Numerical approximation for nonlinear stochastic pantograph equations with Markovian switching

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Abstract The main aim of the paper is to prove that the implicit numerical approximation can converge to the true solution to highly nonlinear hybrid stochastic pantograph differential equation. After providing the boundedness of the exact solution, the paper proves that the backward Euler-Maruyama numerical method can preserve boundedness of moments, and the numerical approximation converges strongly to the true solution. Finally, the exponential stability criterion on the backward EM is given, and a high order example is provided to illustrate the main result.

Key words Strong convergence; Polynomial growth conditions; Moment boundedness; Backward Euler-Maruyama method; Markovian switching; Exponential stability

1 Introduction

Stochastic unbounded delay systems play an important role in a variety of application areas, including biology, epidemiology, mechanic, economics and finance. The systems provide powerful models, such as, infinite delay Kolmogorov-type systems in mathematic biology [7,25], stochastic neural networks [8,10,15], stochastic pantograph equations in science and engineering. So such systems have received an increasing attention [3,6,20,21,24]. The pantograph equation which is a very special unbounded delay equation was used by Ockendon and Taylor [18] in 1971 to study how the electric current is collected by the pantograph of an electric locomotive, from where it gets its name. Research efforts have been devoted to existence-and-uniqueness and stability of the analytical solution for stochastic pantograph equation [1,2,16].

However, most of stochastic pantograph differential equations cannot be solved explicitly. Especially, explicit solutions can rarely be obtained for nonlinear stochastic pantograph equations, so numerical methods have recently received more and more attention. Fan et al. [4] investigated the convergence of the semi-implicit Euler methods for stochastic pantograph equations under the Lipschitz condition and the linear growth condition. Fan et al. [5] investigated the numerical approximation for the linear stochastic pantograph equation by using the Razumikhin technique. Milošević et al. [17] studied a Taylor polynomial approximate solution for stochastic pantograph equation with Markovian switching under the linear growth condition.

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The existing results of convergence derived by these conditions are somewhat restrictive for the purpose of practical applications, because there are many stochastic pantograph differential equations that only satisfy the local Lipschitz condition. Li et al.[9] studied convergence in probability of the Euler approximation solution for stochastic pantograph equations with Markovian switching under weaker conditions. Zhou et al.[22] studied convergence in probability of the Euler-Maruyama approximate solution of stochastic pantograph differential equation. However, to the best of the authors' knowledge, the strong convergence remains essentially unsolved for the important class of stochastic pantograph equations with super-linearly growing coefficients, owing to no result on the moment bound for the numerical solution under the polynomial growth conditions, which is the main topic of the present paper. This paper, which is natural extension of the paper [22], shall prove that a appropriate implicit approximation converges strongly to the exact solution.

This is a continuation of the first author's earlier paper [23], in which a criterion on almost surely exponential stability of numerical simulations is concerned with nonlinear stochastic pantograph equation. The paper shall develop a criterion on strong convergence of the backward Euler-Maruyama method for highly nonlinear hybrid stochastic pantograph equation. The classical linear growth condition is replaced by one-sided polynomial growth conditions, under which there exists a unique global solution and the solution is bounded. On basis of a series of lemmas, the paper shows that the backward Euler-Maruyama approximate solution converges strongly to the analytic solution. Finally, a highly nonlinear example is provided to illustrate the main theory.

The structure of the paper is as follows: In the next section, we introduce polynomial growth conditions under which stochastic pantograph systems have a unique global solutions, and prove the p th moment boundedness of the solution. Section 3 shows that the 2nd moment of the backward EM can be bounded under one-sided polynomial growth conditions, then proposes the forward-backward EM and proves a strong convergence theorem. Section 4 gives the exponential stability criterion on the backward EM approximation. Finally, a highly nonlinear example is considered to illustrate the main theory, which implies that the results in the paper are very general, so that a wider class of nonlinear systems can satisfy these conditions.

2 The global solution

Throughout this paper, unless otherwise specified, let $|x|$ be the Euclidean norm in $x \in \mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T . If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\mathbb{R}_+ = [0, +\infty)$. Let $w(t) = (w_1(t), \dots, w_m(t))$ be an $\bar{m}$ -dimensional Brownian motion.

Let $r(t)(t \geq 0)$ be a right-continuous Markov chain on the probability space taking values in

a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$.

Consider an n -dimensional stochastic pantograph equation with Markovian switching

$$dx(t) = f(x(t), x(qt), r(t))dt + g(x(t), x(qt), r(t))dw(t), \quad 0 < q < 1 \quad (2.1)$$

on $t \geq 0$ with the initial data $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are locally Lipschitz continuous.

(H1) (The Local Lipschitz Condition) For each integer $R \geq 1$, there exists a positive constant C_R such that

$$|f(x_1, y_1, i) - f(x_2, y_2, i)| \vee |g(x_1, y_1, i) - g(x_2, y_2, i)| \leq C_R(|x_1 - x_2| + |y_1 - y_2|)$$

for all $i \in \mathbb{S}$ and $x_k, y_k \in \mathbb{R}^n$ with $|x_k| \vee |y_k| \leq R$ ($k = 1, 2$).

(H2) (Polynomial growth conditions) For all $i \in S$ and any $x \in \mathbb{R}^n$, there exist positive constants $\alpha, a, \bar{a}, a_0, b, \bar{b}, b_0$ such that

$$2\langle x(s), f(x(s), x(qs), i) \rangle \leq -a|x(s)|^{\alpha+2} + \bar{a}|x(qs)|^{\alpha+2} - a_0|x(s)|^2,$$

$$|g(x(s), x(qs), i)|^2 \leq b|x(s)|^{\alpha+2} + \bar{b}|x(qs)|^{\alpha+2} + b_0|x(s)|^2.$$

Theorem 2.1 For $p \geq 2$, let (H1) and (H2) hold with $a_0 > (p-1)b_0$, $a > (p-1)b + \frac{p-2+q^{-1}(\alpha+2)}{\alpha+p}(\bar{a} + (p-1)\bar{b})$. Then for any initial data x_0 , there almost surely exists a unique global solution $x(t)$ to Eq.(2.1) on $t \geq -\tau$. Moreover,

$$\mathbb{E}|x(t)|^p \leq \mathbb{E}|x_0|^p.$$

Proof Under (H1), applying the standing truncation technique [see Mao[11], Theorem 3.2.2, P₉₅] to Eq.(2.1) for any initial data ξ , there exists a unique maximal local strong solution to Eq.(2.1) on $-\tau < t < \nu_e$, where ν_e is the explosion time. To show this solution is global, we only need to show that $\nu_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large such that $k_0 > |x_0|$. For each integer $k \geq k_0$, define the stopping time

$$\nu_k = \inf\{t \in [0, \nu_e) : |x(t)| \geq k\}, (k \in \mathbb{N}),$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual, $\emptyset$ =the empty set). By the definition of the stopping time ν_k , it is obvious that ν_k is an increasing function with k , so $\nu_k \rightarrow \nu_\infty \leq \nu_e$ ($k \rightarrow \infty$) a.s. If we can show that $\nu_\infty = \infty$ a.s., then $\nu_e = \infty$ a.s. which implies that $x(t)$ is global. In other words, we only prove that $\mathbb{P}(\nu_k \leq t) \rightarrow 0$ ($k \rightarrow \infty, t > 0$). Define $V(x) = |x|^p$, since $P(\nu_k \leq t)V(x(\nu_k)) \leq \mathbb{E}V(x(t \wedge \nu_k))$, that is, we only need to prove that $\mathbb{E}V(x(t \wedge \nu_k)) < +\infty$ since $V(x(\nu_k)) = |x(\nu_k)|^p = k^p \rightarrow \infty$. By the Itô formula, we obtain

$$V(x(t)) = V(\xi(0)) + \int_0^t LV(x(s), x(qs), r(s))ds + \int_0^t V_x(x(s))g(x(s), x(qs), r(s))dw(s), \quad (2.2)$$

where

$$LV(x(s), x(qs), r(s)) \leq p|x(s)|^{p-2}x^T(s)f(x(s), x(qs), r(s)) + \frac{p(p-1)}{2}|x(s)|^{p-2}|g(x(s), x(qs), r(s))|^2.$$

Making use of (H2), one may compute

$$\begin{aligned} LV(x(s), x(qs), r(s)) &\leq \frac{p}{2}|x(s)|^{p-2}[-a|x(s)|^{\alpha+2} + \bar{a}|x(qs)|^{\alpha+2} - a_0|x(s)|^2] \\ &\quad + \frac{p(p-1)}{2}|x(s)|^{p-2}[b|x(s)|^{\alpha+2} + \bar{b}|x(qs)|^{\alpha+2} + b_0|x(s)|^2] \\ &\leq \frac{p}{2}|x(s)|^{p-2}[-(a_0 - (p-1)b_0)|x(s)|^2 - (a - (p-1)b)|x(s)|^{\alpha+2}] \\ &\quad + \frac{p}{2}(\bar{a} + (p-1)\bar{b})|x(s)|^{p-2}|x(qs)|^{\alpha+2}. \end{aligned} \quad (2.3)$$

The inequality $a^p b^q \leq \frac{p}{p+q}a^{p+q} + \frac{q}{p+q}b^{p+q}$, $a, b, p, q > 0$, implies

$$\begin{aligned} |x(s)|^{p-2}|x(qs)|^{\alpha+2} &\leq \frac{p-2}{\alpha+p}|x(s)|^{\alpha+p} + \frac{\alpha+2}{\alpha+p}|x(qs)|^{\alpha+p} \\ &= \frac{\alpha+2}{\alpha+p}(|x(qs)|^{\alpha+p} - \frac{1}{q}|x(s)|^{\alpha+p}) + \frac{p-2+q^{-1}(\alpha+2)}{\alpha+p}|x(s)|^{\alpha+p}. \end{aligned} \quad (2.4)$$

Substituting (2.4) into (2.3), yields

$$\begin{aligned} LV(x(s), x(qs)) &\leq \frac{p}{2}|x(s)|^{p-2}[-(a_0 - (p-1)b_0)|x(s)|^2 - (a - (p-1)b)|x(s)|^{\alpha+2}] \\ &\quad + \frac{p}{2}(\bar{a} + (p-1)\bar{b})[\frac{\alpha+2}{\alpha+p}(|x(qs)|^{\alpha+p} - \frac{1}{q}|x(s)|^{\alpha+p}) + \frac{p-2+q^{-1}(\alpha+2)}{\alpha+p}|x(s)|^{\alpha+p}] \\ &\leq \frac{p(\alpha+2)}{2(\alpha+p)}(\bar{a} + (p-1)\bar{b})(|x(qs)|^{\alpha+p} - \frac{1}{q}|x(s)|^{\alpha+p}) - I_0(x), \end{aligned} \quad (2.5)$$

where

$$I_0(x(s)) = \frac{p}{2}(a_0 - (p-1)b_0)|x(s)|^p + \frac{p}{2}[a - (p-1)b - \frac{p-2+q^{-1}(\alpha+2)}{\alpha+p}(\bar{a} + (p-1)\bar{b})]|x(s)|^{\alpha+p}.$$

Recalling that $a_0 > (p-1)b_0$, $a > (p-1)b + \frac{p-2+q^{-1}(\alpha+2)}{\alpha+p}(\bar{a} + (p-1)\bar{b})$, there exists a positive constant c_0 such that

$$I_0(x(s)) \geq c_0|x(s)|^p.$$

Substituting this into (2.5), (2.5) into (2.2), yields

$$\begin{aligned} V(x(t)) &\leq V(\xi(0)) + \frac{p}{2}\frac{\alpha+2}{\alpha+p}(\bar{a} + (p-1)\bar{b})\int_0^t(|x(qs)|^{\alpha+p} - \frac{1}{q}|x(s)|^{\alpha+p})ds \\ &\quad - c_0\int_0^t|x(s)|^p ds + \int_0^t V_x(x(s))g(x(s), x(qs), r(s))dw(s). \end{aligned} \quad (2.6)$$

Applying the substitution technique of integral, one may get

$$\int_0^t(|x(qs)|^{\alpha+p} - \frac{1}{q}|x(s)|^{\alpha+p})ds \leq \frac{1}{q}\int_0^{qt}|x(s)|^{\alpha+p}ds - \frac{1}{q}\int_0^t|x(s)|^{\alpha+p}ds \leq -\frac{1}{q}\int_{qt}^t|x(s)|^{\alpha+p}ds. \quad (2.7)$$

Substituting (2.7) into (2.6), taking expectation and noting that $\int_0^{t \wedge \nu_k} V_x(x(s))g(x(s), x(qs), r(s))dw(s)$ is local martingale, so we have

$$\mathbb{E}V(x(t \wedge \nu_k)) \leq \mathbb{E}V(\xi) - \frac{p(\alpha+2)}{2q(\alpha+p)}(\bar{a} + (p-1)\bar{b})\mathbb{E}\int_{qt}^t|x(s)|^{\alpha+p}ds - c_0\mathbb{E}\int_0^t|x(s)|^p ds \leq \mathbb{E}|x_0|^p.$$

Letting $k \rightarrow \infty$, which implies the required result.

3 Strong convergence

In the section, our main objective is to prove that the numerical approximation of Eq.(2.1) strongly converges to the true solution. After introducing the backward EM scheme and its continuous form of numerical method, we prove the p th moment bound and the main strong convergence theorem is proved.

Given a step size $\Delta \in (0, 1)$, and for $t \in [0, T]$, $M\Delta = T$ for some positive integer M . Let $t_k = k\Delta$ ($k \geq 0$), $[t/\Delta]$ be the integer part of t/Δ .

Lemma 3.1(see [30]) Let $r_k^\Delta = r(t_k)$ for $\Delta > 0$ and $k \geq 0$, then $\{r_k^\Delta, k = 0, 1, 2, \dots\}$ is discrete Markovian chain with the one-step transition probability matrix

$$P(\Delta) = (P_{ij}(\Delta))_{N \times N} = e^{\Delta \Gamma}.$$

Since γ_{ij} are independent of x , the paths of r can be generated independently of x and, in fact, before computing x .

The discrete Markov chain $\{r_k^\Delta, k = 0, 1, 2, \dots\}$ can be simulated as follows: Compute the one-step transition probability matrix $P(\Delta)$. Let $r_0^\Delta = i_0$ and generate a random number ξ_1 which is uniformly distributed in $[0, 1]$. Define

$$r_1^\Delta = \begin{cases} i_1 & \text{if } i_1 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} P_{i_0,j}(\Delta) \leq \xi_1, \end{cases}$$

where we set $\sum_{j=1}^0 P_{i_0,j}(\Delta) = 0$ as usual. Generate independently a new random number ξ_2 which is again uniformly distributed in $[0, 1]$ and then define

$$r_2^\Delta = \begin{cases} i_2 & \text{if } i_2 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_2-1} P_{r_1^\Delta,j}(\Delta) \leq \xi_2 < \sum_{j=1}^{i_2} P_{r_1^\Delta,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} P_{r_1^\Delta,j}(\Delta) \leq \xi_2. \end{cases}$$

Repeating this procedure a trajectory of $\{r_k^\Delta, k = 1, 2, \dots\}$ can be generated. This procedure can be carried out independently to obtain more trajectories. After explaining how to simulate the discrete Markovian chain $\{r_k^\Delta, k = 0, 1, 2, \dots\}$, we can now define the backward Euler-Maruyama approximate solution for Eq.(2.1). Compute the discrete approximations $X(t_k) \approx x(t_k)$ by setting $X_0 = x_0, r_0^\Delta = i_0$ and performing

$$X(t_{k+1}) = X(t_k) + f(X(t_{k+1}), X(t_{[q(k+1)]}), r(t_{k+1}))\Delta + g(X(t_k), X(t_{[qk]}), r(t_k))\Delta w(t_k). \quad (3.1)$$

Let $t_k = k\Delta$ ($k \geq 0$). $X(t_k)$ is an approximation to $x(t_k)$. Moreover, the increments $\Delta w_{t_k} = w(t_{k+1}) - w(t_k), k = 1, 2, \dots$, are independent $N(0, \Delta)$ -distributed Gaussian random variables $\mathcal{F}_{t_k}$ -measurable at the mesh-point t_k . Once we compute the discrete values $X(t_k)$ from the backward EM scheme (3.1), then we define the discrete forward backward EM scheme (see [13,14,19])

$$\hat{X}(t_{k+1}) = \hat{X}(t_k) + f(X(t_k), X(t_{[qk]}), r(t_k))\Delta + g(X(t_k), X(t_{[qk]}), r(t_k))\Delta w_{t_k}, \quad (3.2)$$

where $\hat{X}(0) = X(0) = x_0$. Define

$$\eta(t) = t_k, \quad t \in [t_k, t_{k+1}), \quad k \geq 0. \quad (3.3)$$

The continuous forward backward EM scheme is defined by

$$\bar{X}(t) = \bar{X}(0) + \int_0^t f(X(\eta(s)), X(\eta(qs)), r(\eta(s)))ds + \int_0^t g(X(\eta(s)), X(\eta(qs)), r(\eta(s)))dw(s), \quad t \geq 0. \quad (3.4)$$

Note that the continuous and discrete forward backward EM scheme coincide at the grid point, $\bar{X}(t_k) = \hat{X}(t_k)$.

To guarantee that this method (3.1) is well defined, we impose the following one-sided Lipschitz condition on the drift coefficient $f(x, y, i)$.

(H3) (One-sided Lipschitz condition) There exists a positive constant $\lambda > 0$ such that for any $x_i \in \mathbb{R}^n (i = 1, 2)$

$$\langle x_1 - x_2, f(x_1, y, i) - f(x_2, y, i) \rangle \leq \lambda |x_1 - x_2|^2.$$

Applying a fixed point theorem one can prove that Eq.(3.1) has a unique solution $X(t_{k+1})$ given $X(t_k)$ if $\lambda\Delta < 1$, then the backward EM scheme (3.1) is well defined (see e.g.[12]). From now on we always assume that $\Delta < \lambda^{-1}$.

For convenience, let integer $R > 0$ be arbitrary and define the sequence of stopping time

$$\sigma_R = \inf\{k : |X(t_k)| > R\}. \quad (3.5)$$

Clearly, when $k \in [0, \sigma_R(w)]$, $|X(t_i)(w)| \leq R, i = 1, \dots, k-1$, but we may have $|X(t_k)(w)| > R$.

Lemma 3.1 Let (H1),(H2) and (H3) hold. Then for any $p \geq 2$ and sufficiently large integer R , there exists a constant $C(R, p)$ such that

$$\mathbb{E}[|X(t_k)|^p I_{[0, \sigma_R]}(k)] < C(R, p), \forall k \geq 0.$$

Proof By the inequality $\langle a, b \rangle \leq |a||b| \leq \frac{1}{2}(|a|^2 + |b|^2)$, one may get

$$\begin{aligned} |X(t_k)|^2 &= \langle X(t_k), X(t_{k-1}) + f(X(t_k), X(t_{[qk]}), r(t_k))\Delta + g(X(t_{k-1}), X(t_{[q(k-1)]}), r(t_{k-1}))\Delta w_{t_{k-1}} \rangle \\ &= \langle X(t_k), f(X(t_k), X(t_{[qk]}), r(t_k)) \rangle \Delta + \langle X(t_k), X(t_{k-1}) \\ &\quad + g(X(t_{k-1}), X(t_{[q(k-1)]}), r(t_{k-1}))\Delta w_{t_{k-1}} \rangle \\ &\leq -\frac{a}{2}|X(t_k)|^{\alpha+2}\Delta + \frac{a}{2}|X(t_{[qk]})|^{\alpha+2}\Delta - \frac{a_0}{2}|X(t_k)|^2\Delta + \frac{1}{2}|X(t_k)|^2 \\ &\quad + \frac{1}{2}|X(t_{k-1}) + g(X(t_{k-1}), X(t_{[q(k-1)]}), r(t_{k-1}))\Delta w_{t_{k-1}}|^2. \end{aligned} \quad (3.6)$$

That is,

$$|X(t_k)|^2 \leq \bar{a}|X(t_{[qk]})|^{\alpha+2}\Delta + |X(t_{k-1}) + g(X(t_{k-1}), X(t_{[q(k-1)]}), r(t_{k-1}))\Delta w_{t_{k-1}}|^2.$$

Applying inequality $|a + b + c|^{\frac{p}{2}} \leq 3^{\frac{p}{2}-1}(a^{\frac{p}{2}} + b^{\frac{p}{2}} + c^{\frac{p}{2}})$, $a, b, c > 0$, yields

$$|X(t_k)|^p \leq 3^{\frac{p}{2}-1}[(\bar{a}\Delta)^{\frac{p}{2}}|X(t_{[qk]})|^{\frac{p(\alpha+2)}{2}} + 2^{\frac{p}{2}}|X(t_{k-1})|^p + 2^{\frac{p}{2}}|g(X(t_{k-1}), X(t_{[q(k-1)]}), r(t_{k-1}))\Delta w_{t_{k-1}}|^p]. \quad (3.7)$$

Hence

$$\begin{aligned} \mathbb{E}[|X(t_k)|^p \mathbf{1}_{[0, \sigma_R]}(k)] &\leq 3^{\frac{p}{2}-1}\{(\bar{a}\Delta)^{\frac{p}{2}}\mathbb{E}[|X(t_{[qk]})|^{\frac{p(\alpha+2)}{2}} \mathbf{1}_{[0, \sigma_R]}(k)] + 2^{\frac{p}{2}}\mathbb{E}[|X(t_{k-1})|^p \mathbf{1}_{[0, \sigma_R]}(k)] \\ &\quad + 2^{\frac{p}{2}}\mathbb{E}[|g(X(t_{k-1}), X(t_{[q(k-1)]}), r(t_{k-1}))\Delta w_{t_{k-1}}|^p \mathbf{1}_{[0, \sigma_R]}(k)]\}. \end{aligned} \quad (3.8)$$

It is easy to obtain that $|g(X(t_{k-1}), X(t_{[q(k-1)]}))|, |X(t_{[qk]})|, |X(t_{k-1})|$ are bound for $|X(t_i)(\omega)| < R, i \leq k-1$ if $k \in [0, \sigma_R(\omega)]$. By the Hölder inequality, then we may compute

$$\begin{aligned} & \mathbb{E}[|g(X(t_{k-1}), X(t_{[q(k-1)]}))\Delta w_{t_{k-1}}|^p \mathbf{1}_{[0, \sigma_R]}(k)] \\ & \leq (\mathbb{E}[|g(X(t_{k-1}), X(t_{[q(k-1)]}))r(t_{k-1})|^{2p} \mathbf{1}_{[0, \sigma_R]}(k)])^{\frac{1}{2}} (\mathbb{E}|\Delta w_{t_{k-1}}|^{2p})^{\frac{1}{2}}. \end{aligned} \quad (3.9)$$

According to the property of normal distributed, and noting that $\Delta w_{t_{k-1}}$ is $\bar{m}$ -dimensional Brownian motion, it is easy to see that there exists a constant $c(p)$ such that

$$\mathbb{E}|\Delta w_{t_{k-1}}|^{2p} = c(p)\Delta^p.$$

Substituting this into (3.9), then there exists a positive constant $C(R, p)$ such that

$$\mathbb{E}[|g(X(t_{k-1}), X(t_{[q(k-1)]}))r(t_{k-1})\Delta w_{t_{k-1}}|^p \mathbf{1}_{[0, \sigma_R]}(k)] \leq C(R, p)\Delta^{\frac{p}{2}}.$$

Substituting for this into (3.8) yields

$$\mathbb{E}[|X(t_k)|^p \mathbf{1}_{[0, \sigma_R]}(k)] \leq C(R, p).$$

To avoid complicate notations, the product of $C(R, p)$ and another constant is still denoted by $C(R, p)$. In the following, other constants is similar.

Lemma 3.2 Let (H1), (H2) and (H3) hold with $a_0 > mb_0, a > \bar{a} + mb + m\bar{b}$. Then

$$\mathbb{E}|X(t_k)|^2 \leq (1 + a_0\Delta)\mathbb{E}|X_0|^2 + mb|X_0|^{\alpha+2}\Delta.$$

Proof By (3.1) and (H2), we have

$$\begin{aligned} |X(t_{k+1})|^2 & \leq -\frac{a}{2}|X(t_{k+1})|^{\alpha+2}\Delta + \frac{\bar{a}}{2}|X(t_{[q(k+1)]})|^{\alpha+2}\Delta \frac{a_0}{2}|X(t_{k+1})|^2\Delta \\ & \quad + \frac{1}{2}|X(t_{k+1})|^2 + \frac{1}{2}|X(t_k) + g(X(t_k), X(t_{[qk]}), r(t_k))\Delta w_{t_k}|^2 \\ & \leq -\frac{a}{2}|X(t_{k+1})|^{\alpha+2}\Delta + \frac{\bar{a}}{2}|X(t_{[q(k+1)]})|^{\alpha+2}\Delta - \frac{a_0}{2}|X(t_{k+1})|^2\Delta \\ & \quad + \frac{1}{2}|X(t_{k+1})|^2 + \frac{1}{2}|g(X(t_k), X(t_{[qk]}), r(t_k))\Delta w_{t_k}|^2 \\ & \quad + \frac{1}{2}|X(t_k)|^2 + \langle X(t_k), g(X(t_k), X(t_{[qk]}), r(t_k))\Delta w_{t_k} \rangle. \end{aligned}$$

Then

$$\begin{aligned} (1 + a_0\Delta)|X(t_{k+1})|^2 & \leq -a|X(t_{k+1})|^{\alpha+2}\Delta + \bar{a}|X(t_{[q(k+1)]})|^{\alpha+2}\Delta + |X(t_k)|^2 \\ & \quad + m(b|X(t_k)|^{\alpha+2} + \bar{b}|X(t_{[qk]})|^{\alpha+2} + b_0|X(t_k)|^2)\Delta + s_k. \end{aligned}$$

where

$$s_k = |g(X(t_k), X(t_{[qk]}))|^2(|\Delta w_{t_k}|^2 - m\Delta) + 2\langle X(t_k), g(X(t_k), X(t_{[qk]}), r(t_k))\Delta w_{t_k} \rangle.$$

Then

$$\begin{aligned} (1 + a_0\Delta)(|X(t_{k+1})|^2 - |X(t_k)|^2) & \leq (mb_0 - a_0)\Delta|X(t_k)|^2 + \bar{a}|X(t_{[q(k+1)]})|^{\alpha+2}\Delta - a|X(t_{k+1})|^{\alpha+2}\Delta \\ & \quad + m(b|X(t_k)|^{\alpha+2} + \bar{b}|X(t_{[qk]})|^{\alpha+2})\Delta + s_k. \end{aligned} \quad (3.10)$$

Summing up both sides of inequality (3.10) from 0 to $k-1$, yields

$$(1 + a_0\Delta)|X(t_k)|^2 \leq (1 + a_0\Delta)|X(t_0)|^2 + (mb_0 - a_0)\Delta \sum_{i=0}^{k-1} |X(t_i)|^2 + \bar{a} \sum_{i=0}^{k-1} |X(t_{[q(i+1)])}|^{\alpha+2} \Delta - a \sum_{i=0}^{k-1} |X(t_{i+1})|^{\alpha+2} \Delta + m \sum_{i=0}^{k-1} (b|X(t_i)|^{\alpha+2} + \bar{b}|X(t_{[qi]})|^{\alpha+2}) \Delta + S_k. \quad (3.11)$$

where $S_k = \sum_{i=0}^{k-1} s_i$. Noting that

$$\sum_{i=0}^{k-1} |X(t_{[qi]})|^{\alpha+2} \leq \sum_{i=0}^{[q(k-1)]+1} |X(t_i)|^{\alpha+2}.$$

Similarly,

$$\sum_{i=0}^{k-1} |X(t_{[q(i+1)])}|^{\alpha+2} = \sum_{i=1}^k |X(t_{[qi]})|^{\alpha+2} \leq \sum_{i=0}^{[qk]+1} |X(t_i)|^{\alpha+2}.$$

Substituting these into (3.11), yields

$$\begin{aligned} |X(t_k)|^2 &\leq (1 + a_0\Delta)|X(t_0)|^2 + (mb_0 - a_0) \sum_{i=0}^{k-1} |X(t_i)|^2 \Delta + \bar{a} \sum_{i=0}^{[qk]+1} |X(t_i)|^{\alpha+2} \Delta \\ &\quad - a \sum_{i=1}^k |X(t_i)|^{\alpha+2} \Delta + mb \sum_{i=0}^{k-1} |X(t_i)|^{\alpha+2} \Delta + m\bar{b} \sum_{i=0}^{[q(k-1)]+1} |X(t_i)|^{\alpha+2} \Delta + S_k. \end{aligned}$$

Replacing k by $k \wedge \sigma_R$ and taking expectation on two sides of the above inequality, yields

$$\begin{aligned} \mathbb{E}|X(t_{k \wedge \sigma_R})|^2 &\leq (1 + a_0\Delta)\mathbb{E}|X(t_0)|^2 + (mb_0 - a_0)\Delta \sum_{i=0}^{k-1} \mathbb{E}|X(t_{i \wedge \sigma_R})|^2 \\ &\quad + \bar{a} \sum_{i=0}^{[qk]+1} \mathbb{E}|X(t_{i \wedge \sigma_R})|^{\alpha+2} \Delta - a \sum_{i=1}^k \mathbb{E}|X(t_{i \wedge \sigma_R})|^{\alpha+2} \Delta \\ &\quad + mb \sum_{i=0}^{k-1} \mathbb{E}|X(t_{i \wedge \sigma_R})|^{\alpha+2} \Delta + m\bar{b} \sum_{i=0}^{[q(k-1)]+1} \mathbb{E}|X(t_{i \wedge \sigma_R})|^{\alpha+2} \Delta + \mathbb{E}S_{k \wedge \sigma_R}. \end{aligned}$$

Noting that $S_{k \wedge \sigma_R}$ is local martingale by Lemma 3.1, and recalling that $a_0 > mb_0, a > a_d + mb_d + mb$, then

$$\begin{aligned} \mathbb{E}|X(t_{k \wedge \sigma_R})|^2 &\leq (1 + a_0\Delta)\mathbb{E}|X(t_0)|^2 - (a_0 - mb_0) \sum_{i=0}^{k-1} \mathbb{E}|X(t_{i \wedge \sigma_R})|^2 \\ &\quad - (a - \bar{a} - m\bar{b} - mb) \sum_{i=1}^k \mathbb{E}|X(t_{i \wedge \sigma_R})|^{\alpha+2} \Delta + mb|X(t_0)|^{\alpha+2} \Delta + \mathbb{E}S_{k \wedge \sigma_R} \\ &\leq (1 + a_0\Delta)\mathbb{E}|X_0|^2 + mb|X_0|^{\alpha+2} \Delta + \mathbb{E}S_{k \wedge \sigma_R}, \end{aligned}$$

noting that the right bound is independent of R . Therefore, letting $R \rightarrow \infty$ and applying the Fatou's lemma, there exists a positive constant such that

$$\mathbb{E}|X(t_k)|^2 \leq (1 + a_0\Delta)\mathbb{E}|X_0|^2 + mb|X_0|^{\alpha+2} \Delta.$$

The proof is complete.

Lemma 3.3 Let (H1), (H2) and (H3) hold, and assume that for any $x, y \in \mathbb{R}^n$

$$|f(x, y)| \leq c|x| + \bar{c}|y|^{\alpha+1} + c|x|^{\alpha+1}. \quad (3.12)$$

Then, for any integer $p \geq 2$, there exists a positive constant $C(R, p)$ such that

$$\mathbb{E}[|\hat{X}(t_k) - X(t_k)|^p \mathbf{1}_{[0, \sigma_R]}(k)] \leq C(R, p) \Delta^p, \quad \forall k \in \mathbb{N}.$$

Proof Summing up both schemes of the backward EM and forward backward EM, we have

$$\begin{aligned} \hat{X}(t_{k+1}) - X(t_{k+1}) &= \hat{X}(t_k) - X(t_k) + [f(X(t_k), X(t_{[qk]}), r(t_k)) \\ &\quad - f(X(t_{k+1}), X(t_{[q(k+1)]}), r(t_{k+1}))] \Delta. \end{aligned}$$

Summing up the above equality from 0 to k , yields

$$\hat{X}(t_k) - X(t_k) = (f(X_0, X_0, r_0) - f(X(t_k), X(t_{[qk]}), r(t_k))) \Delta.$$

Applying the Hölder inequality, Lemma 3.1 and (3.12), we have

$$\begin{aligned} \mathbb{E}[|\hat{X}(t_k) - X(t_k)|^p \mathbf{1}_{[0, \sigma_R]}(k)] &\leq \mathbb{E}[|f(X_0, X_0, r_0) - f(X(t_k), X(t_{[qk]}), r(t_k))|^p \mathbf{1}_{[0, \sigma_R]}(k)] \Delta^p \\ &\leq C(R, p) \Delta^p. \end{aligned}$$

Theorem 3.4 Assume that (H1), (H2) and (H3) hold with $a_0 > mb_0, a > mb + (\bar{a} + m\bar{b})/q$. Then for any given $\varepsilon > 0$, there exists a sufficiently large positive integer R_0 and sufficiently small Δ^* such that for every $R > R_0$ and $\Delta < \Delta^*$

$$P(\rho_R \leq T) \leq \varepsilon, \quad T > 0,$$

where $\rho_R = \inf\{t > 0, |\bar{X}(t)| \geq R \text{ or } |X(\eta(t))| > R\}$.

Proof By the Itô formula, one may compute

$$\begin{aligned} |\bar{X}(T \wedge \rho_R)|^2 &= |\bar{X}(0)|^2 + \int_0^{T \wedge \rho_R} (2\langle \bar{X}(s), f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) \rangle \\ &\quad + |g(X(\eta(s)), X(\eta(qs)), r(\eta(s)))|^2) ds \\ &\quad + 2 \int_0^{T \wedge \rho_R} \langle \bar{X}(s), g(X(\eta(s)), X(\eta(qs)), r(\eta(s))) \rangle dw(s) \\ &\leq |\bar{X}(0)|^2 + \int_0^{T \wedge \rho_R} (2\langle X(\eta(s)), f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) \rangle \\ &\quad + |g(X(\eta(s)), X(\eta(qs)), r(\eta(s)))|^2) ds \\ &\quad + 2 \int_0^{T \wedge \rho_R} \langle \bar{X}(s) - X(\eta(s)), f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) \rangle ds \\ &\quad + 2 \int_0^{T \wedge \rho_R} \langle \bar{X}(s), g(X(\eta(s)), X(\eta(qs)), r(\eta(s))) \rangle dw(s). \end{aligned} \tag{3.13}$$

For any $x, y \in \mathbb{R}^n, |x| \leq R, |y| \leq R$, by the local Lipschitz condition (H1), we obtain

$$\begin{aligned} |f(x, y, i)|^2 &\leq 2(|f(x, y, i) - f(0, 0, i)|^2 + |f(0, 0, i)|^2) \\ &\leq 2(C_R |x|^2 + C_R |y|^2 + |f(0, 0, i)|^2) \\ &\leq C(R)(1 + |x|^2 + |y|^2) \leq C(R), \end{aligned} \tag{3.14}$$

where $C(R) = 2C_R \vee |f(0, 0, i)|^2 \vee |g(0, 0, i)|^2$. Similarly,

$$|g(x, y, i)|^2 \leq C(R)(1 + |x|^2 + |y|^2) \leq C(R). \tag{3.15}$$

Substituting (H2) into (3.13) and using (3.14) and (3.15), one may compute

$$\begin{aligned} \mathbb{E}|\bar{X}(T \wedge \rho_R)|^2 &\leq \mathbb{E}|\bar{X}(0)|^2 + \mathbb{E} \int_0^{T \wedge \rho_R} [\bar{a}|X(\eta(qs))|^{\alpha+2} - a|X(\eta(s))|^{\alpha+2} \\ &\quad - a_0|X(\eta(s))|^2 + b|X(\eta(s))|^{\alpha+2} + \bar{b}|X(\eta(qs))|^{\alpha+2} \\ &\quad + b_0|X(\eta(s))|^2] ds + 2C(R) \mathbb{E} \int_0^{T \wedge \rho_R} |\bar{X}(s) - X(\eta(s))| ds, \end{aligned} \tag{3.16}$$

noting $\int_0^{T \wedge \rho_R} \langle \bar{X}(s), g(X(\eta(s)), X(\eta(qs)), r(\eta(s))) \rangle dw(s)$ is local martingale. Noting that

$$\mathbb{E} \int_0^{T \wedge \rho_R} |X(\eta(qs))|^{\alpha+2} ds \leq \frac{1}{q} \mathbb{E} \int_0^{T \wedge \rho_R} |X(\eta(s))|^{\alpha+2} ds. \quad (3.17)$$

Moreover,

$$\mathbb{E} \int_0^{T \wedge \rho_R} |\bar{X}(s) - X(\eta(s))| ds \leq \mathbb{E} \int_0^{T \wedge \rho_R} |\bar{X}(s) - \hat{X}(\eta(s))| ds + \mathbb{E} \int_0^{T \wedge \rho_R} |\hat{X}(\eta(s)) - X(\eta(s))| ds. \quad (3.18)$$

Noting that

$$\begin{aligned} |\bar{X}(s) - \hat{X}(\eta(s))| \mathbf{1}_{[t_k, t_{k+1})}(s) &= \left| \int_{t_k}^s f(X(\eta(h)), X(\eta(qh)), r(\eta(h))) dh \right. \\ &\quad \left. + \int_{t_k}^s g(X(\eta(h)), X(\eta(qh)), r(\eta(h))) dw(h) \right| \mathbf{1}_{[t_k, t_{k+1})}(s). \end{aligned}$$

By (3.14), (3.15) and $\mathbb{E}|w(h)| \leq [\mathbb{E}(w(h)^2)]^{\frac{1}{2}} \leq (\bar{m}\Delta)^{\frac{1}{2}}$, then

$$\mathbb{E} |\bar{X}(s) - \hat{X}(\eta(s))| \mathbf{1}_{[t_k, t_{k+1})}(s) \leq C(R)\Delta^{\frac{1}{2}}.$$

Therefore

$$\mathbb{E} \int_0^{T \wedge \rho_R} |\bar{X}(s) - \hat{X}(\eta(s))| ds \leq C(R, T)\Delta^{\frac{1}{2}}. \quad (3.19)$$

By Lemma 3.3, one may get

$$\mathbb{E} \int_0^{T \wedge \rho_R} |\hat{X}(\eta(s)) - X(\eta(s))| ds \leq \mathbb{E} \int_0^T |\hat{X}(\eta(s \wedge \rho_R)) - X(\eta(s \wedge \rho_R))| ds \leq C(R, T)\Delta. \quad (3.20)$$

Substituting (3.19) and (3.20) into (3.18), yields

$$\mathbb{E} \int_0^{T \wedge \rho_R} |\bar{X}(s) - X(\eta(s))| ds \leq C(R, T)\Delta^{\frac{1}{2}} + C(R, T)\Delta = C(R, T)\Delta^{\frac{1}{2}}. \quad (3.21)$$

Substituting for (3.17) and (3.21) into (3.16), noting that $a_0 > b_0, a > b + \frac{1}{q}(\bar{a} + \bar{b})$, yields

$$\begin{aligned} \mathbb{E} |\bar{X}(T \wedge \rho_R)|^2 &\leq \mathbb{E} |\bar{X}(0)|^2 + \mathbb{E} \int_0^{T \wedge \rho_R} \left[\frac{\bar{a}}{q} |X(\eta(s))|^{\alpha+2} - a |X(\eta(s))|^{\alpha+2} - a_0 |X(\eta(s))|^2 \right. \\ &\quad \left. + b |X(\eta(s))|^{\alpha+2} + \frac{\bar{b}}{q} |X(\eta(s))|^{\alpha+2} + b_0 |X(\eta(s))|^2 \right] ds + 2C(R)C(R, T)\Delta^{\frac{1}{2}} \\ &\leq \mathbb{E} |\bar{X}(0)|^2 - (a_0 - b_0) \mathbb{E} \int_0^{T \wedge \rho_R} |X(\eta(s))|^2 ds \\ &\quad - (a - b - \frac{\bar{a} + \bar{b}}{q}) \mathbb{E} \int_0^{T \wedge \rho_R} |X(\eta(s))|^{\alpha} ds + 2C(R)C(R, T)\Delta^{\frac{1}{2}} \\ &\leq \mathbb{E} |\bar{X}(0)|^2 + C(R, T)\Delta^{\frac{1}{2}}. \end{aligned} \quad (3.22)$$

If $\inf\{t > 0 : |\bar{X}(t)| \geq R\} \leq \inf\{t > 0 : |X(\eta(t))| > R\}$, then

$$|\bar{X}(\rho_R)|^2 = R^2. \quad (3.23)$$

If $\inf\{t > 0 : |\bar{X}(t)| \geq R\} > \inf\{t > 0 : |X(\eta(t))| > R\}$, then ρ_R is a grid point, and $|X(\rho_R)|^2 > R^2, |X([q\rho_R])|^2 \leq R^2$. By lemma 3.3, we have

$$\bar{X}(\rho_R) = \hat{X}(\rho_R) = X(\rho_R) - f(X(\rho_R), X([q\rho_R]), r(\rho_R))\Delta + f(X_0, X_0, r_0)\Delta.$$

Therefore by (H2)

$$\begin{aligned} |\bar{X}(\rho_R)|^2 &\geq \frac{1}{2} |X(\rho_R)|^2 - \langle X(\rho_R), f(X(\rho_R), X([q\rho_R]), r(\rho_R))\Delta \rangle \\ &\quad + \frac{1}{2} |f(X(\rho_R), X([q\rho_R]), r(\rho_R))\Delta|^2 - |f(X_0, X_0, r_0)\Delta|^2 \\ &\geq \frac{1}{2} |X(\rho_R)|^2 - \langle X(\rho_R), f(X(\rho_R), X([q\rho_R]), r(\rho_R))\Delta \rangle - |f(X_0, X_0, r_0)\Delta|^2 \\ &\geq \frac{1}{2} |X(\rho_R)|^2 + \frac{a_0}{2} |X(\rho_R)|^2 \Delta - |f(X_0, X_0, r_0)\Delta|^2 - \frac{\bar{a}}{2} |X([q\rho_R])|^{\alpha+2} \Delta + \frac{a}{2} |X(\rho_R)|^{\alpha+2} \Delta, \end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|\bar{X}(\rho_R)|^2 &> \frac{1}{2}R^2 + \frac{a_0}{2}R^2\Delta - \frac{\bar{a}}{2}R^{\alpha+2}\Delta + \frac{a}{2}R^{\alpha+2}\Delta - |f(X_0, X_0, r_0)|^2\Delta^2 \\ &= (\frac{1}{2} + \frac{a_0}{2}\Delta)R^2 + (\frac{a}{2} - \frac{\bar{a}}{2})R^{\alpha+2}\Delta - |f(X_0, X_0, r_0)|^2\Delta^2 \geq c_1R^2 - c_2\Delta,\end{aligned}\tag{3.24}$$

where c_1 and c_2 are positive constants. Therefore according to (3.23) and (3.24), we have

$$\mathbb{E}|\bar{X}(T \wedge \rho_R)|^2 \geq \mathbb{E}[|\bar{X}(\rho_R)|^2 1_{\{\rho_R \leq T\}}] \geq P(\rho_R \leq T)(c_1R^2 - c_2\Delta),$$

which implies that

$$\mathbb{P}(\rho_R \leq T) \leq \frac{\mathbb{E}|\bar{X}(T \wedge \rho_R)|^2}{c_1R^2 - c_2\Delta} \leq \frac{\mathbb{E}|\bar{X}(0)|^2 + C(R, T)\Delta^{\frac{1}{2}}}{c_1R^2 - c_2\Delta},$$

where the second inequality uses (3.22). For any given $\varepsilon > 0$, choose sufficiently large R_0 such that for any $R \geq R_0$

$$\frac{\mathbb{E}|\bar{X}(0)|^2}{c_1R^2 - c_2\Delta} \leq \frac{\varepsilon}{2}.$$

Choose sufficiently small Δ^* , such that for any $\Delta < \Delta^*$

$$\frac{C(R, T)\Delta^{\frac{1}{2}}}{c_1R^2 - c_2\Delta} \leq \frac{\varepsilon}{2},$$

which implies that $P(\rho_R \leq T) \leq \varepsilon$ as required.

Lemma 3.5 Let (H1) hold. Then for every $t \in [0, T \wedge \rho_R]$,

$$\begin{aligned}\mathbb{E} \int_0^{t \wedge \rho_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(X(\eta(s)), X(\eta(qs)), r(s))|^2 ds &\leq 8C_R(1 + 2C)C_0T\Delta, \\ \mathbb{E} \int_0^{t \wedge \rho_R} |g(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - g(X(\eta(s)), X(\eta(qs)), r(s))|^2 ds &\leq 8C_R(1 + 2C)C_0T\Delta,\end{aligned}$$

where C_0 is a positive constant dependent on $\max_{0 \leq i \leq N} (-\gamma_{ii})$, but independent of Δ .

Proof For $t \in [0, T \wedge \rho_R]$, it is easy to see that

$$\begin{aligned}&\mathbb{E} \int_0^{t \wedge \rho_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(X(\eta(s)), X(\eta(qs)), r(s))|^2 ds \\ &= \sum_{k=0}^{M-1} \mathbb{E} \int_{t_k}^{t_{k+1} \wedge \rho_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(X(\eta(s)), X(\eta(qs)), r(s))|^2 ds.\end{aligned}$$

Noting that for $s \in [t_k, t_{k+1} \wedge \rho_R]$, $r(\eta(s)) = r(t_k)$, $X(\eta(s)) = X(t_k)$, $X(\eta(qs)) = X(t_{[qk]})$ and $I_{\{r(s) \neq r(t_k)\}}$ is conditionally independent with respect to the σ -algebra generated by $r(t_k)$, by this and Lemma 3.2, we may compute

$$\begin{aligned}&\mathbb{E} \int_{t_k}^{t_{k+1} \wedge \rho_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(X(\eta(s)), X(\eta(qs)), r(s))|^2 ds \\ &\leq 2\mathbb{E} \int_{t_k}^{t_{k+1} \wedge \rho_R} |f(X(t_k), X(t_{[qk]}), r(t_k)) - f(X(t_k), X(t_{[qk]}), r(s))|^2 I_{\{r(s) \neq r(t_k)\}} ds \\ &\leq 4\mathbb{E} \int_{t_k}^{t_{k+1} \wedge \rho_R} [|f(X(t_k), X(t_{[qk]}), r(t_k))|^2 + |f(X(t_k), X(t_{[qk]}), r(s))|^2] I_{\{r(s) \neq r(t_k)\}} ds \\ &\leq 8C_R \mathbb{E} \int_{t_k}^{t_{k+1} \wedge \rho_R} (1 + |X(t_k)|^2 + |X(t_{[qk]})|^2) I_{\{r(s) \neq r(t_k)\}} ds \\ &\leq 8C_R \int_{t_k}^{t_{k+1}} \mathbb{E}(\mathbb{E}[(1 + |X(t_k)|^2 + |X(t_{[qk]})|^2) I_{\{r(s \wedge \rho_R) \neq r(t_k)\}} |r(t_k)]) ds \\ &\leq 8C_R \int_{t_k}^{t_{k+1}} \mathbb{E}(1 + |X(t_k)|^2 + |X(t_{[qk]})|^2) \mathbb{E}[I_{\{r(s \wedge \rho_R) \neq r(t_k)\}} |r(t_k)] ds \\ &\leq 8C_R(1 + 2C) \int_{t_k}^{t_{k+1}} \mathbb{E}[I_{\{r(s \wedge \rho_R) \neq r(t_k)\}} |r(t_k)] ds.\end{aligned}\tag{3.25}$$

By the Markov property, we have

$$\begin{aligned}
\mathbb{E}[I_{\{r(s \wedge \rho_R) \neq r(t_k)\}} | r(t_k)] &= \sum_{i \in S} I_{\{r(t_k)=i\}} P(r(s \wedge \rho_R) \neq i | r(t_k) = i) \\
&= \sum_{i \in S} I_{\{r(t_k)=i\}} \sum_{j \neq i} (\gamma_{ij}(s \wedge \sigma_R - t_k) + o(s \wedge \rho_R - t_k)) \\
&= (\max_{1 \leq i \leq N} (-\gamma_{ii} \Delta + o(\Delta))) \sum_{i \in S} I_{\{r(t_k)=i\}} \leq C_0 \Delta,
\end{aligned} \tag{3.26}$$

where C_0 is constant independent of Δ . Substituting (3.26) into (3.25), yields

$$\mathbb{E} \int_{t_k}^{t_{k+1} \wedge \rho_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(X(\eta(s)), X(\eta(qs)), r(s))|^2 ds \leq 8C_R(1 + 2C)C_0\Delta^2.$$

The section inequality is proved similarly.

Lemma 3.6 Let (H1), (H2) and (H3) hold with $a_0 > mb_0, a > mb + (\bar{a} + m\bar{b})/q$, for sufficiently large R , there exists a positive constant $C(R, T)$ such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\bar{X}(t \wedge \gamma_R) - x(t \wedge \gamma_R)|^2 \right] \leq C(R, T)\Delta,$$

where $\gamma_R = \nu_R \wedge \rho_R$.

Proof For any $T_1 \in [0, T]$, by the Hölder's and the Burkholder-Davis-Gundy's inequalities

$$\begin{aligned}
&\mathbb{E} \left[\sup_{0 \leq t \leq T_1} |\bar{X}(t \wedge \gamma_R) - x(t \wedge \gamma_R)|^2 \right] \\
&\leq T \mathbb{E} \int_0^{T_1 \wedge \gamma_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(x(s), x(qs), r(s))|^2 ds \\
&\quad + 4 \mathbb{E} \int_0^{T_1 \wedge \gamma_R} |g(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - g(x(s), x(qs), r(s))|^2 ds.
\end{aligned}$$

By the local Lipschitz condition (H1) and Lemma 3.5, we may compute

$$\begin{aligned}
&\mathbb{E} \int_0^{t \wedge \gamma_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(x(s), x(qs), r(s))|^2 ds \\
&\leq 2 \mathbb{E} \int_0^{t \wedge \gamma_R} |f(X(\eta(s)), X(\eta(qs)), r(\eta(s))) - f(X(\eta(s)), X(\eta(qs)), r(s))|^2 ds \\
&\quad + 2 \mathbb{E} \int_0^{t \wedge \gamma_R} |f(X(\eta(s)), X(\eta(qs)), r(s)) - f(x(s), x(qs), r(s))|^2 ds \\
&\leq 16C_R(1 + 2C_1)C_0T\Delta + 2 \mathbb{E} \int_0^{t \wedge \gamma_R} |f(X(\eta(s)), X(\eta(qs)), r(s)) - f(x(s), x(qs), r(s))|^2 ds.
\end{aligned} \tag{3.27}$$

According to (H1) and (3.21), we may compute

$$\begin{aligned}
&\mathbb{E} \int_0^{t \wedge \gamma_R} |f(X(\eta(s)), X(\eta(qs)), r(s)) - f(x(s), x(qs), r(s))|^2 ds \\
&\leq C_R \mathbb{E} \int_0^{t \wedge \gamma_R} |X(\eta(s)) - x(s)|^2 + |X(\eta(qs)) - x(qs)|^2 ds \\
&\leq 2C_R \mathbb{E} \int_0^{t \wedge \gamma_R} |X(\eta(s)) - x(s)|^2 ds \\
&\leq 2C_R \mathbb{E} \int_0^{t \wedge \gamma_R} [|X(\eta(s)) - \bar{X}(s)|^2 + |\bar{X}(s) - x(s)|^2] ds \\
&\leq 2C_R(C(R, T)\Delta + \mathbb{E} \int_0^{t \wedge \gamma_R} |\bar{X}(s) - x(s)|^2 ds).
\end{aligned}$$

This, together with (3.27), yields

$$\begin{aligned}
&\mathbb{E} \left[\sup_{0 \leq t \leq T_1} |\bar{X}(t \wedge \gamma_R) - x(t \wedge \gamma_R)|^2 \right] \\
&\leq C(R, T)\Delta + 4C_R(T + 4) \int_0^{T_1} \mathbb{E} \left[\sup_{0 \leq t \leq s} |\bar{X}(s \wedge \gamma_R) - x(s \wedge \gamma_R)|^2 \right] ds.
\end{aligned}$$

The Gronwall inequality implies that the conclusion holds.

Theorem 3.7 Let (H1)-(H3) hold with $a_0 > mb_0, a > mb + (\bar{a} + m\bar{b})/q$, For any given $T = N\Delta > 0, s \in [1, 2)$, the backward EM scheme (3.1) has the property

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|X(T) - x(T)|^s = 0.$$

Proof Denoted by $e(T) = X(T) - x(T)$. Applying the Young's inequality

$$a^s b \leq \frac{\sigma s}{2} b^2 + \frac{2-s}{2\sigma^{\frac{2}{2-s}}} b^{\frac{2}{2-s}}, \forall a, b, \sigma > 0,$$

yields

$$\begin{aligned} \mathbb{E}|e(T)|^s &= \mathbb{E}[|e(T)|^s \mathbf{1}_{\{\nu_R > T, \rho_R > T\}}] + \mathbb{E}[|e(T)|^s \mathbf{1}_{\{\nu_R \leq T \text{ or } \rho_R \leq T\}}] \\ &\leq 2^{s-1} \left[\mathbb{E}[|X(T) - \bar{X}(T)|^s \mathbf{1}_{\{\nu_R > T, \rho_R > T\}}] + \mathbb{E}[|\bar{X}(T) - x(T)|^s \mathbf{1}_{\{\nu_R > T, \rho_R > T\}}] \right] \\ &\quad + \frac{\sigma s}{2} \mathbb{E}[|e(T)|^2] + \frac{2-s}{2\sigma^{\frac{2}{2-s}}} \mathbb{P}(\nu_R \leq T \text{ or } \rho_R \leq T). \end{aligned} \quad (3.28)$$

The Hölder inequality and Lemma 3.3 implies that

$$\mathbb{E}[|X(T) - \bar{X}(T)|^s \mathbf{1}_{\{\nu_R > T, \rho_R > T\}}] \leq (\mathbb{E}[|X(T) - \hat{X}(T)|^p \mathbf{1}_{\{\nu_R > T, \rho_R > T\}}])^{\frac{s}{p}} \leq C(p, R) \Delta^s. \quad (3.29)$$

The Hölder inequality, Lemma 3.6 implies that

$$\mathbb{E}[|\bar{X}(T) - x(T)|^s \mathbf{1}_{\{\nu_R > T, \rho_R > T\}}] \leq (\mathbb{E}[|\bar{X}(T) - x(T)|^2 \mathbf{1}_{\{\nu_R > T, \rho_R > T\}}])^{\frac{s}{2}} \leq C(R, T) \Delta^{\frac{s}{2}}. \quad (3.30)$$

For any given $\varepsilon > 0$, by the Hölder inequality and Theorem 2.1 and Lemma 3.2, we choose $\sigma > 0$ such that

$$\frac{\sigma s}{2} \mathbb{E}[|e(T)|^2] \leq \sigma s \mathbb{E}[|x(T)|^2 + |X(T)|^2] \leq \frac{\varepsilon}{3}. \quad (3.31)$$

By Theorem 2.1, there exists sufficiently large R_0 such that for $R \geq R_0$

$$\frac{2-s}{2\sigma^{\frac{2}{2-s}}} \mathbb{P}(\nu_R \leq T) \leq \frac{\varepsilon}{3}. \quad (3.32)$$

Substituting (3.29)-(3.32) into (3.28), yields

$$\mathbb{E}|e(T)|^s \leq 2^{s-1} [C(p, R) \Delta^s + C(R, T) \Delta^{\frac{s}{2}}] + \frac{2-s}{2\delta^{\frac{2}{2-s}}} \mathbb{P}(\rho_R \leq T) + \frac{2\varepsilon}{3}.$$

Applying Lemma 3.4 and choosing sufficiently small Δ such that

$$2^{s-1} [C(p, R) \Delta^s + C(R, T) \Delta^{\frac{s}{2}}] + \frac{2-s}{2\delta^{\frac{2}{2-s}}} \mathbb{P}(\rho_R \leq T) \leq \frac{\varepsilon}{3}.$$

The conclusion holds.

4 Exponential Stability of numerical solution

In the section, we shall prove that the backward EM method can preserve the almost surely exponential stability of the true solution of highly nonlinear stochastic pantograph differential equation.

(H4)(The polynomial growth conditions) For any $x \in \mathbb{R}^d$, there exist positive constants $\alpha, a, \bar{a}, \tilde{a}, b, \tilde{b}, \bar{b}$ such that

$$\langle x(s), f(x(s), x(qs), i) \rangle \leq -a|x(s)|^{\alpha+2} + \bar{a}e^{-(1-q)\varepsilon s} |x(qs)|^{\alpha+2} - a_0|x(s)|^2, \quad (4.1)$$

$$|g(x(s), x(qs), i)|^2 \leq b|x(s)|^{\alpha+2} + \bar{b}e^{-(1-q)\varepsilon s}|x(qs)|^{\alpha+2} + b_0|x(s)|^2. \quad (4.2)$$

Under (H4), Eq.(2.1) has a unique global solution and the solution is almost surely exponentially stable(see[26]).

Theorem 4.1 Let (H1),(H3) and (H4) hold with $\alpha \geq \beta \vee \gamma$, $2a_0 > b_0$, $2a > 2\bar{a}(1 + 1/q) + b + \bar{b}/q$. Then for any initial data ξ , the solution $x(t)$ is pth moment and almost surely exponentially stable, i.e.

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log|x(t)| \leq -\frac{\varepsilon}{2}$$

where $\varepsilon \leq 2a_0 - b_0$.

The proof is similar to Theorem 3 in Zhou[26].

Theorem 4.2 Let (H1),(H3) and (H4) hold with $2a_0 > mb_0$, $2a > mb + (2\bar{a} + m\bar{b})/q$, there exists a sufficiently small $\Delta^* \in (0, 1)$ such that the approximate solution $\{X_k\}$ defined by (3.1) satisfies

$$\limsup_{k \rightarrow \infty} \frac{1}{k\Delta} \log|X_k| \leq -\frac{\varepsilon}{2} \text{ a.s.}$$

where $\varepsilon < 2a_0 - mb_0$.

The proof is similar to Theorem 7 in Zhou[26].

Figure 1: Computer simulation of the path $x(t)$ for Eq.(4.1) with step size $\Delta = 2^{-8}$.

In the following, we shall consider a stochastic pantograph equation to illustrate our theory established in the previous sections.

Example 4.3 Consider the following nonlinear scalar stochastic pantograph equation

$$dx(t) = [-a(r(t))x(t) - b(r(t))x^5(t) + x^5(qt)]dt + [c(r(t))x^2(t)x(qt) + x^3(t)]dw(t), \quad (4.3)$$

with the initial condition $x_0 = 1, q = 0.5$, where $w(t)$ is scalar Brownian motion.

Define $f(x, y, i) = -a_i x - b_i x^5 + y^5$, $g(x, y) = c_i x^2 y + x^3$. Compute

$$\begin{aligned} f(x_1, y, i) - f(x_2, y, i) &\leq -a_i(x_1 - x_2) - b_i(x_1^5 - x_2^5) \\ &\leq -a_i(x_1 - x_2) - b_i(x_1 - x_2)(x_1^4 + x_1^3 x_2 + x_2^2 x_2^2 + x_1 x_2^3 + x_2^4) \\ &\leq -(x_1 - x_2)[a_i + b_i(x_1^4 + x_1^3 x_2 + x_2^2 x_2^2 + x_1 x_2^3 + x_2^4)]. \end{aligned}$$

Noting that $a^2 + b^2 \geq (a + b)^2/2$, compute

$$\begin{aligned}
x_1^4 + x_1^3 x_2 + x_2^2 x_2^2 + x_1 x_2^3 + x_2^4 &\geq (x_1^4 + x_2^4) + x_1 x_2 (x_1^2 + x_2^2) + (x_1 x_2)^2 \\
&\geq \frac{(x_1^2 + x_2^2)^2}{2} + x_1 x_2 (x_1^2 + x_2^2) + (x_1 x_2)^2 \\
&\geq \frac{(x_1^2 + x_2^2)^2}{4} + x_1 x_2 (x_1^2 + x_2^2) + (x_1 x_2)^2 \\
&\geq \left[\frac{x_1^2 + x_2^2}{2} + x_1 x_2 \right]^2,
\end{aligned}$$

which implies

$$\langle x_1 - x_2, f(x_1, y, i) - f(x_2, y, i) \rangle \leq -a_i (x_1 - x_2)^2.$$

This implies that $f(x, y, i)$ satisfies the one-sided Lipschitz condition (H3). By using the inequality $a^p b^q \leq \frac{p}{p+q} a^{p+q} + \frac{q}{p+q} b^{p+q}$, it is easy to compute

$$2\langle x, f(x, y, i) \rangle \leq -2a_i x^2 - 2b_i x^6 + 2y^5 x \leq -2a_i x^2 - 2b_i x^6 + 5y^6/3 + x^6/3 \leq -2a_i x^2 + 5y^6/36 - (2b_i - 1/3)x^6,$$

$$|g(x, y, i)|^2 = (c_i x^2 y + x^3)^2 \leq 2c_i^2 x^4 y^2 + 2x^6 \leq 2c_i^2 y^6/3 + (4c_i^2/3 + 2)x^6.$$

Applying Theorem 2.1 with $\alpha = 4, a_0 = 2 \min_{1 \leq i \leq N} a_i, \bar{a} = 5/3, a = \min_{1 \leq i \leq N} b_i - 1/3, b_0 = 0, \bar{b} = 2 \min_{1 \leq i \leq N} c_i^2/3, b = 4 \min_{1 \leq i \leq N} c_i^2/3 + 2$. Clearly, $a_0 > b_0, a > b + (\bar{a} + \bar{b})/q$ with $p = 2$, then Eq.(4.3) has a unique global solution. For example, choose $a_1 = 0.5, b_1 = 4, c_1 = 0.5$ and $a_2 = 0.2, b_2 = 3, c_2 = 0.2$, then $a_0 = 0.4, \bar{a} = 5/3, a = 17/3, b_0 = 0, \bar{b} = 2/75, b = 154/75$, clearly, $a_0 > b_0, a > b + (\bar{a} + \bar{b})/q$ holds, so Eq.(4.3) has a unique global solution.

Moreover, the discrete backward EM solution is defined by

$$\begin{aligned}
X(t_{k+1}) &= X(t_k) + [-0.5X(t_{k+1}) - 4X^5(t_{k+1}) + X^5(t_{[q(k+1)]})]\Delta \\
&\quad + [0.5X^2(t_k)X(t_{[qk]}) + X^3(t_k)]d\Delta w_{t_k}.
\end{aligned}$$

Obviously, $a > b + \frac{1}{q}(\bar{a} + \bar{b})$, on the basis of Theorem 4.4 the backward approximate solution converges to the true solution. In Figs.1, we plot the Euler-Maruyama method of Eq.(4.3) for step size $\Delta = 1/256$. The figure illustrates that the numerical solution converges strongly to zero.

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