Solutions to unsolved problems on the minimal energies of two classes of trees

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Abstract

The energy of a graph is defined as the sum of the absolute values of all eigenvalues of the graph. Let $\mathbb{T}_{n,p}$, $\mathcal{T}_{n,d}$ be the set of all trees of order n with p pendent vertices, diameter d , respectively. In this paper, we completely characterize the trees with second-minimal and third-minimal energy in $\mathbb{T}_{n,p}$ ($\mathcal{T}_{n,d}$, respectively) for $4 \leq p \leq n - 9$ ($10 \leq d \leq n - 3$, respectively), which solves the problems left in [20].

Keywords: Minimal energy; Tree; Pendent vertex; Diameter

1 Introduction

Let G be a simple graph with n vertices, and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of its adjacency matrix. Then the energy of G is defined as

$$E = E(G) = \sum_{i=1}^n |\lambda_i|.$$

Let G be an acyclic graph of order n . Then $E(G)$ can be expressed as the Coulson integral formula [11]

$$E(G) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left[\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(G, k) x^{2k} \right] dx,$$

where $m(G, k)$ denotes the number of k -matchings in G and $m(G, 0) = 1$. Follow [15], denote

$$m^+(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(G, k) x^{2k}.$$

Then for a tree T with n vertices, we have

$$E(T) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log m^+(T, x) dx. \quad (1)$$

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Since the energy of a graph can be used to approximate the total π -electron energy of the molecular, it has been intensively studied. For details on graph energy, we refer the readers to the book [18] and reviews [8, 10]. One of the fundamental question that is encountered in the study of graph energy is which graphs (from a given class) have minimal and maximal energies. A large number of papers were published on such extremal problems, see Chapter 7 in [18] and some recent papers [9,12,13,15-17,19-30]. There are many other kinds of graph energies, such as matching energy [1, 2, 3], Laplacian energy [4], distance energy [5], Randić energy [6, 14], incidence energy [6], etc.

For terminology and notation not defined here, we refer to [20]. A caterpillar is a tree in which a removal of all pendent vertices makes a path. Let $T(n, d; n_1, n_2, \dots, n_{d-1})$ be a caterpillar obtained from a path $v_0 v_1 \dots v_d$ by adding n_i ($n_i \geq 0$) pendent edges to v_i ($i = 1, \dots, d-1$). For $1 \leq i \leq d-1$, denote $T_{n,d,i} = T(n, d; 0, \dots, 0, n_i = n-d-1, 0, \dots, 0)$.

Gutman [7] proved that the star S_n and the path P_n have minimal and maximal energy among all trees of order n , respectively. Let $\mathcal{T}_{n,d}$ be the set of all trees with n vertices and diameter d , where $2 \leq d \leq n-1$. The trees with minimal and second-minimal energies in $\mathcal{T}_{n,d}$ have been considered by [27, 30, 16, 17, 26, 24]. Let $\mathbb{T}_{n,p}$ be the set of all trees of order n with p pendent vertices, where $2 \leq p \leq n-1$. The tree with minimal energy in $\mathbb{T}_{n,p}$ has been determined by [29] and [28] independently. By the results in [27], [29] and [28], the trees with minimal energy in $\mathcal{T}_{n,d}$ and $\mathbb{T}_{n,n-d+1}$ are the same for all $2 \leq d \leq n-1$. In [20], one of the present authors gave the further relations on the ordering of trees by minimal energies between $\mathcal{T}_{n,d}$ and $\mathbb{T}_{n,n-d+1}$, and completely characterized the trees with second-minimal and third-minimal energy in $\mathbb{T}_{n,p}$ ($\mathcal{T}_{n,d}$, respectively) except for the case $4 \leq p \leq n-9$ ($10 \leq d \leq n-3$, respectively). By [20], the trees with second-minimal and third-minimal energy in $\mathbb{T}_{n,p}$ and $\mathcal{T}_{n,n-p+1}$ are the same, respectively, for $4 \leq p \leq n-9$, and the results on $\mathbb{T}_{n,p}$ are the following:

Theorem 1.1. [20] *Suppose that $4 \leq p \leq n-9$. Then the second-minimal energy tree in $\mathbb{T}_{n,p}$ is one of the two trees $T_{n,n-p+1,3}$ and $T(n, n-p+1; p-3, 0, \dots, 0, 1)$.*

Theorem 1.2. [20] *Suppose that $4 \leq p \leq n-9$. If $E(T_{n,n-p+1,3}) > E(T(n, n-p+1; p-3, 0, \dots, 0, 1))$, then $T_{n,n-p+1,3}$ is the unique tree with third-minimal energy in $\mathbb{T}_{n,p}$; If $E(T_{n,n-p+1,3}) < E(T(n, n-p+1; p-3, 0, \dots, 0, 1))$, then the third-minimal energy tree in $\mathbb{T}_{n,p}$ is one of the two trees $T_{n,n-p+1,5}$ and $T(n, n-p+1; p-3, 0, \dots, 0, 1)$.*

From Theorems 1.1 and 1.2, one can see that for $4 \leq p \leq n-9$, the author of [20] could not determine the unique extremal graph, respectively. In this paper, we will use the Coulson integral formula to solve the above unsolved problems. We only state our results on $\mathbb{T}_{n,p}$ instead of $\mathcal{T}_{n,d}$.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we characterize the trees with second-minimal energy in $\mathbb{T}_{n,p}$ for $4 \leq p \leq n - 9$. In Section 4, we determine the trees with third-minimal energy in $\mathbb{T}_{n,p}$ for $4 \leq p \leq n - 9$.

2 Preliminaries

The following five results on properties of $m^+(G, x)$ appeared in [15].

Lemma 2.1. *Let v be a vertex of graph G and $N(v) = \{v_1, v_2, \dots, v_j\}$ the set of all neighbors of v in G . Then*

$$m^+(G, x) = m^+(G - v, x) + x^2 \sum_{v_i \in N(v)} m^+(G - v - v_i, x).$$

Lemma 2.2. *Let P_t be the path on t vertices. Then*

- (1) $m^+(P_t, x) = m^+(P_{t-1}, x) + x^2 m^+(P_{t-2}, x)$, for any $t \geq 1$,
- (2) $m^+(P_t, x) = (1 + x^2) m^+(P_{t-2}, x) + x^2 m^+(P_{t-3}, x)$, for any $t \geq 2$.

The initials are $m^+(P_0, x) = m^+(P_1, x) = 1$, and we define $m^+(P_{-1}, x) = 0$.

Corollary 2.3. *Suppose $t \geq 1$. Then for any real number x ,*

$$m^+(P_{t-1}, x) \leq m^+(P_t, x) \leq (1 + x^2) m^+(P_{t-1}, x).$$

Lemma 2.4. *For $t \geq -1$, the polynomial $m^+(P_t, x)$ has the following form*

$$m^+(P_t, x) = \frac{1}{\sqrt{1 + 4x^2}} (\lambda_1^{t+1} - \lambda_2^{t+1}),$$

where $\lambda_1 = \frac{1 + \sqrt{1 + 4x^2}}{2}$ and $\lambda_2 = \frac{1 - \sqrt{1 + 4x^2}}{2}$.

Lemma 2.5. *Suppose $t \geq 0$. If t is even, then*

$$\frac{2}{1 + \sqrt{1 + 4x^2}} \leq \frac{m^+(P_t, x)}{m^+(P_{t+1}, x)} \leq 1.$$

If t is odd, then

$$\frac{1}{1 + x^2} \leq \frac{m^+(P_t, x)}{m^+(P_{t+1}, x)} \leq \frac{2}{1 + \sqrt{1 + 4x^2}}.$$

The following result on real analysis is needed in the next sections.

Lemma 2.6. [31] *For any real number $X > -1$, we have*

$$\frac{X}{1 + X} \leq \log(1 + X) \leq X.$$

3 The trees with second-minimal energy in $\mathbb{T}_{n,p}$

In this section, we will characterize the trees with second-minimal energy in $\mathbb{T}_{n,p}$ for $4 \leq p \leq n-9$. For convenience, we define $T_1 = T(n, n-p+1; p-3, 0, \dots, 0, 1)$, $T_2 = T_{n,n-p+1,3}$ and $T_3 = T_{n,n-p+1,5}$.

By Lemmas 2.1 and 2.2, we can easily get the following result.

Lemma 3.1. *Suppose that $4 \leq p \leq n-9$. Then we have*

$$\begin{aligned} m^+(T_1, x) &= A_1 m^+(P_{n-p-2}, x) + A_2 m^+(P_{n-p-3}, x), \\ m^+(T_2, x) &= B_1 m^+(P_{n-p-2}, x) + B_2 m^+(P_{n-p-3}, x), \end{aligned}$$

where $A_1 = 1 + (p+1)x^2 + 2(p-2)x^4$, $A_2 = x^2 + px^4$, $B_1 = 1 + (p+1)x^2 + (2p-3)x^4$, and $B_2 = x^2 + 2x^4$.

Theorem 3.2. (i) *If $5 \leq p \leq n-9$, then T_2 is the unique tree with second-minimal energy in $\mathbb{T}_{n,p}$.*

(ii) *T_2 if $n = 13$, T_1 if $n \geq 14$ is the unique tree with second-minimal energy in $\mathbb{T}_{n,4}$.*

Proof. From Eq. (1), we have

$$\begin{aligned} E(T_1) - E(T_2) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \frac{m^+(T_1, x)}{m^+(T_2, x)} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{m^+(T_1, x) - m^+(T_2, x)}{m^+(T_2, x)} \right) dx. \end{aligned} \quad (2)$$

(i) Suppose that $5 \leq p \leq n-9$. By Lemmas 2.6, 3.1 and Corollary 2.3, we have

$$\begin{aligned} E(T_1) - E(T_2) &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \cdot \frac{m^+(T_1, x) - m^+(T_2, x)}{m^+(T_1, x)} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{-x^2 m^+(P_{n-p-2}, x) + (p-2)x^2 m^+(P_{n-p-3}, x)}{A_1 m^+(P_{n-p-2}, x) + A_2 m^+(P_{n-p-3}, x)} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{(p-2)x^2}{A_2 + A_1 \cdot \frac{m^+(P_{n-p-2}, x)}{m^+(P_{n-p-3}, x)}} dx - \frac{2}{\pi} \int_0^{+\infty} \frac{x^2}{A_1 + A_2 \cdot \frac{m^+(P_{n-p-3}, x)}{m^+(P_{n-p-2}, x)}} dx \\ &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{(p-2)x^2}{A_2 + A_1 \cdot (1+x^2)} dx - \frac{2}{\pi} \int_0^{+\infty} \frac{x^2}{A_1 + A_2 \cdot \frac{1}{1+x^2}} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{(p-3-x^2)x^2}{1 + (p+3)x^2 + (4p-3)x^4 + 2(p-2)x^6} dx \\ &\triangleq F(p). \end{aligned}$$

For a fixed $x > 0$, let

$$f(p) = \frac{(p-3-x^2)x^2}{1+(p+3)x^2+(4p-3)x^4+2(p-2)x^6}.$$

Then we have

$$f'(p) = \frac{x^2 + 6x^4 + 10x^6 + 6x^8 + 2x^{10}}{[1+(p+3)x^2+(4p-3)x^4+2(p-2)x^6]^2} > 0.$$

So $f(p)$ is a strictly increase function of p . Since $F(5) \doteq \frac{2}{\pi} \cdot 0.0091327 > 0$, we can obtain that for $p \geq 5$,

$$E(T_1) - E(T_2) \geq F(p) \geq F(5) > 0.$$

(ii) Now suppose that $p = 4$. Then by Eq. (2), we have

$$\begin{aligned} & E(T_1) - E(T_2) \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{2x^4}{B_2 + B_1 \cdot \frac{m^+(P_{n-p-2}, x)}{m^+(P_{n-p-3}, x)}} - \frac{x^4}{B_1 + B_2 \cdot \frac{m^+(P_{n-p-3}, x)}{m^+(P_{n-p-2}, x)}} \right) dx, \end{aligned} \quad (3)$$

where $B_1 = 1 + 5x^2 + 5x^4$ and $B_2 = x^2 + 2x^4$ by Lemma 3.1.

We first consider the case that n is even. By Eq. (3) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_2) &\leq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{2x^4}{B_2 + B_1 \cdot \frac{1+\sqrt{1+4x^2}}{2}} - \frac{x^4}{B_1 + B_2 \cdot \frac{2}{1+\sqrt{1+4x^2}}} \right) dx \\ &\doteq \frac{2}{\pi} \cdot (-0.018552) < 0. \end{aligned}$$

If n is odd, we want to find n and x satisfying that

$$\frac{m^+(P_{n-p-3}, x)}{m^+(P_{n-p-2}, x)} < \frac{2.01}{1 + \sqrt{1+4x^2}}. \quad (4)$$

By Lemma 2.4, it is equivalent to solve

$$\frac{\lambda_1^{n-p-2} - \lambda_2^{n-p-2}}{\lambda_1^{n-p-1} - \lambda_2^{n-p-1}} < \frac{2.01}{2\lambda_1},$$

which means to solve

$$\left(\frac{\lambda_1}{-\lambda_2} \right)^{n-p-2} > \frac{2\lambda_1 - 2.01\lambda_2}{0.01\lambda_1},$$

that is,

$$\left(\frac{1 + \sqrt{1+4x^2}}{2x} \right)^{2(n-p-2)} > 401 - \frac{402}{1 + \sqrt{1+4x^2}},$$

or equivalently,

$$2(n - p - 2) > \log_{\frac{1+\sqrt{1+4x^2}}{2x}} \left(401 - \frac{402}{1 + \sqrt{1+4x^2}} \right).$$

Note that for $x \in (0, +\infty)$, $\log_{\frac{1+\sqrt{1+4x^2}}{2x}} \left(401 - \frac{402}{1+\sqrt{1+4x^2}} \right)$ is increasing. Therefore if $x \in (0, 19.5]$, then

$$\log_{\frac{1+\sqrt{1+4x^2}}{2x}} \left(401 - \frac{402}{1+\sqrt{1+4x^2}} \right) \leq \log_{\frac{1+\sqrt{1+4 \cdot (19.5)^2}}{2 \cdot 19.5}} \left(401 - \frac{402}{1+\sqrt{1+4 \cdot (19.5)^2}} \right) < 233.$$

Thus when $n \geq 123$ and $x \in (0, 19.5]$, Ineq. (4) holds. Hence if n is odd and $n \geq 123$, from Eq. (3) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_2) &\leq \frac{2}{\pi} \int_{19.5}^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{2x^4}{B_2 + B_1 \cdot 1} - \frac{x^4}{B_1 + B_2 \cdot 1} \right) dx \\ &\quad + \frac{2}{\pi} \int_0^{19.5} \frac{1}{x^2} \log \left(1 + \frac{2x^4}{B_2 + B_1 \cdot \frac{1+\sqrt{1+4x^2}}{2.01}} - \frac{x^4}{B_1 + B_2 \cdot \frac{2.01}{1+\sqrt{1+4x^2}}} \right) dx, \\ &\doteq \frac{2}{\pi} \cdot (-8.1799 \times 10^{-5}) < 0. \end{aligned}$$

For any odd n with $13 \leq n \leq 121$, by computing the energies of the two graphs directly, we get that $E(T_1) < E(T_2)$ for $n \geq 15$ and $E(T_1) > E(T_2)$ for $n = 13$.

The proof is thus complete by Theorem 1.1. ■

4 The trees with third-minimal energy in $\mathbb{T}_{n,p}$

In this section, we will characterize the trees with third-minimal energy in $\mathbb{T}_{n,p}$ for $4 \leq p \leq n - 9$. For the case $p = 4$, we have the following theorem.

Theorem 4.1. *T_1 if $n = 13$, T_2 if $n \geq 14$ is the unique tree with third-minimal energy tree in $\mathbb{T}_{n,4}$.*

Proof. By direct computation, we have $E(T_3) > E(T_1)$ when $p = 4$ and $n = 13$. Hence we can get the result directly from Theorems 1.2 and 3.2. ■

In the following, we consider the case $5 \leq p \leq n - 9$. First, by Lemmas 2.1 and 2.2, we can obtain the following result.

Lemma 4.2. *Suppose that $5 \leq p \leq n - 9$. Then we have*

$$\begin{aligned} m^+(T_1, x) &= C_1 m^+(P_{n-p-4}, x) + C_2 m^+(P_{n-p-5}, x), \\ m^+(T_3, x) &= D_1 m^+(P_{n-p-4}, x) + D_2 m^+(P_{n-p-5}, x), \end{aligned}$$

where $C_1 = 1 + (p+3)x^2 + (4p-3)x^4 + 2(p-2)x^6$, $C_2 = x^2[1 + (p+2)x^2 + (3p-4)x^4]$, $D_1 = 1 + (p+3)x^2 + (4p-2)x^4 + (3p-5)x^6$, and $D_2 = x^2(1 + 4x^2 + 3x^4)$.

Theorem 4.3. Suppose that $5 \leq p \leq n - 9$. Then we have:

- (i) If $7 \leq p \leq n - 9$, then T_3 is the unique tree with third-minimal energy in $\mathbb{T}_{n,p}$.
- (ii) T_3 if $n = 15, 17, 19$ and $n \geq 21$, T_1 if $n = 16, 18, 20$ is the unique tree with third-minimal energy in $\mathbb{T}_{n,6}$.
- (iii) T_3 if $n = 14, 16$, T_1 if $n = 15$ and $n \geq 17$ is the unique tree with third-minimal energy in $\mathbb{T}_{n,5}$.

Proof. By Theorems 1.2 and 3.2, we only need to compare the energies of the two graphs of T_1 and T_3 . From Eq. (1), we have

$$\begin{aligned} E(T_1) - E(T_3) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \frac{m^+(T_1, x)}{m^+(T_3, x)} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{m^+(T_1, x) - m^+(T_3, x)}{m^+(T_3, x)} \right) dx. \end{aligned} \quad (5)$$

We consider the following cases separately:

(1) $12 \leq p \leq n - 9$.

By Lemmas 2.6, 4.2 and Corollary 2.3, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \cdot \frac{m^+(T_1, x) - m^+(T_3, x)}{m^+(T_1, x)} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{[(1-p)x^4 - x^2]m^+(P_{n-p-4}, x) + [(p-2)x^2 + (3p-7)x^4]m^+(P_{n-p-5}, x)}{C_1 m^+(P_{n-p-4}, x) + C_2 m^+(P_{n-p-5}, x)} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{(p-2)x^2 + (3p-7)x^4}{C_2 + C_1 \cdot \frac{m^+(P_{n-p-4}, x)}{m^+(P_{n-p-5}, x)}} dx - \frac{2}{\pi} \int_0^{+\infty} \frac{x^2 + (p-1)x^4}{C_1 + C_2 \cdot \frac{m^+(P_{n-p-5}, x)}{m^+(P_{n-p-4}, x)}} dx \\ &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{(p-2)x^2 + (3p-7)x^4}{C_2 + C_1(1+x^2)} dx - \frac{2}{\pi} \int_0^{+\infty} \frac{x^2 + (p-1)x^4}{C_1 + C_2 \cdot \frac{1}{1+x^2}} dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{(p-3)x^2 + (2p-7)x^4 - (p-1)x^6}{1 + (p+5)x^2 + (6p+2)x^4 + (9p-11)x^6 + (2p-4)x^8} dx \\ &\triangleq G(p). \end{aligned}$$

For a fixed $x > 0$, let

$$g(p) = \frac{(p-3)x^2 + (2p-7)x^4 - (p-1)x^6}{1 + (p+5)x^2 + (6p+2)x^4 + (9p-11)x^6 + (2p-4)x^8} dx.$$

Then we have

$$g'(p) = \frac{x^2 + 10x^4 + 36x^6 + 56x^8 + 35x^{10} + 8x^{12} + 2x^{14}}{[1 + (p+5)x^2 + (6p+2)x^4 + (9p-11)x^6 + (2p-4)x^8]^2} > 0.$$

So $g(p)$ is a strictly increase function of p . Since $G(12) \doteq \frac{2}{\pi} \cdot 0.0028451 > 0$, we can obtain that for $p \geq 12$,

$$E(T_1) - E(T_3) \geq G(p) \geq G(12) > 0.$$

(2) $p = 10$ or 11 .

It follows from Eq. (5) that

$$\begin{aligned} & E(T_1) - E(T_3) \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4[p-2+(3p-7)x^2]}{D_2 + D_1 \cdot \frac{m^+(P_{n-p-4}, x)}{m^+(P_{n-p-5}, x)}} - \frac{x^4[1+(p-1)x^2]}{D_1 + D_2 \cdot \frac{m^+(P_{n-p-5}, x)}{m^+(P_{n-p-4}, x)}} \right) dx. \end{aligned} \quad (6)$$

By Corollary 2.3 and Lemma 4.2, we have

$$\begin{aligned} & E(T_1) - E(T_3) \\ &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4[p-2+(3p-7)x^2]}{D_2 + D_1 \cdot (1+x^2)} - \frac{x^4[1+(p-1)x^2]}{D_1 + D_2 \cdot \frac{1}{1+x^2}} \right) dx \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(p-3+(2p-7)x^2-(p-1)x^4)}{(1+x^2) \cdot [1+(p+4)x^2+(4p+1)x^4+(3p-5)x^6]} \right) dx \\ &\triangleq H(p). \end{aligned}$$

By direct computation, we can obtain that $H(11) \doteq \frac{2}{\pi} \cdot 0.023681 > 0$ and $H(10) \doteq \frac{2}{\pi} \cdot 0.0067362 > 0$.

(3) $p = 9$.

If n is even, by Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(7+20x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{2}} - \frac{x^4(1+8x^2)}{D_1 + D_2 \cdot \frac{2}{1+\sqrt{1+4x^2}}} \right) dx \\ &\doteq \frac{2}{\pi} \cdot 0.11251 > 0. \end{aligned}$$

If n is odd, we want to find n and x satisfying that

$$\frac{m^+(P_{n-p-5}, x)}{m^+(P_{n-p-4}, x)} > \frac{1.99}{1 + \sqrt{1+4x^2}}. \quad (7)$$

By Lemma 2.4, it is equivalent to solve

$$\frac{\lambda_1^{n-p-4} - \lambda_2^{n-p-4}}{\lambda_1^{n-p-3} - \lambda_2^{n-p-3}} > \frac{1.99}{2\lambda_1},$$

which means to solve

$$\left(\frac{\lambda_1}{\lambda_2} \right)^{n-p-4} > \frac{2\lambda_1 - 1.99\lambda_2}{0.01\lambda_1},$$

that is,

$$\left(\frac{1 + \sqrt{1+4x^2}}{2x} \right)^{2(n-p-4)} > 399 - \frac{398}{1 + \sqrt{1+4x^2}},$$

or equivalently,

$$2(n - p - 4) > \log_{\frac{1+\sqrt{1+4x^2}}{2x}} \left(399 - \frac{398}{1 + \sqrt{1+4x^2}} \right).$$

Note that for $x \in (0, +\infty)$, $\log_{\frac{1+\sqrt{1+4x^2}}{2x}} \left(399 - \frac{398}{1 + \sqrt{1+4x^2}} \right)$ is increasing. Therefore if $x \in (0, 1]$, then $\log_{\frac{1+\sqrt{1+4x^2}}{2x}} \left(399 - \frac{398}{1 + \sqrt{1+4x^2}} \right) \leq \log_{\frac{1+\sqrt{1+4}}{2}} \left(399 - \frac{398}{1 + \sqrt{1+4}} \right) < 12$. Thus when $n \geq 19$ and $x \in (0, 1]$, Ineq. (7) holds. Hence if n is odd and $n \geq 19$, from Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_1^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(7 + 20x^2)}{D_2 + D_1 \cdot (1 + x^2)} - \frac{x^4(1 + 8x^2)}{D_1 + D_2 \cdot \frac{1}{1+x^2}} \right) dx \\ &\quad + \frac{2}{\pi} \int_0^1 \frac{1}{x^2} \log \left(1 + \frac{x^4(7 + 20x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{1.99}} - \frac{x^4(1 + 8x^2)}{D_1 + D_2 \cdot \frac{1.99}{1+\sqrt{1+4x^2}}} \right) dx, \\ &\doteq \frac{2}{\pi} \cdot (5.0665 \times 10^{-4}) > 0. \end{aligned}$$

(4) $p = 8$.

If n is odd, by Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(6 + 17x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{2}} - \frac{x^4(1 + 7x^2)}{D_1 + D_2 \cdot \frac{2}{1+\sqrt{1+4x^2}}} \right) dx \\ &\doteq \frac{2}{\pi} \cdot 0.086741 > 0. \end{aligned}$$

If n is even, similar to the proof in (3), we can show that when $n \geq 22$ and $x \in (0, 1.6]$, Ineq. (7) holds. Hence if n is even and $n \geq 22$, from Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_{1.6}^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(6 + 17x^2)}{D_2 + D_1 \cdot (1 + x^2)} - \frac{x^4(1 + 7x^2)}{D_1 + D_2 \cdot \frac{1}{1+x^2}} \right) dx \\ &\quad + \frac{2}{\pi} \int_0^{1.6} \frac{1}{x^2} \log \left(1 + \frac{x^4(6 + 17x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{1.99}} - \frac{x^4(1 + 7x^2)}{D_1 + D_2 \cdot \frac{1.99}{1+\sqrt{1+4x^2}}} \right) dx, \\ &\doteq \frac{2}{\pi} \cdot (5.2403 \times 10^{-4}) > 0. \end{aligned}$$

By direct computation, we get that $E(T_1) > E(T_3)$ for $n = 18$ and 20 .

(5) $p = 7$.

If n is even, by Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(5 + 14x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{2}} - \frac{x^4(1 + 6x^2)}{D_1 + D_2 \cdot \frac{2}{1+\sqrt{1+4x^2}}} \right) dx \\ &\doteq \frac{2}{\pi} \cdot 0.055170 > 0. \end{aligned}$$

If n is odd, similar to the proof in (3), we can show that when $n \geq 27$ and $x \in (0, 2.6]$, Ineq. (7) holds. Hence if n is odd and $n \geq 27$, from Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_{2.6}^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(5 + 14x^2)}{D_2 + D_1 \cdot (1 + x^2)} - \frac{x^4(1 + 6x^2)}{D_1 + D_2 \cdot \frac{1}{1+x^2}} \right) dx \\ &\quad + \frac{2}{\pi} \int_0^{2.6} \frac{1}{x^2} \log \left(1 + \frac{x^4(5 + 14x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{1.99}} - \frac{x^4(1 + 6x^2)}{D_1 + D_2 \cdot \frac{1.99}{1+\sqrt{1+4x^2}}} \right) dx, \\ &\doteq \frac{2}{\pi} \cdot (1.1834 \times 10^{-3}) > 0. \end{aligned}$$

By direct computation, we get that $E(T_1) > E(T_3)$ for $n = 17, 19, 21, 23$ and 25 .

(6) $p = 6$.

If n is odd, by Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(4 + 11x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{2}} - \frac{x^4(1 + 5x^2)}{D_1 + D_2 \cdot \frac{2}{1+\sqrt{1+4x^2}}} \right) dx \\ &\doteq \frac{2}{\pi} \cdot 0.015038 > 0. \end{aligned}$$

If n is even, similar to the proof in (3), we can show that when $n \geq 50$ and $x \in (0, 6.6]$, Ineq. (7) holds. Hence if n is even and $n \geq 50$, from Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\geq \frac{2}{\pi} \int_{6.6}^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(4 + 11x^2)}{D_2 + D_1 \cdot (1 + x^2)} - \frac{x^4(1 + 5x^2)}{D_1 + D_2 \cdot \frac{1}{1+x^2}} \right) dx \\ &\quad + \frac{2}{\pi} \int_0^{6.6} \frac{1}{x^2} \log \left(1 + \frac{x^4(4 + 11x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{1.99}} - \frac{x^4(1 + 5x^2)}{D_1 + D_2 \cdot \frac{1.99}{1+\sqrt{1+4x^2}}} \right) dx, \\ &\doteq \frac{2}{\pi} \cdot (8.9163 \times 10^{-5}) > 0. \end{aligned}$$

For any even n with $16 \leq n \leq 48$, by direct computation, we get that $E(T_1) < E(T_3)$ for $n = 16, 18, 20$, and $E(T_1) > E(T_3)$ for the other cases.

(7) $p = 5$.

If n is odd, by Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\leq \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(3 + 8x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{2}} - \frac{x^4(1 + 4x^2)}{D_1 + D_2 \cdot \frac{2}{1+\sqrt{1+4x^2}}} \right) dx \\ &\doteq \frac{2}{\pi} \cdot (-0.038859) < 0. \end{aligned}$$

If n is even, similar to the proof in Theorem 3.2, we can show that when $n \geq 130$ and $x \in (0, 20]$, the following inequality holds:

$$\frac{m^+(P_{n-p-5}, x)}{m^+(P_{n-p-4}, x)} < \frac{2.01}{1 + \sqrt{1 + 4x^2}}.$$

Hence if n is even and $n \geq 130$, from Eq. (6) and Lemma 2.5, we have

$$\begin{aligned} E(T_1) - E(T_3) &\leq \frac{2}{\pi} \int_{20}^{+\infty} \frac{1}{x^2} \log \left(1 + \frac{x^4(3+8x^2)}{D_2 + D_1 \cdot 1} - \frac{x^4(1+4x^2)}{D_1 + D_2 \cdot 1} \right) dx \\ &\quad + \frac{2}{\pi} \int_0^{20} \frac{1}{x^2} \log \left(1 + \frac{x^4(3+8x^2)}{D_2 + D_1 \cdot \frac{1+\sqrt{1+4x^2}}{2.01}} - \frac{x^4(1+4x^2)}{D_1 + D_2 \cdot \frac{2.01}{1+\sqrt{1+4x^2}}} \right) dx, \\ &\doteq \frac{2}{\pi} \cdot (-1.4442 \times 10^{-4}) < 0. \end{aligned}$$

For any even n with $14 \leq n \leq 128$, by direct computation, we get that $E(T_1) > E(T_3)$ for $n = 14, 16$, and $E(T_1) < E(T_3)$ for the other cases.

The proof is thus complete. ■

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