An Analytical and Numerical Study of Long Wave Run-up in U-shaped and V-shaped Bays

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Abstract

By assuming the flow is uniform along the narrow long bays, the 2-D nonlinear shallow-water equations are reduced to a linear semi-axis variable-coefficient 1-D wave equation via the generalized Carrier-Greenspan transformation. The run-up of long waves in constantly sloping U-shaped and V-shaped bays is studied both analytically and numerically within the framework of the 1-D nonlinear shallow-water theory. An analytic solution, in the form of a double integral, to the resulting linear wave equation is obtained by utilizing the Hankel transform, and consequently the solution to the tsunami run-up problem is developed by applying the inverse generalized Carrier-Greenspan transform. The presented solution is a generalization of the solutions found by Carrier et al. [7] (2003) and Didenkulova and Pelinovsky [9, 10] (2011) for the case of a plane beach and a parabolic bay respectively. The shoreline dynamics in U-shaped and V-shaped bays are computed via a double integral through standard integration techniques.

Keywords: Long wave run-up, Shallow water wave equations, Carrier-Greenspan transformation, Analytic solution

1. Introduction

Tsunamis pose danger to communities near coastal fjords and inlets [10, 16, 20] and understanding the tsunami run-up potential is crucial to saving lives and preventing damages.

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In many geophysical conditions, the tsunami generated by an earthquake can be considered as a long wave [22]. The first analytic solution to the long wave run-up problem was derived for a plane sloping beach [6]. To obtain this analytic solution, Carrier and Greenspan [6] introduced a new coordinate system in terms of the wave height and the bathymetry, and then defined a nonlinear coordinate transform now known as the classical Carrier-Greenspan (CG) transform, from cross-sectionally averaged shallow water wave equations into a linear variable velocity wave equation on the half line. The resulting wave equation was then solved through the use of a Hankel transform [5].

Later, Synolakis [21] expanded the work of Carrier and Greenspan to model wave run-up in a piecewise defined sloping plane beach formed by two different sloping regions. In 1994, Tadepalli and Synolakis considered the run-up of N-waves in a piecewise defined sloping plane beach [23]. This study resulted in run-up laws for N-waves. The run-up laws imply that a long wave with a leading depression has larger wave run-up than a similar long wave with a leading crest and also that the first wave run-up is not necessarily the largest one. By expanding the work of Carrier and Greenspan, Carrier et al. [7] found a Green's function for the run-up problem with a non-zero initial velocity. This solution required the computation of highly singular elliptic integrals and used unnecessary linearizations of the CG transform. This drawback was addressed by Kanoglu and Synolakis [14] and a new Green's function that does not involve singular integrals was developed.

Until recently, no generalization of the CG transform for fjords and inlets existed. In 2011, the classical CG transformation was generalized by Didenkulova and Pelinovsky to the case of sloping bays with parabolic cross-sections [9] and then to sloping bays with U-shaped and V-shaped cross-sections [10]. Examples for U-shapes and V-shaped bays are shown in Figures 1 and 2. For such bays, the water flow can be assumed to be uniform in the cross-section and thus the conservation of mass and linear momentum principles become

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(uS) = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial H}{\partial x} = g \frac{dh}{dx}. \quad (2)$$

where $u = u(x, t)$ is the cross-sectional average velocity, $H = H(x, t)$ and $h = h(x)$ are, respectively, the total water depth and unperturbed water depth along the main axis of the bay, g is the acceleration due to gravity, and $S(x, t)$ is the cross-sectional area of the bay under water at the point (x, t) . It is assumed that S is a function of total depth H only. We denote the water displacement as $\eta(x, t) = H(x, t) - h(x)$.

The generalizations of the CG transform for parabolic, U-shaped and V-shaped bays have led to the development of a d'Alembert solution for a parabolic bay of infinite length [10], an analytic solution for a parabolic bay of finite length [3], and an analytic solution for U-shaped and V-shaped bays of finite length [11].

Following the 2011 generalization, Rybkin et al. [19] further generalized the CG transform by assuming a sloping bay with a constant but arbitrary cross section (*i.e.* the baythemetry has the form $f(y) + \alpha x$ where $f(y)$ is arbitrary and α is a positive constant). The usefulness of this generalization as a numerical tool was studied by Harris et al. [12] in the case of a sloping bay with trapezoidal cross-section. The results of the work by Harris et al. [12] show that the generalized CG transform can be used as a fast and accurate numerical model for computing the run-up of long waves.

In this paper we present an analytic solution that is valid for U-shaped and V-shaped bays. In particular we derive an exact solution for bays with cross-section given by $\gamma|y|^{2/3}$ for positive γ and then another exact solution for bays with cross-section given by $\gamma|y|^m$ where m and γ are positive numbers. For convenience of notation we will denote a bay with cross-section $\gamma|y|^m$ by $\mathcal{B}_m$. The solution in $\mathcal{B}_{2/3}$ bays is developed by following the methodology of Courant and Hilbert [8] and is written in terms of elementary functions whereas the solution for any $\mathcal{B}_m$ bay is derived by generalizing the work of Carrier et al. [7] and is written in terms

75 of a double integral involving Bessel functions. Neither solution is without its advantages:
 76 our solution for $\mathcal{B}_{2/3}$ bays is trivial to compute, and has the potential to be an analytic tool
 77 to discover run-up properties for various waves and the presented solution for any $\mathcal{B}_m$ bay is
 78 valid for a large number of bays. Following the development of our solutions, we demonstrate
 79 that our solution for any $\mathcal{B}_m$ bay agrees with the solutions found by Carrier et al. [7] in a
 80 plane beach and Didenkulova and Pelinovsky [10] in a parabolic bay. We then detail the
 81 accurate evaluation of our solution for $\mathcal{B}_m$ bays. Finally, we compare our solution for some
 82 $\mathcal{B}_m$ bays to the numerical model developed by Harris et al. [12] for both U-shaped and V-
 83 shaped bays. In particular, we examine the run-up details of a solitary Gaussian wave in a
 84 $\mathcal{B}_3$ bay, a $\mathcal{B}_1$ bay and a $\mathcal{B}_{2/3}$ bay. The results of our comparisons show that the differences
 85 in wave dynamics between our solution and the Harris' numerical model are negligible for
 86 the $\mathcal{B}_m$ bays we tested.

87 **2. Carrier-Greenspan transform**

88 The generalized CG transform is one of the main tools for deriving the analytic solutions
 89 presented in this manuscript. Following Rybkin et al. [19], Didenkulova and Pelinovsky [10],
 90 we state the generalized CG transform that is valid for $\mathcal{B}_m$ bays. For such bays the cross-
 91 sectional area $S(x, t)$ can be written explicitly in terms of $H(x, t)$ as $S(H) = \frac{m}{m+1} H^{(m+1)/m}$,
 92 and hence the generalized CG transformation from the coordinate system $(u(x, t), \eta(x, t), x, t)$
 93 to the new coordinate system $(\Phi_\sigma, \Phi_\lambda, \sigma, \lambda)$ is defined by

$$94 \quad u = \left(1 + \frac{1}{m}\right) \frac{\Phi_\sigma}{\sigma}, \quad \eta = \frac{1}{2g} (\Phi_\lambda - u^2), \quad x = \frac{1}{\alpha} \left(\eta - \frac{\sigma^2}{4g} \left(1 - \frac{1}{m+1}\right) \right), \quad t = \frac{\lambda - u}{\alpha g}. \quad (3)$$

95 In the new coordinate system, $(\Phi_\sigma, \Phi_\lambda, \sigma, \lambda)$, the nonlinear system (1-2) becomes the linear
 96 second order equation

$$97 \quad \Phi_{\lambda\lambda} = \Phi_{\sigma\sigma} + \frac{m+2}{m\sigma} \Phi_\sigma. \quad (4)$$

98 Furthermore, both of the new coordinate variables λ and σ are defined on the half line and
 99 the shoreline is identified by points of the form $(\sigma = 0, \lambda)$ [11, 12, 19]. For more information

about this transform the reader should consult Harris et al. [11, 12], Rybkin et al. [19] and references therein.

To solve (4) it is necessary to specify two initial conditions and two boundary conditions. For our initial conditions we assume that the initial velocity $u(x, 0) = 0$ and that the water displacement is given by some initial profile $\eta_0(x) \equiv \eta(x, 0)$. These assumptions have been extensively used in literature [7, 9, 10, 11, 12, 19]. Under the CG transform (3), the initial conditions become

$$\Phi(\sigma, 0) = 0 \quad \text{and} \quad \Phi_\lambda(\sigma, \lambda)|_{\lambda=0} = 2g \eta_0(x(\sigma, \lambda = 0)), \quad (5)$$

where $x(\sigma, \lambda = 0)$ is implicitly defined through the third equation in (3). For a discussion of how to relax the non-zero initial velocity assumption, the reader is referred to Kanoglu and Synolakis [14].

For boundary conditions, we suppose that the velocity u is bounded at the moving shoreline $\sigma = 0$ and that $\eta(x, t)$ vanishes as σ approaches infinity. Under (3), our physical boundary conditions become

$$\Phi_\sigma(\sigma, \lambda)|_{\sigma=0} = 0 \quad \text{and} \quad \lim_{\sigma \rightarrow \infty} \Phi(\sigma, \lambda) = 0. \quad (6)$$

We note that these assumptions have been used extensively in literature [3, 7, 9, 10, 11, 12, 14, 19]. Finally, note that (4) along with conditions (5-6) form a consistent system.

3. Analytic solution for $\mathcal{B}_{2/3}$ bays

Before deriving a solution to the run-up problem that is valid for any U-shaped and V-shaped bay, we derive an analytic solution for $\mathcal{B}_{2/3}$ bays. Recall here that a $\mathcal{B}_{2/3}$ bay is a constantly sloping bay with cross-section $\gamma|y|^{2/3}$, for $\gamma > 0$. The main advantage of this solution over the analytic solution for any m , which we derive later in this manuscript, is its simplicity.

In the case of a $\mathcal{B}_{2/3}$ bay equation (4) simplifies to

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{4}{\sigma} \Phi_{\sigma} = 0. \quad (7)$$

Following Courant and Hilbert [8], we introduce the substitution

$$R(\sigma, \lambda) = \frac{1}{\sigma} \left(\sigma^3 \Phi(\sigma, \lambda) \right)_{\sigma}.$$

Under this substitution equation (7) becomes

$$R_{\lambda\lambda} - R_{\sigma\sigma} = 0, \quad (8)$$

and our initial conditions given in equation (5) turn into

$$R(\sigma, 0) = 0 \quad \text{and} \quad R_{\lambda}(\sigma, 0) = \frac{2g}{\sigma} \left(\sigma^3 \eta_0(\sigma) \right)_{\sigma}.$$

Utilizing the d'Alembert formula for the linear wave equation [4], it follows that

$$R(\sigma, \lambda) = \frac{1}{2} \int_{|\lambda-\sigma|}^{\lambda+\sigma} R_{\lambda}(s, 0) ds$$

solves (8) and satisfies the above initial conditions. Consequently, by using the definition of

R and solving for the potential $\Phi(\sigma, \lambda)$ we obtain

$$\Phi(\sigma, \lambda) = \sigma^{-3} \left(\sigma^3 \Phi(\sigma, \lambda) \right)_{\sigma=0} + \sigma^{-3} \int_0^{\sigma} \sigma' R(\sigma', \lambda) d\sigma'.$$

Applying shoreline boundary condition $\Phi(0, \lambda) = 0$, we have

$$\begin{aligned} \Phi(\sigma, \lambda) &= \frac{1}{\sigma^3} \int_0^{\sigma} \sigma' R(\sigma', \lambda) d\sigma' \\ &= \frac{1}{2\sigma^3} \int_0^{\sigma} \int_{|\lambda-\sigma'|}^{\lambda+\sigma'} R_{\lambda}(s, 0) ds \sigma' d\sigma' \\ &= \frac{g}{\sigma^3} \int_0^{\sigma} \int_{|\lambda-\sigma'|}^{\lambda+\sigma'} s^{-1} \left(s^3 \eta_0(s) \right)_s ds \sigma' d\sigma'. \end{aligned}$$

We now interchange the orders of integration, integrate and simplify to write

$$\Phi(\sigma, \lambda) = \frac{g}{2\sigma^3} \int_{|\lambda-\sigma|}^{\lambda+\sigma} (s^2 + \sigma^2 - \lambda^2) \eta_0(s) s ds. \quad (9)$$

The potential $\Phi(\sigma, \lambda)$ in equation (9) is the potential that solves the linear wave equation (4) and satisfies the conditions (5) and (6). The water dynamics away from the shoreline can now be computed using equation (9) and the generalized CG transform (3).

Note that equation (9) can be used as a tool to analytically study the details of tsunami run-up in $\mathcal{B}_{2/3}$ bays. For example, the number of run-ups of a given wave profile $\eta_0(x)$ can be analytically computed by using (9) to find the shoreline velocity via (3) and then finding the zeros of the velocity. Such an application of our solution is a topic for future research.

4. Analytic solution in U-shaped and V-shaped bays of infinite length

In this section, we present an analytic solution to the run-up problem for any $\mathcal{B}_m$ bay. To this end we first define $\Omega(s, \lambda)$ to be the image of $\sigma^{1/m}\Phi(\sigma, \lambda)$ under the Hankel (Fourier-Bessel) transform [5]. Explicitly, we let

$$\Omega(s, \lambda) = \int_0^\infty \sigma' J_{\frac{1}{m}}(\sigma' s) \left(\sigma'^{\frac{1}{m}} \Phi(\sigma', \lambda) \right) d\sigma' \quad (10)$$

where $J_{1/m}(z)$ is the Bessel function of the first kind [2]. We then multiply (4) by $\sigma^{(m+1)/m} J_{1/m}(\sigma s)$, integrate with respect to σ , apply integration by parts and finally use the boundary conditions (6) to obtain

$$\Omega_{\lambda\lambda} + s^2\Omega = 0. \quad (11)$$

Under the Hankel (Fourier-Bessel) transform, the initial conditions (5) become

$$\Omega(s, 0) = 0 \quad \text{and} \quad \Omega_\lambda(s, \lambda)|_{\lambda=0} = 2g \int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda=0)) J_{\frac{1}{m}}(\sigma' s) d\sigma'. \quad (12)$$

Solving the elementary ordinary differential equation (11) with the conditions (12), we note that

$$\Omega(s, \lambda) = \frac{2g}{s} \int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda=0)) J_{\frac{1}{m}}(\sigma' s) d\sigma' \sin(\lambda s) \quad (13)$$

solves (11) and satisfies (12). Since Ω is defined as the Hankel transform of $\sigma^{1/m} \Phi(\sigma, \lambda)$, we apply the inverse Hankel transform to (13) to obtain

$$\Phi(\sigma, \lambda) = 2g \sigma^{-\frac{1}{m}} \int_0^\infty \left(\int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda=0)) J_{\frac{1}{m}}(\sigma' s) d\sigma' \right) J_{\frac{1}{m}}(\sigma s) \sin(\lambda s) ds. \quad (14)$$

Equivalently, by swapping the order of integration, we have that

$$\Phi(\sigma, \lambda) = 2g \sigma^{-\frac{1}{m}} \int_0^\infty \left(\int_0^\infty J_{\frac{1}{m}}(\sigma' s) J_{\frac{1}{m}}(\sigma s) \sin(\lambda s) ds \right) \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda = 0)) d\sigma' \quad (15)$$

solves (4) and satisfies conditions (5-6). We emphasize that the presented integral form is not singular, unlike the work of Carrier et al. [7].

Finally, to compute the wave dynamics via the generalized CG transform, we need to evaluate $\Phi_\sigma(\sigma, \lambda)$ and $\Phi_\lambda(\sigma, \lambda)$. These derivatives are readily computed from (14) as

$$\Phi_\lambda(\sigma, \lambda) = 2g \sigma^{-\frac{1}{m}} \int_0^\infty \left(\int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda = 0)) J_{\frac{1}{m}}(\sigma' s) d\sigma' \right) J_{\frac{1}{m}}(\sigma s) \cos(\lambda s) s ds \quad (16)$$

and

$$\begin{aligned} \Phi_\sigma(\sigma, \lambda) = & g \sigma^{-\frac{1}{m}} \int_0^\infty \left(\int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda = 0)) J_{\frac{1}{m}}(\sigma' s) d\sigma' \right) (J_{\frac{1}{m}-1}(\sigma s) - J_{\frac{1}{m}+1}(\sigma s)) \sin(\lambda s) s ds \\ & - \frac{2g}{m} \sigma^{-\frac{1}{m}-1} \int_0^\infty \left(\int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda = 0)) J_{\frac{1}{m}}(\sigma' s) d\sigma' \right) J_{\frac{1}{m}}(\sigma s) \sin(\lambda s) ds \end{aligned} \quad (17)$$

respectively. The required integrals for non-shoreline points, $(\sigma > 0, \lambda)$, are easily computable with one of Matlab's [17] many built in integration methods (*e.g.* `integral2()`, `quad()`, etc.). Despite the availability of standard methods, we recommend using the algorithm presented in Appendix A due to the dramatic decrease in runtime over the use of Matlab's `quad()` subroutine.

5. Comparison of our analytic solution for $\mathcal{B}_m$ bays to two previously known solutions

To check the consistency between the presented analytic solution (14) and prior research, we now demonstrate that (14) simplifies to the solution in Carrier et al. [7] for a plane beach ($\mathcal{B}_\infty$ bays), and to the solution found by Didenkulova and Pelinovsky [10] for a parabolic bay ($\mathcal{B}_2$ bays).

Recall that (15) is valid for all $\mathcal{B}_m$ bays and that the Carrier-Greenspan solution is valid for $m \rightarrow \infty$. Thus we need to show that $\lim_{m \rightarrow \infty} \Phi(\sigma, \lambda)$, where $\Phi(\sigma, \lambda)$ is defined in (15) is

the solution given in Carrier et al. [7]. A direct evaluation of the limit yields

$$\lim_{m \rightarrow \infty} \Phi(\sigma, \lambda) = 2g \int_0^\infty \left(\int_0^\infty J_0(\sigma' s) J_0(\sigma s) \sin(\lambda s) ds \right) \sigma' \eta_0(x(\sigma', \lambda = 0)) d\sigma',$$

which is the solution derived by Carrier et al. [7] once the zero initial velocity condition is imposed.

Similarly, we show that our solution agrees with the analytic solution found by Didenkulova and Pelinovsky [10]. By using the identity $J_{1/2}(z) = \sqrt{2/(\pi z)} \sin(z)$ [1, 437-438], equation (15) becomes

$$\Phi(\sigma, \lambda) = \frac{4g}{\pi\sigma} \int_0^\infty \sigma' \eta_0(x(\sigma', \lambda = 0)) \left(\int_0^\infty \frac{\sin(\sigma s) \sin(\lambda s) \sin(\sigma' s)}{s} ds \right) d\sigma'. \quad (18)$$

Consequently, by employing the identity $\int_0^\infty \sin(as)/s ds = (\pi/2)\text{sgn}(a)$ and trigonometric identities, we note

$$\begin{aligned} \int_0^\infty \frac{\sin(\sigma s) \sin(\lambda s) \sin(\sigma' s)}{s} ds &= \frac{\pi}{8} (\text{sgn}(\lambda - \sigma + \sigma') - \text{sgn}(\lambda - \sigma - \sigma') - \text{sgn}(\lambda + \sigma + \sigma') + \text{sgn}(\lambda + \sigma - \sigma')) \\ &= \begin{cases} \frac{\pi}{4}, & |\sigma - \lambda| < \sigma' < \sigma + \lambda \\ 0, & \text{else} \end{cases}. \end{aligned}$$

Hence equation (18) simplifies to

$$\Phi(\sigma, \lambda) = \frac{g}{\sigma} \left(\int_{\lambda-\sigma}^{\lambda+\sigma} \sigma' \eta_0(x(\sigma', \lambda = 0)) d\sigma' \mathcal{U}(\lambda - \sigma) + \int_{\sigma-\lambda}^{\lambda+\sigma} \sigma' \eta_0(x(\sigma', \lambda = 0)) d\sigma' \mathcal{U}(\sigma - \lambda) \right).$$

Here $\mathcal{U}$ denotes the Heaviside function [1]. Changing notations, we write

$$\Phi(\sigma, \lambda) = \frac{[\Theta(\lambda + \sigma) - \Theta(\lambda - \sigma)] \mathcal{U}(\lambda - \sigma) + [\Theta(\lambda + \sigma) - \Theta(\sigma - \lambda)] \mathcal{U}(\sigma - \lambda)}{\sigma} \quad (19)$$

where

$$\Theta(\xi) = g \int_0^\xi \eta_0(x(\sigma, \lambda = 0)) \sigma d\sigma = g \int_0^\xi \left(\frac{\sigma^2}{6g} + \alpha x(\sigma, \lambda = 0) \right) \sigma d\sigma = \frac{\xi^4}{24} + g\alpha \int_0^\xi x(\sigma, \lambda = 0) \sigma d\sigma.$$

Equation (19) agrees with the solution found by Didenkulova and Pelinovsky [10] up to a sign difference and coefficient. These differences are due to two minor differences between

Rybkin et al. [19] and Didenkulova and Pelinovsky [10]: a factor of $3/2$ in the definition of the potential $\Phi(\sigma, \lambda)$ and the choice in the direction of x .

The results of this section show that our solution is in fact a generalization of two previously known analytic solutions. We now turn our attention to the computation of the shoreline dynamics.

6. Computing the exact shoreline dynamics

The shoreline dynamics for the long wave run-up problem are of great interest in many applications. Since both (14) and the inverse transform (3) have singularities at $\sigma = 0$, regularization is required to find the shoreline dynamics. To this end, we present the needed regularizations and examine how to compute the shoreline dynamics using our solution (14).

We first use the continuity of $\Phi(\sigma, \lambda)$ and the identity $\lim_{z \rightarrow 0} \frac{J_\nu(z)}{z^\nu} = \frac{1}{2^\nu \Gamma(1+\nu)}$ [1] to note that

$$\Phi(0, \lambda) = \frac{2g}{2^{\frac{1}{m}} \Gamma(1 + \frac{1}{m})} \int_0^\infty \int_0^\infty \sigma'^{1+\frac{1}{m}} \eta_0(x(\sigma'), \lambda = 0) J_{\frac{1}{m}}(\sigma' s) d\sigma' \sin(\lambda s) s^{\frac{1}{m}} ds. \quad (20)$$

The only part of the CG transform (3) that contains a singularity at $\sigma = 0$ is

$$u(\sigma, \lambda) = \left(1 + \frac{1}{m}\right) \frac{\Phi_\sigma(\sigma, \lambda)}{\sigma},$$

we only need to regularize the computation of the shoreline velocity. To this effect we apply L'Hopital's rule along with continuity to obtain

$$u(0, \lambda) = \left(1 + \frac{1}{m}\right) \lim_{\sigma \rightarrow 0} \frac{\Phi_\sigma(\sigma, \lambda)}{\sigma} = \left(1 + \frac{1}{m}\right) \Phi_{\sigma\sigma}(0, \lambda).$$

Furthermore, equation (4) yields

$$\Phi_{\lambda\lambda}(0, \lambda) = \Phi_{\sigma\sigma}(0, \lambda) + \frac{m+2}{m} \lim_{\sigma \rightarrow 0} \left(\frac{\Phi_\sigma(\sigma, \lambda)}{\sigma} \right) = 2 \frac{m+1}{m} \Phi_{\sigma\sigma}(0, \lambda),$$

and hence the shoreline velocity is given by

$$u(0, \lambda) = \frac{1}{2} \Phi_{\lambda\lambda}(0, \lambda). \quad (21)$$

Thus the shoreline wave height and velocity are computable with knowledge of the derivatives $\Phi_\lambda(\sigma, \lambda)$ and $\Phi_{\lambda\lambda}(\sigma, \lambda)$. Computing the needed derivatives from (20) yields

$$\Phi_\lambda(0, \lambda) = \frac{2g}{2^{\frac{1}{m}} \Gamma(1 + \frac{1}{m})} \int_0^\infty \int_0^\infty \sigma'^{1+\frac{1}{m}} \eta_0(x(\sigma'), \lambda = 0) J_{\frac{1}{m}}(\sigma' s) d\sigma' \cos(\lambda s) s^{\frac{1}{m}+1} ds \quad (22)$$

and

$$\Phi_{\lambda\lambda}(0, \lambda) = \frac{-2g}{2^{\frac{1}{m}} \Gamma(1 + \frac{1}{m})} \int_0^\infty \int_0^\infty \sigma'^{1+\frac{1}{m}} \eta_0(x(\sigma'), \lambda = 0) J_{\frac{1}{m}}(\sigma' s) d\sigma' \sin(\lambda s) s^{\frac{1}{m}+2} ds. \quad (23)$$

The shoreline dynamics can now be found by computing (21-23), using (21) to find the shoreline velocity and then directly computing the rest of the CG transform through (3). As with the non-shoreline points, we recommend using the algorithm presented in Appendix A to compute the required double integrals.

7. Comparison to a numerical solution

In this section we compare our analytic solution to a numerical method for the run-up of a solitary Gaussian wave in three different bays each having slope $\alpha = 1/50$. The particular bays we choose to examine are $\mathcal{B}_3$, $\mathcal{B}_1$ and $\mathcal{B}_{2/3}$ bays. These bays are shown in Figure 2. Our choice of these bays facilitates the study of run-up in V-shaped bays while emphasizing that our solution is valid in U-shaped bays. We note that $\mathcal{B}_1$ bays are the transitional bays between U-shaped (*i.e.* $\mathcal{B}_m$ where $m > 1$) and V-shaped bays (*i.e.* $\mathcal{B}_m$ where $0 < m \leq 1$), and are also a degenerate case of the trapezoidal bays studied by Harris et al. [12].

For all of our tests we use the initial condition given by the Gaussian profile

$$\tilde{\eta}_0(x) = 0.03 \exp \left(- \left(\frac{x - 1500}{500} \right)^2 \right) \quad (24)$$

shown in Figure 3. The initial wave is 0.03 meters tall at its maximum point and is centered 1500 meters away from the shore. Note that the depth of the bay at the point of maximal initial height is 30 meters.

The numerical model used in our comparison is developed by Harris et al. [12] and solves the cross-sectionally averaged shallow water wave equations using the CG transform. Note that the numerical model by Harris et al. [12] solves (4) and finds the wave dynamics by implementing the CG transform (3). In particular, equation (4) is turned into a system of equations where $\Phi_\sigma(\sigma, \lambda)$ and $\Phi_\lambda(\sigma, \lambda)$ are the unknown functions and the resulting system is solved via an implicit finite difference method. This model has been verified for parabolic bays (*i.e.* $\mathcal{B}_2$ bays) via comparison with the analytic solution found by Didenkulova and Pelinovsky [10] and has been validated against a 2-D fully numerical scheme for the nonlinear shallow water wave equations in the case of a constantly sloping bay with trapezoidal cross-sections [12]. For convenience, we refer to the solution obtained by the Harris et al. [12] numeric method as the *numerical model* and denote it by $\eta_n(x, t)$. Likewise, we denote our solution (15) by $\eta_s(x, t)$.

The shoreline plots for our initial Gaussian wave are shown in Figure 4. In $\mathcal{B}_3$ bays, we note that there is only one run-up and one run-down. This pattern for run-up of a solitary Gaussian wave has been extensively seen in literature [3, 6, 7, 13, 15, 18, 21, 23]. For $\mathcal{B}_1$ bays, we also see two run-ups and one run-down but the second run-up is smaller than the first one. The extra run-up is interesting but since the second run-up is smaller than the first one, this pattern of run-up is not atypical. Finally, in $\mathcal{B}_{2/3}$ bays we see that the Gaussian wave produces two run-ups and one run-down. Most notably, the second run-up is larger than the first run-up. This behavior is unlike what has been previously seen in studies concerning solitary wave run-up [3, 6, 7, 13, 15, 18, 21, 23]. Finally, we would like to note that in all three bays the Gaussian wave has a amplification factor, the ratio of the maximum run-up to the initial max height, of between 4.0 and 14.5.

For all three bays our solution is in excellent agreement with the numerical model as shown in Figure 4. To quantify the error between the models we examine the relative error, $|(\eta_s(x, t) - \eta_n(x, t))/\eta_s(x, t)|$, at the points of maximum run-up and minimum run-down. To

this effect, the relative error in the maximum run-up and in the minimum run-down for $\mathcal{B}_3$ bays are 0.08% and 0.17% respectively; the relative error in the maximum run-up and in the minimum run-down for $\mathcal{B}_1$ bays are 0.14% and 0.23% respectively; and the relative error in the maximum run-up and in the minimum run-down for $\mathcal{B}_{2/3}$ bays are 0.39% and 0.27% respectively. These relative errors can be further reduced (at the expense of a longer runtime) by decreasing the grid size used in the numerical model and by using a stricter convergence criterion for computing the analytic solution. Note that with the exception of $\mathcal{B}_{2/3}$ bays, the relative error in the maximum run-up is smaller than the relative error in the minimum run-down. We believe this phenomena is due to a combination of the fact that the largest run-up in $\mathcal{B}_{2/3}$ bays occurs after the minimum run-down and the fact that the error in the numerical solution increases as a function of time.

In the rest of this section, we examine the space-time behavior of both our solution and the numerical model in $\mathcal{B}_{2/3}$ bays. We choose $\mathcal{B}_{2/3}$ bays since such bays have an interesting run-up pattern and since we believe a discussion of the space-time behavior details in the other bays does not add anything to the content of this paper. Figure 5 shows the near shore waves at the selected times of 151, 205, 257, 300 and 332 seconds. the shoreline wave height at these times are shown in Figure 4 a-e. Furthermore, the space-time behavior at the times in question are shown in Figure 5 a-e. Two solutions are visually indistinguishable. Note that the maximum run-ups are shown in plots c and e and that the minimum run-down is shown in plot d.

8. Conclusion

In this article we developed an analytic solution for the long wave run-up problem that is valid in $\mathcal{B}_{2/3}$ bays and an analytic solution that is valid in U-shaped and V-shaped bays. The first solution was developed by applying the technique of Courant and Hilbert [8] and the more general solution is found by applying the Fourier-Bessel (Hankel) transform to the linear variable velocity wave equation that results from the generalized Carrier-Greenspan

transform. We show that our analytic solution for any U-shaped and V-shaped bays is a generalization of two previously known solutions by proving that our solution simplifies to the exact solutions found by Carrier et al. [7] and Didenkulova and Pelinovsky [10]. The simplicity of the form of our solution for $\mathcal{B}_{2/3}$ bays allows for an easier examination of the run-up properties of a given wave than our analytic solution.

The presented analytic solution is tested via a comparison of our analytic solution to a numerical model developed by Harris et al. [12]. The results of these examinations are that both our analytic solution and the numerical model developed by Harris et al. [12] agree to a high level of accuracy (less than 0.5% relative error). During our validation process, we discovered a Gaussian wave which when used as an initial profile for the run-up problem in a $\mathcal{B}_{2/3}$ bay, produces two wave run-ups where the first run-up is not the largest one.

Our analytic solution allows for future work in the relaxation of the non-zero initial velocity assumption and the potential for the development of new run-up laws for various initial wave profiles. Furthermore, the applicability of our solution to all non-breaking initial profiles is another topic for future work.

Appendix A Numerical algorithm to evaluate (16), (17), (22) and (23)

In this appendix, we present a method to quickly evaluate the double integrals in equations (16), (17), (22) and (23). For simplicity, we demonstrate our method on the double integral in equation (16),

$$I(\sigma, \lambda) = \int_0^\infty \left[\int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda = 0)) J_{\frac{1}{m}}(\sigma' s) d\sigma' \right] J_{\frac{1}{m}}(\sigma s) \cos(\lambda s) s ds.$$

We first approximate the outermost integral by a second order integration method (although the use of higher order methods is possible) with a fixed step size and some finite upper limit. The value for the upper limit is chosen based on how accurate the computed initial condition approximates the actual initial condition. We then approximate the outermost integral by

$$I(\sigma, \lambda; N, \Delta s, \{w_i\}_{i=0}^N) = \sum_{i=0}^N \left[\int_0^\infty \sigma'^{\frac{m+1}{m}} \eta_0(x(\sigma', \lambda = 0)) J_{\frac{1}{m}}(\sigma' i \Delta s) d\sigma' \right] J_{\frac{1}{m}}(\sigma i \Delta s) \cos(\lambda i \Delta s) i \Delta s w_i \Delta s.$$

where $\{w_i\}_{i=0}^N$ are the weights for some numerical integration method, Δs is the fixed step size and N is some integer. For the approximation of the inner integral, we use Matlabs' `quad()` function [17]. Note that the function $x(\sigma, \lambda = 0)$ is computable by applying Newtons' method to the third equation in (3).

A straightforward analysis shows that $I(\sigma, \lambda; N, \Delta s, \{w_i\}_{i=0}^N)$ is periodic in λ with period $2\pi/\Delta s$. Since $I(\sigma, \lambda)$ is not periodic, we must impose $\pi/\Delta s > \lambda_{\max}$, where $\lambda_{\max}$ is the largest value for λ under consideration. We note that a smaller Δs leads to a better approximation of $I(\sigma, \lambda)$ by $I(\sigma, \lambda; N, \Delta s, \{w_i\}_{i=0}^N)$. To this effect, Δs should be chosen so that the integrand is well sampled. An examination of the integrand reveals that there are two oscillatory factors in the outermost integral, $\cos(\lambda s)$ and $J_{1/m}(\sigma s)$. We thus impose that

$$\Delta s \leq \frac{1}{K} \min_{i \in \{1, \dots, L\}} \left\{ \frac{j_{1/m, i+1} - j_{1/m, i}}{\sigma_{\max}}, \frac{\pi}{\lambda_{\max}} \right\}$$

where $j_{1/m, i}$ is the i^{th} root of $J_{1/m}(z)$, L is the smallest integer such that $j_{1/m, L} > N\Delta s$ and $\sigma_{\max}$ and $\lambda_{\max}$ are the largest values for which we wish to compute σ and λ . The integer K determines how many points should be sampled for the smallest half period of $\cos(\lambda s)$ and $J_{1/m}(\sigma s)$. By construction, the above bound for Δs ensures that there at least K sample values of our integrand per half period. Thus we let

$$\Delta s = \min \left\{ \frac{\pi}{\lambda_{\max}}, \frac{1}{K} \min_{i \in \{1, \dots, L\}} \left\{ \frac{j_{1/m, i+1} - j_{1/m, i}}{\sigma_{\max}}, \frac{\pi}{\lambda_{\max}} \right\} \right\}$$

where K is provided by the user.

Our approximation $I(\sigma, \lambda; N, \Delta s, \{w_i\}_{i=0}^N)$ now depends only on the values of K and N . To compute our analytic solution we therefore first pick some positive integer K and then compute $I(\sigma, \lambda; N, \Delta s, \{w_i\}_{i=0}^N)$ for $N = 1, 2, \dots$ until either $I(\sigma, \lambda; N, \Delta s, \{w_i\}_{i=0}^N)$ agrees with the initial condition to the desired accuracy or fails to converge. In the case where $I(\sigma, \lambda; N, \Delta s, \{w_i\}_{i=0}^N)$ fails to converge, the parameter K is increased and the test for the accuracy in computing the initial condition repeats. For the models considered in this paper the value $\lambda_{\max} = 200$ was always used and $K = 50$ was sufficient for our desired level of agreement with the initial condition.

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Figure 1: Top Left: a (x, z) cross-sectional view of an N -wave in a constantly sloping parabolic or $\mathcal{B}_2$ bay. Top Right: a (y, z) cross-sectional view of a parabolic or $\mathcal{B}_2$ bay with water displacement $\eta(x)$. Bottom: a 3-D view of the bay and N -wave shown the two cross-sectional views above. Note that the bathymetry that is used is given by the function $z \sim |y|^2 + 0.01x$.

Figure 2: 3-D views of a Gaussian wave in constantly sloping bays with various cross-sections. Top Left: The $\mathcal{B}_{2/3}$ bay with cross-section $|y|^{2/3}$. Top Right: The $\mathcal{B}_1$ bay with cross-section $|y|$. Bottom: The $\mathcal{B}_3$ bay with cross-section $|y|^3$.

Figure 3: The initial condition $\tilde{\eta}_0(x)$ where $\tilde{\eta}(x)$ is defined in (24). Note that this wave is centered 1500 meters from the shore, has a maximum height of 0.03 meters (at a depth of 30 meters) and has not yet reached the unperturbed shoreline. This initial condition was used in all comparisons between the analytic solution and the numerical model by Harris et al. [12].

Figure 4: The run-up of the Gaussian wave $\tilde{\eta}_0(x)$ defined in (24) for three different bays. In each of the shoreline plots to the right our analytic solution is given in red and the results of the numerical model developed by Harris et al. [12] are shown in blue. Recall that $\mathcal{B}_m$ denotes a constantly sloping bay with cross-section given by $\gamma|y|^m$ for some positive γ . Top Left: The cross-section of the $\mathcal{B}_3$ bay with $\gamma = 1$. Top Right: The shoreline wave height for this bay. Here we see that the initial wave has only one run-up and run-down. Middle Left: The cross-section of the $\mathcal{B}_1$ bay with $\gamma = 1$. Middle Right: The shoreline wave height for this bay. Here we see that the initial wave has two run-ups with a run-down in between. Note that the second run-up is smaller than the first run-up. Top Left: The cross-section of the $\mathcal{B}_{2/3}$ bay with $\gamma = 1$. Top Right: The shoreline wave height for this bay. Here we see that the initial wave has two run-ups with a run-down in between. Note that the second run-up is larger than the first run-up.

Figure 5: A comparison of our analytic solution (14) to the numerical model developed by Harris et al. [12] for the run-up of a Gaussian wave in a $\mathcal{B}_{2/3}$ bay. Throughout each plot, the solid black line represents the bathymetry, the black star represents the unperturbed shoreline position and the black dashed line shows the unperturbed water level. In plot a, we see the left moving part of the Gaussian hump as it approaches the shore. Plot b is taken at 205 seconds and shows the point of maximum run-up. Plot c shows the point of minimum run-down. The wave obtains another maximum run-up in plot d. Note that the run-up in plot d is larger than the run-up in plot b. Finally, in plot e we see the wave as the water level at the shore approaches the unperturbed water level.