A fifth-order finite volume weighted compact scheme for solving one-dimensional Burgers' equation

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Abstract

In the present paper, a high-order finite volume compact scheme is proposed to solve one dimensional Burgers' equation. The nonlinear advective terms are computed by the fifth-order finite volume weighted upwind compact scheme, in which the nonlinear Weighted Essentially Non-Oscillatory weights are coupled with lower order compact stencils. The diffusive terms are discretized by using the finite volume six-order Padé scheme. The strong stability preserving third-order Runge-Kutta time discretizations is used in this work. Numerical results are compared with the exact and some existing numerical solutions to demonstrate the essentially non-oscillatory and high resolution of the proposed method. The numerical results are shown to be more accurate than some numerical results given in the literature.

Keywords: Burgers' equation, finite volume method, compact schemes, weighted essentially non-oscillatory scheme, Padé schemes

1. Introduction

In this paper, the following one-dimensional Burgers' equation is considered

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, x \in (a, b), t \in (0, T], \quad (1)$$

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4 with initial condition

$$u(x, 0) = f(x), x \in [a, b], \quad (2)$$

5 and boundary condition

$$u(a, t) = \alpha, u(b, t) = \beta, t \in (0, T], \quad (3)$$

6 where $\nu = \frac{1}{Re}$ and Re called the Reynolds number.

7 The Burgers' equation is the simplest well-known nonlinear model equa-
8 tion from fluid dynamics. It is a simplified form of the Navier-Stokes equa-
9 tions that contain nonlinear advection and diffusion terms. The analytical
10 solutions were obtained by using the Hopf-Cole transformation [1, 2]. Ben-
11 ton and Platzman have surveyed the exact solutions of the one-dimensional
12 Burgers' equation with different boundary and initial conditions in [3]. For
13 some special initial and boundary conditions, the analytic solutions involve
14 infinite series, which may converge very slowly for large Reynolds number.

15 In the past few years, the numerical solutions of (1) have attracted a lot
16 of attention, and a variety of numerical methods have been proposed for this
17 equation. A review of numerical methods for nonlinear Burgers' equation
18 can be found in [4], such as finite difference method [5, 6, 7, 8, 9], B-spline
19 method [10, 11] and cubic Hermite collocation method [12]. Jiwari and Mittal
20 proposed lots of numerical methods for Burgers' equation, such as weighted
21 average differential quadrature method [13], haar wavelet quasilinearization
22 approach [14], hybrid numerical scheme [15], modified cubic B-splines colloca-
23 tion method [16], numerical scheme based on differential quadrature method
24 [17] and differential quadrature method [18]. Recently, Mukundan proposed
25 efficient numerical techniques for Burgers' equation [19] and Kumar proposed
26 a composite numerical scheme for the numerical simulation of coupled Burg-
27 ers' equation [20]. The present numerical results are also compared with some
28 other numerical solutions, such as local discontinuous galerkin finite element
29 method [21], automatic differentiation method [22], meshless method of lines
30 [23] and polynomial based differential quadrature method[24].

31 In 1992, Lele developed high-order linear compact difference schemes
32 based on implicit interpolations in [25]. These implicit schemes are very
33 accurate in smooth regions, and have spectral-like resolution properties by
34 using the global grid. However, these classical linear compact schemes nec-
35 essarily produce non-physical oscillations when they are applied directly to
36 flow with discontinuities. In recent years, there were a variety of methods to
37 overcome this problem. Hybrid schemes which coupling a fifth-order compact

38 upwind algorithm in conservation form with WENO (weighted essentially
 39 non-oscillatory) scheme [26] were proposed in [27] for shock-turbulence in-
 40 teraction. A new class of linear and hybrid weighted nonlinear schemes were
 41 recently proposed in [28, 29] based on the cell-centered compact scheme. An-
 42 other methodology for capturing discontinuities has been proposed by Deng
 43 [30]. They developed nonlinear compact schemes, where adaptive compact
 44 stencils are used. Deng and Zhang [31] also proposed weighted compact non-
 45 linear schemes (WCNS) based on the weighted technique. According to the
 46 idea of the WENO schemes, Jiang [32] proposed weighted compact schemes
 47 where the final compact scheme was a weighted combination of compact sub-
 48 stencils. Ghosh employed the idea that is described in [32], and developed
 49 a class of upwind finite difference compact-reconstruction WENO schemes
 50 [33], where lower order compact stencils are identified at each interface and
 51 combined using the WENO weights. In recent years, a lot of papers dealing
 52 with compact schemes that were based on finite volume method can be found
 53 in the literature. Kobayashi proposed a class of Padé finite volume methods
 54 for linear equations in [34]. In [35], Piller proposed a high-order compact
 55 finite volume method for the scalar advection-diffusion equation, where ad-
 56 vective fluxes are computed by the compact scheme introduced in [27] and
 57 the diffusive fluxes are discretized by using the Padé methods [34].

58 In this paper, a high-order finite volume compact scheme is proposed
 59 for solving Burgers' equation, where a fifth-order upwind weighted compact-
 60 WENO scheme is used for the numerical approximation of advective terms,
 61 while the sixth-order Padé finite volume method proposed by Pirozzoli [27]
 62 is applied to approximate the diffusive terms. Comparing with classical non-
 63 compact WENO schemes, the proposed compact schemes have spectral-like
 64 resolution properties by using the global grid. In recent years, the compact
 65 finite difference schemes have been successfully used for solving Burgers'
 66 equation [5, 6, 8, 9]. The classical linear central compact scheme for inviscid
 67 nonlinear Burgers' equation often yields oscillatory solutions across discon-
 68 tinuities. To achieve the essentially non-oscillatory properties of the present
 69 compact scheme, lower order compact stencils are combined with WENO
 70 weights to yield a fifth-order compact weighted interpolation. Then, the
 71 present scheme can maintain high order accuracy without oscillation. The
 72 accuracy of the proposed scheme will be checked by comparing the numer-
 73 ical against the exact solution. Comparisons of the numerical results with
 74 the exact solutions show that the proposed scheme is capable of achieving
 75 fifth-order accuracy. A number of tests are presented to show high resolution

property of the proposed approach. Numerical comparisons are also made with some available results given in the literature, the numerical results show that the proposed method produces highly accurate results.

This paper is organized as follows. In Section 2, the present finite volume compact-WENO scheme is described. Comparisons of the numerical solutions with exact solutions and other numerical results are given in Section 3, and the accuracy of the method is also investigated in this section. Some concluding remarks are given in Section 4.

2. Finite volume compact-WENO schemes for Burgers' equation

To construct the conservative finite volume scheme, we rewrite (1) as follows

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, x \in [a, b], t \in (0, T], \quad (1)$$

where $f(u) = \frac{u^2}{2}$. The computational domain $[a, b]$ can be divided into N cells as follows:

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b.$$

We denote cells by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. For simplicity, we consider an uniform grid with $h = \frac{b-a}{N}$. In the finite volume approach, (1) is integrated over all the mesh cells. For cell I_j with cell faces $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$, we can obtain the following equation

$$\frac{\partial}{\partial t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u dx + f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) = \nu(u_x(x_{j+\frac{1}{2}}, t) - u_x(x_{j-\frac{1}{2}}, t)). \quad (2)$$

The cell-averaged value of cell j is denoted as $\bar{u}_j$, which can be defined as

$$\bar{u}(x_j, t) = \frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) dx.$$

We approximate (2) by the following scheme

$$\frac{d\bar{u}_j(t)}{dt} = -\frac{1}{h}((\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}) - \nu((\hat{u}_x)_{j+\frac{1}{2}} - (\hat{u}_x)_{j-\frac{1}{2}})), \quad (3)$$

with a global Lax-Friedrichs flux defined by

$$\hat{f}_{j+\frac{1}{2}} = h(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) = \frac{1}{2}[f(u_{j+\frac{1}{2}}^+) + f(u_{j+\frac{1}{2}}^-) - \alpha(u_{j+\frac{1}{2}}^- - u_{j+\frac{1}{2}}^+)], \quad (4)$$

93 where $\alpha = \max_u |f'(u)|$, $u_{j+\frac{1}{2}}^-$ and $u_{j+\frac{1}{2}}^+$ are approximated by a fifth-order
 94 compact-WENO reconstruction. The approximation of $(\hat{u}_x)_{j+\frac{1}{2}}$ will be ob-
 95 tained by using the sixth-order Padé finite volume scheme [34].

96 2.1. Reconstruction for advective terms

97 In [27], Pirozzoli proposed an fifth-order upwind compact approximation
 98 for $u_{j+\frac{1}{2}}^-$ as follows

$$\frac{3}{10}u_{j-\frac{1}{2}}^- + \frac{6}{10}u_{j+\frac{1}{2}}^- + \frac{1}{10}u_{j+\frac{3}{2}}^- = \frac{1}{30}\bar{u}_{j-1} + \frac{19}{30}\bar{u}_j + \frac{10}{30}\bar{u}_{j+1}. \quad (5)$$

The small length scale compact scheme (5) is very accurate and keeps good resolutions in smooth regions, but non-physical oscillations are generated when they are applied directly to some physical problems with discontinuities. Unfortunately, the amplitude does not decrease if we refine the grid. In this work, we adopt the main idea of the WENO scheme to form the present compact-WENO scheme. It uses a nonlinear convex combination of all candidate stencils to reduce spurious non-physical oscillations. In order to capture the shock precisely and keep high-order accuracy and high resolution in the smooth area, we construct a fifth order finite volume compact-WENO scheme by using three third-order compact interpolations as candidates. We follow the idea that is described in [33] and rewrite (5) as a weighted combination of three third order compact substencils, they are

$$\frac{2}{3}u_{j-\frac{1}{2}}^- + \frac{1}{3}u_{j+\frac{1}{2}}^- = \frac{1}{6}(\bar{u}_{j-1} + 5\bar{u}_j), \quad (6a)$$

$$\frac{1}{3}u_{j-\frac{1}{2}}^- + \frac{2}{3}u_{j+\frac{1}{2}}^- = \frac{1}{6}(5\bar{u}_j + \bar{u}_{j+1}), \quad (6b)$$

$$\frac{2}{3}u_{j+\frac{1}{2}}^- + \frac{1}{3}u_{j+\frac{3}{2}}^- = \frac{1}{6}(\bar{u}_j + 5\bar{u}_{j+1}). \quad (6c)$$

99 We multiply (6) by linear weights $c_0 = \frac{2}{10}, c_1 = \frac{5}{10}, c_2 = \frac{3}{10}$ respectively to
 100 obtain linear compact scheme (5).

The WENO methodology is adopted for solving equations with discontinuous solutions. Then, the optimal linear weights $\{c_0, c_1, c_2\}$ are replaced by nonlinear weights $\{\omega_0, \omega_1, \omega_2\}$. The fifth-order finite volume upwind compact-WENO approximations for advective terms can be obtained [36, 37]. Given the cell averages $\bar{u}_j$ for each cell I_j , we obtain fifth-order compact approximations for $u_{j+\frac{1}{2}}^-$ based on stencils $\{x_{j-1}, x_j, x_{j+1}\}$, resulting in an implicit

system given by

$$\begin{aligned} & \frac{2\omega_0 + \omega_1}{3} u_{j-\frac{1}{2}}^- + \frac{\omega_0 + 2(\omega_1 + \omega_2)}{3} u_{j+\frac{1}{2}}^- + \frac{1}{3} \omega_2 u_{j+\frac{3}{2}}^- \\ &= \frac{1}{6} \omega_0 \bar{u}_{j-1} + \frac{5(\omega_0 + \omega_1) + \omega_2}{6} \bar{u}_j + \frac{\omega_1 + 5\omega_2}{6} \bar{u}_{j+1}, \end{aligned} \quad (7)$$

101 where nonlinear weights ω_k can be taken as [38, 39]

$$\omega_k = \frac{\alpha_k^z}{\sum_{l=0}^2 \alpha_l^z}, \quad \alpha_k^z = c_k \left(1 + \left(\frac{\tau_5}{\beta_k + \epsilon} \right)^p \right), \quad k = 0, 1, 2, \quad (8)$$

where $\tau_5 = |\beta_2 - \beta_0|$, which can improve the order of accuracy around the smooth extrema as compared to the classic WENO scheme [40]. We set $p = 2$ and $\epsilon = 10^{-13}$ in our numerical computation. The classical smooth indicators β_k ($k = 0, 1, 2$) in [40] are given by

$$\begin{aligned} \beta_0 &= \frac{13}{12} (\bar{u}_{j-2} - 2\bar{u}_{j-1} + \bar{u}_j)^2 + \frac{1}{4} (\bar{u}_{j-2} - 4\bar{u}_{j-1} + 3\bar{u}_j)^2, \\ \beta_1 &= \frac{13}{12} (\bar{u}_{j-1} - 2\bar{u}_j + \bar{u}_{j+1})^2 + \frac{1}{4} (\bar{u}_{j-1} - \bar{u}_{j+1})^2, \\ \beta_2 &= \frac{13}{12} (\bar{u}_j - 2\bar{u}_{j+1} + \bar{u}_{j+2})^2 + \frac{1}{4} (3\bar{u}_j - 4\bar{u}_{j+1} + \bar{u}_{j+2})^2. \end{aligned}$$

102 From (8), we can observe that the nonlinear weights depend on the so-
103 lutions, then we need to solve a tridiagonal system of (7) to get $u_{j+\frac{1}{2}}^-$ at
104 each time step. $u_{j+\frac{1}{2}}^+$ can be obtained similarly. For more details about the
105 scheme, readers can see [37].

106 2.2. Reconstruction for diffusive terms

107 To obtain a high-order scheme for Burgers' equation in smooth regions,
108 sixth-order Padé finite volume scheme introduced in [34] is used for the dis-
109 cretization of the diffusive fluxes. The implicit interpolation formula for
110 interior nodes can be written as

$$\frac{2}{11} \frac{\partial u}{\partial x} \Big|_{j-\frac{1}{2}} + \frac{\partial u}{\partial x} \Big|_{j+\frac{1}{2}} + \frac{2}{11} \frac{\partial u}{\partial x} \Big|_{j+\frac{3}{2}} = \frac{51}{44} \frac{\bar{u}_{j+1} - \bar{u}_j}{\Delta x} + \frac{3}{44} \frac{\bar{u}_{j+2} - \bar{u}_{j-1}}{\Delta x}. \quad (9)$$

111 It can be seen that the high orders of accuracy with relatively small stencils
112 can be achieved.

2.3. Boundary conditions

For the high-order compact scheme, we need to set appropriate boundary closures for both advective and diffusive terms. For the discretization of the advective term near left boundary nodes, a fifth-order approximation formulas is used in the present work. The formulas can be written as

$$\frac{1}{5}\tilde{u}_{j-\frac{1}{2}} + \frac{4}{5}\tilde{u}_{j+\frac{1}{2}} = \frac{37}{60}\bar{u}_j + \frac{29}{60}\bar{u}_{j+1} - \frac{7}{60}\bar{u}_{j+2} + \frac{1}{60}\bar{u}_{j+3}. \quad (10)$$

For the diffusive terms at boundary nodes, approximation formulae for the first-order derivatives of non-periodic problems can be derived from one-sided schemes, the 6th-order von Neumann boundary [34] is used in this work. The derived formulae at right boundary point is

$$u_{N+\frac{1}{2}} + \frac{27}{23}u_{N-\frac{1}{2}} = \frac{3h}{23} \left(\frac{\partial u}{\partial x} \Big|_{N+\frac{1}{2}} - 3 \frac{\partial u}{\partial x} \Big|_{N-\frac{1}{2}} \right) + \frac{325}{138}\bar{u}_N - \frac{13}{69}\bar{u}_{N-1} + \frac{1}{138}\bar{u}_{N-2}. \quad (11)$$

The derived formulae at left boundary point can be obtained similarly.

2.4. Temporal discretization

We use a strong stability preserving (SSP) high order Runge-Kutta time discretizations in this work. The third-order SSP Runge-Kutta method is

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}), \end{aligned} \quad (12)$$

where $L(u)$ is the spatial operator. For (3), we set $L = -\frac{1}{\Delta x_j}((\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}) - \nu((\hat{u}_x)_{i+\frac{1}{2}} - (\hat{u}_x)_{i-\frac{1}{2}}))$ in our computation.

The algorithm for the present FVCW scheme solving (1) consists of the following steps:

1. We need to solve the tridiagonal system of (7) to get $u_{j+\frac{1}{2}}^-$ by using nonlinear weights (8) at each time step.
2. (9) coupled with (11) are used to obtain the diffusive fluxes.
3. The global Lax-Friedrichs flux (4) can be obtained based on step 1.
4. The right side of (3) can be computed based on step 2 and step 3.
5. The third-order SSP Runge-Kutta method (12) is used to get the value u_j^{n+1} .

137 *2.5. Convergence and linear stability analysis*

138 We consider the linear stability properties of the third-order SSP Runge-
 139 Kutta method (12) when coupled with FVCW scheme to solve Burgers' equa-
 140 tion (1) by means of the von Neumann analysis. For nonlinear equation (1),
 141 we linearize and freeze coefficients, then the equation (1) can be written as
 142 following

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad (13)$$

143 where $\lambda = \max |u|$. We consider a uniform spatial mesh. The semidiscrete
 144 equation is

$$\frac{d\bar{u}_j(t)}{dt} = -\lambda \frac{1}{h} (\hat{u}_{j+\frac{1}{2}} - \hat{u}_{j-\frac{1}{2}}) + \frac{1}{h} \nu ((\hat{u}_x)_{j+\frac{1}{2}} - (\hat{u}_x)_{j-\frac{1}{2}}). \quad (14)$$

145 Equation (14) can be cast in matrix form as

$$\frac{d\bar{\mathbf{u}}}{dt} = -\lambda \frac{1}{h} \mathbf{C} \hat{\mathbf{u}} + \frac{1}{h} \nu \mathbf{C} \hat{\mathbf{u}}_{\mathbf{x}}, \quad (15)$$

146 where

$$\bar{\mathbf{u}} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_N \end{bmatrix}, \hat{\mathbf{u}} = \begin{bmatrix} \hat{u}_{\frac{3}{2}} \\ \hat{u}_{\frac{5}{2}} \\ \vdots \\ \hat{u}_{N+\frac{1}{2}} \end{bmatrix}, \hat{\mathbf{u}}_{\mathbf{x}} = \begin{bmatrix} \hat{u}_{x\frac{3}{2}} \\ \hat{u}_{x\frac{5}{2}} \\ \vdots \\ \hat{u}_{xN+\frac{1}{2}} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}.$$

147 We consider the inviscid equation $\frac{d\bar{\mathbf{u}}}{dt} = -\lambda \frac{1}{h} \mathbf{C} \hat{\mathbf{u}}$. Equation (5) with periodic
 148 boundary can be written in matrix form as

$$\mathbf{A} \hat{\mathbf{u}} = \mathbf{B} \bar{\mathbf{u}}, \quad (16)$$

149 where

$$\mathbf{A} = \begin{bmatrix} \frac{6}{10} & \frac{1}{10} & & & \frac{3}{10} \\ \frac{3}{10} & \frac{10}{6} & & & \\ \frac{1}{10} & \frac{10}{3} & & & \\ & \frac{10}{6} & \frac{1}{10} & & \\ & & \ddots & \ddots & \ddots \\ & & & \frac{3}{10} & \frac{6}{10} & \frac{1}{10} \\ \frac{1}{10} & & & \frac{6}{10} & \frac{3}{10} & \frac{10}{6} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \frac{19}{30} & \frac{1}{3} & & & \frac{1}{30} \\ \frac{1}{30} & \frac{19}{30} & & & \\ & \frac{1}{30} & \frac{1}{3} & & \\ & & \ddots & \ddots & \ddots \\ & & & \frac{1}{30} & \frac{19}{30} & \frac{1}{3} \\ \frac{1}{3} & & & \frac{19}{30} & \frac{1}{3} & \frac{19}{30} \end{bmatrix}.$$

150 The inviscid equation can be written as

$$\frac{d\bar{\mathbf{u}}}{dt} = -\lambda \frac{1}{h} \mathbf{C} \mathbf{A}^{-1} \mathbf{B} \bar{\mathbf{u}} \quad (17)$$

151 where $\mathbf{C}$, $\mathbf{A}$ and $\mathbf{B}$ are circulant tridiagonal matrices. The eigenvalues of a
152 circulant tridiagonal matrix are [41] presented as

$$\lambda^{(k)} = \beta + (\alpha + \gamma) \cos\left(\frac{2k\pi}{M}\right) - \mathbf{i}(\alpha - \gamma) \sin\left(\frac{2k\pi}{M}\right), k = 0, \dots, M-1, \quad (18)$$

153 where α , β and γ are the subdiagonal, diagonal and superdiagonal terms,
154 respectively. The eigenvalues of $\mathbf{C}$, $\mathbf{A}$ and $\mathbf{B}$ are

$$\begin{aligned} \lambda_{\mathbf{C}}^{(k)} &= 1 - \exp^{-2\pi\mathbf{i}k/M}, \\ \lambda_{\mathbf{A}}^{(k)} &= \frac{6}{10} + \frac{4}{10} \cos\left(\frac{2k\pi}{M}\right) - \mathbf{i} \frac{2}{10} \sin\left(\frac{2k\pi}{M}\right), \\ \lambda_{\mathbf{B}}^{(k)} &= \frac{19}{30} + \frac{11}{30} \cos\left(\frac{2k\pi}{M}\right) - \mathbf{i} \frac{29}{30} \sin\left(\frac{2k\pi}{M}\right), \\ k &= 0, \dots, M-1. \end{aligned} \quad (19)$$

155 The eigenvalues of $\mathbf{C} \mathbf{A}^{-1} \mathbf{B}$ are

$$\lambda_{\mathbf{C} \mathbf{A}^{-1} \mathbf{B}}^{(k)} = \frac{\lambda_{\mathbf{C}}^{(k)} \lambda_{\mathbf{B}}^{(k)}}{\lambda_{\mathbf{A}}^{(k)}}, k = 0, \dots, M-1. \quad (20)$$

156 We consider von Neumann stability analysis of the third-order SSP Runge-
157 Kutta method (12) for the solution of the semidiscrete problem. The ampli-
158 fier matrix can be deduced as follows

$$G = 1 + \sigma \mathbf{S} + \frac{1}{2}(\sigma \mathbf{S})^2 + \frac{1}{6}(\sigma \mathbf{S})^3, \quad (21)$$

159 where $\sigma = \lambda \Delta t / h$, $\mathbf{S} = -\mathbf{C} \mathbf{A}^{-1} \mathbf{B}$. The eigenvalues of the matrix $\mathbf{S}$ satisfy

$$\lambda_{\mathbf{G}}^{(k)} = 1 + \sigma \lambda_{\mathbf{S}}^{(k)} + \frac{1}{2}(\sigma \lambda_{\mathbf{S}}^{(k)})^2 + \frac{1}{6}(\sigma \lambda_{\mathbf{S}}^{(k)})^3, k = 0, \dots, M-1. \quad (22)$$

160 The stability condition for the present scheme with the fourth-order Runge-
161 Kutta method is $\sigma < 1.0$ [27] and third-order SSP Runge-Kutta method (12)
162 is $\sigma < 0.9$ [42].

163 The stability analysis of the diffusion equations for the Runge-Kutta
 164 method with sixth-order Padé finite volume scheme with periodic bound-
 165 ary conditions is discussed in [34].

166 A convergence study of the linear upwind compact (5) for smooth solu-
 167 tions was presented in [27]. The convergence properties of the nonlinear up-
 168 wind scheme (7) is severely influenced by nonlinear weights. The convergence
 169 of the present scheme with different nonlinear weights for conservation law
 170 are discussed in [36]. The present nonlinear weighted compact scheme which
 171 is coupled the fifth-order upwind (7) for advection term with sixth-order Padé
 172 finite volume scheme for diffusion terms can converge at fifth-order.

173 3. Numerical examples

174 In this section, we provide some classical numerical examples with both
 175 smooth initial conditions and initial discontinuous distributions for the fifth-
 176 order finite volume compact WENO scheme with the third order SSP Runge-
 177 Kutta time discreization, which is denoted as the “FVCW” scheme. The
 178 original high order Padé finite volume methods scheme of Kobayashi [34] is
 179 denoted as “HOP”. The discrete L_1 , L_2 and L_∞ error norms are defined as
 180 follows

$$||e_n||_{L_1} = \frac{1}{N} \sum_{j=1}^N |\bar{u}_j - \tilde{u}_j|, ||e_n||_{L_\infty} = \max_{1 \leq j \leq N} |\bar{u}_j - \tilde{u}_j|, ||e_n||_{L_2} = \left(\sum_{j=1}^N |\bar{u}_j - \tilde{u}_j|^2 \Delta x \right)^{\frac{1}{2}}, \quad (1)$$

181 where $\bar{u}$ and $\tilde{u}$ are exact and numerical solution respectively.

182 **Example 1.** In this example, we consider (1) with initial condition in the
 183 following form

$$u(x, 0) = 2\nu \frac{\pi \sin(\pi x)}{\sigma + \cos(\pi x)}, x \in (0, 1), \quad (2)$$

and the boundary conditions

$$u(0, t) = 0, u(1, t) = 0.$$

184 The exact solution of this problem is given in [43]

$$u(x, t) = \frac{2\nu\pi e^{-\pi^2\nu t} \sin(\pi x)}{\sigma + e^{-\pi^2\nu t} \cos(\pi x)}. \quad (3)$$

185 The numerical error and numerical rates of convergence of the present finite
 186 volume compact-WENO scheme can be investigated with this closed form of
 187 the exact solution.

188 In Table 1, the numerical results for $\sigma = 2$. $Re = 2, 10$ are compared
 189 with those obtained in [12], [22] and [16]. We can observe that FVCW
 190 schemes have lower L_2 errors than other three schemes. We compare our
 191 numerical results for $\sigma = 2$. $Re = 10$ with those obtained in Liao [6] and
 192 classical fifth-order Padé finite volume methods in Table 2 and observe that
 193 the present scheme can obtain fifth-order accuracy and the L_2 errors are
 194 lower than that by implicit fourth-order finite-difference scheme [6]. The L_2
 195 errors and orders for fifth-order Padé finite volume scheme, WENO scheme
 196 and the present FVCW scheme for $\sigma = 2$. $Re = 1000$ are given in Table 3.
 197 We observe that the FVCW scheme can obtain fifth-order accuracy and the
 198 L_2 errors are lower than those obtained with WENO scheme. In table 4, we
 199 compare our results with results of Mittal [16] and Jiwari [13] for $\sigma = 200$
 200 and $\Delta t = 0.01$ at $T = 1$. In this case, the fifth-order of accuracy can also
 201 be maintained and the L_∞ and L_2 errors of present schemes are much lower
 202 than the errors of Jiwari [13] and Mittal [16].

203 **Example 2.** We consider (1) with the following initial condition

$$u(x, 0) = \sin(\pi x), x \in (0, 1), \quad (4)$$

and the homogeneous boundary conditions

$$u(0, t) = 0, u(1, t) = 0.$$

204 The exact solution for this example as given in [2]

$$u(x, t) = 2\pi\nu \frac{\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\nu t) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\nu t) \cos(n\pi x)}, \quad (5)$$

with the Fourier coefficients are defined as

$$a_0 = \int_0^1 \exp\{-(2\pi\nu)^{-1}[1 - \cos(\pi x)]\} dx,$$

$$a_n = 2 \int_0^1 \exp\{-(2\pi\nu)^{-1}[1 - \cos(\pi x)]\} \cos(n\pi x) dx, n = 1, 2, 3, \dots$$

Table 1: Comparison of results at $t = 0.001$, $\sigma = 2$, $\Delta t = 0.0001$, $h = 0.025$ for $Re = 2, 10$.

		Exact	FVCW	Ganaie [12]	Asaithambi [22]	Mittal [16]
Re=2	0.1	0.3278695524	0.3278695524	0.327871	0.327874	0.327870
	0.2	0.6550692222	0.6550692220	0.655067	0.655078	0.655071
	0.3	0.9784124992	0.9784124987	0.978418	0.978427	0.978416
	0.4	1.2884634969	1.2884634950	1.288464	1.288485	1.288469
	0.5	1.5630638524	1.5630638350	1.563061	1.563096	1.563074
	0.6	1.7566421091	1.7566419249	1.756648	1.756691	1.756654
	0.7	1.7872063975	1.7872065979	1.787201	1.787281	1.787204
	0.8	1.5376943907	1.5376944007	1.537693	1.537794	1.537649
	0.9	0.9168597988	0.9168597587	0.916880	0.916941	0.916786
	L_∞ error		8.36E-07	2.00E-05		7.44E-05
	L_2 error		2.05E-07	3.54E-06		2.79E-05
Re=10	0.1	0.0657497591	0.0657497591	0.065750	0.065750	0.065750
	0.2	0.1313829355	0.1313829355	0.131383	0.131383	0.131383
	0.3	0.1962808678	0.1962808678	0.196281	0.196281	0.196281
	0.4	0.2585757378	0.2585757375	0.258576	0.258576	0.258576
	0.5	0.3138493556	0.3138493522	0.313848	0.313850	0.313850
	0.6	0.3529718209	0.3529717855	0.352972	0.352972	0.352972
	0.7	0.3594428596	0.3594429072	0.359443	0.359444	0.359443
	0.8	0.3095803849	0.3095803878	0.309580	0.309583	0.309579
	0.9	0.1847537428	0.1847537364	0.184752	0.184756	0.184751
	L_∞ error		1.55E-07	2.00E-06		3.08E-06
	L_2 error		3.60E-08	3.54E-07		1.15E-06

Table 2: Numerical errors and convergence rate for Example 1 with $\sigma = 2$, $Re = 10$, $T = 1$.

	Liao [6], $\Delta t = 0.000001$		FVCW, $\Delta t = C\Delta x^2/Re$		HOP, $\Delta t = C\Delta x^2/Re$	
N	L_∞ error	L_∞ order	L_∞ error	L_∞ order	L_∞ error	L_∞ order
10	1.5763E-05	-	1.033E-05	-	8.094E-06	-
20	9.8231E-07	4.004	3.805E-07	4.76	3.817E-07	4.41
40	6.1273E-08	4.003	1.273E-08	4.90	1.272E-08	4.91
80	3.8303E-09	4.000	4.033E-10	4.98	4.015E-10	4.99
160	2.3954E-10	3.999	1.262E-11	5.00	1.269E-11	4.98

Table 3: Numerical errors and convergence rate for Example 1 with $\sigma = 2$, $Re = 1000$, $T=1$

	HOP, $\Delta t = C\Delta x^2/Re$		WENO, $\Delta t = C\Delta x^2/Re$		FVCW, $\Delta t = C\Delta x^2/Re$	
N	L_2 error	L_2 order	L_2 error	L_2 order	L_2 error	L_2 order
20	5.451E-09	-	2.131E-07	-	6.439E-08	-
40	2.508E-10	4.44	6.673E-09	5.00	6.055E-10	6.73
80	8.456E-12	4.89	1.787E-10	5.22	1.174E-11	5.69
160	2.687E-13	4.98	5.158E-12	5.11	2.782E-13	5.40
320	8.664E-15	4.95	1.564E-13	5.04	8.693E-15	5.00

Table 4: Numerical errors and convergence rate for Example 1 with $\sigma = 100$, $Re = 200$, $T = 1$.

N	Mittal [16], $\Delta t = 0.01$		Jiwari [13], $\Delta t = 0.01$		L_2 order	FVCW, $\Delta t = C\Delta x^2/Re$			
	L_∞ error	L_2 error	L_∞ error	L_2 error		L_∞ error	L_∞ order	L_2 error	L_2 order
10	1.215E-07	8.631E-08	4.708E-08	6.459E-08	-	2.010E-09	-	8.653E-10	-
20	3.062E-08	2.153E-08	1.091E-08	4.465E-09	2.11	7.071E-11	4.83	2.019E-11	5.42
40	7.644E-09	5.378E-09	1.980E-09	2.786E-10	2.46	2.629E-12	4.75	6.460E-13	4.97
80	1.917E-09	1.345E-09	7.182E-09	2.665E-10	1.86	8.941E-14	4.88	2.047E-14	4.98

205 Numerical results obtained by the present finite volume compact-WENO
 206 method and the exact solution for (1) at time $T = 0.1$ for $Re = 1$ and
 207 $h = 0.1, 0.0125, \Delta t = 0.0001, 0.00001$ are shown in Table 5. We compare our
 208 numerical results with those obtained with sixth-order compact finite dif-
 209 ference scheme [9], implicit fourth-order compact finite difference scheme [6]
 210 and local discontinuous Galerkin (LDG) finite element method [21]. These
 211 comparisons show that the present finite volume compact-WENO scheme ob-
 212 tains much better results than linear compact schemes [6] and LDG method
 213 [21]. The numerical results are almost the same as [9] with $h = 0.1$, however
 214 the time step size we set $\Delta t = 0.001$ in our computation while $\Delta t = 0.00001$
 215 in [9]. The numerical solutions with $Re = 100$ at different times at different
 216 points are shown in Table 6. Numerical results show that the present numer-
 217 ical solutions are in good agreement with the exact solutions. Table 6 also
 218 shows that the numerical solutions using the present scheme with $h = 0.0125$
 219 are much more accurate than the numerical solutions using automatic dif-
 220 ferentiation method [22]. The numerical results obtained by the present
 221 scheme with $h = 0.025, \Delta t = 0.001$ are slightly better than the results using
 222 fourth-order finite difference method with $h = 0.0125, \Delta t = 0.001$ [5]. The
 223 numerical solutions produced by the present scheme are much more accu-
 224 rate than the results using the modified cubic B-splines collocation method
 225 with $h = 0.025, \Delta t = 0.0025$ [16]. Figure 1 shows the numerical solutions
 226 for $Re = 1.0, 10.0, 100.0, 10000.0$ at different times which exhibit the correct
 227 physical behavior of the problem. The figure also shows that the propagation
 228 front grow steeper as we set the Reynolds number larger. For $Re = 20000.0$,
 229 the advection term dominates over the dissipation term. From Figure 2, we
 230 can observe that the classical linear fifth-order Padé finite volume yields oscil-
 231 latory solutions across discontinuities, however, the FVCW scheme captures
 232 non-oscillatory solutions.

Table 5: Comparison between exact and numerical solutions of Example 2 for $Re = 1$ and $T = 0.1$

x	FVCW		Sari [9]	Liao [6]	Shao [21]	Exact
	$h = 0.1$	$h = 0.0125$	$h = 0.1$	$h = 0.0125$	$h = 0.1, k = 4$	
	$\Delta t = 0.001$	$\Delta t = 0.00001$	$\Delta t = 0.00001$	$\Delta t = 0.00001$	$\Delta t = 0.00001$	
0.1	0.1095410340	0.1095381513	0.10954	0.10953815119	0.10953815187843	0.1095381513
0.2	0.2097937393	0.2097921489	0.20979	0.20979214867	0.20979215073044	0.2097921489
0.3	0.2918993650	0.2918963509	0.29190	0.29189635032	0.29189635398773	0.2918963508
0.4	0.3479248166	0.3479239124	0.34792	0.34792391150	0.34792391648131	0.3479239124
0.5	0.3715814171	0.3715774762	0.37158	0.37157747490	0.37157748046658	0.3715774761
0.6	0.3590491636	0.3590455801	0.35905	0.35904557846	0.35904558368977	0.3590455800
0.7	0.3099099916	0.3099050008	0.30991	0.30990499905	0.30990500314125	0.3099050006
0.8	0.2278224940	0.2278174068	0.22782	0.22781740532	0.22781740781517	0.2278174066
0.9	0.1206949990	0.1206866913	0.12069	0.12068669034	0.12068669132062	0.1206866911

Table 6: Comparison between exact and numerical solutions of Example 2 for $Re = 100$.

x	T	FVCW		Hassanien [5]	Mittal [16]	Asaithambi [22]	Exact
		$h = 0.025$	$h = 0.0125$	$h = 0.0125$	$h = 0.01$	$h = 0.0125$	
		$\Delta t = 0.005$	$\Delta t = 0.0001$	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.0001$	
0.25	0.4	0.3419148998	0.3419149324	0.34191	0.34192	0.34193	0.3419149324
	0.6	0.2689648148	0.2689648453	0.26896	0.26896	0.26898	0.2689648453
	0.8	0.2214818903	0.2214819145	0.22148	0.22148	0.22149	0.2214819145
	1.0	0.1881939425	0.1881939614	0.18819	0.18819	0.18820	0.1881939614
	3.0	0.0751140806	0.0751140839	0.07511	0.07511	0.07512	0.0751140839
0.50	0.4	0.6607111017	0.6607109708	0.66071	0.66071	0.66076	0.6607109710
	0.6	0.5294183037	0.5294182636	0.52942	0.52942	0.52945	0.5294182637
	0.8	0.4391382575	0.4391382506	0.43914	0.43914	0.43916	0.4391382507
	1.0	0.3744200327	0.3744200376	0.37442	0.37442	0.37443	0.3744200376
	3.0	0.1501789931	0.1501790052	0.15218	0.15018	0.15018	0.1501790052
0.75	0.4	0.9102646129	0.9102645487	0.91027	0.91027	0.91046	0.9102645462
	0.6	0.7672501164	0.7672432886	0.76724	0.76724	0.76733	0.7672432826
	0.8	0.6474031323	0.6473952374	0.64740	0.64740	0.64744	0.6473952348
	1.0	0.5560569314	0.5560506935	0.55605	0.55605	0.55608	0.5560507045
	3.0	0.2248087625	0.2248112673	0.22481	0.22483	0.22484	0.2248112482

(a) $\text{Re}=1, \Delta t = 0.001, h = 0.025$

(b) $\text{Re}=10, \Delta t = 0.01, h = 0.025$

(c) $\text{Re}=100, \Delta t = 0.01, h = 0.025$

(d) $\text{Re}=10000, \Delta t = 0.01, h = 0.025$

Figure 1: Numerical solutions for Example 2

(a) $Re=20000, T=0.6$

Figure 2: Numerical solutions for Example 2

233 **Example 3.** The initial condition for this example is

$$u(x, 0) = 4x(1 - x), x \in (0, 1), \quad (6)$$

and the boundary conditions are

$$u(0, t) = 0, u(1, t) = 0.$$

The exact solution of this example is also given by Eq.(5), but in this case the Fourier coefficients are

$$a_0 = \int_0^1 \exp\left(-\frac{-3x^2 - 2x^3}{3\nu}\right) dx,$$

$$a_n = \int_0^1 \exp\left(-\frac{-3x^2 - 2x^3}{3\nu}\right) \cos(n\pi x) dx, n = 1, 2, 3, \dots$$

234 Table 7 shows that computed solutions for this example with $Re = 100$ are
 235 in good agreement with exact solutions. The numerical solutions obtained by
 236 using the present compact-WENO scheme are compared with those obtained
 237 by using compact finite difference scheme [6, 8] and modified cubic B-splines
 238 collocation method [16] in Table 7. The numerical comparisons show that
 239 the present scheme produces better results than Liao [6] and Mittal [16].

Table 7: Comparison between exact and numerical solutions of Example 3 for $Re = 100$.

x	T	FVCW $h = 0.025$ $\Delta t = 0.001$	Liao [6] $h = 0.0125$ $\Delta t = 0.01$	Zhang [8] $h = 0.025$ $\Delta t = 0.001$	Mittal [16] $h = 0.025$ $\Delta t = 0.001$	Exact
0.25	0.4	0.36226	0.36226	0.36226	0.36225	0.36226
	0.6	0.28204	0.28204	0.28203	0.28202	0.28204
	0.8	0.23045	0.23045	-	0.23044	0.23045
	1.0	0.19469	0.19469	0.19469	0.19468	0.19469
	3.0	0.07613	0.07606	0.07613	0.07613	0.07613
0.50	0.4	0.68368	0.68369	0.68367	0.68368	0.68368
	0.6	0.54832	0.54832	0.54831	0.54832	0.54832
	0.8	0.45371	0.45372	-	0.45371	0.45371
	1.0	0.38568	0.38568	0.38567	0.38567	0.38568
	3.0	0.15218	0.15203	0.15218	0.15218	0.15218
0.75	0.4	0.92050	0.92040	0.92051	0.92052	0.92050
	0.6	0.78300	0.78304	0.78301	0.78300	0.78299
	0.8	0.66273	0.66276	-	0.66272	0.66272
	1.0	0.56933	0.56934	0.56932	0.56932	0.56932
	3.0	0.22774	0.22751	0.22774	0.22782	0.22774

Figure 3: Numerical solutions for Example 3, $Re=100, \Delta t = 0.001, h = 0.025$

Table 8: Comparison between exact and numerical solutions of Example 4 for $Re = 200$.

T	Korkmaz [24] $h = 0.02, \Delta t = 0.01$		Haq [23] $h = 0.01, \Delta t = 0.001$		FVCW $h = 0.02, \Delta t = 0.01$		$h = 0.01, \Delta t = 0.001$	
	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2
1.7	2.3E-05	1.0E-05	5.6554E-04	9.51E-06	5.929E-05	1.173E-05	1.623E-06	2.976E-07
2.4	3.5E-05	1.3E-05	3.3077E-04	2.96E-06	2.199E-05	4.518E-06	5.072E-07	1.114E-07
3.1	4.2E-04	4.8E-03	1.7990E-05	1.951E-05	7.19E-06	2.073E-06	4.830E-06	6.028E-07

Example 4. In this example, we consider the Shock-like solution of (1), and the analytical solution given in [44] is

$$u(x, t) = \frac{\frac{x}{t}}{1 + \sqrt{\frac{t}{t_0}} \exp\left(\frac{x^2}{4\nu t}\right)}, t \geq 1, \quad (7)$$

where $t_0 = \exp\left(\frac{1}{8\nu}\right)$. Initial condition is obtained from the analytical solution when $t = 1$ is used. Boundary conditions are set to be $u(a, t) = u(b, t) = 0$.

In our computation, we set $Re = 200, 2000, a = 0, b = 1, \Delta t = 0.01, 0.001$ and $h = 0.02, 0.01$. The L_∞ and L_2 errors at various times are compared with results using differential quadrature method [24] and meshless method of lines [23]. Table 8 shows that the present scheme produces better results than [24] at times $T = 2.4, 3.1$. However, the results at time $T = 1.7$ obtained in [24] are slightly better than that using the present scheme. It is evident from Table 8 that the present compact-WENO scheme gives much better results as compared to meshless method of lines [23]. The numerical results with $Re = 200$ and $Re = 2000$ at various times are depicted in Figure 4. It can be seen that for the smaller viscosity value the propagation front is steeper.

Example 5. The initial and boundary conditions for this example are

$$u(x, 0) = \begin{cases} 1, & \text{if } 0 \leq x < 5, \\ 6 - x, & \text{if } 5 \leq x < 6, \\ 0, & \text{if } 6 \leq x < 12, \end{cases} \quad (8)$$

and $u(0, t) = 1, u(12, t) = 0$.

The numerical solutions of this example have been studied by Mittal [16], Asaithambi [22] as an example of a shock wave approaching steady state.

Figure 4: Numerical solutions for Example 4

259 Figure 5 shows that correct physical behavior for different values of Reynolds
 260 number can be captured by the present method with $h = 0.1$. Figure 5 also
 261 shows that the propagation front grow steeper as we set the Reynolds number
 262 larger.

263 **Example 6.** As the last test example we consider the particular solution of
 264 (1) as given in [45]

$$u(x, t) = \frac{\alpha + \mu + (\mu - \alpha) \exp \eta}{1 + \exp \eta}, 0 \leq x \leq 1, t \geq 0, \quad (9)$$

where $\eta = \frac{\alpha(x - \mu t - \gamma)}{\nu}$, and α, μ and γ are chosen as $\alpha = 0.4, \mu = 0.6$ and $\gamma = 0.125$. The initial condition is selected as the analytic solution at time $t = 0$. The boundary conditions are chosen as

$$u(0, t) = 1, u(1, t) = 0.2.$$

265 To compare with Xu [11], Dağ [10] and Mittal [16], simulations are carried
 266 out using the parameters $Re = 100, h = 1/36$ and $\Delta t = 0.001$, and numerical
 267 results are shown in Table 9. It can be seen that the numerical results
 268 obtained by the present compact-WENO scheme are much better than the
 269 results in [11], [10] and [16]. Numerical solutions for $Re = 100$ at different
 270 times which exhibit the correct physical behavior of the problem are shown
 271 in Figure 6.

(a) $\text{Re}=1, \Delta t = 0.001$

(b) $\text{Re}=10, \Delta t = 0.01$

(c) $\text{Re}=100, \Delta t = 0.01$

(d) $\text{Re}=1000, \Delta t = 0.01$

Figure 5: Numerical solutions for Example 5 with $h = 0.1$

Table 9: Comparison between exact and numerical solutions of Example 6 at $T = 0.5$ for $Re = 100$.

x	FVCW $h = 1/36$ $\Delta t = 0.01$	Xu [11] $h = 1/36$ $\Delta t = 0.001$	Dağ [10] $h = 1/36$ $\Delta t = 0.025$	Mittal [16] $h = 1/36$ $\Delta t = 0.01$	Exact
0.000	1.000	1.000	1.000	1.000	1.000
0.056	1.000	1.000	1.000	1.000	1.000
0.111	1.000	1.000	1.000	1.000	1.000
0.167	1.000	1.000	1.000	1.000	1.000
0.222	1.000	1.000	1.000	1.000	1.000
0.278	0.998	0.998	0.999	0.998	0.998
0.333	0.980	0.980	0.986	0.982	0.980
0.389	0.848	0.859	0.850	0.844	0.847
0.444	0.454	0.451	0.448	0.458	0.452
0.500	0.238	0.237	0.236	0.238	0.238
0.556	0.204	0.204	0.204	0.203	0.204
0.611	0.200	0.200	0.200	0.200	0.200
0.667	0.200	0.200	0.200	0.200	0.200
0.722	0.200	0.200	0.200	0.200	0.200
0.778	0.200	0.200	0.200	0.200	0.200
0.833	0.200	0.200	0.200	0.200	0.200
0.889	0.200	0.200	0.200	0.200	0.200
0.944	0.200	0.200	0.200	0.200	0.200
1.000	0.200	0.200	0.200	0.200	0.200
L_1 error	1.897E-04	1.2E-03	8.3E-03	5.97E-03	
L_2 error	4.655E-04	2.9E-03	1.2E-03	1.45E-03

Figure 6: Numerical solutions for Example 6, $Re=100, \Delta t = 0.001, h = 0.025$

272 4. Conclusions

273 In this paper, we develop a high-order finite volume compact scheme
 274 in space with a third-order SSP Runge-Kutta scheme in time for solving
 275 nonlinear Burgers' equation. In the present scheme, advective terms are
 276 computed by the fifth-order finite volume upwind compact-WENO scheme,
 277 while the diffusive terms are discretized by using the six-order Padé finite
 278 volume methods. Comparing with classical WENO schemes, the proposed
 279 compact schemes have spectral like resolution properties by using the global
 280 grid. We need to calculate the nonlinear weights at each time stage, then
 281 the disadvantage of high computational cost for compact schemes becomes
 282 even worse for the present nonlinear compact-WENO scheme. To achieve the
 283 essentially non-oscillatory properties of the present compact scheme, lower
 284 order compact stencils are combined with the nonlinear WENO weights to
 285 yield a fifth-order upwind compact interpolation. Some examples are tested,
 286 the numerical results show that the present scheme has better resolution
 287 compared to the classical finite volume non-compact WENO scheme. The
 288 present scheme can also produce essentially non-oscillatory solutions com-
 289 pared to the classical finite volume compact scheme. The numerical results
 290 show the good performance and high resolution property of the proposed
 291 FVCW scheme. The application of the scheme on non-uniform grids will be
 292 studied in our future work.

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