Post-buckling response of non-uniform cross-section bilaterally constrained beams

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Abstract

Buckling and post-buckling behaviors of structural elements have been widely used to create multi-stable mechanisms that have shown a great efficiency in many applications such as sensing, actuation and energy harvesting. Under an increasing axial loading, the strain energy stored in a buckled bilaterally constrained elastica is suddenly released, through a snap-through transition. These transitions can be used to convert low-rate and low-frequency excitations into high-rate motions that are converted into electrical signals using piezoelectric transducers. However, for efficient sensing and energy harvesting, these transitions have to be controlled. It has been shown that the spacing between the transitions cannot be controlled just by tuning the geometry properties of a uniform cross-section beam. This paper investigates the effect of different non-uniform cross-sections on the post-buckling response of a bilaterally constrained beam. An energy-based model that takes into account the shape variation is herein presented. The variation of the beam's cross-section area can either be continuous or piecewise continuous. The total potential energy of the system is minimized under constraints that represent the physical confinement of the beam between the lateral boundaries. Results demonstrate that the spacing ratio between buckling-mode transitions can be efficiently controlled by the beam's shape and geometry dimensions.

Keywords: Beams; control; non-uniform cross-section; snap-through transitions.

1. Introduction

For over a half of a century, the buckling of elastic structural elements has been considered as a failure limit state. Most research on buckling and post-buckling has intended to identify its characteristics to prevent it from happening [1-4]. However, the potential of using these elastic instabilities for different functional purposes has been recently explored and shown of a great practical importance [5-7]. Buckled elements have been used to create mono-stable, bi-stable and multi-stable mechanisms that have shown a great efficiency in many applications such as sensing, actuation and energy harvesting [8-13].

One of the major objectives of energy harvesting is to convert ambient energies into electrical energy that powers remote wireless sensors without the limitations of wiring harness or battery lifetime [14, 15]. Piezoelectric harvesters are one of the most promising technologies for deployment in structures given the size limitations and the possibility of being embedded within the construction material [16]. However, vibration-based piezoelectric harvesters suffer a narrow range of frequency input for optimal performance. Even though many methods to broaden the generator's operating bandwidth have been proposed [17-21], harnessing the efficiency for vibration-based piezoelectric transducers for quasi-static (< 1 Hz) input has remained elusive.

Using post-buckling snap-through instabilities of bilaterally constrained beams as a mechanical trigger to release the stored strain energy, our group has recently developed a mechanism that is able to harvest energy under quasi-static excitations [11, 22]. The mechanism has been

integrated with a Piezo-Floating-Gate sensor to open the doors to self-powered sensing of quasi-static events [12]. The snap-through buckling mode transitions efficiently transform the global quasi-static input excitations into localized high acceleration motions that can be used by piezoelectric transducers to convert the kinetic energy into electric power [22]. The harvested energy is stored in a capacitor bank. When the capacitor voltage meets a preset threshold, the converted electric power is transferred to the objective wireless sensors. Through the process, part of the converted energy is used to bias the electronic circuit. Since the strain energy release is sudden, it is critical to control the spacing between the buckling mode transitions in order to reduce the electronic leakage and, therefore, increase the energy utilization efficiency.

In a previous work [23] we showed that the spacing between buckling mode transitions can be controlled by using a combination of multiple slender elastic beams, stacked in a parallel configuration. However, by increasing the number of stacked beams, the system becomes voluminous. Therefore, it is of interest to control the spacing between the buckling snap-through events of a single beam. Recently, Borchani et al. [24-25] presented an energy-based theoretical model for bilaterally constrained beams subjected to an axial quasi-static load. Using small deformation assumptions, the model accurately predicts the post-buckling response of a uniform cross-section beam subjected to a gradually increasing axial force. This model was used to investigate the effect of tuning the geometry (length, width, thickness and gap between the lateral boundaries) and material properties (elastic modulus) of the elastica on its post-buckling response. Results showed that changing the geometry and material properties of a uniform cross-section beam has no influence on the spacing ratio between the

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transitions. However, the use of a non-uniform cross-section to control the buckling mode transitions has never been addressed.

This paper investigates the effect of different non-uniform cross-sections on the post-buckling response of a bilaterally constrained beam. The main hypothesis behind this work is that the beam shape can be designed to achieve a specific spacing between its buckling mode transitions. The theoretical static model proposed in [25] is herein modified to account for non-uniform cross-sections. The buckling mode shapes are assumed to be the same as uniform cross-section beams. Both continuous and piecewise continuous variations are considered. The total potential energy is minimized under constraints that represent the physical confinement of the beam between the lateral boundaries. Taking into account the geometry variation, the model allows for accurate prediction of the post-buckling mode transitions. Finally, the theoretical model is used to design for non-uniform cross-section beams with controlled transition spacing.

2. Static analysis of a bilaterally constrained beam with random geometry properties

2.1 Theoretical formulation

Fig. 1. Example of (a) a beam with non-uniform piecewise continuous cross-section and (b) buckling configurations.

The problem under consideration consists of a clamped-clamped prismatic homogeneous straight beam subjected to an increasing axial load, $\hat{p}$. The beam is composed of multiple discontinuous non-uniform cross-sections as shown in Fig. 1(a). Each continuous region has a length L_k , thickness, $t_k(x)$, width $b_k(x)$, non-uniform cross-section area, $A_k(x)$, moment of inertia $I_k(x)$ and modulus of elasticity E_k , where x represents the longitudinal coordinate. The slender beam is confined between two frictionless rigid walls spaced a distance h_0 (as shown in Fig. 1(b)). The net gap between the lateral constraints and the beam varies as a function of x and is defined by $h_k(x) = h_0 - t_k(x)$. $\hat{w}(x)$ denotes the transverse deflection. Euler-Bernoulli beam theory is used under small deformation assumptions to model the beam.

The non-dimensional governing equations of the buckling behavior of a fixed-fixed non-uniform cross-section beam subjected to an axial load, $\hat{p}$, are

$$\frac{d^4 W(X)}{dX^4} + N^2 \frac{d^2 W(X)}{dX^2} = 0 \quad (1a)$$

$$\begin{cases} W(0) = W(1) = 0, \\ \left. \frac{dW(X)}{dX} \right|_{X=0} = \left. \frac{dW(X)}{dX} \right|_{X=1} = 0 \end{cases} \quad (1b)$$

where, the non-dimensional variables are defined as

$$X = \frac{x}{L}, \quad W(X) = \frac{\hat{w}(X L)}{h_k(X L)}, \quad N^2 = \frac{\hat{p} L^2}{E_k I_k(X L)} \quad (2)$$

$$- \frac{h_k(X L)^2 A_k(X L)}{2 I_k(X L)} \int_{\frac{L_{k-1}}{L}}^{\frac{L_k}{L}} \left(\frac{dW(X)}{dX} \right)^2 dX$$

The symmetric and antisymmetric mode shapes of the general solution can be expressed, respectively, as

$$\begin{cases} W^s(X) = 1 - \cos(N_i X) \\ N_i = (i + 1)\pi \end{cases} \quad i=1,3, \dots \quad (3a)$$

$$\begin{cases} W^a(X) = 1 - 2X - \cos(N_j X) + \frac{2 \sin(N_j X)}{N_j} \\ N_j = 2.86\pi, 4.92\pi, 6.94\pi, 8.95\pi, \dots \end{cases} \quad j=2,4, \dots \quad (3b)$$

Since the buckling modes described in Eq. (3) generate an orthogonal basis, the mode superposition method is used to express the beam's deflection as a linear combination of buckling modes. Therefore, the general solution can be expressed as,

$$W(X) = \sum_{i=1,3,5,\dots}^{\infty} A_i W_i^s(X) + \sum_{j=2,4,6,\dots}^{\infty} A_j W_j^a(X) \quad (4)$$

where A_i ($i=1..\infty$) are weight coefficients to be determined.

In this study, an energy method is used to investigate the post-buckling response of a beam with non-uniform cross-sections (as shown in Fig. 1(a)). As noted above, the beam is composed of multiple discontinuous cross-sections. Due to the discontinuity of thickness and/or width, the potential energy is computed in each segment, and then the total potential energy is the sum of the partial energies. The bending and compression strain energies u_k^b and u_k^c , respectively, and the work of the external work, u_k^p , in the k^{th} segment, can be written, in terms of the axial force, $\hat{p}$, as,

$$u_k^b = \frac{1}{2} E_k \int_{L_{k-1}}^{L_k} I_k(x) \left[\frac{d^2 \hat{w}(x)}{dx^2} \right]^2 dx \quad (5a)$$

$$u_k^c = \frac{1}{2} \hat{p} \Delta_k^c \quad (5b)$$

$$u_k^p = \frac{1}{2} \hat{p} \Delta_k \quad (5c)$$

where Δ_k^c is the length shortening of the portion k due to compression only and Δ_k is its total length shortening due to both compression and section rotation. Their expression is given by

$$\Delta_k^c = \frac{\hat{p}}{E_k} \int_{L_{k-1}}^{L_k} \frac{1}{b_k(x) t_k(x)} dx \quad (6a)$$

$$\Delta_k = \Delta_k^c + \frac{1}{2} \int_{L_{k-1}}^{L_k} \left[\frac{d\hat{w}(x)}{dx} \right]^2 dx \quad (6b)$$

Substituting Eq. (6) into Eq. (5), the potential energy of the k^{th} segment can be expressed as,

$$\begin{aligned}\Pi_k &= u_k^b + u_k^c - u_k^p \\ &= \frac{1}{2} E_k \int_{L_{k-1}}^{L_k} I_k(x) \left[\frac{d^2 \hat{w}(x)}{dx^2} \right]^2 dx - \frac{1}{4} \hat{p} \int_{L_{k-1}}^{L_k} \left[\frac{d \hat{w}(x)}{dx} \right]^2 dx\end{aligned}\quad (7)$$

Substituting Eqs. (2) - (4) into Eq. (7), the partial potential energy is obtained as,

$$\begin{aligned}\Pi_k &= \frac{E_k L}{24} \int_{L_{k-1}}^{L_k} h_k(X)^2 b_k(X) t_k(X)^3 \times \\ &\quad \left(\sum_{i=1,3,5,\dots}^{\infty} \left(A_i \frac{d^2 W_i^s}{dX^2} \right)^2 + \sum_{j=2,4,6,\dots}^{\infty} \left(A_j \frac{d^2 W_j^a}{dX^2} \right)^2 \right. \\ &\quad + 2 \sum_{i=1,3,5,\dots}^{\infty} \sum_{j=2,4,6,\dots}^{\infty} A_i A_j \frac{d^2 W_i^s}{dX^2} \frac{d^2 W_j^a}{dX^2} \\ &\quad \left. + 2 \sum_{i=4,6,8,\dots}^{\infty} \sum_{j=2,4,6,\dots}^{\infty} A_i A_j \frac{d^2 W_i^a}{dX^2} \frac{d^2 W_j^a}{dX^2} \right) dX \\ &\quad - \frac{\hat{p}}{4L} \int_{L_{k-1}}^{L_k} h_k(X)^2 \times \\ &\quad \left(\sum_{i=1,3,5,\dots}^{\infty} \left(A_i \frac{dW_i^s}{dX} \right)^2 + \sum_{j=2,4,6,\dots}^{\infty} \left(A_j \frac{dW_j^a}{dX} \right)^2 \right. \\ &\quad + 2 \sum_{i=1,3,5,\dots}^{\infty} \sum_{j=2,4,6,\dots}^{\infty} A_i A_j \frac{dW_i^s}{dX} \frac{dW_j^a}{dX} \\ &\quad \left. + 2 \sum_{i=4,6,8,\dots}^{\infty} \sum_{j=2,4,6,\dots}^{\infty} A_i A_j \frac{dW_i^a}{dX} \frac{dW_j^a}{dX} \right) dX\end{aligned}\quad (8)$$

and the total potential energy is expressed by

$$\Pi(A_i) = \sum_{k=1}^n \Pi_k(A_i), \quad (i = 1.. \infty) \quad (9)$$

The unknown coefficients A_i can then be determined by solving the constrained minimization problem presented in Eq. (10).

$$\begin{cases} \text{Min } \Pi(A_i) \\ 0 \leq W(X) \leq 1 \end{cases} \quad (10)$$

Since the objective function is nonlinear, the minimization problem was numerically solved using Nelder–Mead algorithm. For a function of n variables, the algorithm maintains a set of $n+1$ points forming the vertices of a polytope in n -dimensional space. At each iteration, $n+1$ points $x_1, x_2, \dots, x_{n+1}$ form a polytope. The points are ordered so that $f(x_1) \leq f(x_2) \leq \dots \leq f(x_{n+1})$. A new point is then generated to replace the worst point x_{n+1} .

The centroid of the polytope consisting of the best n points is computed by $c = \frac{1}{n} \sum_{i=1}^n x_i$. A trial point x_i is generated by reflecting the worst point through the centroid, $x_i = c + \alpha(c - x_{n+1})$, where $\alpha > 0$ is a parameter.

If the new point x_i is neither a new worst point nor a new best point, $f(x_1) \leq f(x_i) \leq f(x_n)$, x_i replaces x_{n+1} . If the new point x_i is better than the best point, $f(x_i) \leq f(x_1)$, the reflection is very successful and can be carried out further to $x_e = c + \beta(x_i - r)$, where $\beta > 1$ is a parameter to expand the polytope. If the expansion is successful, $f(x_e) < f(x_i)$, x_e replaces x_{n+1} ; otherwise the expansion failed, and x_i replaces x_{n+1} . If the new point x_i is worse than the second worst point, $f(x_i) > f(x_n)$, the polytope is assumed to be too large and needs to be contracted. A new trial point is defined as

$$x_e = \begin{cases} c + \gamma(x_{n+1} - c), & \text{if } f(x_i) \geq f(x_{n+1}), \\ c + \gamma(x_i - c), & \text{if } f(x_i) < f(x_{n+1}), \end{cases} \quad (11)$$

where $0 < \gamma < 1$ is a parameter. If $f(x_e) < \text{Min}(f(x_{n+1}), f(x_i))$, the contraction is successful, and x_e replaces x_{n+1} . Otherwise a further contraction is carried out.

Strictly speaking, Nelder–Mead is not a true global optimization algorithm; however, in practice it tends to work reasonably well for problems that do not have many local minima.

2.2 Validation of the model

Validation with published results of a uniform cross-section beam

Fig. 2. Comparison between the current and the previously published models for uniform cross-section beams.

In order to verify the current analytical formulation, the model was first simplified into a uniform cross-section and compared with the previously published results [21]. The beam consisted of a length of 250 mm, a thickness of 2.34 mm, a width of 30 mm, a Young's modulus of 2.31 GPa, and a gap of 4 mm. The beam was partitioned into three segments with equal widths and thicknesses. The potential energy in each segment was computed. Then the total potential energy was minimized under the confinement constraints. Fig. 2 presents the response of a uniform cross-section beam under a gradually increasing axial force, simulated by both formulations: current and previously published models. The labels $\Phi 3$, $\Phi 5$, and $\Phi 7$ refer to the dominant buckling modes to which the beam transitions after the snap-through events. Both responses coincide with a maximum difference of 7.69% at transition $\Phi 5$. The

discrepancy between the two curves is due to a numerical error in the minimization of the total potential energy in Eq. (10). Since the current model addresses non-uniform beams, a more complicated total potential energy is minimized in this study than the previous work.

Validation with experimental results

The post-buckling response of a uniform section bilaterally constrained beam has been experimentally investigated in previous studies [11, 22]. The experimental setup presented in [22] was used to investigate the force-displacement response of the non-uniform section beam. A polycarbonate ($E=2.31$ GPa) beam with fixed end supports was placed between rigid continuous aluminum frames. The constraining walls were separated by a gap (h_0), which was 4 mm for the tested setup in this work. The beam was placed inside the gap, adjacent to one of the constraining walls. The net gap, h , for traverse deformations of the beam is the difference between the wall gap and the beam thickness, t . The thickness of the beam was constant and equal to 2.3 mm. However, the width was linearly variable as shown in Fig. 3(a). The figure also displays the geometry properties of the tested sample. The testing procedure consisted of applying a gradually increasing compressive force on the polycarbonate beam. The axial loading was applied using a mechanical test system (MTS model Flextest 40 with series 370 load unit), as shown in Fig. 3(b). The top edge of the beam was moving at a speed of 0.16 mm/s. Fig. 3(c) displays an example of the beam deformed in its seventh buckling mode configuration, $\Phi 7$.

Fig. 4 presents the theoretical and experimental responses of the beam presented in Fig. 3(a). Experimentally measured axial forces were slightly lower than theoretical predictions because the presented model does not take into account the friction between the beam and its lateral constraints. Nevertheless, the theoretical model is still able to predict the post-buckling response of the constrained beam with a maximum error of 5.26%, recorded at $\Phi 3$. Therefore it can be used to further investigate the influence of different beam shape scenarios on its buckling response and spacing between its transition events.

Fig. 3. (a) A sketch of the tested beam, (b) the experimental setup and (c) an example of a loaded configuration of the beam.

Fig. 4. Theoretical and experimental post-buckling responses of a beam with linear width variation.

Table 1. Variable parameter in each of the studied cases.

Case	Group	Variable	Size (mm)
(a)	Piecewise constant	b_{top}	15; 20; 25; 30
(b)		L_{top}	0; 50; 100
(c)		t_{top}	1.54; 1.74; 1.94; 2.34
(d)	Linear	b_{top}	15; 20; 25; 30
(e)		t_{top}	1.54; 1.74; 2.04; 2.34
(f)	Sinusoidal	b_{min}	15; 20; 25

3. Effect of a beam shape on its post-buckling response

The theoretical model presented in section 2 was used to investigate the influence of the beam's geometry on its elastic post-buckling behavior. As mentioned above, the objective of this work is to control the spacing between the transition events by using non-uniform cross-sections. The main parameters that contribute to the cross-section variation are the thickness and width. Their variation can be either continuous along the beam (as shown in Fig. 5(d)-(f)), therefore the total potential energy is computed over the whole beam, or piecewise continuous (Fig. 5(a)-(c)) and hence, the total potential energy is the sum of partial potential energies in each segment. In the latter case, the length of each segment is considered an important parameter as well. The total length and the elastic modulus of the beam and the gap separating the lateral constraints were fixed at 250 mm, 2.31 GPa and 4 mm, respectively. The variation of the width, thickness and segments length is detailed in Table 1. The studied cases are divided into two groups. The first group consists of piecewise constant cross-sections where either the thickness or width varies between the segments. While, the second group consists of continuous linear or sinusoidal variation of either width or thickness. In all cases, the width and thickness at the bottom edge are fixed at 30 mm and 2.3 mm, respectively.

Piecewise constant cross-sections

The first case presented in Fig. 5 (a) investigates the effect of the top width on the post-buckling response of the beam. The beam is divided into two equi-length segments. The top width is varied between 15 mm and 30 mm as shown in Table 1. The second case, presented in Fig. 5(b), studies the sensitivity of the response to the segments length. The top width is fixed at 15 mm (half of the bottom edge width) and the length of the top segment is varied between 0 and 100 mm. The third case, presented in Fig. 5(c), studies the sensitivity of the response to the top thickness. The top thickness is varied between 1.54 mm and 2.34 mm. The beam is divided

into two equi-length segments and the top thickness is varied between 1.54 mm and 2.34 mm as detailed in Table 1. The change in the response of the three cases is presented in Fig. 6. The stiffness (slope within the same buckling modes) and the axial force at which the buckling transitions happened are drastically affected by the change of the studied parameters. More importantly, the spacing between the transition events, noted by $\Phi 3$, $\Phi 5$ and $\Phi 7$, is very sensitive to the geometry dimensions. Fig. 6(a) shows that the smaller is the top width of the beam, the larger is the spacing between $\Phi 3$ and $\Phi 5$ but the lesser is the spacing between $\Phi 5$ and $\Phi 7$. The ratio between the two spacing is defined as $R = (F_7 - F_5)/(F_5 - F_3)$, where F_i denotes the axial force at which transition to buckling mode i occurs. Results in Fig. 6(b) and Fig. 6(c) demonstrate similar effect of top length and thickness on the stiffness and spacing ratio between different buckling mode transitions. However, the ratio between the two spacing is found inversely proportional to both length and thickness of the top section.

Fig. 5. Studied cases (not to scale) comprising (a) piecewise constant width, (b) piecewise constant section with different segment lengths, (c) piecewise constant thickness (d) linear width, (e) linear thickness and (f) sinusoidal width.

Fig. 6. Effect of (a) top width, (b) top segment length and (c) top thickness on the transition events of a piecewise constant cross-section beam.

3.1 Linear and sinusoidal variation

The beam width and thickness can be defined as continuous linear or non-linear functions in terms of the longitudinal coordinate, x . The linear functions defining the beam width and thickness shown in Fig. 5 (d) and (e), respectively, are given by

$$b(x) = b_{bot} - \frac{b_{bot} - b_{top}}{L}x \quad (11)$$

and

$$t(x) = t_{bot} - \frac{t_{bot} - t_{top}}{L}x. \quad (12)$$

The sinusoidal function related to the width geometry presented in Fig. 5(f) is expressed as

$$b(x) = b_{bot} + 2A \sin\left(2\pi m \frac{x}{L}\right), \quad (13)$$

where b_{bot} and t_{bot} are fixed at 30 mm and 2.34 mm, respectively, and the top thickness, t_{top} , top width, b_{top} , sine amplitude, A , and number of waves, m , are variable.

Fig. 7(a) and (b) present the response of a beam with linear width and thickness, respectively. The width of the top edge varies between 10 mm and 30 mm and the thickness between 1.54 mm and 2.34 mm as detailed in Table 1. Results show that the effect of these two parameters on the beam's response is similar to the piecewise constant scenarios presented above. The effect of the sine amplitude on the response of a beam with sinusoidal width variation is presented in Fig. 7(c). The wave number, m , was fixed at 1.

Results show that an increase in the sine amplitude (i.e. reduction of the least section width) results in a larger spacing between $\Phi 5$ and $\Phi 7$ but a lesser spacing between $\Phi 3$ and $\Phi 5$. Therefore the sine amplitude has an opposite effect to the top width in piecewise constant sections.

Fig. 7. Effect of (a) top width, (b) top thickness and (c) sine amplitude on the transition events of a linearly/sinusoidally varied cross-section beam.

4. Design of non-uniform cross-section beams with controlled spacing between mode transitions

The main objective of this work is to attain elastic post-buckling responses with controlled spacing between buckling mode transitions. The primary parameters controlling the post-buckling behavior of a beam are its width, thickness and length of its segments in the case of piecewise constant cross-sections. Therefore, it is crucial to investigate their effect on the spacing ratio, R . Fig. 8 displays the spacing ratio variation with respect to the beam's top width and thickness in both piecewise constant and linear scenarios (Fig. 5(a), (c), (d) and (e)), top segment length in piecewise uniform cross-sections (Fig. 5(b)) and minimum cross-section width in a sinusoidal width variation scenario (Fig. 5(f)). Results in Fig. 8(a) and (b) show that piecewise constant width and/or thickness allows more control over the spacing between transitions than linear width or thickness variation as it covers a wider range of spacing ratio values. High values can be attained by reducing the top thickness and adopting either piecewise constant or linear cross-sections. On the other hand, low

ratio values can be easily achieved either by increasing the length of the top segment (Fig. 8(c)) or adopt a sinusoidal shape (Fig. 8(d)). It should be mentioned that a unit spacing ratio is attainable by many geometry configurations such as sinusoidal or even uniform cross-section beams.

Fig. 8. Variation of the spacing ratio with respect to (a) top width in piecewise constant and linear scenarios, (b) top thickness in piecewise constant and linear scenarios, (c) top segment length in a piecewise constant cross-section scenario and (d) minimum width in a sinusoidal scenario.

Results in Fig. 8 prove that a large range of spacing ratio can be covered by varying the shape and/or geometry dimensions. Fig. 9 presents examples of beam geometries with different spacing ratios that span between 0.25 and 1.5. Their force-displacement responses are presented in Fig. 10. As shown in the figure, the shape and dimension properties of the beam not only affect its stiffness but more importantly, the ratio between its transition events. It can be designed in a way to control the exact axial force values at which transitions occur. For instance, the beam shape presented in Fig. 9(a) was designed to achieve a spacing ratio of 0.25 as shown in Fig. 10(a). The top width is fixed at 15 mm and the length of the top segment was determined to be 200 mm. Similarly, the target spacing ratios of 0.5 and 1 are obtained by reducing the minimum width of the sinusoidal beam to 13 mm and 10 mm, as shown in Fig. 9(b) and (d), respectively. Likewise, a ratio of 1.25 was achieved by adjusting the top thickness to 1.45 mm and adopting a linear thickness variation (as shown in Fig. 9(e)), while a ratio of 1.5 is attained by a piecewise constant thickness configuration with a top thickness of 1.7 mm.

To validate the designs presented in Fig. 9, the geometry configuration that corresponds to a spacing ratio of 0.75 (presented in Fig. 9(c)) was experimentally tested. This configuration was chosen for its ease of construction. The experimental and theoretical post-buckling responses of the beam are displayed in Fig. 11. Results show a good agreement between the theoretical and experimental results. Table 2 summarizes the theoretical and experimental forces at which mode transitions occur and the spacing ratio between the transitions. It can be seen that the theoretical model accurately predicted the buckling events of the

designed beam with a maximum error of 3.36% in terms of the axial force recorded at transition $\Phi 7$.

Fig. 9. Beam configurations with specific spacing ratio (a) $R=0.25$, (b) $R=0.5$, (c) $R=0.75$, (d) $R=1.0$, (e) $R=1.25$ and (f) $R=1.5$.

Fig. 10. Force-displacement responses with transition spacing ratios (a) $R=0.25$, (b) $R=0.5$, (c) $R=0.75$, (d) $R=1.0$, (e) $R=1.25$ and (f) $R=1.5$.

Table 2. Comparison between theoretical and experimental results.

	Force at mode transitions			R
	F_3 (N)	F_5 (N)	F_7 (N)	
Theoretical	400	1400	2160	0.76
Experimental	410.6	1362.4	2087.5	0.762
Error (%)	0.49	1.74	3.36	0.26

Fig. 11. Theoretical and experimental post-buckling responses of the beam designed with a 0.75 spacing ratio.

5. Conclusions

This paper investigated the effect of the shape of a beam on its snap-through buckling transitions. The beam is bilaterally constrained and subjected to a quasi-static axial load. A theoretical static model that accounts for non-uniform cross-sections was used to study the post-buckling response of beams with different non-uniform cross-section variations: continuous and piecewise continuous. Results showed that the spacing between the transition events is very sensitive to the adopted shape and its geometry dimensions. The major parameters that control the buckling spacing ratio are the thickness and width of the beam. The accuracy of the theoretical model was experimentally validated. Therefore it was used to design for non-uniform cross-section beams with controlled spacing between transitions.

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