Convergence of Runge-Kutta methods for neutral delay integro-differential equations

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Abstract

The present paper is concerned with the convergence of numerical methods to solve initial value problems of neutral delay integro-differential equations (NDIDEs). The error estimation of Runge-Kutta methods for a class of nonlinear systems of NDIDEs are derived. Some numerical experiments are given which confirm the theoretical results.

Key words: neutral delay integro-differential equations; Runge-Kutta methods; stability; convergence; error estimation; numerical integration

1 Introduction

Let $\mathbb{R}^d$ be a d dimensional real Euclidian space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$, the inner product and norm on $(\mathbb{R}^d)^p = \mathbb{R}^{dp}$ ($p \geq 1$) be defined by

$$\langle U, V \rangle = \sum_{i=1}^p \langle u_i, v_i \rangle, \|U\| = \sqrt{\langle U, U \rangle},$$

where $U = (u_1^T, u_2^T, \dots, u_p^T)^T$, $V = (v_1^T, v_2^T, \dots, v_p^T)^T \in \mathbb{R}^{dp}$, each $u_i, v_i \in \mathbb{R}^d$. In addition, for any non-negative diagonal matrix $D = \text{diag}(d_1, d_2, \dots, d_p)$, we define a pseudo inner product on $\mathbb{R}^{dp}$ by

$$\langle U, V \rangle_D = \sum_{i=1}^p d_i \langle u_i, v_i \rangle$$

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** This work is supported by NSF of China (No. 11371302, 11571291) and scientific research fund of the department of education, Hunan province, China (15A184)).

and the corresponding pseudo norm by

$$\|U\|_D = \sqrt{\langle U, U \rangle_D}.$$

For any given $p \times q$ real matrix $A = (a_{ij})$, we can define a linear mapping $\tilde{A} : \mathbb{R}^{dq} \rightarrow \mathbb{R}^{dp}$, such that for any given $U = (u_1^T, u_2^T, \dots, u_q^T)^T \in \mathbb{R}^{dq}$

$$\tilde{A}U = V = (v_1^T, v_2^T, \dots, v_p^T)^T \in \mathbb{R}^{dp},$$

where $v_i = \sum_{j=1}^q a_{ij}u_j$, $i = 1, 2, \dots, p$. For simplicity, we shall use the same symbol A to denote the linear mapping $\tilde{A}$ unless otherwise stated, and symbol $\|A\|$ denote the norm of the linear mapping $\tilde{A}$, i.e. the spectral norm of the matrix A (see [19]). Furthermore, for a symmetric matrix A , the symbols $\lambda_{\max}^A$ and $\lambda_{\min}^A$ denote the maximum and minimum eigenvalues of A respectively, and the symbol $A > 0$ (resp. $A \geq 0$) means that the matrix A is positive definite (resp. non-negative definite). For a vector $b = (b_1, b_2, \dots, b_d)^T \in \mathbb{R}^d$, the symbol $b > 0$ (resp. $b \geq 0$) means that each component $b_i > 0$ (resp. $b_i \geq 0$), $i = 1, 2, \dots, d$.

Consider the following initial value problems (IVPs) of nonlinear neutral delay integro-differential equations (NDIDEs) of Hale type ([1])

$$\begin{cases} \frac{d}{dt}[y(t) - Ny(t - \tau)] = f(t, y(t), y(t - \tau), K(t)), & t \in [0, T], \\ y(t) = \varphi(t), & -\tau \leq t \leq 0, \end{cases} \quad (1.1)$$

where $\tau > 0$ and $T > 0$ are given constants, $N \in \mathbb{R}^{d \times d}$ stands for a constant matrix with $\|N\| < 1$, $\varphi : [-\tau, 0] \rightarrow \mathbb{R}^d$ is a continuous mapping, and

$$K(t) = \int_{t-\tau}^t g(t, \xi, y(\xi)) d\xi, \quad (1.2)$$

$f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $g : [0, T] \times [-\tau, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are continuous mappings and satisfy the following conditions, respectively,

$$\begin{aligned} & \langle (u_1 - u_2) - N(v_1 - v_2), f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \rangle \\ & \leq \alpha_1 \|u_1 - u_2\|^2 + \alpha_2 \|v_1 - v_2\|^2 + \beta \|w_1 - w_2\|^2, \\ & \forall t \in [0, T], \quad u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}^d, \end{aligned} \quad (1.3)$$

$$\begin{aligned} & \|g(t, \theta, s_1) - g(t, \theta, s_2)\| \leq \gamma \|s_1 - s_2\|, \\ & \forall t \in [0, T], \quad t - \tau \leq \theta \leq t, \quad s_1, s_2 \in \mathbb{R}^d, \end{aligned} \quad (1.4)$$

where $\alpha_1, \alpha_2, \beta, \gamma$ are real constants and $\beta \geq 0, \gamma \geq 0$.

For convenience, we shall always use the symbol $\mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$ to denote the problem class consisting of all the problems (1.1) satisfying the conditions (1.3)-(1.4).

It is obvious that the problem class $\mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$ is more general than those of delay differential equations (DDEs), delay integro-differential equations (DIDEs), neutral delay differential equations (NDDEs) and so on, which have been in-depth studied by many authors and a great deal of interesting results have been obtained, as reflected in the recent monograph by Bellen and Zennaro [14], Brunner [15], Kuang [16] and in many references such as Enright [2], Zennaro [3], Vermiglio [7], Huang [4,5], Zhang [8–10], Zhao [11], Gan [12], Tian [13], Koto [6] etc.

In 2007, Yu et al. [24] investigated the stability of Runge-Kutta methods for $(1.1) \in \mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$. An improve version of the results in paper [24] refers also to [25]. Furthermore, for solving (1.1), Zhang et al.[26] extended the Pouzet-Runge-Kutta methods, Zhang and He [27] extended the one-leg methods, and they studied the stability of these extended methods. Qin et al. [28] and Wang [37] gave the results of nonlinear stability of one-leg methods for a class of functional integro-differential equations and neutral Volterra delay-integro-differential equations, respectively.

On the other hand, many dynamical systems in physics and engineering are characterized by the property of a bounded absorbing set in which all trajectories enter in a finite time and thereafter remain inside. This asymptotic behavior of the system is called "dissipativity" [29]. In the study of dynamical systems, the dissipativity of the system is often interested and it is highly desirable to have numerical methods that retain the dissipativity of the underlying system. In 2010, the dissipativity results of the theoretical solution of NDIDEs (1.1) are obtained by Wen [30]. And then, the numerical dissipativity results of Runge-Kutta methods, one-leg methods and θ -methods for solving (1.1) are given by Wen [31], Liu [32] and He [33], respectively.

As far as we know, there are a few papers dealing with the convergence of numerical methods for NDIDEs (see [34–36]). These important convergence results are based on the classical Lipschitz conditions. The studies about convergence of the numerical method for nonlinear NDIDEs based on a one-sided Lipschitz condition are proceed only by Wang [38,39]. However, up to now, the convergence results, which concern the numerical method for nonlinear NDIDEs $(1.1) \in \mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$, have not yet been seen in the literature.

In this paper, our objective is to discuss the error behavior of Runge-Kutta methods for solving initial value problems $(1.1) \in \mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$.

Remark 1.1 It is well known that the solution of DDEs (also NDIDEs) is possibly not smooth even discontinuous. In general, the derivatives of the so-

lution of (1.1) have the jump discontinuities at points $t = j\tau$ ($j = 0, 1, 2, \dots$). About the definition, existence and uniqueness of the solution of IVPs (1.1), the reader may refer to Baker and Paul's paper [40]. According to [40] point of view, we assume $y(t) - Ny(t)$ is sufficiently differentiable.

In order to make the error analysis feasible, for processing the jump discontinuous of derivatives of the solution, we refer to the technique which is given by Zhang etc. [41] and [42]. We always assume that problem (1.1) has a unique solution $y(t)$ which is smooth enough on each interval of the form $(t_0 + (j-1)\tau, t_0 + j\tau)$ (j is a positive integer) and the left limitations of its derivatives used later exist at each jump point $t_0 + j\tau$. Furthermore, we suppose that

$$\|D^{(i)}y(t)\| \leq M_i, \quad (1.5)$$

where

$$D^{(i)}y(t) = \begin{cases} y^{(i)}(t), & t \in (t_0 + (j-1)\tau, t_0 + j\tau), \\ y^{(i)}(t_0 + j\tau - 0), & t = t_0 + j\tau, \end{cases}$$

For function $g(t, \xi, y(\xi))$, we always make the assumption that all its partial derivatives used later are continuous and satisfy

$$\left\| \frac{\partial^i g(t, \xi, y(\xi))}{\partial \xi^i} \right\| \leq N_i, i = 1, 2, \dots, t \in [0, T], \xi \in [t - \tau, t], \quad (1.6)$$

where we use above same means for jump discontinuous of partial derivatives of $g(t, \xi, y(\xi))$.

Remark 1.2 In order to ease of notation we assume that T is such that $T = (M+1)\tau$ for some positive integer M . It is well know that the solution $y(t)$ of the IVP for (1.1) is $(q+1)$ -times continuously differentiable on each subinterval $(i\tau, (i+1)\tau]$, $i = 0, 1, \dots, M$, and has a bounded first derivative on $[0, T]$; At the primary discontinuity points $\{\xi_i\}$ defined by $\xi_i = i\tau$ ($i = 0, 1, \dots, \min\{q, M\}$) we have

$$\lim_{t \rightarrow \xi_i^-} y^{(i)}(t) = \lim_{t \rightarrow \xi_i^+} y^{(i)}(t),$$

while the $(i+1)$ -th derivative of y is in general not continuous at $t = \xi_i$. When we solve (1.1) with numerical methods, we let the step size h satisfy $\tau = mh$ with some positive integer m . Thus these primary discontinuity points $\{\xi_i\}$ are included in the mesh. At the primary discontinuity points, we can exploit the approach to restarting. Therefore, without loss of generality, we always assume that the $y(t)$ is sufficiently differentiable and satisfies (1.5).

This paper is organized as follows. In Section 2, the descriptions of the

numerical methods is given. In Sections 3, the convergence results of Runge-Kutta methods are established. In section 4, some numerical examples are given to confirm our theoretical results.

2 Runge-Kutta methods for NDIDEs

An s -stage Runge-Kutta method (A, b, c) for ordinary differential equations (ODEs) can be expressed as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array} \quad (2.1)$$

where $A = (a_{ij}) \in \mathbb{R}^{s \times s}$, $b = (b_1, b_2, \dots, b_s)^T \in \mathbb{R}^s$ and $c = (c_1, c_2, \dots, c_s)^T \in \mathbb{R}^s$. In this paper we will always assume that the method is consistent, which implies that

$$\sum_{j=1}^s b_j = 1, c_i = \sum_{j=1}^s a_{i,j} \in [0, 1], i = 1, 2, \dots, s.$$

Definition 2.1 (see [19]) A Runge-Kutta method (A, b, c) is said to be diagonal stable if there exists an $s \times s$ diagonal matrix $D = \text{diag}(d_1, d_2, \dots, d_s) > 0$ such that

$$E := DA + A^T D > 0.$$

Remark 2.2 It is easily seen that the method (2.1) is diagonal stable, then the matrix A is non-singular and there exists a constant $l > 0$ depending only on the method such that

$$E_l := A^{-T} E A^{-1} - 2lD > 0. \quad (2.2)$$

Definition 2.3 (see [20]) A Runge-Kutta method (A, b, c) is said to be algebraically stable if

$$b \geq 0, \quad M := \text{diag}(b)A + A^T \text{diag}(b) - bb^T \geq 0.$$

Definition 2.4 A Runge-Kutta method (A, b, c) is said to have stage order p if p is the largest nonnegative integer such that the following simplifying conditions ([21]) are holden,

$$B(p) : b^T c^{j-1} = \frac{1}{j}, j = 1, 2, \dots, p,$$

$$C(p) : Ac^{j-1} = \frac{1}{j}c^j, j = 1, 2, \dots, p,$$

with $c^j = (c_1^j, c_2^j, \dots, c_s^j)^T$.

In order to solve the NDIDEs (1.1), we let step size $h = \tau/m$ with an arbitrarily given positive integer m and

$$t_n = nh, \quad t_{n,j} = t_n + c_j h, j = 1, 2, \dots, s.$$

According to the idea of the paper [22], the adaptation of the method (2.1) for solving the problem (1.1) leads to

$$\begin{cases} Y_i^{(n)} - NY_i^{(n-m)} = y_n - Ny_{n-m} \\ \quad + h \sum_{j=1}^s a_{ij} f(t_{n,j}, Y_j^{(n)}, Y_j^{(n-m)}, K_j^{(n)}), \quad i = 1, 2, \dots, s, \\ y_{n+1} - Ny_{n+1-m} = y_n - Ny_{n-m} \\ \quad + h \sum_{i=1}^s b_i f(t_{n,i}, Y_i^{(n)}, Y_i^{(n-m)}, K_i^{(n)}), \end{cases} \quad (2.3)$$

where y_n and $Y_i^{(n)}$ denote approximations to $y(t_n)$ and $y(t_{n,i})$, respectively, $y_n = \varphi(t_n)$ for $n \leq 0$, $Y_i^{(n)} = \varphi(t_{n,i})$ for $t_{n,i} \leq 0$, $K_i^{(n)}$ is an approximation to $\int_{t_{n,i}-\tau}^{t_{n,i}} g(t_{n,i}, \xi, y(\xi))d\xi$, it can be computed by an appropriate quadrature rule

$$K_i^{(n)} = h \sum_{j=0}^m p_j g(t_{n,i}, t_{n-j,i}, Y_i^{(n-j)}), \quad i = 1, 2, \dots, s. \quad (2.4)$$

Remark 2.5 As for the quadrature rule (2.4), we usually adopt the repeated trapezoidal rule, the repeated Simpson's rule or other repeated Newton-Cotes rules, etc., according to the requirement of the convergence of the method (2.3). On the other hand, in order to ensure convergence of the quadrature rule (see Theorem 2.1.1 in [23]), we also assume that there exists a finite constant η , independent of m , such that

$$h \sqrt{(m+1) \sum_{j=0}^m |p_j|^2} < \eta \quad \text{with} \quad mh = \tau. \quad (2.5)$$

Remark 2.6 In this paper, we assume that the quadrature formula (2.4) is an m -panel repeated r -point rule based on a rule of polynomial degree of q . Thus we have

$$\|K(t_{n,i}) - h \sum_{j=0}^m p_j g(t_{n,i}, t_{n-j,i}, y(t_{n-j,i}))\| \leq c_q h^{q+1}, \quad i = 1, 2, \dots, s. \quad (2.6)$$

where c_q depends only on the quadrature rule, τ and N_{q+1} .

Remark 2.7 For any given $\varphi \in \mathbf{C}_m[-\tau, 0]$, $y_1, y_2, \dots, y_n \in \mathbf{R}^m$ and $h > 0$ sufficiently small, we can in general compute y_{n+1} from the equation (2.3)-(2.4) and thus complete a integration step $(\varphi, y_1, y_2, \dots, y_n) \rightarrow (\varphi, y_1, y_2, \dots, y_{n+1})$.

3 Convergence analysis of Runge-Kutta methods

In this section, we focus on the error analysis of algebraically stable Runge-Kutta methods for the NDIDEs (1.1).

For any $n \geq 0$, let $R_i^{(n)}$, $i = 1, \dots, s$ and $R_0^{(n)}$ be defined by

$$\begin{cases} R_i^{(n)} = y(t_{n,i}) - Ny(t_{n-m,i}) - (y(t_n) - Ny(t_{n-m})) \\ \quad - h \sum_{j=1}^s a_{ij} f(t_{n,j}, y(t_{n,j}), y(t_{n-m,j}), K(t_{n,j})), \quad i = 1, 2, \dots, s, \\ R_0^{(n)} = y(t_{n+1}) - Ny(t_{n+1-m}) - (y(t_n) - Ny(t_{n-m})) \\ \quad - h \sum_{i=1}^s b_i f(t_{n,i}, y(t_{n,i}), y(t_{n-m,i}), K(t_{n,i})). \end{cases} \quad (3.1)$$

Let $R^{(n)} = (R_1^{(n)T}, R_2^{(n)T}, \dots, R_s^{(n)T})^T$. Using the Taylor expansion and conditions $B(p)$ and $C(p)$, it is easy to prove that there exist constants $\tilde{d}_1$ and $\tilde{d}_0$, which depend only on the method and bounds M_i for certain derivatives of the true solution $y(t)$, such that

$$\|R^{(n)}\| \leq \tilde{d}_1 h^{p+1}, \quad \|R_0\| \leq \tilde{d}_0 h^{p+1}. \quad (3.2)$$

Now we consider a fictitious integral step $(t_n, \varphi(t), y(t_1), y(t_2), \dots, y(t_n)) \rightarrow (t_{n+1}, \varphi(t), y(t_1), y(t_2), \dots, y(t_n), z_{n+1})$:

$$\begin{cases} Z_i^{(n)} - Ny(t_{n-m,i}) = y(t_n) - Ny(t_{n-m}) \\ \quad + h \sum_{j=1}^s a_{ij} f(t_{n,j}, Z_j^{(n)}, y(t_{n-m,j}), \tilde{K}_j^{(n)}), \quad i = 1, 2, \dots, s, \\ z_{n+1} - Ny(t_{n+1-m}) = y(t_n) - Ny(t_{n-m}) \\ \quad + h \sum_{i=1}^s b_i f(t_{n,i}, Z_i^{(n)}, y(t_{n-m,i}), \tilde{K}_i^{(n)}), \end{cases} \quad (3.3)$$

where

$$\tilde{K}_i^{(n)} = h \sum_{j=1}^m p_j g(t_{n,i}, t_{n-j,i}, y(t_{n-j,i})) + hp_0 g(t_{n,i}, t_{n,i}, Z_i^{(n)}), \quad i = 1, 2, \dots, s. \quad (3.4)$$

Let

$$\Delta K_i^{(n)} = K(t_{n,i}) - h \sum_{j=0}^m p_j g(t_{n,i}, t_{n-j,i}, y(t_{n-j,i})), \quad i = 1, 2, \dots, s,$$

and

$$\Delta K^{(n)} = (\Delta K_1^{(n)T}, \Delta K_2^{(n)T}, \dots, \Delta K_s^{(n)T})^T.$$

Then from (3.4) and (1.4) we have

$$\|K(t_{n,i}) - \tilde{K}_i^{(n)}\| \leq \|\Delta K_i^{(n)}\| + h|p_0|\gamma\|y(t_{n,i}) - Z_i^{(n)}\|,$$

or further

$$\|K(t_{n,i}) - \tilde{K}_i^{(n)}\|^2 \leq 2\|\Delta K_i^{(n)}\|^2 + 2h^2 p_0^2 \gamma^2 \|y(t_{n,i}) - Z_i^{(n)}\|^2. \quad (3.5)$$

In order to derive global error bounds of method (2.3) with (2.4), we will divide our work in four steps.

3.1 Local error estimate

First, we establish a local error estimate and we have following conclusion.

Theorem 3.1 Assume that a diagonally stable Runge-Kutta method (A, b, c) is used to solve the problem which belongs to the class $\mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$. Then there exist constants $\check{c}_1, \check{c}_2, h_1$, which depend only on the method (including the quadrature rule (2.4)), α_1, β and γ such that

$$\begin{aligned} \|y(t_{n+1}) - z_{n+1}\| &\leq \check{c}_1(\|R^{(n)}\| + \|R_0^{(n)}\|) \\ &\quad + \check{c}_2 \sqrt{h} \|\Delta K^{(n)}\|, \quad h \in (0, h_1], \end{aligned} \quad (3.6)$$

where the meaning of each quantity are given by the above.

Proof It follows from (3.1) and (3.3) that

$$\begin{cases} y(t_{n,i}) - Z_i^{(n)} = \sum_{j=1}^s a_{ij} Q_j^{(n)} + R_i^{(n)}, \quad i = 1, 2, \dots, s \\ y(t_{n+1}) - z_{n+1} = \sum_{j=1}^s b_j Q_j^{(n)} + R_0^{(n)}, \end{cases} \quad (3.7)$$

where

$$Q_j^{(n)} = h(f(t_{n,j}, y(t_{n,j}), y(t_{n-m,j}), K(t_{n,j})) - f(t_{n,j}, Z_j^{(n)}, y(t_{n-m,j}), \tilde{K}_j^{(n)})).$$

For simplicity, we denote

$$\begin{cases} \hat{Y}^{(n)} = (y(t_{n,1})^T, y(t_{n,2})^T, \dots, y(t_{n,s})^T)^T \\ Z^{(n)} = (Z_1^{(n)T}, Z_2^{(n)T}, \dots, Z_s^{(n)T})^T, \\ W^{(n)} = (W_1^{(n)T}, W_2^{(n)T}, \dots, W_s^{(n)T})^T = \hat{Y}^{(n)} - Z^{(n)}, \\ Q^{(n)} = (Q_1^{(n)T}, Q_2^{(n)T}, \dots, Q_s^{(n)T})^T. \end{cases}$$

Then (3.7) can be written in the more compact form

$$\begin{cases} W^{(n)} = AQ^{(n)} + R^{(n)}, \\ y(t_{n+1}) - z_{n+1} = b^T Q^{(n)} + R_0^{(n)}. \end{cases} \quad (3.8)$$

Using the condition (1.3) and inequality (3.5) we can obtain that

$$\langle W^{(n)}, DQ^{(n)} \rangle \leq h\alpha_1 \|W^{(n)}\|_D^2 + 2h\beta(\|\Delta K^{(n)}\|_D^2 + h^2 p_0^2 \gamma^2 \|W^{(n)}\|_D^2). \quad (3.9)$$

On the other hand,

$$\begin{aligned} \langle W^{(n)}, A^{-T}EA^{-1}W^{(n)} \rangle &= \langle W^{(n)}, A^{-T}(DA + A^T D)A^{-1}W^{(n)} \rangle \\ &= 2\langle DW^{(n)}, Q^{(n)} + A^{-1}R^{(n)} \rangle \\ &= 2\langle W^{(n)}, DQ^{(n)} \rangle + 2\langle W^{(n)}, DA^{-1}R^{(n)} \rangle, \end{aligned} \quad (3.10)$$

where matrix E is introduced in Definition 2.1. Therefore, from (3.9), (3.10) and (2.2) we have

$$\begin{aligned} \langle W^{(n)}, E_l W^{(n)} \rangle &= 2\langle W^{(n)}, DQ^{(n)} \rangle + 2\langle W^{(n)}, DA^{-1}R^{(n)} \rangle - 2l\langle W^{(n)}, DW^{(n)} \rangle \\ &\leq 2(h(\alpha_1 + 2\beta p_0^2 \gamma^2) - l) \|W^{(n)}\|_D^2 + 4h\beta d_0 \|\Delta K^{(n)}\|^2 \\ &\quad + \frac{\lambda_{min}^{E_l}}{2} \|W^{(n)}\|^2 + \frac{2d_0^2 \|A^{-1}\|^2}{\lambda_{min}^{E_l}} \|R^{(n)}\|^2. \end{aligned} \quad (3.11)$$

Here we assume $h \leq 1$ and put $d_0 = \max\{d_1, d_2, \dots, d_s\}$. When

$$h(\alpha_1 + 2\beta p_0^2 \gamma^2) \leq l,$$

it is clear that

$$\langle W^{(n)}, E_l W^{(n)} \rangle \leq 4hd_0\beta \|\Delta K^{(n)}\|^2 + \frac{2d_0^2 \|A^{-1}\|^2}{\lambda_{min}^{E_l}} \|R^{(n)}\|^2 + \frac{\lambda_{min}^{E_l}}{2} \|W^{(n)}\|^2,$$

which gives

$$\|W^{(n)}\|^2 \leq \frac{2}{\lambda_{min}^{E_l}} \left(4hd_0\beta \|\Delta K^{(n)}\|^2 + \frac{2d_0^2 \|A^{-1}\|^2}{\lambda_{min}^{E_l}} \|R^{(n)}\|^2 \right). \quad (3.12)$$

Thus we further obtain

$$\|W^{(n)}\| \leq 2\sqrt{h}\sqrt{\frac{2\beta d_0}{\lambda_{\min}^{E_l}}}\|\Delta K^{(n)}\| + \frac{2d_0\|A^{-1}\|}{\lambda_{\min}^{E_l}}\|R^{(n)}\|.$$

From the second form of (3.7), it follows immediately that

$$\begin{aligned}\|y(t_{n+1}) - z_{n+1}\| &= \|b^T Q^{(n)} + R_0^{(n)}\| = \|b^T A^{-1}(W^{(n)} - R^{(n)}) + R_0^{(n)}\| \\ &\leq \|b^T A^{-1}\| (\|W^{(n)}\| + \|R^{(n)}\|) + \|R_0^{(n)}\| \\ &\leq \check{c}_1(\|R^{(n)}\| + \|R_0^{(n)}\|) + \check{c}_2\sqrt{h}\|\Delta K^{(n)}\|, h \in (0, h_1],\end{aligned}$$

where

$$\check{c}_1 = \max\{1, \|b^T A^{-1}\| (\frac{2d_0\|A^{-1}\|}{\lambda_{\min}^{E_l}} + 1)\}, \check{c}_2 = 2\sqrt{\frac{2\beta d_0}{\lambda_{\min}^{E_l}}}\|b^T A^{-1}\|,$$

$$h_1 = \begin{cases} 1, & \alpha_1 + 2\beta p_0^2 \gamma^2 \leq 0, \\ \min\{1, \frac{l}{\alpha_1 + 2\beta p_0^2 \gamma^2}\}, & \alpha_1 + 2\beta p_0^2 \gamma^2 > 0. \end{cases}$$

This completes the proof of Theorem 3.1. $\square$

3.2 Local error estimate with respect to inner stage

The next thing is to estimate $Y_i^{(n)} - Z_i^{(n)}$ given by (2.3) and (3.3). We first observe (2.3) and (3.3). For simplicity, we write

$$\begin{cases} U_i^{(n)} = Z_i^{(n)} - Y_i^{(n)}, & \Omega_i^{(n)} = y(t_{n,i}) - Y_i^{(n)} \\ \hat{Q}_i^{(n)} = h \left(f(t_{n,i}, Z_i^{(n)}, y(t_{n-m,i}), \tilde{K}_i^{(n)}) - f(t_{n,i}, Y_i^{(n)}, Y_i^{(n-m)}, K_i^{(n)}) \right), \\ w_n = z_n - y_n, & \varepsilon_n = y(t_n) - y_n, \end{cases}$$

where $i = 1, 2, \dots, s$. A combination of (2.3) and (3.3) leads to

$$\begin{cases} U_i^{(n)} - N\Omega_i^{(n-m)} = \varepsilon_n - N\varepsilon_{n-m} + \sum_{j=1}^s a_{ij}\hat{Q}_j^{(n)}, i = 1, 2, \dots, s \\ w_{n+1} - N\varepsilon_{n-m+1} = \varepsilon_n - N\varepsilon_{n-m} + \sum_{j=1}^s b_j\hat{Q}_j^{(n)}. \end{cases} \quad (3.13)$$

We further denote

$$\begin{cases} U^{(n)} = (U_1^{(n)T}, U_2^{(n)T}, \dots, U_s^{(n)T})^T, \Omega^{(n)} = (\Omega_1^{(n)T}, \Omega_2^{(n)T}, \dots, \Omega_s^{(n)T})^T, \\ \hat{Q}^{(n)} = (\hat{Q}_1^{(n)T}, \hat{Q}_2^{(n)T}, \dots, \hat{Q}_s^{(n)T})^T, K^{(n)} = (K_1^{(n)T}, K_2^{(n)T}, \dots, K_s^{(n)T})^T, \\ \tilde{K}^{(n)} = (\tilde{K}_1^{(n)T}, \tilde{K}_2^{(n)T}, \dots, \tilde{K}_s^{(n)T})^T, \Delta\tilde{K}^{(n)} = \tilde{K}^{(n)} - K^{(n)}. \end{cases}$$

Then (3.13) can be written as follows,

$$\begin{cases} U^{(n)} - N\Omega^{(n-m)} = e(\varepsilon_n - N\varepsilon_{n-m}) + A\hat{Q}^{(n)}, \\ w_{n+1} - N\varepsilon_{n-m+1} = \varepsilon_n - N\varepsilon_{n-m} + b^T\hat{Q}^{(n)}. \end{cases} \quad (3.14)$$

Here and later, the notation $N\Omega^{(k)}$ would be understood as $(I \otimes N)\Omega^{(k)}$, where $I \otimes N$ means the Kronecker product of I and N (cf. p.93 in [17]).

According to (3.14), we can obtain following theorem.

Theorem 3.2 Assume that a diagonally stable Runge-Kutta method (A, b, c) is used to solve the problem which belongs to the class $\mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$. Then there exist constants $\check{c}_3, \check{c}_4, \check{c}_5, h_2$, which depend only on the method (including the quadrature rule (2.4)), α_1, β and γ such that

$$\|U^{(n)}\|^2 \leq \check{c}_3 \max_{0 \leq q \leq m} \|\Omega^{(n-q)}\|^2 + \check{c}_4 \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + h^3 \check{c}_5 \|W^{(n)}\|^2, h \in (0, h_2]. \quad (3.15)$$

Proof Using the condition (1.3), similar to (3.9) we can obtain that

$$\begin{aligned} \langle U^{(n)} - N\Omega^{(n-m)}, \hat{Q}^{(n)} \rangle_D &\leq h\alpha_1 \|U^{(n)}\|_D^2 + h\alpha_2 \|\Omega^{(n-m)}\|_D^2 \\ &\quad + h\beta \|\Delta\tilde{K}^{(n)}\|_D^2 \end{aligned} \quad (3.16)$$

On the other hand, from (2.4) and (3.4), by applying the conditions (1.4), (2.5) and Hölder inequality, we have

$$\begin{aligned} \|\Delta\tilde{K}_i^{(n)}\| &= h \left\| \sum_{j=0}^m p_j (g(t_{n,i}, t_{n-j,i}, y(t_{n-j,i})) - g(t_{n,i}, t_{n-j,i}, Y_i^{(n-j)})) \right\| \\ &\quad + h|p_0| \|g(t_{n,i}, t_{n,i}, Z_i^{(n)}) - g(t_{n,i}, t_{n,i}, y(t_{n,i}))\| \\ &\leq h\gamma \sum_{j=0}^m |p_j| \|\Omega_i^{(n-j)}\| + h\gamma|p_0| \|W_i^{(n)}\| \\ &\leq h\gamma \sqrt{\sum_{j=0}^m |p_j|^2} \sqrt{\sum_{j=0}^m \|\Omega_i^{(n-j)}\|^2} + h\gamma|p_0| \|W_i^{(n)}\| \\ &\leq \frac{\gamma\eta}{\sqrt{m+1}} \sqrt{\sum_{j=0}^m \|\Omega_i^{(n-j)}\|^2} + h\gamma|p_0| \|W_i^{(n)}\|. \end{aligned}$$

Therefore we have

$$\begin{aligned} \|\Delta\tilde{K}^{(n)}\|_D^2 &\leq \frac{2(\gamma\eta)^2}{m+1} \sum_{j=0}^m \|\Omega^{(n-j)}\|_D^2 + 2h^2\gamma^2|p_0|^2 \|W^{(n)}\|_D^2 \\ &\leq 2d_0(\gamma\eta)^2 \max_{0 \leq j \leq m} \|\Omega^{(n-j)}\|^2 + 2h^2d_0\gamma^2|p_0|^2 \|W^{(n)}\|^2. \end{aligned} \quad (3.17)$$

An argument similar to the one used in (3.11) and a combination of (3.16)

and (3.17) give that

$$\begin{aligned}
& \langle U^{(n)} - N\Omega^{(n-m)}, E_l(U^{(n)} - N\Omega^{(n-m)}) \rangle \\
& \leq 2h\alpha_1 \|U^{(n)}\|_D^2 + 2h\alpha_2 \|\Omega^{(n-m)}\|_D^2 + 2h\beta \|\Delta \tilde{K}^{(n)}\|_D^2 \\
& + 2\|U^{(n)} - N\Omega^{(n-m)}\|_D \|A^{-1}e(\varepsilon_n - N\varepsilon_{n-m})\|_D \\
& - 2l\|U^{(n)} - N\Omega^{(n-m)}\|_D^2 \\
& \leq 2h\alpha_1 \|U^{(n)}\|_D^2 + 2h\alpha_2 \|\Omega^{(n-m)}\|_D^2 + 2h\beta \|\Delta \tilde{K}^{(n)}\|_D^2 \\
& + \frac{1}{2l} \|A^{-1}e(\varepsilon_n - N\varepsilon_{n-m})\|_D^2,
\end{aligned}$$

which gives

$$\begin{aligned}
\|U^{(n)} - N\Omega^{(n-m)}\|^2 & \leq \frac{2h}{\lambda_{min}^{E_l}} (\alpha_1 \|U^{(n)}\|_D^2 + \alpha_2 \|\Omega^{(n-m)}\|_D^2 + 2d_0\beta(\gamma\eta)^2 \max_{0 \leq q \leq m} \|\Omega^{(n-q)}\|^2) \\
& + \frac{sd_0 \|A^{-1}\|^2}{2l\lambda_{min}^{E_l}} \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + \frac{2h}{\lambda_{min}^{E_l}} 2h^2 d_0 \gamma^2 |p_0|^2 \|W^{(n)}\|^2.
\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}
\|U^{(n)}\|_D^2 & \leq d_0 \|U^{(n)}\|^2 \leq 2d_0 \|U^{(n)} - N\Omega^{(n-m)}\|^2 + 2d_0 \|N\Omega^{(n-m)}\|^2 \\
& \leq \frac{4d_0 h}{\lambda_{min}^{E_l}} (\alpha_1 \|U^{(n)}\|_D^2 + \alpha_2 \|\Omega^{(n-m)}\|_D^2 + 2d_0\beta(\gamma\eta)^2 \max_{0 \leq q \leq m} \|\Omega^{(n-q)}\|^2) \\
& + \frac{sd_0^2 \|A^{-1}\|^2}{l\lambda_{min}^{E_l}} \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + 2d_0 \|N\|^2 \|\Omega^{(n-m)}\|^2 + 2d_0 \frac{2h}{\lambda_{min}^{E_l}} 2h^2 d_0 \gamma^2 |p_0|^2 \|W^{(n)}\|^2
\end{aligned}$$

For any given constant $c_0 \in (0, 1)$, when $\frac{4d_0\alpha_1}{\lambda_{min}^{E_l}} h \leq c_0$, we further obtain

$$\|U^{(n)}\|^2 \leq \check{c}_3 \max_{0 \leq q \leq m} \|\Omega^{(n-q)}\|^2 + \check{c}_4 \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + h^3 \check{c}_5 \|W^{(n)}\|^2, \quad h \in (0, h_2],$$

where

$$\begin{aligned}
\check{c}_3 &= \frac{1}{\tilde{d}_0(1-c_0)} \left(\frac{4d_0^2 h}{\lambda_{min}^{E_l}} (\alpha_2^+ + 2\beta(\gamma\eta)^2) + 2d_0 \|N\|^2 \right), \\
\check{c}_4 &= \frac{1}{\tilde{d}_0(1-c_0)} \frac{sd_0^2 \|A^{-1}\|^2}{l\lambda_{min}^{E_l}}, \quad \check{c}_5 = \frac{1}{\tilde{d}_0(1-c_0)} \frac{8d_0^2 \gamma^2 |p_0|^2}{\lambda_{min}^{E_l}}, \\
h_2 &= \begin{cases} 1, & \alpha_1 \leq 0, \\ \min\{1, \frac{\lambda_{min}^{E_l} c_0}{4d_0 \alpha_1}\}, & \alpha_1 > 0, \end{cases} \\
\tilde{d}_0 &= \min\{d_1, d_2, \dots, d_s\}, \quad \alpha_2^+ = \max\{0, \alpha_2\}.
\end{aligned}$$

This completes the proof of Theorem 3.2. $\square$

3.3 Global error estimate with respect to inner stage

Another step in our work is to estimate the term $\|\Omega^{(n)}\|$. For this purpose, we consider a combination of (2.3) and (3.1):

$$\begin{cases} \Omega^{(n)} - N\Omega^{(n-m)} = e(\varepsilon_n - N\varepsilon_{n-m}) + A\bar{Q}^{(n)} + R^{(n)}, \\ \varepsilon_{n+1} - N\varepsilon_{n-m+1} = \varepsilon_n - N\varepsilon_{n-m} + b^T\bar{Q}^{(n)} + R_0^{(n)}, \end{cases} \quad (3.19)$$

where

$$\bar{Q}_j^{(n)} = f(t_{n,j}, y(t_{n,j}), y(t_{n-m,j}), K(t_{n,j})) - f(t_{n,j}, Y_j^{(n)}, Y_j^{(n-m)}, K_j^{(n)}),$$

and

$$\bar{Q}^{(n)} = (\bar{Q}_1^{(n)T}, \bar{Q}_2^{(n)T}, \dots, \bar{Q}_s^{(n)T})^T, \quad \Delta\bar{K}^{(n)} = \bar{K}^{(n)} - K^{(n)}.$$

$$\bar{K}^{(n)} = (K(t_{n,1})^T, K(t_{n,2})^T, \dots, K(t_{n,s})^T)^T, \quad K^{(n)} = (K_1^{(n)T}, K_2^{(n)T}, \dots, K_s^{(n)T})^T.$$

Similar manipulations to Theorem 3.2 lead to following conclusion.

Theorem 3.3 Assume that a diagonally stable Runge-Kutta method (A, b, c) is used to solve the problem which belongs to the class $\mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$. Then there exist constants $\check{c}_6, \check{c}_7, \check{c}_8, h_3$, which depend only on the method (including the quadrature rule (2.4)), $\alpha_1, \alpha_2, \beta$ and γ such that

$$\begin{aligned} \max_{0 \leq k \leq n} \|\Omega^{(k)}\| &\leq \check{c}_6 \sqrt{h} \max_{0 \leq i \leq n} \|\Delta K^{(i)}\| + \check{c}_7 \max_{0 \leq k \leq n} \|R^{(k)}\| \\ &\quad + \check{c}_8 \max_{0 \leq k \leq n} \|\varepsilon_k - N\varepsilon_{k-m}\|, \quad h \in (0, h_3], \end{aligned} \quad (3.20)$$

Proof We are now almost ready to invoke the derivation of (3.15) as follows.

First, similar to (3.16) we have

$$\langle \Omega^{(n)} - N\Omega^{(n-m)}, \bar{Q}^{(n)} \rangle_D \leq \alpha_1 h \|\Omega^{(n)}\|_D^2 + \alpha_2 h \|\Omega^{(n-m)}\|_D^2 + h\beta \|\Delta\bar{K}^{(n)}\|_D^2.$$

Secondly, in contrast to (3.17) we obtain

$$\begin{aligned} \|\bar{K}_i^{(n)} - K_i^{(n)}\| &= \|K(t_{n,i}) - h \sum_{j=0}^m p_j g(t_{n,i}, t_{n-j,i}, Y_i^{(n-j)})\| \\ &\leq \|K(t_{n,i}) - h \sum_{j=0}^m p_j g(t_{n,i}, t_{n-j,i}, y(t_{n-j,i}))\| \\ &\quad + \|h \sum_{j=0}^m p_j (g(t_{n,i}, t_{n-j,i}, y(t_{n-j,i})) - g(t_{n,i}, t_{n-j,i}, Y_i^{(n-j)}))\| \\ &\leq \|\Delta K_i^{(n)}\| + h\gamma \sum_{j=0}^m |p_j| \|\Omega_i^{(n-j)}\|. \end{aligned}$$

Then we further acquire

$$\|\Delta \bar{K}^{(n)}\|_D^2 \leq 2\|\Delta K^{(n)}\|_D^2 + 2(\gamma\eta)^2 \max_{0 \leq j \leq m} \|\Omega^{(n-j)}\|_D^2.$$

Therefore, we have following estimation (in contrast to (3.18)).

$$\begin{aligned} & \langle \Omega^{(n)} - N\Omega^{(n-m)}, E_l(\Omega^{(n)} - N\Omega^{(n-m)}) \rangle \\ & \leq 2h\alpha_1 \|\Omega^{(n)}\|_D^2 + 2h\alpha_2 \|\Omega^{(n-m)}\|_D^2 + 2h\beta \|\bar{K}^{(n)} - K^{(n)}\|_D^2 \\ & \quad + 2\|\Omega^{(n)} - N\Omega^{(n-m)}\|_D \|A^{-1}e(\varepsilon_n - N\varepsilon_{n-m}) + A^{-1}R^{(n)}\|_D \\ & \quad - 2l\|\Omega^{(n)} - N\Omega^{(n-m)}\|_D^2 \\ & \leq 2h\alpha_1 \|\Omega^{(n)}\|_D^2 + 2h\alpha_2 \|\Omega^{(n-m)}\|_D^2 + 2h\beta \|\bar{K}^{(n)} - K^{(n)}\|_D^2 \\ & \quad + \frac{1}{2l} \|A^{-1}e(\varepsilon_n - N\varepsilon_{n-m}) + A^{-1}R^{(n)}\|_D^2, \end{aligned}$$

which leads to

$$\begin{aligned} \|\Omega^{(n)} - N\Omega^{(n-m)}\|^2 & \leq \frac{2h}{\lambda_{min}^{E_l}} (\alpha_1 \|\Omega^{(n)}\|_D^2 + \alpha_2 \|\Omega^{(n-m)}\|_D^2 + 2\beta(\gamma\eta)^2 \max_{0 \leq q \leq m} \|\Omega^{(n-q)}\|_D^2) \\ & \quad + \frac{sd_0 \|A^{-1}\|^2}{l\lambda_{min}^{E_l}} \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + \frac{d_0 \|A^{-1}\|^2}{l\lambda_{min}^{E_l}} \|R^{(n)}\|^2 + \frac{4d_0 h\beta}{\lambda_{min}^{E_l}} \|\Delta K^{(n)}\|^2. \end{aligned}$$

Then

$$\begin{aligned} \|\Omega^{(n)} - N\Omega^{(n-m)}\| & \leq \sqrt{h}\tilde{c}_1 \max_{0 \leq j \leq m} \|\Omega^{(n-j)}\| + \sqrt{h}\tilde{c}_2 \|\Delta K^{(n)}\| \\ & \quad + \tilde{c}_3 \|\varepsilon_n - N\varepsilon_{n-m}\| + \tilde{c}_4 \|R^{(n)}\|, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \tilde{c}_1 & = \sqrt{\frac{2d_0}{\lambda_{min}^{E_l}} (\alpha_1^+ + \alpha_2^+ + 2\beta(\gamma\eta)^2)}, \quad \tilde{c}_2 = 2\sqrt{d_0\beta/\lambda_{min}^{E_l}}, \\ \tilde{c}_3 & = \|A^{-1}\| \sqrt{\frac{sd_0}{l\lambda_{min}^{E_l}}}, \quad \tilde{c}_4 = \|A^{-1}\| \sqrt{\frac{d_0}{l\lambda_{min}^{E_l}}}, \quad \alpha_1^+ = \max\{0, \alpha_1\}. \end{aligned}$$

Our task now is to establish the estimation of $\Omega^{(n)}$.

In view of (3.21), we have

$$\begin{aligned} \|\Omega^{(n)}\| & \leq \|\Omega^{(n)} - N\Omega^{(n-m)}\| + \|N\| \|\Omega^{(n-m)} - N\Omega^{(n-2m)}\| \\ & \quad + \|N\|^2 \|\Omega^{(n-2m)} - N\Omega^{(n-3m)}\| + \dots + \|N\|^k \|\Omega^{(n-km)} - N\Omega^{(n-k(m+1))}\| \\ & \leq \frac{\tilde{c}_1 \sqrt{h}}{1-\|N\|} \max_{0 \leq i \leq n} \|\Omega^{(i)}\| + \frac{\tilde{c}_2 \sqrt{h}}{1-\|N\|} \max_{0 \leq i \leq n} \|\Delta K^{(i)}\| \\ & \quad + \frac{\tilde{c}_3}{1-\|N\|} \max_{0 \leq i \leq n} \|\varepsilon_i - N\varepsilon_{i-m}\| + \frac{\tilde{c}_4}{1-\|N\|} \max_{0 \leq i \leq n} \|R^{(i)}\|, \end{aligned}$$

where $k = \lfloor \frac{n}{m} \rfloor$. Therefore,

$$\begin{aligned} \max_{0 \leq i \leq n} \|\Omega^{(i)}\| &\leq \frac{\tilde{c}_1 \sqrt{h}}{1 - \|N\|} \max_{0 \leq i \leq n} \|\Omega^{(i)}\| + \frac{\tilde{c}_2 \sqrt{h}}{1 - \|N\|} \max_{0 \leq i \leq n} \|\Delta K^{(i)}\| \\ &\quad + \frac{\tilde{c}_3}{1 - \|N\|} \max_{0 \leq i \leq n} \|\varepsilon_i - N\varepsilon_{i-m}\| + \frac{\tilde{c}_4}{1 - \|N\|} \max_{0 \leq i \leq n} \|R^{(i)}\|. \end{aligned} \quad (3.22)$$

For the constant c_0 given by above, when $\frac{\tilde{c}_1 \sqrt{h}}{1 - \|N\|} \leq c_0$, (3.22) gives

$$\begin{aligned} \max_{0 \leq k \leq n} \|\Omega^{(k)}\| &\leq \check{c}_6 \sqrt{h} \max_{0 \leq i \leq n} \|\Delta K^{(i)}\| + \check{c}_7 \max_{0 \leq k \leq n} \|R^{(k)}\| \\ &\quad + \check{c}_8 \max_{0 \leq k \leq n} \|\varepsilon_k - N\varepsilon_{k-m}\|, \quad h \in (0, h_3], \end{aligned}$$

where

$$\check{c}_6 = \frac{\tilde{c}_2}{(1 - c_0)(1 - \|N\|)}, \quad \check{c}_7 = \frac{\tilde{c}_4}{(1 - c_0)(1 - \|N\|)}, \quad \check{c}_8 = \frac{\tilde{c}_3}{(1 - c_0)(1 - \|N\|)}, \quad h_3 = \left(\frac{c_0(1 - \|N\|)}{c_1}\right)^2.$$

This completes the proof of Theorem 3.3. $\square$

3.4 Global error estimate

Finally, we have to derive global error bounds of methods (2.3) for problem (1.1) and we can give following result.

Theorem 3.4 Suppose that the Runge-Kutta method (A, b, c) is algebraically stable and diagonally stable, it has stage order p , the quadrature rule (2.4) satisfies (2.6) with certain positive integer q . Then the method (2.3)-(2.4) applied to problem (1.1) $\in \mathbf{R}(\alpha_1, \alpha_2, \beta, \gamma)$ is convergent of order at least $\min\{p, q + \frac{1}{2}\}$ and the global error satisfies the following inequality:

$$\|y(t_n) - y_n\| \leq C(t_n) h^{\min\{p, q + \frac{1}{2}\}}, \quad h \in (0, h_0], \quad (3.23)$$

where the function $C(t)$ and the maximum stepsize h_0 depend only on the method, some of the bounds M_i, N_i , the parameters $\alpha_1, \alpha_2, \beta, \gamma$.

Proof From (3.14) and in view of algebraically stability of the method, we have (cf. [18])

$$\|w_{n+1} - N\varepsilon_{n-m+1}\|^2 \leq \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + 2\langle U^{(n)} - N\Omega^{(n-m)}, \hat{Q}^{(n)} \rangle_B, \quad (3.24)$$

where $B = \text{diag}(b_1, b_2, \dots, b_s)$ is nonnegative definite.

Similar to (3.16) and (3.17) we can obtain

$$\begin{aligned}
\langle U^{(n)} - N\Omega^{(n-m)}, \hat{Q}^{(n)} \rangle_B &\leq \alpha_1 h \|U^{(n)}\|_B^2 + \alpha_2 h \|\Omega^{(n-m)}\|_B^2 + h\beta \|\Delta \tilde{K}^{(n)}\|_B^2 \\
&\leq \alpha_1 h \|U^{(n)}\|_B^2 + \alpha_2 h \|\Omega^{(n-m)}\|_B^2 \\
&\quad + 2h\beta((\gamma\eta)^2 \max_{0 \leq j \leq m} \|\Omega^{(n-j)}\|_B^2 + h^2 b_0 \gamma^2 |p_0|^2 \|W^{(n)}\|^2),
\end{aligned} \tag{3.25}$$

where $b_0 = \max\{b_1, b_2, \dots, b_s\}$.

We now turn to the estimation of $\|w_{n+1} - N\varepsilon_{n-m+1}\|$ from (3.24). Substituting (3.25) and (3.15) into (3.24), we have

$$\begin{aligned}
\|w_{n+1} - N\varepsilon_{n-m+1}\|^2 &\leq \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + 2\alpha_2 h \|\Omega^{(n-m)}\|_B^2 \\
&\quad + 2h\alpha_1^+ b_0 (\check{c}_3 \max_{0 \leq j \leq m} \|\Omega^{(n-j)}\|^2 + \check{c}_4 \|\varepsilon_n - N\varepsilon_{n-m}\|^2 + h^3 \check{c}_5 \|W^{(n)}\|^2) \\
&\quad + 2h\beta b_0 ((\gamma\eta)^2 \max_{0 \leq j \leq m} \|\Omega^{(n-j)}\|^2 + h^2 \gamma^2 |p_0|^2 \|W^{(n)}\|^2) \\
&\leq (1 + 2h\alpha_1^+ b_0 \check{c}_4) \|\varepsilon_n - N\varepsilon_{n-m}\|^2 \\
&\quad + 2hb_0 (\alpha_2^+ + \alpha_1^+ \check{c}_3 + \beta(\gamma\eta)^2) \max_{0 \leq j \leq m} \|\Omega^{(n-j)}\|^2 \\
&\quad + 2hb_0 (\alpha_1^+ h^3 \check{c}_5 + h^2 \beta \gamma^2 |p_0|^2) \|W^{(n)}\|^2, \quad h \in (0, h_2].
\end{aligned}$$

Substituting the previous formula with (3.12) and (3.20) leads to

$$\begin{aligned}
\|w_{n+1} - N\varepsilon_{n-m+1}\|^2 &\leq (1 + h\hat{d}_1) \max_{0 \leq k \leq n} \|\varepsilon_k - N\varepsilon_{k-m}\|^2 + \hat{d}_2 h^2 \max_{0 \leq k \leq n} \|\Delta K^{(k)}\|^2 \\
&\quad + \hat{d}_3 h \max_{0 \leq k \leq n} \|R^{(k)}\|^2, \quad h \in (0, h_4],
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
\hat{d}_1 &= 2b_0(\alpha_1^+ \check{c}_4 + 3(\alpha_2^+ + \alpha_1^+ \check{c}_3 + \beta(\gamma\eta)^2) \check{c}_8^2), \\
\hat{d}_2 &= 6b_0(\alpha_2^+ + \alpha_1^+ \check{c}_3 + \beta(\gamma\eta)^2) \check{c}_6^2 + 16b_0(\alpha_1^+ h^3 \check{c}_5 + h^2 \beta \gamma^2 |p_0|^2) \frac{d_0 \beta}{\lambda_{min}^{E_l}}, \\
\hat{d}_3 &= 6b_0 \check{c}_7^2 (\alpha_2^+ + \alpha_1^+ \check{c}_3 + \beta(\gamma\eta)^2) + 2b_0(\alpha_1^+ h^3 \check{c}_5 + h^2 \beta \gamma^2 |p_0|^2) \left(\frac{2d_0 \|A^{-1}\|}{\lambda_{min}^{E_l}} \right)^2, \\
h_4 &= \min\{h_2, h_3\}.
\end{aligned}$$

On the other hand,

$$\|\varepsilon_{n+1} - N\varepsilon_{n-m+1}\| \leq \|y(t_{n+1}) - z_{n+1}\| + \|w_{n+1} - N\varepsilon_{n-m+1}\|.$$

Hence

$$\begin{aligned}
\|\varepsilon_{n+1} - N\varepsilon_{n-m+1}\|^2 &\leq (1 + h)h^{-1} \|y(t_{n+1}) - z_{n+1}\|^2 \\
&\quad + (1 + h) \|w_{n+1} - N\varepsilon_{n-m+1}\|^2.
\end{aligned} \tag{3.27}$$

A combination of (3.6), (3.26) and (3.27) leads to

$$\begin{aligned}
\|\varepsilon_{n+1} - N\varepsilon_{n-m+1}\|^2 &\leq (1+h)h^{-1}(\check{c}_1(\|R^{(n)}\| + \|R_0^{(n)}\|) + \check{c}_2\sqrt{h}\|\Delta K^{(n)}\|)^2 \\
&\quad + (1+h)((1+h\hat{d}_1)\max_{0\leq k\leq n}\|\varepsilon_k - N\varepsilon_{k-m}\|^2 + \hat{d}_2h^2\max_{0\leq k\leq n}\|\Delta K^{(k)}\|^2 + \hat{d}_3h\max_{0\leq k\leq n}\|R^{(k)}\|^2). \\
&\leq (1+\check{d}_1h)\max_{0\leq k\leq n}\|\varepsilon_k - N\varepsilon_{k-m}\|^2 + (1+h)h\hat{d}_3\max_{0\leq k\leq n}\|R^{(k)}\|^2 \\
&\quad + 2(1+h)h^{-1}\check{c}_1^2(\|R^{(n)}\| + \|R_0^{(n)}\|)^2 + (1+h)(2\check{c}_2^2 + \hat{d}_2h^2)\max_{0\leq k\leq n}\|\Delta K^{(k)}\|^2
\end{aligned}$$

where $\check{d}_1 = 1 + \hat{d}_1 + h\hat{d}_1$. By induction, we get

$$\begin{aligned}
\|\varepsilon_{n+1} - N\varepsilon_{n-m+1}\|^2 &\leq [\check{d}_2\max_{0\leq k\leq n}\|R^{(k)}\|^2 + \check{d}_3h^{-2}\max_{0\leq k\leq n}(\|R^{(k)}\| + \|R_0^{(k)}\|)^2 \\
&\quad + \check{d}_4h^{-1}\max_{0\leq k\leq n}\|\Delta K^{(k)}\|^2]\exp(\check{d}_1t_{n+1}), h \in [0, h_5],
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
\check{d}_2 &= \frac{(1+h_5)\hat{d}_3}{\check{d}_1}, \check{d}_3 = \frac{2(1+h_5)\check{c}_1^2}{\check{d}_1}, \check{d}_4 = \frac{(1+h_5)(2\check{c}_2^2 + \hat{d}_2h_5^2)}{\check{d}_1} \\
h_5 &= \min\{h_1, h_2, h_3\}.
\end{aligned}$$

Thus (3.28) gives

$$\begin{aligned}
\|\varepsilon_n - N\varepsilon_{n-m}\| &\leq [\sqrt{\check{d}_2}\max_{0\leq k\leq n-1}\|R^{(k)}\| + \sqrt{\check{d}_3}h^{-1}\max_{0\leq k\leq n-1}(\|R^{(k)}\| + \|R_0^{(k)}\|) \\
&\quad + \sqrt{\check{d}_4}h^{-1/2}\max_{0\leq k\leq n}\|\Delta K^{(k)}\|]\exp(\tfrac{1}{2}\check{d}_1t_n), h \in [0, h_5].
\end{aligned} \tag{3.29}$$

Substituting (2.6) and (3.2) into (3.28), we further obtain

$$\|\varepsilon_n - N\varepsilon_{n-m}\| \leq \check{d}_5\exp(\tfrac{1}{2}\check{d}_1t_n)h^{\min\{p, q+1/2\}}, h \in [0, h_5], \tag{3.30}$$

where $\check{d}_5 = \tilde{d}_1h_5\sqrt{\check{d}_2} + (\tilde{d}_1 + \tilde{d}_0)\sqrt{\check{d}_3} + c_q\sqrt{\check{d}_4}$.

Therefore we have

$$\begin{aligned}
\|\varepsilon_n\| &\leq \|\varepsilon_n - N\varepsilon_{n-m}\| + \|N\|\|\varepsilon_{n-m} - N\varepsilon_{n-2m}\| + \cdots + \|N\|^k\|\varepsilon_{n-(k-1)m} - N\varepsilon_{n-km}\| \\
&\leq \check{d}_5h^{\min\{p, q\}}\exp(\tfrac{1}{2}\check{d}_1t_n)\left(1 + \|N\|\exp(\tfrac{1}{2}\check{d}_1t_{-m}) + \cdots + \|N\|^k\exp(\tfrac{1}{2}\check{d}_1t_{-km})\right) \\
&\leq \frac{\check{d}_5\exp(\tfrac{1}{2}\check{d}_1t_n)}{1 - \|N\|\exp(-\tfrac{1}{2}\check{d}_1\tau)}h^{\min\{p, q+1/2\}}, h \in (0, h_0],
\end{aligned} \tag{3.31}$$

where $h_0 = h_5$. This completes the proof of Theorem 3.4. $\square$

Remark 3.5 Any one of the s-stage ($s > 1$) Gauss, Radau IA, Radau IIA Runge-Kutta methods of the form (2.1) is algebraically stable and diagonally

stable. Therefore, when any one of these methods with an appropriate quadrature rule is applied to the problem (1.1) satisfying (1.3)-(1.4), the numerical solution is convergent.

Remark 3.6 If the integral term in problem (1.1) has a weakly singular kernel, the method (2.3) may not be used directly. In this case, we may have to modify the approximation of integral item.

4 Numerical experiments

Consider the following initial value problem of neutral delay integro-differential equations

$$\begin{cases} \frac{d}{dt}[y(t) - \frac{1}{10}y(t-1)] = -100y(t) + \frac{y^2(t-1)}{1+y^2(t-1)} + \frac{1}{10} \int_{t-1}^t y(\xi)d\xi \\ \quad + \frac{991}{10} \exp(-t) - \frac{\exp(-2t+2)}{1+\exp(-2t+2)}, \quad t \geq 0, \\ y(t) = \exp(-t), \quad t \in [-1, 0], \end{cases} \quad (4.1)$$

It can be easily verified that the problem (4.1) belongs to the class $\mathbf{R}(-94.45, 5.61, 0.06, 1)$, where the inner product is standard inner product.

To provide some insight in this matter, the present section is devoted to a number of numerical experiments with five Runge-Kutta schemes, viz. the 2-stage and 3-stage Gauss, 3-stage Radau IA, 2-stage and 3-stage Radau IIA method. Below we have tabulated their stage orders p .

Table 4.1 The stage order of Runge-Kutta schemes

method	stage order p
2-stage Gauss	2
3-stage Gauss	3
3-stage Radau IA	2
2-stage Radau IIA	2
3-stage Radau IIA	3

It is well know that these methods are all algebraically stable and diagonally stable, so according to the results of the theorem 3.4, the corresponding method (for NDIDEs) is convergent of order at least $\min\{p, q + \frac{1}{2}\}$.

Let $E(h, T)$ denote the Euclidean norm of the absolute error at endpoint T obtained with step size h and p_r be the observed order of computed errors, which is defined as

$$p_r = \log(E(h; T)/E(h/2; T))/\log 2.$$

In Table 4.2 and Table 4.3, the numerically computed errors $E(h, T)$ and the observed orders are given for 2-stage Gauss method and 2-stage Radau IIA method with compound trapezoidal formula and with compound Simpson formula applied to problem (4.1), respectively. In Table 4.4 we list the numerically computed errors $E(h, T)$ and the observed orders which are given by applying 3-stage Gauss method and 3-stage Radau IIA method with compound Simpson formula to problem (4.1).

These results confirms the theoretical conclusion of Theorem 3.4. At the same time these results also show that the order result which is given in Theorem 3.4 can be ulteriorly improved. This is also our motivation of further study on this subject in the future.

Table 4.2 Global errors $E(h, T)$ of the numerical results and observation orders p_r of 2-stage Gauss method and 2-stage Radau IIA method with compound trapezoidal formula when applied to problem (4.1) with $T = 10$.

	Gauss(2)		Radau IIA (2)	
h	$E(h; T)$	p_r	$E(h; T)$	p_r
0.1	$0.16976424 \times 10^{-8}$		$0.19860224 \times 10^{-9}$	
0.05	$0.01094254 \times 10^{-8}$	3.9555	$0.03337375 \times 10^{-9}$	2.5730
0.025	$0.00097888 \times 10^{-8}$	3.4826	$0.00432741 \times 10^{-9}$	2.9471
0.0125	$0.00013784 \times 10^{-8}$	2.8278	$0.00025976 \times 10^{-9}$	4.0582
0.00625	$0.00002783 \times 10^{-8}$	2.3084	$0.00007567 \times 10^{-9}$	1.7793
0.003125	$0.00000654 \times 10^{-8}$	2.0884	$0.00003998 \times 10^{-9}$	0.9202
0.0015625	$0.00000161 \times 10^{-8}$	3.4826	$0.00001290 \times 10^{-9}$	1.6318
0.00078125	$0.00000040 \times 10^{-8}$	2.8278	$0.00000360 \times 10^{-9}$	1.8383
0.000390625	$0.00000010 \times 10^{-8}$	2.3084	$0.00000095 \times 10^{-9}$	1.9235
0.0001953125	$0.00000002 \times 10^{-8}$	2.0884	$0.00000024 \times 10^{-9}$	1.9629

Table 4.3 Global errors $E(h, T)$ of the numerical results and observation orders p_r of 3-stage Gauss method and 3-stage Radau IIA method with compound trapezoidal formula when applied to problem (4.1) with $T = 10$.

	Gauss(3)		Radau IIA (3)	
h	$E(h; T)$	p_r	$E(h; T)$	p_r
0.1	$0.61894138 \times 10^{-10}$		$0.19860224 \times 10^{-9}$	
0.05	$0.16266798 \times 10^{-10}$	1.9278	$0.03337375 \times 10^{-9}$	2.5730
0.025	$0.04096847 \times 10^{-10}$	1.9893	$0.00432741 \times 10^{-9}$	2.9471
0.0125	$0.01024928 \times 10^{-10}$	1.9989	$0.00025976 \times 10^{-9}$	4.0582
0.00625	$0.00256245 \times 10^{-10}$	1.9999	$0.00007567 \times 10^{-9}$	1.7793
0.003125	$0.00064061 \times 10^{-10}$	1.9999	$0.00003998 \times 10^{-9}$	0.9202
0.0015625	$0.00016015 \times 10^{-10}$	1.9999	$0.00001290 \times 10^{-9}$	1.6318
0.00078125	$0.00004003 \times 10^{-10}$	2.0000	$0.00000360 \times 10^{-9}$	1.8383
0.000390625	$0.00001001 \times 10^{-10}$	1.9999	$0.00000095 \times 10^{-9}$	1.9235
0.0001953125	$0.00000250 \times 10^{-10}$	2.0000	$0.00000024 \times 10^{-9}$	1.9629

Table 4.4 Global errors $E(h, T)$ of the numerical results and observation orders p_r of 3-stage Radau IA method and 3-stage Radau IIA method with compound Simpson formula when applied to problem (4.1) with $T = 10$.

	Radau IA(3)		Radau IIA (3)	
h	$E(h; T)$	p_r	$E(h; T)$	p_r
0.1	$0.451243640503 \times 10^{-9}$		$0.25159128591 \times 10^{-11}$	
0.05	$0.030579656482 \times 10^{-9}$	3.8832	$0.01703566893 \times 10^{-11}$	3.8844
0.025	$0.001514895454 \times 10^{-9}$	4.3352	$0.00084768550 \times 10^{-11}$	4.3288
0.0125	$0.000059454611 \times 10^{-9}$	4.6712	$0.00003366515 \times 10^{-11}$	4.6542
0.00625	$0.000002063318 \times 10^{-9}$	4.8487	$0.00000119668 \times 10^{-11}$	4.8141
0.003125	$0.000000067701 \times 10^{-9}$	4.9296	$0.00000004201 \times 10^{-11}$	4.8320
0.0015625	$0.000000002175 \times 10^{-9}$	4.9599	$0.00000000135 \times 10^{-11}$	4.9541
0.00078125	$0.000000000054 \times 10^{-9}$	5.3264	$0.000000000015 \times 10^{-11}$	3.1258

5 Conclusions

In the present paper, we study the convergence of Runge-Kutta methods for Hale-type neutral delay integro-differential equations (1.1). By extending the B-theory of numerical methods for stiff ordinary differential, several convergence results of Runge-Kutta methods for (1.1) are obtained. Numerical experiments further verified the theoretical results. Our future work will research the convergence of other methods for (1.1), such as one-leg methods, multistep Runge-Kutta methods, general linear methods.

Acknowledgments:

We are grateful to the two anonymous referees for useful comments and suggestions. This work is supported by NSF of China (No. 11371302, 11571291) and scientific research fund of the department of education, Hunan province, China (15A184).

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