A Partitioned PSB Method for Partially Separable Unconstrained Optimization Problems[☆]

Huiping Cao^{a,*}, Lan Yao^b

^a*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*

^b*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*

Abstract

In this paper, we propose a partitioned PSB method for solving partially separable unconstrained optimization problems. By using a projection technique, we construct a sufficient descent direction. Under appropriate conditions, we show that the partitioned PSB method with projected direction is globally and superlinearly convergent for uniformly convex problems. In particular, the unit step length is accepted after finitely many iterations. Finally, some numerical results are presented, which show that the partitioned PSB is effective and competitive.

Keywords: partially separable optimization problems, partitioned PSB method, projected PSB method, global convergence, superlinear convergence

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1. Introduction

We consider the following unconstrained optimization problem

$$\min f(x), \quad x \in R^n, \quad (1.1)$$

where $f : R^n \rightarrow R$ is twice continuously differentiable. We use $g(x)$, $G(x)$ to denote the gradient of $f(x)$ and the Hessian of $f(x)$, respectively.

^{*}Corresponding author

Email address: huiping_cao@hnu.edu.cn (Huiping Cao)

Quasi-Newton methods are particularly welcome for solving small and medium scale problems due to their superlinear convergence and no necessity to calculate the Hessian of the objective function. A quasi-Newton method typically generates a sequence of iterates $\{x_k\}$ by

$$x_{k+1} = x_k - B_k^{-1}g(x_k)$$

with an approximate Hessian B_k . The approximation B_k usually satisfies the secant condition:

$$B_{k+1}s_k = y_k,$$

where $s_k = x_{k+1} - x_k$ and $y_k = g(x_{k+1}) - g(x_k)$. When B_k is positive definite, the quasi-Newton direction $d_k = -B_k^{-1}g(x_k)$ is a descent direction for f at x_k . In this case, it is possible to globalize the method by adopting some line search techniques (see e.g., [1, 2, 3, 4]). In this paper, we focus on the PSB method. If some trust region strategies are adopted, the PSB method has been proved to be globally and superlinearly convergent. But for the line search type PSB method, there is little work about the global convergence, which may result from the non positive definiteness of the approximate Hessian. Recently, An, Li and Xiao [5] introduced a projection technique and proposed a projected PSB method, which can generate a sufficient descent direction for the objective function. Under certain conditions, the method is globally and superlinearly convergent.

In this paper, we will study the application of the PSB method for solving large-scale problems. A large number of numerical experiments have shown that the approximation B_k generated by a quasi-Newton method is generally dense. So the direct use of the quasi-Newton methods is impossible due to the storage difficulty for the matrix B_k . In order to solve large scale problems, several methods have been proposed. The Limited memory BFGS (L-BFGS) method [6] is an efficient extension of BFGS method for large-scale problems. The method constructs a matrix H_k , an approximate Hessian of $\nabla^2 f(x_k)^{-1}$ by updating the basic matrix H_0 m times using the BFGS formula with previous m iterations. It has been shown that the L-BFGS method converges linearly.

Another important kind of sparse quasi-Newton method is that investigating the sparsity of Hessian $\nabla^2 f(x)$. Suppose that for all $x \in R^n$,

$$(\nabla^2 f(x))_{i,j} = 0, (i, j) \in F,$$

where F is the subset of $I \times I$ and $I = \{1, \dots, n\}$. Toint [7] and Fletcher [8] have studied the sparse quasi-Newton methods which can satisfies the secant equation and sparse condition simultaneously. However, as pointed out in [9], at each step of the method in [8] or [10], one must solve a convex program or
30 a linear system. In addition, the approximate Hessian may be ill-posed if some component of s_k is zero.

Based on the positive definite matrix completion in semidefinite programming, Yamashita [11] proposed a new sparse quasi-Newton update, called Matrix Completion Quasi-Newton (MCQN) update, for unconstrained optimization
35 problems with sparse Hessian structures. Dai and Yamashita [9] established the local and superlinear convergence for the MCQN update with Broyden's convex family under appropriate assumptions. Besides, the authors gave the related analysis for global convergence assuming that the objective function is uniformly convex and its dimension is only two.

For many large scale problems, the function f has the following partially separable form

$$\min f(x) = \sum_{i=1}^m f_i(x), \quad (1.2)$$

40 where each element function $f_i, i = 1, \dots, m$ depends only on a few variables. In this case, the partitioned quasi-Newton method [12, 13] performs well in practice. The partitioned BFGS method generates the approximate Hessian B_k^i for each function f_i by the BFGS update. Then quasi-Newton matrix B_k , as an approximation to the Hessian of $f(x)$ at x_k , is the summation of $B_k^{(i)}, i = 1, \dots, m$.
45 In 1986, Toint [14] established the global convergence of the partitioned BFGS method for solving the partially separable convex problems. Griewank [15] established a stronger global convergence of the partitioned BFGS method. Moreover, it does ensure superlinear convergence with the step sizes going to 1.

In this paper, we study the partitioned quasi-Newton method with PSB
50 update for solving the partially separable problem (1.2). Taking into account
that the approximate Hessian matrix generated by the PSB update may not
be positive definite, which will lead the quasi-Newton direction to be a non-
descent direction, we adopt a projection technique similar to that in [5] so that
the quasi-Newton direction is descent for f . Under appropriate conditions, we
55 will show that the method is globally and superlinearly convergent. Moreover,
the unit step length will be accepted after finitely many iterations. In another
words, the method locally reduces to the undamped PSB method.

The remainder of the paper is organized as follows. In Section 2, we propose
a partitioned PSB update and give its local convergence properties. In Section
60 3, we construct a sufficient descent for the objective function and propose a
projected partitioned PSB method for solving (1.2). Moreover, we establish its
global and superlinear convergence. In Section 4, we present some preliminary
numerical results. Finally, we give the remarks.

2. The Partitioned PSB Update and Its Properties

65 For the partially separable problem (1.2). we suppose that for each $i =$
 $1, 2, \dots, m$, function f_i is twice continuously differentiable. For each $i = 1, 2, \dots,$
 m , we let $g_i(x)$ and $G_i(x)$ be the gradient and Hessian of f_i at x . Similar to
[14], we suppose that for each $i = 1, \dots, m$, the null space of $G_i(x)$, denoted by
 $\mathcal{N}_i$, is independent of x . We let $P^{(i)} : R^n \rightarrow R^n$ be the orthogonal projection
70 onto $\mathcal{N}_i^\perp$. Then we can rewrite $f_i(x)$ as $f_i(x^{(i)})$, where $x^{(i)} = P^{(i)}x$, whose the
 j -the component can be specified by

$$x_j^{(i)} = \begin{cases} x_j, & \text{if } f_i(x) \text{ depends on } x_j; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that

$$g(x) = \sum_{i=1}^m g_i(x) = \sum_{i=1}^m g_i(x^{(i)}), G(x) = \sum_{i=1}^m G_i(x) = \sum_{i=1}^m G_i(x^{(i)}).$$

Consequently, we have for any $d \in R^n$,

$$G_i(x)d = G_i(x^{(i)})d^{(i)},$$

where $d^{(i)} = P^{(i)}d$ has the same form as $x^{(i)}$. In a partitioned quasi-Newton method, the approximate Hessian matrix B_k takes the form

$$B_k = \sum_{i=1}^m B_k^{(i)},$$

with each $B_k^{(i)}, i = 1, \dots, m$, an approximation to $G_i(x_k)$ that satisfies the secant equation

$$B_{k+1}^{(i)} s_k^{(i)} = y_k^{(i)},$$

where $s_k^{(i)} = x_{k+1}^{(i)} - x_k^{(i)}$ and $y_k^{(i)} = g_i(x_{k+1}^{(i)}) - g_i(x_k^{(i)})$. Matrices $B_k^{(i)}, i = 1, \dots, m$ can be updated by any existing update formula. In this paper, we focus on the PSB update. Specifically, for each $i = 1, \dots, m$, we update $B_k^{(i)}$ by

75 the following PSB update:

$$\begin{aligned} B_{k+1}^{(i)} = & B_k^{(i)} + \frac{(y_k^{(i)} - B_k^{(i)} s_k^{(i)}) s_k^{(i)T} + s_k^{(i)} (y_k^{(i)} - B_k^{(i)} s_k^{(i)})^T}{\|s_k^{(i)}\|^2} \\ & - \frac{(y_k^{(i)} - B_k^{(i)} s_k^{(i)})^T s_k^{(i)}}{\|s_k^{(i)}\|^2} \cdot \frac{s_k^{(i)} s_k^{(i)T}}{\|s_k^{(i)}\|^2}. \end{aligned}$$

If for some $i = 1, 2, \dots, m$, $s_k^{(i)} = 0$, we let $B_{k+1}^{(i)} = B_k^{(i)}$. These matrices $B_k^{(i)}, i = 1, \dots, m$, are then accumulated to generate the search direction d_k by solving the following system of linear equations

$$\left(\sum_{i=1}^m B_k^{(i)} \right) d + g(x_k) = 0. \quad (2.1)$$

Then x_{k+1} is determined by

$$x_{k+1} = x_k + \alpha_k d_k,$$

where α_k is a step length that is determined by some line search. When $\alpha_k \equiv 1$, we call the above procedure full step partitioned PSB method. The above partitioned PSB method is very similar to the partitioned BFGS method except for the update formula.

In what follows, we will give several lemmas, which show that the full step partitioned PSB method can retain some good properties of the ordinary PSB method. For simplicity, we denote

$$A_{k+1} = \sum_{i=1}^m A_{k+1}^{(i)}, \text{ with } A_{k+1}^{(i)} = \int_0^1 G_i(x_k^{(i)} + \tau s_k^{(i)}) d\tau, \quad i = 1, 2, \dots, m.$$

It is clear that

$$A_{k+1}^{(i)} s_k^{(i)} = y_k^{(i)}, i = 1, \dots, m.$$

For each i and any k , we let $\delta_k^{(i)} = 0$ if $s_k^{(i)} = 0$, and

$$\delta_k^{(i)} = \frac{\|y_k^{(i)} - B_k^{(i)} s_k^{(i)}\|}{\|s_k^{(i)}\|} = \frac{\|(A_{k+1}^{(i)} - B_k^{(i)}) s_k^{(i)}\|}{\|s_k^{(i)}\|}, \text{ if } s_k^{(i)} \neq 0.$$

We also let

$$\delta_k = \frac{\|y_k - B_k s_k\|}{\|s_k\|}.$$

80 It is obvious that

$$\delta_k \leq \sum_{i=1}^m \delta_k^{(i)}. \quad (2.2)$$

The following lemma is obvious.

Lemma 2.1. *Suppose that for each $i = 1, 2, \dots, m$, $G_i(x)$ is Lipschitz continuous with a common Lipschitz constant $L > 0$, i.e.,*

$$\|G_i(x) - G_i(y)\|_F \leq L\|x - y\|, \forall x, y, i = 1, 2, \dots, m,$$

where $\|\cdot\|_F$ denotes the Frobenius norm of matrices. Then we have

$$\|A_{k+1}^{(i)} - A_k^{(i)}\|_F \leq \frac{1}{2}L(\|s_k^{(i)}\| + \|s_{k-1}^{(i)}\|), \quad i = 1, \dots, m. \quad (2.3)$$

The following lemma can be proved according to Lemma 3.2 in [5].

Lemma 2.2. *Suppose that for each $i = 1, \dots, m$, $G_i(x)$ is Lipschitz continuous with a common Lipschitz constant $L > 0$. Let the sequence $\{x_k\}$ be generated by the full step partitioned PSB method. If*

$$\sum_{k=0}^{\infty} \|s_k\|^2 < \infty,$$

then one has

$$\lim_{t \rightarrow \infty} \frac{1}{t+1} \sum_{k=0}^t \sum_{i=1}^m \delta_k^{(i)^2} = 0. \quad (2.4)$$

In particular, there is an infinite index set K independent of i such that the subsequence $\{\delta_k^{(i)}\}_K$ converges to zero for any $i = 1, \dots, m$.

If s_k satisfies the condition

$$\sum_{k=0}^{\infty} \|s_k\| < \infty,$$

then one has

$$\sum_{k=0}^{\infty} \sum_{i=1}^m \delta_k^{(i)^2} < \infty. \quad (2.5)$$

85 In particular, for any $i = 1, \dots, m$, the whole sequence $\{\delta_k^{(i)}\}$ converges to zero.

Lemma 2.3. Suppose that for each $i = 1, \dots, m$, $G_i(x)$ is Lipschitz continuous with a common Lipschitz constant $L > 0$ and $\sum_{k=0}^{\infty} \|s_k\|^2 < \infty$. Let the index set K be specified in Lemma 2.2. If there is an accumulation point x^* of $\{x_k\}_K$ at which $G(x^*)$ is nonsingular, then there is a constant $M_1 > 0$ such that the following inequality holds for all $k \in K$ sufficiently large

$$\|d_k\| \leq M_1 \|g(x_k)\|. \quad (2.6)$$

Proof. Without loss of generality, we suppose $\{x_k\}_K \rightarrow x^*$. Since $\sum_{k=0}^{\infty} \|s_k\|^2 < \infty$, it is clear that when $k \in K$ is sufficiently large, A_{k+1} is nonsingular. Moreover, there is a constant $M_2 > 0$ such that the inequality $\|A_{k+1}^{-1}\| \leq M_2$ holds for all $k \in K$ sufficiently large. It then follows from Lemma 2.2 and (2.2) that
 90 $\{\delta_k\}_K \rightarrow 0$. Therefore, the inequality $\delta_k \leq \frac{1}{2M_2}$ is satisfied for all $k \in K$ sufficiently large. Consequently, we get that for all $k \in K$ sufficiently large,

$$\begin{aligned} \|d_k\| &\leq \|A_{k+1}^{-1}\| (\|A_{k+1}d_k + g(x_k)\| + \|g(x_k)\|) \\ &\leq M_2 (\|g(x_k)\| + \delta_k \|d_k\|) \\ &\leq M_2 (\|g(x_k)\| + \frac{1}{2M_2} \|d_k\|) \\ &\leq M_2 \|g(x_k)\| + \frac{1}{2} \|d_k\|. \end{aligned} \quad (2.7)$$

The last inequality implies (2.6) with $M_1 = 2M_2$. $\square$

The following theorem gives the local and superlinear convergence of the full step partitioned PSB method.

95 **Theorem 2.1.** *Suppose that for each $i = 1, \dots, m$, $G_i(x)$ is Lipschitz continuous with a common Lipschitz constant $L > 0$. Let x^* be a solution of the unconstrained optimization problem (1.2) at which $G(x^*)$ is nonsingular. For any $r \in (0, 1)$, there exist constants $\epsilon > 0$ and $\delta > 0$ such that if $\|x_0 - x^*\| \leq \epsilon$ and $\|B_0^{(i)} - G_i(x^*)\| \leq \delta$ for all $i, i = 1, 2, \dots, m$, then the sequence generated*
100 *by the full step partitioned PSB method remains in $U_\epsilon(x^*) \triangleq \{x \mid \|x - x^*\| \leq \epsilon\}$. Moreover, the convergence rate of $\{x_k\}$ is superlinear.*

3. A Projected Partitioned PSB Method

In this section, we will globalize the partitioned PSB method. Like the ordinary PSB method, the matrix B_k generated by the partitioned PSB method may not be positive definite. As a result, the generated direction d_k may not be a descent direction for f at x_k . In order to get a descent direction, we adopt a projection technique similar to that in [5]. Suppose d_k^{PSB} is the direction generated by the partitioned PSB method. We let d_k be defined by

$$d_k = -g(x_k) + \lambda_k \left(I - \frac{g(x_k)g(x_k)^T}{\|g(x_k)\|^2} \right) d_k^{PSB}. \quad (3.1)$$

It is a linear combination of the steepest descent direction $-g(x_k)$ and the projection of d_k^{PSB} onto the subspace orthogonal to $-g(x_k)$. It follows from (3.1) that

$$g(x_k)^T d_k = -\|g(x_k)\|^2.$$

Consequently, d_k is a sufficiently descent direction for function f at x_k . In the case where $g(x_k)^T d_k^{PSB} < 0$, we can take $\lambda_k = -\frac{\|g(x_k)\|^2}{g(x_k)^T d_k^{PSB}}$ so that we get a descent direction parallel to d_k^{PSB} , i.e., it holds that

$$d_k = \lambda_k d_k^{PSB}.$$

As we have had a descent direction, we can determine the step length α_k by letting it satisfy the Armijo condition or the Wolfe conditions. The Armijo

line search is find a step length $\alpha_k = \max\{\beta_k \rho^i, i = 0, 1, \dots\}$ such the following inequality

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \sigma_1 \alpha_k g(x_k)^T d_k, k = 1, 2, \dots \quad (3.2)$$

holds, where $\rho \in (0, 1), \beta_k \geq \beta^*, \beta^* > 0$ and $\sigma_1 \in (0, 1)$. The Wolfe conditions is find a step length α_k satisfying

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \sigma_1 \alpha_k g(x_k)^T d_k, \\ g(x_k + \alpha_k d_k)^T d_k \geq \sigma_2 g(x_k)^T d_k, \end{cases} \quad (3.3)$$

where $0 < \sigma_1 < \sigma_2 < 1, \sigma_1 < 1/2$.

Based on the above arguments, we propose a damped version of the partitioned PSB method as follows.

Algorithm 3.1. Projected Partitioned PSB Method (P-PSB)

Step 0. Given constants $\lambda > 0$ and $\varepsilon > 0$. Chosen initial point $x_0 \in R^n$ and initial symmetric matrices $B_0^{(i)}, i = 1, 2, \dots, m$. Set

$$B_0 = \sum_{i=1}^m B_0^{(i)}.$$

Let $k := 0$.

Step 1. Stop, if $\|g(x_k)\| \leq \varepsilon$. Otherwise, solve the system of linear equations (2.1) to get d_k^{PSB} .

Step 2. Compute d_k by (3.1), where $\lambda_k \in R^n$ is determined by

$$\lambda_k = \begin{cases} 0, & \text{if } g(x_k)^T d_k^{PSB} = 0, \\ \min\{\lambda k^{1/4}, \frac{\|g(x_k)\|^2}{|g(x_k)^T d_k^{PSB}|}\}, & \text{otherwise.} \end{cases}$$

Step 3. Let α_k be determined by the Armijo line search (3.2) or Wolfe line search (3.3). Let $x_{k+1} := x_k + \alpha_k d_k$.

Step 4. If for some $i = 1, 2, \dots, m, \bar{s}_k^{(i)} = 0$, we let $B_{k+1}^{(i)} = B_k^{(i)}$. Other-

wise, update $B_k^{(i)}$ by the PSB update formula

$$B_{k+1}^{(i)} = B_k^{(i)} + \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)}) \bar{s}_k^{(i)T} + \bar{s}_k^{(i)} (\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} - \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T \bar{s}_k^{(i)}}{\|\bar{s}_k^{(i)}\|^2} \cdot \frac{\bar{s}_k^{(i)} \bar{s}_k^{(i)T}}{\|\bar{s}_k^{(i)}\|^2}, \quad (3.4)$$

where $\bar{s}_k^{(i)} = \bar{\alpha}_k d_k^{PSB(i)}$, $\bar{\alpha}_k = \lambda_k \alpha_k$ and $\bar{y}_k^{(i)} = g_i(x_k^{(i)} + \bar{s}_k^{(i)}) - g_i(x_k^{(i)})$. Let B_{k+1} be determined by

$$B_{k+1} = \sum_{i=0}^m B_{k+1}^{(i)}. \quad (3.5)$$

115 **Step 5.** Let $k := k + 1$. Go to step 1.

In the Armijo line search, β_k is determined by

$$\beta_k = \begin{cases} 1, & \text{if } g(x_k)^T d_k^{PSB} \geq 0, \\ \max\{\beta^* \|g(x_k)\|^\tau, \frac{\|g(x_k)\|^2}{|g(x_k)^T d_k^{PSB}|}\}, & \text{otherwise,} \end{cases}$$

where $\beta^* > 0$ and $\tau > 0$ are constants.

Remark. Matrix B_{k+1} determined by (3.4) and (3.5) may be singular even if B_k is nonsingular. To overcome this drawback, we can adopt the same ideas
120 in [5, 16] to give some analysis of nonsingular partitioned PSB update.

In what follows, for tridiagonal partially separable function, i.e., the Hessian of function f is tridiagonal matrix, one has $m = n - 1$. In this case, we modify the partitioned PSB update as follows. We rewrite B_{k+1} as

$$\begin{aligned} B_{k+1} &= \sum_{i=1}^{n-1} B_{k+1}^{(i)} = \sum_{i=1}^{n-1} \left(B_k^{(i)} + \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)}) \bar{s}_k^{(i)T} + \bar{s}_k^{(i)} (\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \right. \\ &\quad \left. - \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T \bar{s}_k^{(i)}}{\|\bar{s}_k^{(i)}\|^2} \cdot \frac{\bar{s}_k^{(i)} (\bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \right) \\ &= B_k + \sum_{i=1}^{n-1} \left(\frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)}) \bar{s}_k^{(i)T} + \bar{s}_k^{(i)} (\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \right. \\ &\quad \left. - \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T \bar{s}_k^{(i)}}{\|\bar{s}_k^{(i)}\|^2} \cdot \frac{\bar{s}_k^{(i)} (\bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \right), \end{aligned} \quad (3.6)$$

where for $i = 1$, we let

$$B_k^{(1)} = \begin{pmatrix} B_k(1,1) & B_k(1,2) & 0 & \cdots & 0 \\ B_k(2,1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and for $i = 2, \dots, n-1$, we let

$$B_k^{(i)} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & B_k(i,i) & B_k(i,i+1) & \cdots & 0 \\ 0 & \cdots & B_k(i+1,i) & B_k(i+1,i+1) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Set $C_0 = B_k$ and define for $i = 1, 2, \dots, n-1$

$$\begin{aligned} C_i &= C_0 + \sum_{j=1}^i \theta_j \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)}) \bar{s}_k^{(i)T} + \bar{s}_k^{(i)} (\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \\ &\quad - \theta_j^2 \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T \bar{s}_k^{(i)}}{\|\bar{s}_k^{(i)}\|^2} \cdot \frac{\bar{s}_k^{(i)} (\bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \\ &= C_{i-1} + \theta_i \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)}) \bar{s}_k^{(i)T} + \bar{s}_k^{(i)} (\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \\ &\quad - \theta_i^2 \frac{(\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)})^T \bar{s}_k^{(i)}}{\|\bar{s}_k^{(i)}\|^2} \cdot \frac{\bar{s}_k^{(i)} (\bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \end{aligned}$$

125 Note that $C_0^{(i)} = C_1^{(i)} = \dots = C_{i-1}^{(i)}$, then

$$\begin{aligned} C_i &= C_{i-1} + \theta_i \frac{(\bar{y}_k^{(i)} - C_{i-1}^{(i)} \bar{s}_k^{(i)}) \bar{s}_k^{(i)T} + \bar{s}_k^{(i)} (\bar{y}_k^{(i)} - C_{i-1}^{(i)} \bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \\ &\quad - \theta_i^2 \frac{(\bar{y}_k^{(i)} - C_{i-1}^{(i)} \bar{s}_k^{(i)})^T \bar{s}_k^{(i)}}{\|\bar{s}_k^{(i)}\|^2} \cdot \frac{\bar{s}_k^{(i)} (\bar{s}_k^{(i)})^T}{\|\bar{s}_k^{(i)}\|^2} \end{aligned}$$

For a scalar $\alpha \in (0, 1)$, θ_i can be chosen such that

$$|\det C_i| \geq |\sqrt[n]{\alpha}| \det C_{i-1}|, \theta_i \in \left[\frac{1 - \sqrt[n]{\alpha}}{1 + \sqrt[n]{\alpha}}, 1 \right]. \quad (3.7)$$

therefore we can get $|\det(B_{k+1})| \geq \alpha |\det B_k|$, and θ_i can be chosen so that

$$B_{k+1} \text{ is nonsingular, and } |\theta_i - 1| \leq \hat{\theta} < 1. \quad (3.8)$$

The dependence of θ_i on the iteration k is suppressed, but note that $\hat{\theta}$ is independent of k [17].

To prove the global convergence of Algorithm 3.1, we need the following assumptions.

Assumption 3.1. (a) Function $f(x)$ is twice continuously differentiable in the level set $\Omega = \{x | f(x) \leq f(x_0)\}$, and for each $i = 1, 2, \dots, m$, $G_i(x)$ is Lipschitz continuous with a common constant $L > 0$ such that

$$\|G_i(x) - G_i(y)\|_F \leq L\|x - y\|, \quad \forall x, y \in \Omega.$$

(b) Function $f(x)$ is uniformly convex, that is, there exist constants $M \geq m > 0$ such that

$$m\|p\|^2 \leq p^T G(x)p \leq M\|p\|^2, \quad \forall p \in R^n, \forall x \in \Omega.$$

Let x^* be the unique solution of problem (1.2). It can be obtained according to the conditions of Assumption 3.1 that there are constants $M \geq m > 0$ such that [2]

$$\frac{1}{2}m\|x - x^*\|^2 \leq f(x) - f(x^*) \leq \frac{1}{2}M\|x - x^*\|^2, \quad \forall x \in \Omega,$$

and

$$\|g(x)\| \geq m\|x - x^*\|, \quad \forall x \in \Omega.$$

130 Firstly, similar to Lemma 3.1 in [5], we have the following lemma.

Lemma 3.1. Let the conditions in Assumption 3.1 hold and the sequence $\{x_k\}$ be generated by Algorithm 3.1. Then one has

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\|^2 = \sum_{k=0}^{\infty} \|s_k\|^2 < \infty, \quad (3.9)$$

which implies

$$\sum_{k=0}^{\infty} \|\bar{s}_k\|^2 < \infty. \quad (3.10)$$

For the sake of convenience, for each i and any k , we let $\xi_k^{(i)} = 0$ if $\bar{s}_k^{(i)} = 0$,

$$\xi_k^{(i)} = \frac{\|\bar{y}_k^{(i)} - B_k^{(i)} \bar{s}_k^{(i)}\|}{\|\bar{s}_k^{(i)}\|} = \frac{\|(\bar{A}_{k+1}^{(i)} - B_k^{(i)}) d_k^{PSB(i)}\|}{\|d_k^{PSB(i)}\|}, \text{ if } \bar{s}_k^{(i)} \neq 0. \quad (3.11)$$

and

$$\xi_k = \frac{\|\bar{y}_k - B_k \bar{s}_k\|}{\|\bar{s}_k\|} = \frac{\|(\bar{A}_{k+1} - B_k) \bar{s}_k\|}{\|\bar{s}_k\|} = \frac{\|(\bar{A}_{k+1} - B_k) d_k^{PSB}\|}{\|d_k^{PSB}\|},$$

where $\bar{A}_{k+1} = \int_0^1 G(x_k + t\bar{s}_k)$. By the definitions of ξ_k and $\xi_k^{(i)}$, $i = 1, \dots, m$, we can derive

$$\xi_k \leq \sum_{i=1}^m \xi_k^{(i)}. \quad (3.12)$$

The following result can be proved in a way similar to Lemma 3.3 in [5].

Lemma 3.2. *Let the conditions in Assumption 3.1 hold and the sequence $\{x_k\}$ be generated by Algorithm 3.1. Then one has*

$$\lim_{t \rightarrow \infty} \frac{1}{t+1} \sum_{k=0}^t \sum_{i=1}^m \xi_k^{(i)^2} = 0. \quad (3.13)$$

In particular, there is an infinite index set K independent of i such that the
135 *subsequence of $\{\xi_k^{(i)}\}_K$ converges to zero for any $i = 1, \dots, m$.*

If $\{s_k\}$ satisfies the condition

$$\sum_{k=0}^{\infty} \|\bar{s}_k\| < \infty,$$

then one has

$$\sum_{k=0}^{\infty} \sum_{i=1}^m \xi_k^{(i)^2} < \infty. \quad (3.14)$$

In particular, for any $i = 1, \dots, m$, the whole sequence $\{\xi_k^{(i)}\}$ converges to zero.

The following theorem establishes the global convergence of Algorithm 3.1.

Theorem 3.1. *Let the conditions in Assumption 3.1 hold and the sequence $\{x_k\}$ be generated by Algorithm 3.1. Then $\{x_k\}$ converges to the unique solution of*
140 *(1.2) in Ω .*

Proof. According to Lemma 3.2, there is a subsequence $\{\xi_k\}_{k \in K}$ of $\{\xi_k\}$ defined by (3.11) converges to zero. Since $\{x_k\}_{k \in K} \subseteq \Omega$ is bounded, without loss of generality, we assume that $\{x_k\}_{k \in K}$ itself converges to some point $\bar{x}$. Because $\|\bar{s}_k\| \rightarrow 0$ as $k \rightarrow \infty$, so $\{\bar{A}_{k+1}\}_{k \in K} \rightarrow G(\bar{x})$. In particular, there exists a constant $\bar{M}_2 > 0$ such that $\|\bar{A}_{k+1}^{-1}\| \leq \bar{M}_2$ for all $k \in K$ sufficiently large. Thus we have

$$\begin{aligned} \|d_k^{PSB}\| &\leq \|\bar{A}_{k+1}^{-1} ((\bar{A}_{k+1} - B_k)d_k^{PSB} - g(x_k))\| \\ &\leq \|\bar{A}_{k+1}^{-1}\| (\|(\bar{A}_{k+1} - B_k)d_k^{PSB}\| + \|g(x_k)\|) \\ &\leq \bar{M}_2(\xi_k \|d_k^{PSB}\| + \|g(x_k)\|). \end{aligned}$$

Since $\{\xi_k\}_{k \in K} \rightarrow 0$ as $k \rightarrow \infty$ with $k \in K$, the last inequality implies that there is a constant $\bar{M}_1 > 0$ such that the following inequality

$$\|d_k^{PSB}\| \leq \bar{M}_1 \|g(x_k)\|. \quad (3.15)$$

is satisfied for all $k \in K$ sufficiently large.

By the definition of d_k , we have

$$\begin{aligned} \|d_k + g(x_k)\|^2 &= \lambda_k^2 \left\| \left(I - \frac{g(x_k)g(x_k)^T}{\|g(x_k)\|^2} \right) d_k^{PSB} \right\|^2 \\ &\leq \lambda_k^2 \|d_k^{PSB}\|^2 \\ &\leq \lambda^2 \bar{M}_1^2 k^{1/2} \|g(x_k)\|^2, \end{aligned} \quad (3.16)$$

where the last inequality follows from the definition of λ_k and (3.15).

If $\liminf_{k \rightarrow \infty} \|g(x_k)\| \neq 0$, then there is a constant $\eta > 0$ such that $\|g(x_k)\| \geq \eta$ for all k . Consequently, the inequality (3.16) yields

$$\|d_k + g(x_k)\|^{-2} \geq \lambda^{-2} \bar{M}_1^{-2} \eta^{-2} k^{-1/2},$$

which implies

$$\sum_{k=1}^{\infty} \|d_k + g(x_k)\|^{-2} = +\infty.$$

This yields a contradiction with Theorem 2.2 in [5]. Consequently, it must hold that

$$\liminf \|g(x_k)\| = 0.$$

150 This means, there is a limit point $\bar{x}$ of $\{x_k\}$ that solves the problem (1.2). By the uniqueness of the solution of (1.2), we have $\bar{x} = x^*$. Since the sequence of the function values $\{f(x_k)\}$ is decreasing and f is uniformly convex, the whole sequence $\{x_k\}$ converges to x^* . $\square$

The reminder part of the section is devoted to the superlinear convergence
155 of Algorithm 3.1. First, in a way similar to the proofs of Lemmas 3.5-3.6 in [5], we can prove the following results. We omit the proof.

Lemma 3.3. *Let the conditions in Assumption 3.1 hold and the sequence $\{x_k\}$ be generated by Algorithm 3.1 with α_k determined by Armijo line search (3.2) with $0 < \sigma_1 < 1/2$. Then there is a constant $\epsilon > 0$ such that if $\xi_k \leq \epsilon$, then
160 we have $g(x_k)^T d_k^{PSB} < 0$, $\lambda_k = -\frac{\|g(x_k)\|^2}{g(x_k)^T d_k^{PSB}}$ and $d_k = -\frac{\|g(x_k)\|^2}{g(x_k)^T d_k^{PSB}} d_k^{PSB}$. Moreover, the step length $\alpha_k = \beta_k$ is accepted.*

Lemma 3.4. *Let the conditions in Assumption 3.1 hold and the sequence $\{x_k\}$ be generated by Algorithm 3.1 with α_k determined by Armijo line search (3.2) with $0 < \sigma_1 < 1/2$. Then we have*

$$\sum_{k=0}^{\infty} \|x_k - x^*\| < \infty. \quad (3.17)$$

As a direct result of (3.17), we have

$$\sum_{k=0}^{\infty} \|\bar{s}_k\| \leq \sum_{k=0}^{\infty} \|s_k\| = \sum_{k=0}^{\infty} \|x_{k+1} - x_k\| \leq \sum_{k=0}^{\infty} (\|x_{k+1} - x^*\| + \|x_k - x^*\|) < \infty.$$

Therefore, we get from Lemma 3.2 that

$$\sum_{k=0}^{\infty} \sum_{i=1}^m \xi_k^{(i)2} < \infty.$$

In particular, for any $i = 1, 2, \dots$, the whole sequence $\{\xi_k^{(i)}\}$ converges to zero.

In what follows, we give the superlinear convergence of Algorithm 3.1.

Theorem 3.2. *Let the conditions in Assumption 3.1 hold and the sequence
165 $\{x_k\}$ be generated by Algorithm 3.1 with α_k determined by Armijo line search (3.2) with $0 < \sigma_1 < 1/2$. Then $\{x_k\}$ converges to x^* superlinearly.*

4. Numerical Experiments

In this section, we test the performance of the proposed partitioned PSB (P-PSB) method on 30 problems [18, 19, 20], where the Hessians of the tested problems have various sparse structures. We also compare the P-PSB method with the projected PSB (PSB) method [5], the partitioned BFGS (P-BFGS) method and limited memory BFGS (L-BFGS) method. The numerical experiments are done by using MATLAB v7.10 on a Core (TM) 2 PC. The parameters in Algorithm 3.1 are specified as follows:

$$\lambda = 10^3, \beta^* = 10^{-7}, \tau = \frac{1}{8}, \sigma_1 = 10^{-4}, \sigma_2 = 0.9.$$

For the L-BFGS method, $m = 12$, which is the number of stored vectors of L-BFGS, is set, and the scaling factor $s_{k-1}^T y_{k-1} / \|y_{k-1}\|^2$ is employed.

In the experiments, we employ the following termination criterion:

$$\|\nabla f(x_k)\|_2 < 10^{-5} \cdot \max(1, \|x_k\|_2) \text{ or } k \geq 5000,$$

170 where the second criterion implies that the method fails to obtain a solution.

To show the advantage of the P-PSB method in computation, we test the P-PSB method, the P-BFGS method and the L-BFGS method on 30 partially separable problems with different dimensions, where the size varies from 10 to 2×10^6 . Noticing that the number of iterations is virtually independent of the
175 problem size, so we just list a small set of sizes to illustrate the behaviour of the three methods. We give the numerical results in Tables 1, 2 and 3, where the meaning of each column is listed below.

Pro: the number of the test problem;

n: the dimension of the problem;

180 Ite: the total number of iterations;

fcnt: the total number of function evaluations;

grad: the total number of gradient evaluations;

time: the total CPU time (sec).

Results presented in Tables 1, 2 and 3 advocate the proposed P-PSB method.

185 Take account of the total number of iterations, P-PSB is the most efficient on 14 problems (1, 4, 7, 9, 10, 12, 16, 17, 18, 19, 23, 25, 26, 28), while equivalent to L-BFGS on 5 problems (5, 11, 13, 14, 24) and equivalent to P-BFGS on 2 problems (24, 27). It's obviously that the P-PSB method outperforms P-BFGS and L-BFGS in terms of iterations count. For the total number of function

190 evaluations, P-PSB has the best performance on 15 problems (1, 4, 5, 9, 10, 12, 13, 14, 16, 18, 22, 23, 25, 26, 28), while P-BFGS has a satisfying performance on 9 problems (2, 3, 6, 7, 8, 11, 17, 19, 30), and both methods are equivalent on 2 problems (24, 27). It suggests that the P-PSB method is competitive with the P-BFGS method. As for the total number of gradient evaluations, P-PSB works

195 well on 15 problems (1, 4, 5, 7, 9, 10, 12, 16, 17, 18, 19, 22, 23, 25, 28), L-BFGS performs well on 10 problems (3, 6, 8, 11, 13, 14, 15, 20, 21, 29), and the both methods are equivalent on 2 problems (24, 26). Compared with the L-BFGS the proposed P-PSB method used less gradient evaluations. At last, we pay our attention to the CPU time. P-PSB uses the least time on 16 problems (1, 4, 5,

200 6, 10, 12, 13, 14, 16, 18, 22, 25, 26, 27, 28, 30), and L-BFGS uses the least time on 7 problems (8, 11, 15, 19, 20, 21, 29). It seems that the advantage on CPU time for the proposed method is not obvious.

In order to analyse the efficiency of proposed method more precisely, we adopt a notion of performance profile [21], which is a distribution function for a performance metric to evaluate and compare the performance of the set of solvers S on a test set P . Suppose that there exist N_s solvers and N_p problems, for each problem p and solver s , Dolan and Moré [21] defined $t_{p,s}$ the number of iterations (the number of function evaluations or others) required to solve problem p by solver s . Requiring a baseline for comparisons, they compared the performance on problem p by solver s with the best performance by any solver on this problem; i.e., using the performance ratio

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s} : s \in S\}}.$$

Suppose that a parameter $r_M \geq r_{p,s}$ for all p, s is chosen, and $r_{p,s} = r_M$ if and only if solver s does not solve problem p . In order to obtain an overall assessment of the performance of the solver, they defined

$$\rho_s(t) = \frac{1}{N_p} \text{size}\{p \in P : r_{p,s} \leq t\},$$

thus $\rho_s : R \rightarrow [0, 1]$ was the probability for solver $s \in S$ that a performance ratio $r_{p,s}$ was within a factor $t \in R$ of the best possible ratio.

205 In Figure 1, the performance of the two methods: P-PSB and P-BFGS, relative to the number of iteration, the number of function evaluations, the number of gradient evaluations and CPU time, are evaluated using the performance profiles, respectively. Figure 1 indicates that the P-PSB method is better than P-BFGS on all aspects of performance.

210 Then, we compared the proposed P-PSB method with L-BFGS method based on the performance profile. The L-BFGS method constructs a matrix H_k , an approximate Hessian of $\nabla^2 f(x_k)^{-1}$ by updating the basic matrix H_0 m times using the BFGS formula with previous m iterations. It has been shown that the L-BFGS method converges linearly. Since the L-BFGS method does
215 not use much information of the Hessian, it is now widely used in practice. While, it converges very slowly for ill-conditioned problems [11]. In Figure 2, we give the comparison of P-PSB method and L-BFGS method relative to the number of iteration, the number of function evaluations, the number of gradient evaluations and CPU time. The top curves in the figures correspond to the
220 P-PSB, which shows that the P-PSB has a satisfying performance as well as L-BFGS

At last, we compare the proposed P-PSB method with its dense version, i.e., the projected PSB method in Table 4. The preliminary numerical results show that the P-PSB method can terminate at solutions of the problems successfully.
225 While, the projected PSB fails for several problems. Moreover, it's obvious that the P-PSB method has better performance, especially in terms of CPU time. The P-PSB method can obtain solutions of problems up to 2×10^6 dimension, while the projected PSB method can only get solutions with dimension no more

than 500 in a short period of time. Taking into account that the subproblem
 230 of a sparse quasi-Newton method is easier to solve than its dense version, so
 P-PSB method is desirable to solve large-scale problems. The results further
 positively supported the proposed method.

5. Final Remarks

We have proposed a partitioned PSB quasi-Newton method for solving par-
 235 tially separable unconstrained optimization problems. Under appropriate con-
 ditions, local and superlinear convergence has been well established. By the
 use of a projection technique, we globalized the proposed method and proved
 its global and superlinear convergence. Our preliminary numerical results have
 shown that the proposed method worked well. It can solve large-scale partially
 240 separable problems successfully.

It should be pointed out that just like the standard PSB update, the quasi-
 Newton matrix B_{k+1} generated by the partitioned PSB method may be singular
 even if B_k is nonsingular. We have given some analysis about the nonsingular
 partitioned PSB update in Section 3. However, the analysis is based on selecting
 245 suitable sub-matrices of the previous iteration, which is influenced by the sparse
 structure of the Hessian matrices. Thus the issue will be an question which needs
 further study.

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Table 1: Comparison on Problems 1-10

Pro	n	P-PSB				P-BFGS				L-BFGS			
		Ite	fcnt	grad	time	Ite	fcnt	grad	time	Ite	fcnt	grad	time
1	1000	14	46	29	0.0000	33	86	68	0.0313	30	123	63	0.0313
	10000	14	59	30	0.0938	33	86	68	0.2031	30	123	63	0.2656
	100000	14	59	30	0.9844	33	86	68	1.1563	30	123	63	2.8281
	1000000	14	59	30	4.8438	33	86	68	9.4063	30	123	63	25.3125
2	1000	36	127	125	0.1563	21	89	45	0.0313	20	97	62	0.0625
	10000	36	127	125	0.8750	21	89	45	0.2031	20	97	62	0.4688
	100000	36	127	125	5.9531	21	89	45	1.9531	20	97	62	3.7500
	1000000	36	127	125	65.7969	21	89	45	20.0469	20	97	62	29.1563
3	1000	47	485	102	0.0625	33	86	68	0.0000	30	112	63	0.0313
	10000	44	431	95	0.3438	33	86	68	0.1563	30	112	63	0.2500
	100000	50	290	107	2.1719	33	86	68	1.2656	30	112	63	2.4688
	1000000	49	285	104	21.3906	33	86	68	9.4688	30	112	63	23.0469
4	1000	2	9	5	0.0000	18	39	37	0.0313	14	45	29	0.0313
	10000	2	9	5	0.0000	18	39	37	0.2813	14	45	29	0.1875
	100000	2	9	5	0.1563	18	39	37	1.4844	14	45	29	1.6563
	1000000	2	9	5	1.1719	18	39	37	13.7656	14	45	29	14.1250
5	1000	8	22	17	0.0313	11	41	23	0.0313	8	28	17	0.0000
	10000	8	22	17	0.0938	11	41	23	0.1563	8	28	17	0.0938
	100000	8	22	17	0.4219	11	41	23	1.2656	8	28	17	0.7813
	1000000	9	25	19	4.5938	11	41	23	7.1094	9	58	35	16.9844
6	1000	8	33	18	0.0000	9	31	20	0.0313	7	33	15	0.0000
	10000	8	33	18	0.0313	9	31	20	0.0625	7	33	15	0.0313
	100000	8	33	18	0.2656	9	31	20	0.5625	7	33	15	0.3750
	1000000	8	33	18	2.5938	9	31	20	3.0625	7	33	15	2.8750
7	1000	12	35	26	0.0000	13	34	28	0.0000	13	47	27	0.0000
	10000	12	35	26	0.1250	13	34	28	0.0938	13	47	27	0.0938
	100000	12	35	26	1.1875	13	34	28	0.7031	13	47	27	1.2969
	1000000	12	35	26	6.4844	13	34	28	5.9063	13	47	27	8.7500
8	1000	15	79	34	0.0000	17	114	36	0.0781	10	42	23	0.0000
	10000	12	63	27	0.1250	17	114	36	0.7656	10	42	23	0.0781
	100000	21	116	49	1.2031	17	114	36	4.0156	10	42	23	0.8594
	1000000	25	125	54	12.3125	17	114	36	37.0000	10	42	23	5.8594
9	1000	54	172	124	0.0313	66	243	142	0.0313	60	212	128	0.0938
	10000	54	172	124	0.3750	66	243	142	0.3438	60	212	128	0.5313
	100000	54	172	124	2.2969	66	243	142	1.2656	61	217	130	4.7031
	1000000	56	196	131	20.3750	66	243	142	14.9219	61	217	130	58.1250
10	1000	9	101	56	0.0313	38	184	132	0.0313	21	241	175	0.0313
	10000	9	101	56	0.1875	37	182	130	0.3438	18	235	172	0.3125
	100000	9	101	56	0.8281	37	182	130	1.8594	17	226	167	1.8438
	1000000	9	101	56	9.0469	37	182	130	18.9063	17	228	167	21.5469

Table 2: Comparison on Problems 11-20

Pro	n	P-PSB				P-BFGS				L-BFGS			
		Ite	fcnt	grad	time	Ite	fcnt	grad	time	Ite	fcnt	grad	time
11	1000	4	16	10	0.0000	5	26	11	0.0000	4	20	9	0.0000
	10000	4	16	10	0.0313	5	26	11	0.0938	4	20	9	0.0313
	100000	4	16	10	0.4063	5	26	11	0.5781	4	20	9	0.2500
	1000000	4	16	10	2.0781	5	26	11	3.0156	4	20	9	1.6250
12	1000	2	7	5	0.0000	7	17	16	0.0000	7	24	15	0.0000
	10000	2	7	5	0.0000	7	17	16	0.0156	7	24	15	0.0000
	100000	2	7	5	0.1250	7	17	16	0.4219	7	24	15	0.2188
	1000000	2	7	5	0.8281	7	17	16	1.9844	7	24	15	2.1719
13	1000	10	35	23	0.0000	11	68	29	0.0313	10	45	22	0.0000
	10000	10	35	23	0.0938	11	68	29	0.0938	10	45	22	0.0469
	100000	10	35	23	0.7344	11	68	29	1.1563	10	45	22	0.4844
	1000000	10	35	23	4.0938	11	68	29	6.1563	10	45	22	4.7500
14	1000	7	23	25	0.0000	11	25	26	0.0313	7	30	24	0.0313
	10000	7	23	25	0.0625	11	25	26	0.1250	7	30	24	0.0469
	100000	7	23	25	0.8906	11	25	26	1.1875	7	30	24	0.4219
	1000000	7	23	25	4.7813	11	25	26	6.5625	7	30	24	5.6250
15	1000	15	70	32	0.0313	92	208	185	0.1563	7	24	15	0.0000
	10000	15	70	32	0.2188	92	208	185	1.3750	7	24	15	0.0313
	100000	15	70	32	1.6719	92	208	185	8.7188	7	24	15	0.3438
	1000000	15	70	32	16.6875	92	208	185	86.0313	7	24	15	2.5781
16	1000	8	23	18	0.0000	12	52	27	0.0313	40	646	115	0.2188
	10000	8	23	18	0.1563	12	52	27	0.2188	59	1253	137	1.8906
	100000	8	23	18	0.7969	12	52	27	1.3125	59	1253	137	15.4688
	1000000	8	23	18	7.6250	12	52	27	12.8594	59	1254	138	158.9844
17	1000	39	275	85	0.0313	75	207	162	0.0625	71	269	155	0.0625
	10000	76	789	159	0.6094	75	207	162	0.3750	71	269	156	0.3906
	100000	60	604	127	3.0781	75	207	162	2.7813	70	264	152	2.6563
	1000000	62	648	130	28.8906	75	207	162	23.1719	71	270	156	29.7031
18	1000	21	71	55	0.0313	71	522	146	0.0625	22	96	56	0.0000
	10000	21	71	55	0.0938	67	527	139	0.5313	22	96	56	0.1250
	100000	21	71	55	0.8281	67	527	138	3.3594	22	96	56	0.8125
	1000000	21	71	55	7.1094	66	520	136	30.6094	22	96	56	9.3750
19	1000	49	186	100	1.9063	41	105	85	0.8750	41	144	83	0.0938
	10000	38	164	78	8.8594	41	105	85	5.7813	42	147	85	0.7188
	100000	37	162	76	74.0938	45	113	93	60.9531	42	147	85	5.1094
	1000000	36	160	74	670.6406	45	113	93	575.7188	54	196	111	72.8125
20	1000	36	110	75	0.6875	49	209	118	1.2500	27	101	55	0.0313
	10000	36	110	75	4.6875	49	209	118	9.1563	27	101	55	0.1875
	100000	36	110	75	19.3961	49	209	118	71.4526	27	101	55	2.0625
	1000000	36	110	75	26.5590	49	209	118	91.7188	27	101	55	20.1875

Table 3: Comparison on Problems 21-30

Pro	n	P-PSB				P-BFGS				L-BFGS			
		Ite	fent	grad	time	Ite	fent	grad	time	Ite	fent	grad	time
21	1000	76	1440	205	0.4063	129	2317	143	0.2500	16	56	34	0.0313
	10000	93	1443	197	2.1250	121	2102	187	1.5781	14	49	29	0.1875
	100000	77	1322	160	12.0469	97	1715	151	9.7031	12	43	25	0.5313
	1000000	87	1192	113	82.6094	71	1318	178	93.2344	11	40	23	6.6875
22	1000	10	26	24	0.0625	10	27	23	0.0000	8	35	19	0.0313
	10000	22	51	27	0.3125	12	30	27	0.0938	7	32	17	0.0625
	100000	6	19	15	0.9688	11	28	25	0.7813	7	30	17	0.3906
	1000000	4	14	11	3.6719	10	26	23	5.8438	15	54	33	9.6719
23	1000	6	12	13	0.0000	7	14	15	0.0000	7	21	15	0.0000
	10000	6	12	13	0.0313	7	14	15	0.0313	7	21	15	0.0313
	100000	6	12	13	0.3438	7	14	15	0.3281	7	21	15	0.2813
	1000000	6	12	13	1.9063	8	16	17	1.8750	8	24	17	3.3438
24	1000	5	11	11	0.0313	5	11	11	0.0000	5	16	11	0.0000
	10000	5	11	11	0.0313	5	11	11	0.0313	5	16	11	0.0313
	100000	5	11	11	0.3438	5	11	11	0.3125	5	16	11	0.2500
	1000000	5	11	11	1.8438	5	11	11	1.6094	5	16	11	2.0938
25	1000	4	8	9	0.0000	5	10	11	0.0313	5	15	11	0.0313
	10000	4	8	9	0.0313	5	10	11	0.0625	5	15	11	0.0625
	100000	4	8	9	0.4375	5	10	11	0.4844	5	15	11	0.5313
	1000000	4	8	9	2.0000	5	10	11	2.4375	5	15	11	3.4844
26	1000	6	13	13	0.0000	6	13	13	0.0313	6	19	13	0.0000
	10000	6	13	13	0.0313	6	13	13	0.0313	6	19	13	0.0313
	100000	6	13	13	0.3438	7	15	15	0.3125	7	22	15	0.2188
	1000000	6	13	13	1.7656	7	15	15	1.9375	7	22	15	3.1406
27	1000	13	26	27	0.0000	13	26	27	0.0313	243	3593	395	0.7188
	10000	13	26	27	0.0938	13	26	27	0.0938	90	1163	156	1.4844
	100000	13	26	27	0.5781	13	26	27	0.6250	59	853	100	8.1875
	1000000	13	26	27	4.7813	13	26	27	5.4531	36	487	66	50.6875
28	1000	2	10	5	0.0000	3	18	7	0.0000	9	33	19	0.0313
	10000	2	10	5	0.0313	3	18	7	0.0000	9	33	19	0.0625
	100000	2	10	5	0.0781	3	18	7	0.1563	10	36	21	0.8906
	1000000	2	10	5	0.3906	3	18	7	0.7656	10	36	21	7.2188
29	1000	15	40	31	0.0000	32	67	65	0.0625	18	58	38	0.0313
	10000	13	35	27	0.1563	26	55	53	0.2344	13	41	27	0.0625
	100000	27	39	55	1.8281	20	43	41	1.1875	8	26	17	0.3594
	1000000	28	52	68	19.7031	14	31	29	8.1563	6	20	13	2.4375
30	1000	55	215	112	0.0625	18	44	38	0.0313	27	87	57	0.0313
	10000	41	303	84	0.5469	18	44	38	0.1875	31	101	63	0.3125
	100000	21	132	43	1.3906	18	44	38	1.0156	33	112	70	2.7344
	1000000	10	50	21	3.8125	18	44	38	10.0781	42	147	86	38.1719

Table 4: Numerical Results of PSB Method

Pro 4					Pro 5				Pro 6				Pro 7			
n	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time
10	2	9	5	0.0000	15	55	31	0.0156	10	45	23	0.0000	18	52	38	0.0313
50	2	9	5	0.0313	19	75	39	0.0625	65	446	133	0.2188	12	35	26	0.0313
100	2	9	5	0.0313	19	75	39	0.3281	68	471	141	1.0625	12	35	26	0.1875
500	2	9	5	2.3281	15	53	31	17.5313	119	562	256	17.5623	12	35	26	14.1406
Pro 8					Pro 10				Pro 12				Pro 13			
n	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time
10	38	172	79	0.0313	9	101	56	0.0000	5	26	11	0.0000	2	7	5	0.0000
50	40	219	82	0.1563	11	111	60	0.0625	5	26	11	0.0313	2	7	5	0.0000
100	38	164	79	0.6094	11	111	60	0.1875	5	26	11	0.0625	2	7	5	0.0625
500	64	312	131	75.0625	13	128	64	16.0313	5	26	11	5.9531	2	7	5	2.4531
Pro 15					Pro 16				Pro 21				Pro 22			
n	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time
10	11	25	26	0.0313	9	30	21	0.0000	32	169	64	0.0313	10	26	20	0.0000
50	11	25	26	0.0313	8	28	19	0.0313	127	889	260	0.5313	22	39	47	0.8125
100	11	25	26	0.2500	8	28	19	0.1094	181	1428	366	3.2188	18	53	36	1.3750
500	11	25	26	14.1719	8	28	19	10.2656	226	1986	530	16.3988	14	64	30	17.2031
Pro 23					Pro 24				Pro 25				Pro 26			
n	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time
10	25	161	50	0.0000	26	106	52	0.0313	7	41	14	0.0000	5	11	11	0.0000
50	74	732	150	0.2969	35	142	70	0.1250	9	58	18	0.0313	5	11	11	0.0313
100	129	1384	266	2.1094	36	151	72	0.6094	9	55	18	0.1563	5	11	11	0.0938
500	344	3541	697	408.5938	39	185	80	46.4063	9	61	18	11.3906	5	11	11	6.0156
Pro 27					Pro 28				Pro 29				Pro 30			
n	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time	Ite	fcnt	grad	Time
10	4	14	9	0.0313	6	40	12	0.0313	42	111	84	0.2813	6	51	13	0.0938
50	4	14	9	0.0000	6	40	12	0.0000	179	358	359	0.7813	6	51	13	0.6094
100	4	14	9	0.0781	6	40	12	0.1719	441	882	883	7.8438	6	51	13	2.7031
500	4	14	9	4.7813	8	71	15	10.0781	563	1026	1183	15.3620	6	51	12	7.1094

Figure 1: Performance profiles for P-PSB and P-BFGS

(a) Results comparison on the number of iteration, (b) Results comparison on the number of function evaluations, (c) Results comparison on the number of gradient evaluation, (d) Results comparison on CPU time.

Figure 2: Performance profiles for P-PSB and L-BFGS

(a) Results comparison on the number of iteration, (b) Results comparison on the number of function evaluations, (c) Results comparison on the number of gradient evaluation, (d) Results comparison on CPU time.