The coefficient of normal restitution for the Hertzian contact of two rough spheres colliding in a viscous fluid

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Abstract

We use previous theoretical results for the added mass, history and lubrication forces between two spheres colliding in a fluid with viscosity ν to investigate the effect of viscous dissipation on the coefficient of restitution during contact. We assume that the mechanical interaction is governed by Hertzian mechanical contact of small duration τ and that the minimum approach distance between particles is approximately equal to the height σ of surface micro-asperities. A non-dimensionalization of the equation of motion indicates that the contact dynamics is governed by two parameters - the ratio ϵ of the surface roughness σ and the sphere radius a , and a contact Stokes number defined as $St_c = \sigma^2/\nu\tau$. An asymptotic solution of the equation of motion in the limit of small ϵ/St_c is used to obtain an explicit expression for the coefficient of restitution during contact and the latter is compared with estimates based on numerical solutions of the non-linear equation of motion.

Keywords: particle collisions, coefficient of restitution, Hertzian contact, lubrication force

1. Introduction

The coefficient of restitution e is a key parameter describing the dynamics of particle collisions and the dissipation of kinetic energy in granular and particle-laden flow. While the coefficient of restitution for collisions in viscous fluids has been the subject of many investigations, most of the attention has been directed to the hydrodynamic damping during the approach of the particles and the question of viscous dissipation during mechanical contact has received little attention. Experimental investigations [1, 2, 3, 4] have found that lubrication forces generated by the drainage of the viscous fluid film in the gap of approaching spheres results in a significant damping of the sphere momentum (and a reduction of e) for collisions with low impact Stokes number $St = \frac{m_* W_\infty}{6\pi\rho\nu a_*^2}$, where $a_* = a_1 a_2 / (a_1 + a_2)$ and $m_* = m_1 m_2 / (m_1 + m_2)$ are the sphere reduced radius and mass, W_∞ is the "impact" velocity (at separations where lubrication effects are negligible), and ν and ρ are the fluid viscosity and density. The laboratory experiments of sphere-wall collisions in a viscous fluid [1, 2, 3] have focused on estimating an effective coefficient of restitu-

tion e_{wet} which includes the strong hydrodynamic interaction of the particles prior to making contact. The laboratory experiments have established that e_{wet} decreases at low impact velocities and is a well-defined function of the collision Stokes number St , such that e_{wet} becomes zero below a critical $St \approx 10$. Our focus here is on the contact dynamics during viscous collisions of small particles in industrial and marine sediment transport flows where the impact Stokes numbers are typically less than 100 and where the coefficient of restitution is reduced significantly below $e = 1$.

Existing estimates of e_{wet} based on integrating Newton's second law for the particle motion (e.g Barnocky and Davis [1]) typically neglect the hydrodynamic forces during mechanical contact, and assume that all the viscous dissipation occurs before and after the particles make contact. For mechanical contact characterized with a coefficient of restitution e_{dry} , a simple prediction

$$e_{wet} = e_{dry} - (1 + e_{dry})St^{-1}\log(\sigma_0/\sigma) \quad (1)$$

was obtained in Joseph et al. [3] by integrating Newton's equation for the motion of a sphere subject to lubrication force from some small initial separation σ_0 to a minimum separation distance corresponding to the size σ of surface microasperities. It was found that the

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model prediction is consistent with the observed dependence of e_{wet} on W_∞ . However, we note that the comparisons of equation (1) with measurements (e.g. Joseph et al. [3], Yang and Hunt [4]) utilize $e_{dry} < 1$ which implies that additional dissipation occurs during contact. Marshall [5] extended the e_{wet} model of Barnocky and Davis by including viscous damping during contact based on an approximate model for the viscous corner flow associated with the lateral expansion of the contact region. The purpose of the present paper is to investigate the effect of hydrodynamic dissipation during contact based on an asymptotic solution for the viscous gap flow [6] and to estimate the corresponding reduction of the coefficient of restitution during Hertzian contact, hereinafter denoted by e_{con} , below the theoretical value $e_{con} = 1$. The goal of this investigation is to determine the dependence of the coefficient of restitution on the particle roughness, the impact velocity, and the material properties of the solids and the interstitial fluid.

2. Coefficient of restitution in dry contacts

The modeling of collisions begins with understanding the purely mechanical interaction during dry contacts where hydrodynamic forces are negligible. Extensive laboratory studies have demonstrated (see Goldsmith [7]) that the Hertzian elastic theory of contact is adequate for describing the dynamics of the mechanical contact during moderate collisions of hard spheres in air where hydrodynamic dissipation is negligible. It is well-known that the Hertzian force between two spherical particles is proportional to the 3/2-power of the overlap distance Δ at the point of contact [7]

$$F_H = -k_H \Delta^{3/2}, \quad (2a)$$

where the Hertzian stiffness parameter k_H is related to the Young's moduli E_1 and E_2 , and the Poisson ratios λ_1 and λ_2 of the two solids as follows

$$k_H = \frac{2}{3} E_* a_*^{1/2}; \quad (2b)$$

$$E_* = \frac{2}{\frac{1-\lambda_1^2}{E_1} + \frac{1-\lambda_2^2}{E_2}}; \quad (2c)$$

For given impact velocity $\bar{W}$, the maximum Hertzian deformation of the particle boundaries is

$$\delta_H = \bar{W} \tau_* \quad (3)$$

where τ_* is the Hertzian time scale

$$\tau_* = \left[\frac{25m_*^2}{16\bar{W}k_H^2} \right]^{1/5}. \quad (4)$$

When the impact velocity is greater than a critical value (Johnson [8])

$$v_{yield}^2 \approx 106a_*^3 Y^5 / (m_* E^4) \quad (5)$$

related to the yield strength Y of the material, most of the energy dissipation during dry collisions results from plastic deformation; for most materials, this critical velocity is on the order of a few *cm/s*. When the impact velocity is less than the critical value v_{yield} , only viscoelasticity of the solid is thought to dissipate energy in dry collisions. The general form of the total mechanical force in such viscoelastic collisions is obtained by adding a dissipative force proportional to the strain rate $\dot{\Delta}$ (Kuwabara and Kono [9], Schafer et al. [10], Briliantov et al. [11], Falcon et al. [12]) to the Hertzian force (2a),

$$F = -k_H \Delta^{3/2} - \mu \dot{\Delta} |\Delta|^\gamma, \quad (6)$$

where μ is a parameter related to the internal frictional dissipation in the solid.

The dissipation in such viscoelastic collisions can be analyzed by non-dimensionalizing the equation of motion for the particle pair

$$m_* \frac{dW}{dt} = -k_H \Delta^{3/2} - \mu \dot{\Delta} |\Delta|^\gamma, \quad (7)$$

using the initial impact velocity $\bar{W}$ and the Hertzian time scale to define the non dimensional velocity $w = W/\bar{W}$, non-dimensional time $\tau = t/\tau_*$ and non-dimensional displacement $\delta = \Delta/(\tau_* \bar{W})$. It is then found that the resulting non-dimensional equation

$$\frac{dw}{d\tau} = -\frac{5}{4} \delta^{3/2} - \frac{\mu \tau_* (\bar{W} \tau_*)^\gamma}{m_*} \dot{\delta} |\delta|^\gamma, \quad (8)$$

depends on a single dimensionless parameter $\mu \tau_* m_*^{-1} (\bar{W} \tau_*)^\gamma$ that represents the ratio of the restoring and dissipative mechanical forces. Multiplying by $w = \dot{\delta}$ and integrating the result during the contact period in the limit of weak dissipation $\mu \rightarrow 0$, it is possible to obtain an analytical expression for the coefficient of restitution (see Kuwabara and Kono [9], Falcon et al. [12], also Ray et al. [13])

$$e_{dry} = 1 - \frac{4}{5} \mathcal{B} \left(\frac{3}{2}, \frac{2(\gamma+1)}{5} \right) \frac{\mu \tau_*}{m_*} (\tau_* \bar{W})^\gamma, \quad (9)$$

where $\mathcal{B}$ is the Beta function. The case $\gamma = 1/2$ corresponds to a viscoelastic solid and predicts a dry coefficient of restitution that decreases weakly with the impact velocity proportional to $-\bar{W}^{1/5}$. Verification of the prediction (9) for $\gamma = 1/2$ has been difficult due to lack of measurement data for μ which is related to the solid viscosity coefficients.

For rough surfaces, the range of plastic dissipation extends to much lower velocity as the particle radius in the yield condition (5) is replaced by the effective asperity curvature radius. The Hertzian force model (2a) is valid for rough surfaces only at high loads. At low loads, the presence of surface roughness tends to increase the effective contact area (relative to Hertzian contact of smooth surfaces) and leads to an approximately linear elastic force [20]. The detailed investigation of such complex multi-asperity contact models (see [21] for a recent review) is beyond the scope of the present paper where the focus is on the viscous dissipation due to the interstitial fluid. We will illustrate the effect of viscous dissipation using the Hertzian contact model which is commonly assumed to be valid over the entire range of loads in numerical models of particle-laden flows. The investigation of slow viscous collisions in the following section will show that hydrodynamic effects can give rise to a dissipative term similar to the second term in equation (6).

3. Coefficient of restitution in viscous contacts

The main obstacle to evaluating the effect of hydrodynamic forces during contact has been the lack of theoretical model or measurements of these forces. For that reason, numerical modelers (e.g. Derksen and Sundaresan [14], Simeonov and Calantoni [15]) have used various ad hoc assumptions to limit the magnitude of the lubrication force during contact. Here, we should point out that classic theories for the lubrication force (e.g. Cooley and O'Neill [16]) are based on the steady Stokes equation and, as such, are not appropriate to describe the hydrodynamics during mechanical contact when the particle velocity changes very rapidly and imparts large accelerations on the surrounding fluid.

The unsteady hydrodynamics of colliding particles was recently investigated in [6], where following Barnocky and Davis [1], the particles were assumed to make contact through microasperities of height σ so that the gap between the particles remains finite during contact. This asperity contact model should be contrasted with the elastohydrodynamic theory of Davis et al. [25] for the viscous collision of smooth particles

where the minimum separation between the particles δ_{EHL} becomes so small that the fluid pressure in the gap generated by the squeezed flow becomes sufficiently large to transmit the mechanical stresses without actual contact. For low impact collisions, the minimum approach distance δ_{EHL} is typically much smaller than real particle roughness which is in the range of 0.01-1 μm [3]. For example, using the estimate of Yang and Hunt [26]

$$\delta_{EHL} = \frac{a}{3} \left(\frac{4(1-\lambda^2)}{\pi \sqrt{2}} \right)^{2/5} \left(\frac{\nu \bar{W}}{ac^2} \right)^{2/5}, \quad (10)$$

where $c = \sqrt{E_*/\rho_s}$ is the one-dimensional sound speed, we find $\delta_{EHL} = 1.410^{-6} \text{mm}$ for two 1 mm quartz spheres ($c = 5000 \text{m/s}$) colliding in water with impact velocity $\bar{W} = 1 \text{mm/s}$. Therefore, boundary deformations due to elastohydrodynamic effects are ignored in our collision model focusing on low impact collisions.

Analytical progress in estimating the lubrication forces during the collision of rough spheres is only possible if we ignore the deviation from spherical geometry introduced by the asperities and its effect on the gap flow. In reality, the presence of asperities will further restrict the fluid flow in the gap leading to an increase of the lubrication forces. Thus, the presented analytical model would provide a lower bound estimate of the viscous dissipation during collision of rough spheres. To determine the unsteady hydrodynamic forces during contact, we solved the time-dependent Stokes equations for the viscous flow in the vicinity of two slowly colliding non-rotating spheres whose diameter ratio is κ [6]. The initial value problem was solved using Laplace transform and an initial condition corresponding to the steady state solution of Cooley and O'Neill [16] for the approach of two spheres in a viscous fluid. The solution was obtained using a singular perturbation expansion in the small parameter $\epsilon = \sigma/a$ and the Stokes streamfunction approach of Cooley and O'Neill [16]. Tangent-sphere coordinates were used to evaluate the viscous solution outside the gap between the particles. The asymptotic theory was limited to the interaction of relatively smooth particles such that the viscous time scale in the gap σ^2/ν is of the same order or smaller than the Hertzian contact time scale τ_* . It will be shown below that the ratio of these two time scales, which we refer to as the contact Stokes number

$$St_c = \frac{\sigma^2}{\nu \tau_*}, \quad (11)$$

describes the relative importance of the mechanical forces and the viscous dissipation. Below, we investigate the dependence of the coefficient of restitution on

the particle roughness σ , the material stiffness represented by τ_* or St_c , the diameter ratio κ and the impact velocity $\bar{W}$.

In the related paper mentioned above [6], we obtained the following result for the time-dependent hydrodynamic force during the collisional contact of a sphere of radius a with a fixed motionless sphere of radius a/κ ,

$$f = -\frac{4}{3}\pi\rho a^3 C_M \frac{dW}{dt} - 3\pi\rho \sqrt{\nu} a^2 K \int_0^t \frac{dW}{d\tau} \frac{d\tau}{\sqrt{\pi(t-\tau)}} - \frac{6\pi\rho\nu a^2 W}{(1+\kappa)^2 \sigma}, \quad (12)$$

where $C_M = 3K/4 - 1$ is the added mass coefficient,

$$K(\kappa) = \int_0^\infty \frac{\lambda^2 e^{-\lambda}}{1 - e^{-\lambda(1+\kappa)}} d\lambda \quad (13)$$

is an $O(1)$ function of κ and the first, second and third term in (12) correspond to the added mass, history and lubrication forces, respectively. In the two limits $\kappa = 0$ (wall) and $\kappa \rightarrow \infty$ (single particle), our estimates for the added mass coefficient $C_M = 3\zeta(3)/2 - 1 \approx 0.803$ and $C_M = 1/2$ were found to agree with previous calculations [18, 19, 22, 23, 24]. We should note that the last term in (12) has the same form as the classical leading order result [16] except that the gap separation is replaced by the fixed height of the asperity. This is not necessarily a contradiction because during the Hertzian compression the gap decreases from σ to $\sigma - \delta_H$ where $\delta_H = \bar{W}\tau_* = \sigma\epsilon Re/St_c$ is much smaller than σ for the considered low Reynolds number collisions $Re = a\bar{W}/\nu \ll 1$ provided that $St_c > \epsilon$; the small $O(\epsilon Re/St_c)$ term, as well as other higher order terms, are therefore not included in the leading order force estimate (12).

3.1. Asymptotic estimate for small ϵ/St_c

Using the theoretical estimate in (12), we can now analyze the effect of the hydrodynamic forces on the collision process based on Newton's equation for the particle motion

$$m_K \frac{dW}{dt} = -3\pi\rho \sqrt{\nu} a^2 K \int_0^t \frac{dW}{d\tau} \frac{d\tau}{\sqrt{\pi(t-\tau)}} - \frac{6\pi\rho\nu a^2 W}{(1+\kappa)^2 \sigma} - k_H \Delta^{3/2}, \quad (14)$$

where the added mass force in (12) has been incorporated into the left hand side via the effective mass parameter

$$m_K = \frac{4}{3}\pi\rho_s a^3 \left[1 + C_M \frac{\rho}{\rho_s} \right] \quad (15)$$

The initial conditions correspond to the onset of contact when $W(t=0) = \bar{W}$ and $\delta(t=0) = 0$.

We can again identify the pertinent non-dimensional parameters by non-dimensionalizing the equation of motion using the initial impact velocity $|\bar{W}|$ and the modified time scale $\tau_K = \left(\frac{25m_K^2}{16k_H^2 \bar{W}} \right)^{1/5}$ to define the non dimensional velocity $w = W/\bar{W}$, non-dimensional time $\tau = t/\tau_K$ and non-dimensional displacement $\delta = \Delta/(\tau_K \bar{W})$. The resulting non-dimensional equation

$$\frac{dw}{d\tau} = -3K(1 + \rho C_M/\rho_s)^{-4/5} \frac{3\rho\epsilon}{4\rho_s St_c^{1/2}} \int_0^\tau \frac{dw}{d\tilde{\tau}} \frac{d\tilde{\tau}}{\sqrt{\pi(\tau - \tilde{\tau})}} - (1 + \rho C_M/\rho_s)^{-3/5} \frac{3\rho\epsilon}{4\rho_s St_c} \frac{6w}{(1+\kappa)^2} - \frac{5}{4} \delta^{3/2}, \quad (16)$$

indicates that the two non-dimensional parameters $\epsilon/St_c^{1/2}$ and ϵ/St_c describe the relative magnitude of the history and resistance lubrication force with respect to the $O(1)$ Hertzian force. For small contact Stokes numbers $St_c \ll 1$, the history term is $O(St_c^{1/2})$ smaller than the lubrication force term and the leading order $O(\epsilon/St_c)$ of equation (16) corresponds to equation (8) with $\gamma = 0$ and the coefficient $\mu\tau_*/m_*$ replaced by $(1 + \rho C_M/\rho_s)^{-3/5} \frac{3\rho\epsilon}{4\rho_s St_c} \frac{6}{(1+\kappa)^2}$. The respective asymptotic estimate of the coefficient of restitution for small ϵ/St_c becomes

$$e_{con} = 1 - \frac{4}{5} \mathcal{B}\left(\frac{3}{2}, \frac{2}{5}\right) (1 + \rho C_M/\rho_s)^{-3/5} \frac{3\rho\epsilon}{4\rho_s St_c} \frac{6}{(1+\kappa)^2}. \quad (17)$$

We first note that the above estimate of the coefficient of restitution depends weakly on the added mass coefficient C_M because the exponent and the factor ρ/ρ_s are less than one. Equation (17) also indicates that the effect of viscous damping decreases as the diameter of the target particle decreases (κ increases). The case with the largest reduction of e_{con} corresponds to a collision with a wall ($\kappa = 0$) and will be the focus of the remaining discussion. Our analytical estimate (17) finally shows that $1 - e_{con}$ is inversely proportional to the parameter St_c/ϵ . Assuming $\kappa = 0$ and using equations (11,4,2b), we can obtain the following estimate

$$\frac{St_c}{\epsilon} = \left(\frac{\bar{W}}{4\pi^2 c} \right)^{1/5} \frac{c\sigma}{\nu}. \quad (18)$$

This shows that the leading order estimate of e_{con} in (17) does not depend on the particle radius a and that the decrease of e_{con} from 1.0 is inversely proportional to the

surface roughness σ and the one-fifth power of the impact velocity.

To discuss the relevant range of the parameter St_c/ϵ , we next consider the specific case of quartz particles with $\lambda = 0.1$, $\rho_s = 2.6\rho$ and $c = 5000m/s$ colliding in water with initial impact velocity $\bar{W} = 1mm/s$. Then, for typical surface roughness $0.01\mu m < \sigma < 1\mu m$ of particles used in laboratory experiments [3], we have the following range $1 < St_c/\epsilon < 100$. Noting the weak dependence on $\bar{W}$ (e.g. a 100-fold increase in $\bar{W}$ results in a factor of 2.5 increase in St_c) below we will use the range $1 < St_c/\epsilon < 100$ to investigate the dependence of e_{con} on St_c/ϵ .

3.2. Numerical estimate for arbitrary ϵ/St_c

Next, we will compare the asymptotic estimate in (17) with the numerical integration of the full equation of motion (16). We integrate the equation of motion (16) and the corresponding kinematic condition $d\delta/dt = w$ using a simple forward Euler scheme

$$w_{n+1} = w_n + \dot{w}_n \Delta t \quad \delta_{n+1} = \delta_n + w_n \Delta t, \quad (19)$$

where Δt is the time step and the rate of change of velocity $\dot{w}_n$ at time t_n is estimated from (16). The main complication in the numerical treatment is the history integral which makes equation (16) an implicit integro-differential equation for the acceleration $\dot{w}$. It is however possible to evaluate the history integral at time t_n in terms of data at previous times ($\dot{w}_0, \dots, \dot{w}_{n-1}$), by splitting the history integral in two intervals $(0, t_{n-1})$ and (t_{n-1}, t_n) , using integration by parts in the second interval and evaluating the result with quadratures. For more details and a discussion of the numerical accuracy of the scheme, the reader is referred to Simeonov [6] where it was demonstrated that the numerical error in the predicted coefficient of restitution was less than 0.3% when the non-dimensional time step was $\Delta t = 0.001$. The same time step was used for the simulations presented below.

By integrating numerically equation (16) subject to the initial condition $w = -1$ and $\delta = 0$, we were able to estimate the coefficient of restitution e_{con} which is essentially the value of w at the end of contact when $\delta = 0$ again. For the collision of a quartz sphere with a quartz wall for the following range $1 < St_c/\epsilon < 100$ of the main non-dimensional parameter and for two different values of ϵ . The numerically computed e_{con} is shown as a function of ϵ/St_c in Figure 1 (symbols) along with the analytical estimate (straight line). Figure 1 shows that there is a significant reduction of e_{con} as the parameter ϵ/St_c approaches $O(1)$. As expected, the numer-

Figure 1: The coefficient of restitution during contact as a function of the non-dimensional parameter St_c/ϵ for the normal collision of a quartz sphere with a quartz wall ($\kappa = 0$). The symbols correspond to the numerically computed coefficient of restitution for two different values of ϵ and the straight line corresponds to the analytical solution in equation (17).

ical solution is adequately represented by the asymptotic result at small values of ϵ/St_c . The analytical solution tends to overestimate the reduction of e_{con} only at very large $\epsilon/St_c > 0.2$ corresponding to very smooth (or soft) particles. Therefore, for most practical applications involving hard particles, the analytical estimate of the contact coefficient of restitution should be adequate in describing the viscous dissipation. We should mention that the significant deviation of e_{con} (Figure 1, symbols) at large ϵ/St_c from the linear dependence is not due to the $O(\epsilon/St_c^{1/2})$ history forces, but is rather due to higher order $O(\epsilon/St_c)$ terms in the asymptotic expansion of the non-linear Hertzian force in (16). For example, suppressing the history force in the simulation for $\epsilon/St_c = 0.5$ only increases e_{con} by 10%.

4. Discussion

It may be instructive to use our theory to estimate the effect of viscous dissipation on the contact coefficient of restitution in laboratory experiments of sphere-wall collisions at low Stokes numbers. To make such estimates, we used the experimental conditions from the laboratory experiments of Joseph et al. [3] for the low St collision of nylon, glass and steel spheres with a zerodur wall (Table 1).

Using the data in Table 1, we computed ϵ/St_c and estimated e_{con} from equation (17). The resulting e_{con} given in Table 1 is substantially lower than the measured (Joseph et al. [3]) $e_{dry} = 0.97$ for the relatively

smooth glass and steel spheres with $d = 4.1\text{mm}$ and $d = 6.35\text{mm}$. Our estimates imply that viscous dissipation during contact may be important even for the relatively large laboratory particles when their roughness does not exceed $O(0.1\mu\text{m})$. We note that the values of e_{con} given in Table 1 are based on W_∞ , which may significantly overestimate the impact velocity $\bar{W}$ due to viscous dissipation prior to impact. Therefore, the values of e_{con} given in Table 1 would slightly underestimate the damping by viscous dissipation.

The coefficient of restitution during contact e_{con} is usually taken to be constant and equal to the coefficient of restitution in dry collisions, e_{dry} . Here, we have developed a theoretical model for collisions in a viscous fluid, which suggests that the coefficient of restitution during contact e_{con} depends on two non-dimensional parameters - the non-dimensional particle roughness $\epsilon = \sigma/a$ and St_c/ϵ . In the limit of small ϵ/St_c and small St_c , an asymptotic expression shows that the decrease of e_{con} from 1.0 is inversely proportional to the surface roughness and the one-fifth power of the particle impact velocity. It was found that e_{con} is significantly reduced below $e_{con} = 1$ as the contact Stokes number St_c approaches the non-dimensional particle roughness ϵ . The asymptotic prediction for e_{con} was validated with numerical simulations that retained the fully non-linear aspect of Hertzian contact. The obtained theoretical expression for e_{con} can be used to model the collision of particles in numerical simulations that do not resolve explicitly the thin layer lubrication effects during collisions in a viscous fluid [27, 28, 29, 30, 31].

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Table 1: The coefficient of restitution during viscous contact e_{con} (17) in several low Stokes number collisions of glass, nylon and steel spheres with zerodur wall in water-glycerol solutions (Joseph et al. [3]) Also given is the range of the observed wet coefficient of restitution e_{wet}

d [mm]	E_1 [GPa]	λ_1	ρ_p [$\frac{kg}{m^3}$]	σ [μm]	E_2 [GPa]	λ_2	ν [$\frac{mm^2}{s}$]	W_∞ [$\frac{mm}{s}$]	St	$\epsilon \times 10^5$	$\frac{\gamma_{fc}}{\epsilon}$	e_{wet}	e_{con}
6.35	65	0.23	2540	0.13	91	0.24	5	50	17.9	4.1	34.2	0.15-0.3	0.928
4.1	65	0.23	2540	0.05	91	0.24	1	50	57.9	2.4	13.2	0.4-0.7	0.813
3.0	65	0.23	2540	0.138	91	0.24	1	25	21.2	9.2	31.6	0.2-0.5	0.922
6.35	2.76	0.4	1140	2.0	91	0.24	1	20	16.1	63	219	0.15-0.3	0.979
6.35	190	0.27	7780	0.0236	91	0.24	11.6	40	19	0.74	8.35	0.2	0.894