A modified Perry's conjugate gradient method-based derivative-free method for solving large-scale nonlinear monotone equations *

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Abstract. In this paper, we propose a derivative-free method for solving large-scale nonlinear monotone equations. It combines the modified Perry's conjugate gradient method [19] for unconstrained optimization problems and the hyperplane projection method [8]. We prove that the proposed method converges globally if the equations are monotone and Lipschitz continuous without differentiability requirement on the equations, which makes it possible to solve some nonsmooth equations. Another good property of the proposed method is that it is suitable to solve large-scale nonlinear monotone equations due to its lower storage requirement. Preliminary numerical results show that the proposed method is promising.

Keywords: Monotone equations; Derivative-free method; Global convergence; Projection method.

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1. Introduction

In this paper, we consider solving the following nonlinear equations:

$$F(x) = 0, \quad (1.1)$$

where $F : R^n \rightarrow R^n$ is continuous and monotone. By monotonicity, we mean

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in R^n. \quad (1.2)$$

Nonlinear monotone equations arise in various applications such as subproblems in the generalized proximal algorithms with Bregman distances [1]. Some monotone variational inequality problems can also be converted into systems of nonlinear monotone equations by means of fixed point maps or normal maps if the underlying function satisfies some coercive conditions [2].

Among the iterative methods for solving problem (1.1), the Newton method, quasi-Newton methods, and their variants are attractive because of their fast local convergence property (see

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[3, 4, 5],[6]). The main drawback of these methods, for large values of n , is that they need to solve a linear system of equations at each iteration using the Jacobian matrix or an approximation of it.

In recent years, great efforts have been made to find a solution of problem (1.1) especially for large-scale nonlinear monotone equations. Zhang and Zhou [9] combined the spectral gradient method [7] with the projection method [8] to solve nonlinear monotone equations. The method in [9] is globally convergent if the nonlinear equations are Lipschitz continuous. In addition, Yu et al. [11] proposed a spectral gradient procedure and a projection method, respectively, for nonlinear monotone equations with convex constraints.

Recently, researchers also aim at developing conjugate gradient based method for solving large-scale nonlinear equations. Cheng [22] first introduced a PRP-type method for nonlinear monotone equations, which is a combination of the well-known PRP conjugate gradient method and the hyperplane projection method [8]. Then some derivative-free methods were presented which also belong to the conjugate gradient scheme. Li and Li[17] proposed a three-term PRP based [13] derivative-free iterative method for solving the nonlinear monotone equations (1.1). Combined the three-term HS [14] methods with the projection scheme in Solodov & Svaiter [8], Yan et al. [18] proposed a derivative-free iterative method for solving the nonlinear monotone equations (1.1). In addition, Li [17] and Yan et al.[18] discussed the two-term version derivative-free method in Cheng [21] for solving the nonlinear monotone equations (1.1). Xiao and Zhu [15] presented a modified version of the CG_DESCENT method of Hager and Zhang [12] to solve the constrained nonlinear monotone equations. And under some mild conditions, they proved that their proposed method is globally convergent. More recently, Li [25] extended the modified Liu-Storey method [24] to solve the nonlinear monotone equations.

The conjugate gradient-based derivative-free methods for nonlinear monotone equations possess some nice properties. First, due to the lower storage requirement, they are suitable for solving large-scale equations. Second, the methods are function value-based methods and can be applied to solve nonsmooth equations. Third, global convergence is achieved without the differentiability assumption on the equations. In addition, the methods are globally convergent even when the solution set is not a singleton.

In this paper, we will develop an efficient derivative-free algorithm for solving nonlinear monotone equations (1.1). Our work can be considered as a further research of the modified Perry's conjugate gradient method [19] in unconstrained optimization. Here, we extend it to solve nonlinear monotone equations with some little modifications. The rest of this paper is organized as follows. In Section 2, we first propose the method. In Section 3, we prove the global convergence of the proposed method. In Section 4, we report some numerical results to test the proposed method.

2. Motivation and the algorithm

In this section, we simply introduce modified Perry's conjugate gradient method in [19] for solving large-scale unconstrained optimization problems and describe the hyperplane projection method of Solodov and Svaiter. Combined with the hyperplane projection method, we extend modified Perry's conjugate gradient method [19] to solve large-scale constrained nonlinear equations (1.1).

In [19], Livieris and Pintelas proposed a modified Perry's conjugate gradient method. In order to guarantee that the proposed method generates descent directions, Livieris and Pintelas [19] exploit the idea of the spectral-type modified Fletcher-Reeves method [9, 10] to the well-known Perry's conjugate gradient method[20]. Under suitable conditions, they establish the global convergence of our proposed method provided that the line search satisfy the Wolfe conditions. Numerical results demonstrate that the proposed method is promising.

Motivated by the modified Perry's conjugate gradient method [19] as well as by the projection scheme in Solodov & Svaiter [8], we propose an efficient derivative-free algorithm for solving nonlinear monotone equations (1.1). Firstly, we define the search direction as follows

$$d_k = \begin{cases} -F_0, & \text{if } k = 0, \\ -(I + \beta_k^{\text{MP}} \frac{F_k^T d_{k-1}}{\|F_k\|^2})F_k + \beta_k^{\text{MP}} d_{k-1}, & \text{if } k \geq 1, \end{cases} \quad (2.1)$$

where

$$\beta_k^{\text{MP}} = \frac{F_k^T (y_{k-1} - \bar{s}_{k-1})}{\bar{d}_{k-1}^T w_{k-1}} \quad (2.2)$$

$$w_{k-1} = y_{k-1} + \gamma \bar{s}_{k-1}, \gamma > 0, \quad y_{k-1} = F(z_{k-1}) - F(x_{k-1}), \quad \bar{s}_{k-1} = z_{k-1} - x_{k-1} = \alpha_{k-1} d_{k-1}. \quad (2.3)$$

For convenience, we call the method (2.1) and (2.2) as MPD method.

It is easy to see that the condition

$$F_k^T d_k = -\|F_k\|^2, \quad (2.4)$$

holds, for any line search.

About the hyperplane projection method in [8], note that by the monotonicity of F , for any $\tilde{x}$ such that $F(\tilde{x}) = 0$, we have

$$\langle F(z_k), \tilde{x} - z_k \rangle \leq 0$$

Let x_k be the current iterate. By performing some kind of line search procedure along a direction d_k , a point $z_k = x_k + \alpha_k \bar{d}_k$ can be computed such that

$$\langle F(z_k), x_k - z_k \rangle > 0.$$

Thus the hyperplane

$$H_k = \{x \in R^n | \langle F(z_k), x - z_k \rangle = 0\}.$$

strictly separates the current iterate x_k from the zeros of equation (1.1). Therefore, it is reasonable to let the next iterate x_{k+1} be the projection of x_k onto the hyperplane. Specifically, x_{k+1} is determined by

$$x_{k+1} = x_k - \frac{F(z_k)^T(x_k - z_k)}{\|F(z_k)\|^2} F(z_k). \quad (2.5)$$

In the remainder of this paper, we always assume that F satisfies the following assumptions.

Assumption (A)

(i) Function F is monotone:

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in R^n. \quad (2.6)$$

(ii) Function F is Lipschitz continuous, i.e., there is a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in R^n. \quad (2.7)$$

We state the steps of the algorithm as follows.

Algorithm 2.1.

Step 0: Given an initial point $x_0 \in R^n$, and constants $\beta > 0, \sigma > 0, \rho \in (0, 1)$. Let $k := 0$.

Step 1: Stop if $F(x_k) = 0$. Otherwise, compute d_k by equation (2.1), where β_k is computed by (2.2) and (2.3).

Step 2: Let $\alpha_k = \max\{\beta\rho^i : i = 0, 1, \dots, \}$ be determined by the line search rule

$$-F(x_k + \alpha_k d_k)^T d_k \geq \sigma \alpha_k \|F(x_k + \alpha_k d_k)\| \cdot \|d_k\|^2. \quad (2.8)$$

Step 3: Compute $z_k = x_k + \alpha_k d_k$.

Step 4: Stop if $F(z_k) = 0$. Otherwise, update x_{k+1} by (2.5). Let $k := k + 1$ and go to Step 1.

3. Global convergence analysis

Lemma 3.1 *Let sequences x_k and z_k be generated by Algorithm 2.1. Then, we have*

$$\alpha_k \geq \min\left\{\beta, \frac{\rho}{L + \sigma\|F(z'_k)\|} \frac{\|F_k\|^2}{\|d_k\|^2}\right\}, \quad (3.1)$$

where $z'_k = x_k + \alpha'_k d_k, \alpha'_k = \rho^{-1}\alpha_k, \delta = 1 - \frac{1}{4\mu}$.

Proof. If $\alpha_k \neq \beta$, by the line search process, we have that $\alpha'_k = \rho^{-1}\alpha_k$ does not satisfy (2.8). That is,

$$-F(z'_k)^T d_k < \sigma \alpha'_k \|F(z'_k)\| \cdot \|d_k\|^2.$$

It follows from (2.4) and the Lipschitz continuity of F that

$$\begin{aligned} \|F_k\|^2 &= -F_k^T d_k = (F(z'_k) - F(x_k))^T d_k - F(z'_k)^T d_k \\ &\leq L\alpha'_k \|d_k\|^2 + \sigma \alpha'_k \|F(z'_k)\| \cdot \|d_k\|^2 \\ &\leq \rho^{-1}\alpha_k (L + \sigma\|F(z'_k)\|) \|d_k\|^2. \end{aligned}$$

Hence, we have

$$\alpha_k \geq \frac{\rho}{L + \sigma \|F(z'_k)\|} \frac{\|F_k\|^2}{\|d_k\|^2}.$$

This implies (3.1).

The next lemma comes from Lemma 2.1 in Solodov & Svaiter [8], which also holds for Algorithm 2.1. The proof is similar to that in Solodov & Svaiter [8], and we therefore omit it.

Lemma 3.2 *For any $\bar{x}$ satisfying $F(\bar{x}) = 0$, we have that*

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - \|x_{k+1} - x_k\|^2.$$

In particular, the sequence $\{x_k\}$ is bounded and

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\|^2 < \infty.$$

Remark 1. Lemma 3.2 reveals an interesting property of Algorithm 2.1, which is that the distance from the iterates to the solution set of equation (1.1) decreases with k .

Lemma 3.3 *Suppose that $\{x_k\}$ is generated by Algorithm 2.1, Then, we have*

$$\lim_{k \rightarrow \infty} \alpha_k \|d_k\| = 0. \quad (3.2)$$

Proof. For any k , it follows from (2.5) and (2.8) that

$$\|x_{k+1} - x_k\| = \frac{|F(z_k)^T(x_k - z_k)|}{\|F(z_k)\|} = \frac{-\alpha_k F(z_k)^T d_k}{\|F(z_k)\|} \geq \sigma \alpha_k^2 \|d_k\|^2. \quad (3.3)$$

In particular, we have that

$$\lim_{k \rightarrow \infty} \alpha_k \|d_k\| = 0.$$

The next lemma comes from Lemma 2.4 in Yan et al. [18], which also holds for Algorithm 2.1. Here we omit it.

Lemma 3.4 *Suppose that $\{x_k\}$ is generated by Algorithm 2.1, $\bar{x}$ satisfy $F(\bar{x}) = 0$, $z'_k = x_k + \alpha'_k d_k$, $\alpha'_k = \rho^{-1} \alpha_k$. Then, the sequences $\{\|F_k\|\}$ and $\{\|F(z'_k)\|\}$ are bounded. That is, there exists constant $M \geq 0$, such that*

$$\|F_k\| \leq M, \quad \|F(z'_k)\| \leq M. \quad (3.4)$$

The following theorem establishes the global convergence of our method.

Theorem 3.1 *Let $\{x_k\}$ be generated by Algorithm 2.1. Then, we have*

$$\liminf_{k \rightarrow \infty} \|F_k\| = 0. \quad (3.5)$$

Proof. For the sake of contradiction, we suppose that the conclusion is not true. Then there exists a constant $\varepsilon > 0$ such that $\|F_k\| > \varepsilon$. Since $F_k \neq 0$ and with (2.4), it follows that $d_k \neq 0$. By the monotonicity of F and (2.1), we have,

$$\begin{aligned}\bar{s}_{k-1}^T w_{k-1} &= \langle F(z_{k-1}) - F(x_{k-1}), z_{k-1} - x_{k-1} \rangle + \gamma \bar{s}_{k-1}^T \bar{s}_{k-1} \\ &\geq \gamma \bar{s}_{k-1}^T \bar{s}_{k-1}.\end{aligned}$$

This together with the definition of $\bar{s}_{k-1}$ implies

$$d_{k-1}^T w_{k-1} \geq \gamma \alpha_{k-1} \|d_{k-1}\|^2. \quad (3.6)$$

We have from (2.2), (2.3) and (3.6) that

$$\begin{aligned}|\beta_k^{\text{MP}}| &= \left| \frac{F_k^T (y_{k-1} - \bar{s}_{k-1})}{d_{k-1}^T w_{k-1}} \right| \leq \frac{|F_k^T y_{k-1}| + |F_k^T \bar{s}_{k-1}|}{d_{k-1}^T w_{k-1}} \\ &\leq \frac{L \alpha_{k-1} \|d_{k-1}\| \|F_k\| + \alpha_{k-1} \|d_{k-1}\| \|F_k\|}{\gamma \alpha_{k-1} \|d_{k-1}\|^2} \\ &= \left(\frac{L+1}{\gamma} \right) \frac{\|F_k\|}{\|d_{k-1}\|}.\end{aligned}$$

Therefore, from (2.1) and (3.4), we obtain

$$\begin{aligned}\|d_k\| &\leq \|F_k\| + |\beta_k^{\text{MP}}| \frac{|F_k^T d_{k-1}|}{\|F_k\|^2} \|F_k\| + |\beta_k^{\text{MP}}| \|d_{k-1}\| \\ &\leq \|F_k\| + 2 \left(\frac{L+1}{\gamma} \right) \|F_k\| \\ &= \left(1 + \frac{2(L+1)}{\gamma} \right) M.\end{aligned}$$

Define $C = \left(1 + \frac{2(L+1)}{\gamma} \right)$. We have $\|d_k\| \leq C$.

This together with Lemmas 2.1 and 2.4 and inequalities $\|d_k\| \geq \varepsilon, \|F_k\| \geq \varepsilon$ implies for all k sufficiently large,

$$\begin{aligned}\alpha_k \|d_k\| &\geq \min \left\{ \beta, \frac{\delta \rho}{L + \sigma \|F(z'_k)\|} \frac{\|F_k\|^2}{\|d_k\|^2} \right\} \|d_k\| \\ &\geq \min \left\{ \beta \varepsilon, \frac{\delta \rho \varepsilon^2}{(L + \sigma M) \|C\|} \right\} > 0.\end{aligned}$$

The last inequality yields a contradiction with (3.2). Consequently, (3.5) holds. The proof is complete.

4. Numerical experiments

This section reports some numerical experiments. We test the MPE method and compare it with the modified-PRP-based derivative-free method in Li and Li[17](called MPRPL) and the modified-HS-based derivative-free method in Yan [18](called MHSY).

The methods in Tables have the following meanings:

- "MPD" stands for the modified-Perry's-CG-based derivative-free method with the search direction 2.1
- "MPRPL" stands for the modified-PRP-based derivative-free method with the following search direction

$$d_k = \begin{cases} -F_0, & \text{if } k = 0, \\ -(I + \beta_k^{\text{PRP}} \frac{F_k^T d_{k-1}}{\|F_k\|^2})F_k + \beta_k^{\text{PRP}} d_{k-1}, & \text{if } k \geq 1, \end{cases} \quad (4.1)$$

where

$$y_{k-1} = F_k - F_{k-1}, \quad \beta_k^{\text{PRP}} = \frac{F_k^T y_{k-1}}{\|F_{k-1}\|^2}.$$

- "MHSY" stands for the modified-HS-based derivative-free method with the following search direction

$$d_k = \begin{cases} -F_0, & \text{if } k = 0, \\ -(I + \beta_k^{\text{MHS}} \frac{F_k^T d_{k-1}}{\|F_k\|^2})F_k + \beta_k^{\text{MHS}} d_{k-1}, & \text{if } k \geq 1, \end{cases} \quad (4.2)$$

where

$$y_{k-1} = F_k - F_{k-1}, \quad \beta_k^{\text{MHS}} = \frac{F_k^T w_{k-1}}{d_{k-1}^T w_{k-1}}, \quad , w_k = y_k + t \|F_k\| \bar{s}_k$$

$$\bar{s}_k = z_k - x_k = \alpha_k d_k, \quad t = 1 + \|F_k\|^{-1} \max\left(0, -\frac{y_k^T \bar{s}_k}{\|\bar{s}_k\|^2}\right).$$

For the three methods, we choose the same initial trial steplength as

$$\beta = \left| \frac{F(x_k)^T d_k}{d_k^T (F(x_k + \epsilon d_k) - F(x_k)) / \epsilon} \right| \approx \left| \frac{F(x_k)^T d_k}{d_k^T \nabla F(x_k) d_k} \right|, \quad \epsilon = 10^{-8}. \quad (4.3)$$

In order to improve the efficiency of the three methods in practical computation, we utilize the following line search process.

Line Search Process:

If inequality

$$-F(x_k + \beta d_k)^T d_k \geq \sigma \beta \|F(x_k + \beta d_k)\| \cdot \|d_k\|^2, \quad (4.4)$$

holds, then we let $\alpha_k = \beta$. Otherwise we let α_k be the largest scalar in the set $\{\beta \rho^i, i = 0, 1, \dots\}$ such that inequality (4.4) is satisfied. In addition, if the above β satisfies $\beta < 10^{-4}$, we set $\beta = 1$. In the line search process, $\rho = 0.5, \sigma = 2$.

We stop the iteration if the condition $\min\{\|F(x_k)\|, \|F(z_k)\|\} \leq atol + rtol \|F(x_0)\|$ is satisfied, where $atol = rtol = 10^{-4}$. All of the algorithms are coded in Matlab 7.0 and run on a personal computer with a 2.0 GHZ CPU processor.

We test the performance of MPD, MPRPL, MHSY methods on the following five problems with different initial points and various sizes .

Problem 1 ([23]). The elements of function $F(x)$ are given by

$$\begin{aligned} F_1(x) &= 2x_1 + \sin(x_1) - 1, \\ F_i(x) &= -2x_{i-1} + 2x_i + \sin(x_i) - 1, \quad i = 2, \dots, n-1, \\ F_n(x) &= 2x_n + \sin(x_n) - 1. \end{aligned}$$

Problem 2 ([6]). The elements of $F(x)$ are given by

$$F_i(x) = x_i - \sin(x_i), \quad i = 1, \dots, n.$$

Problem 3 ([23]). The elements of $F(x)$ are given by

$$F_i(x) = 2x_i - \sin(|x_i|), \quad i = 1, \dots, n.$$

Problem 4 ([6]). Function $F(x)$ is given by

$$F(x) = Ax + g(x)$$

where $g(x) = (e^{x_1} - 1, e^{x_2} - 1, \dots, e^{x_n} - 1)^T$

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

Problem 5. Function $F(x)$ is given by

$$F(x) = Ax + |X| - B$$

where $|X| = (|x_1|, |x_2|, \dots, |x_n|)^T$, $B = (1, 1, \dots, 1)^T$, and

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

The results are listed in Tables 1-5, where $x_1 = (0.1, \dots, 0.1)^T$, $x_2 = (1, \dots, 1)^T$, $x_3 = (1, \frac{1}{2}, \dots, \frac{1}{n})^T$, $x_4 = (-10, \dots, -10)^T$, $x_5 = (-0.1, \dots, -0.1)^T$, $x_6 = (-1, \dots, -1)^T$. We will report the following results: the CPU time **Time** (in seconds), the number of iterations **Iter**, and the number of function evaluations **Feval**.

Table 1: Test results for Problem 1 with given initial points.

Initial	n	MPRPL			MHSY			MPD		
		Time	Iter	Feval	Time	Iter	Feval	Time	Iter	Feval
x_1	1000	0.17	196	591	0.19	198	646	0.12	190	580
x_2	1000	0.11	203	612	0.14	204	624	0.17	192	600
x_3	1000	0.19	195	588	0.24	193	201	0.17	192	582
x_4	1000	0.15	243	741	0.18	256	762	0.14	230	709
x_5	1000	0.17	194	585	0.19	209	688	0.18	190	572
x_6	1000	0.16	189	571	0.25	231	657	0.15	182	550
x_1	5000	8.19	487	1465	8.38	523	1547	8.09	468	1428
x_2	5000	12.85	769	2310	13.87	823	2722	12.53	744	2255
x_3	5000	7.98	469	1411	9.01	524	1849	7.68	449	1371
x_4	5000	15.96	929	2840	18.74	1000	3261	15.61	900	2771
x_5	5000	7.59	453	1363	8.05	526	1653	7.29	432	1320
x_6	5000	6.32	371	1122	7.32	436	1476	6.02	348	1073
x_1	10000	34.59	539	1622	37.90	600	2002	10.31	530	1605
x_2	10000	65.06	1023	3072	72.6	1054	3166	67.79	1017	3054
x_3	10000	34.30	516	1553	35.99	533	1569	35.03	516	1554
x_4	10000	109.65	1668	5075	116.8	1742	5668	109.65	1668	5075
x_5	10000	31.57	497	1497	33.63	532	1568	32.02	495	1491
x_6	10000	25.36	396	1202	26.70	405	1206	26.11	394	1196

Table 2: Test results for Problem 2 with given initial points.

Initial	n	MPRPL			MHSY			MPD		
		Time	Iter	Feval	Time	Iter	Feval	Time	Iter	Feval
x_1	1000	0.81	12	39	1.15	17	36	0.62	12	27
x_2	1000	0.81	12	39	1.15	17	36	0.62	12	27
x_3	1000	0.55	8	27	0.75	11	24	0.45	8	19
x_4	1000	0.85	9	42	1.20	14	44	0.71	9	33
x_5	1000	0.53	6	28	1.14	13	42	0.45	6	22
x_6	1000	0.53	6	28	1.14	13	42	0.45	6	22
x_1	5000	5.32	13	42	6.93	17	36	3.90	13	29
x_2	5000	5.01	12	39	6.93	17	36	3.65	12	27
x_3	5000	3.01	7	24	4.19	10	22	2.23	7	17
x_4	5000	7.38	12	60	10.42	19	64	5.99	12	48
x_5	5000	3.40	6	28	6.94	13	42	2.74	6	22
x_6	5000	3.40	6	28	6.94	13	42	2.74	6	22
x_1	10000	21.15	13	42	27.57	17	36	15.61	13	29
x_2	10000	19.65	12	39	27.58	17	36	14.36	12	27
x_3	10000	11.92	7	24	14.94	9	20	8.91	7	17
x_4	10000	35.15	14	72	44.61	20	69	28.98	14	58
x_5	10000	13.86	6	28	29.46	14	45	11.06	6	22
x_6	10000	13.86	6	28	29.46	14	45	11.06	6	22

Table 3 Test results for Problem 3 with given initial points.

Initial	n	MPRPL			MHSY			MPD		
		Time	Iter	Feval	Time	Iter	Feval	Time	Iter	Feval
x_1	1000	0.81	12	39	1.15	17	36	0.62	12	27
x_2	1000	0.81	12	39	1.15	17	36	0.62	12	27
x_3	1000	0.55	8	27	0.75	11	24	0.45	8	19
x_4	1000	0.85	9	42	1.20	14	44	0.71	9	33
x_5	1000	0.53	6	28	1.14	13	42	0.45	6	22
x_6	1000	0.53	6	28	1.14	13	42	0.45	6	22
x_1	5000	5.32	13	42	6.93	17	36	3.90	13	29
x_2	5000	5.01	12	39	6.93	17	36	3.65	12	27
x_3	5000	3.01	7	24	4.19	10	22	2.23	7	17
x_4	5000	7.38	12	60	10.42	19	64	5.99	12	48
x_5	5000	3.40	6	28	6.94	13	42	2.74	6	22
x_6	5000	3.40	6	28	6.94	13	42	2.74	6	22
x_1	10000	21.15	13	42	27.57	17	36	15.61	13	29
x_2	10000	19.65	12	39	27.58	17	36	14.36	12	27
x_3	10000	11.92	7	24	14.94	9	20	8.91	7	17
x_4	10000	35.15	14	72	44.61	20	69	28.98	14	58
x_5	10000	13.86	6	28	29.46	14	45	11.06	6	22
x_6	10000	13.86	6	28	29.46	14	45	11.06	6	22

Table 4: Test results for Problem 4 with given initial points.

Initial	n	MPRPL			MHSY			MPD		
		Time	Iter	Feval	Time	Iter	Feval	Time	Iter	Feval
x_1	1000	0.06	50	219	0.07	59	231	0.05	36	163
x_2	1000	0.06	56	252	0.07	61	263	0.05	36	165
x_3	1000	0.05	34	152	0.08	38	175	0.03	28	127
x_4	1000	0.07	100	421	0.08	113	529	0.06	86	379
x_5	1000	0.06	42	187	0.08	53	203	0.03	37	167
x_6	1000	0.06	60	261	0.07	81	288	0.03	42	189
x_1	5000	1.11	48	209	1.15	63	262	0.95	37	163
x_2	5000	1.19	46	224	1.45	55	282	0.82	31	149
x_3	5000	0.62	27	117	0.65	32	126	0.65	27	121
x_4	5000	2.59	109	483	2.72	116	558	2.39	94	435
x_5	5000	1.21	53	231	1.86	60	278	0.97	40	179
x_6	5000	1.20	54	234	1.97	61	275	1.06	43	193
x_1	10000	3.96	45	195	4.30	78	222	3.63	39	175
x_2	10000	4.27	41	210	5.26	47	234	2.84	28	137
x_3	10000	1.93	22	96	2.12	25	105	2.32	25	112
x_4	10000	10.86	117	533	11.32	123	539	12.05	119	594
x_5	10000	4.34	49	212	5.06	51	232	3.65	40	180
x_6	10000	4.89	56	246	5.65	66	253	3.75	41	185

Table 5: Test results for Problem 5 with given initial points.

Initial	n	MPRPL			MHSY			MPD		
		Time	Iter	Feval	Time	Iter	Feval	Time	Iter	Feval
x_1	1000	0.66	47	199	0.69	49	201	0.62	37	165
x_2	1000	0.45	22	105	0.48	32	123	0.41	21	96
x_3	1000	0.03	48	209	0.09	50	275	0.05	38	117
x_4	1000	0.47	37	165	0.55	49	178	0.31	30	94
x_5	1000	0.05	54	237	0.06	54	237	0.05	41	122
x_6	1000	0.19	274	1146	0.22	325	1222	0.14	155	468
x_1	5000	1.22	51	221	1.32	53	229	0.92	36	161
x_2	5000	0.56	22	105	0.92	28	120	0.54	21	96
x_3	5000	1.11	47	200	1.80	58	262	0.733	38	117
x_4	5000	1.20	48	206	1.50	56	224	0.72	39	121
x_5	5000	1.17	47	206	2.00	56	343	0.73	41	119
x_6	5000	25.06	1108	2607	28.5	1361	2781	10.45	608	1826
x_1	10000	3.90	44	191	3.92	43	190	3.31	36	161
x_2	10000	51.1	22	105	51.86	23	110	47.1	21	96
x_3	10000	4.24	47	204	5.76	50	221	2.53	38	117
x_4	10000	10.5	48	209	11.20	58	227	6.90	40	134
x_5	10000	4.02	45	196	4.50	53	221	2.48	39	114
x_6	10000	183.1	2144	3912	201.4	2040	3876	76.3	1170	3511

From tables 1-5, we can see the MPD method performs better than the MPRPL method in Li and Li [17] and obviously better than the MHSY method in Yan et al. [18], which implies that the proposed method is computationally efficient for solving large-scale nonlinear monotone equations.

5. Conclusions

In this paper, we have proposed a derivative-free method to solve nonlinear monotone equations which is combined the modified Perry's conjugate gradient method [19] for unconstrained optimization problems and the hyperplane projection method [8]. Under appropriate conditions, we prove that the proposed method converges globally. The proposed method is suitable for large-scale monotone equations because of the low memory requirement. The preliminary numerical results show that our method is promising. In addition, the derivative-free methods can solve absolute value equations, for example Problem 5.

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