$