Adjoint design sensitivity analysis and optimization of nonlinear structures using geometrical mapping approach

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Abstract: A new adjoint shape design sensitivity formulation for nonlinear structures subject to contact forces is developed. The method is based on a geometrical mapping approach where shape variation is regarded as a mapping characterized by the shape variation velocity field. An adjoint variable method is developed for performing sensitivity analysis of the average shear strain in the tire belt area, which profile is properly parameterized in function of arcs, allowing explicit design velocities calculated. It can be seen that the present method is much faster than the direct differentiation method and more accurate than the classical finite difference scheme.

Keywords: shape optimization; adjoint variable method; nonlinear structures with contact forces; sensitivity analysis; geometrical mapping approach

1. Introduction

Sensitivity analysis in shape optimization problems is complicated by the fact that the profile of the structure may change sharply in the optimization process [1]. This issue becomes even more critical in highly nonlinear problems such as, for example, optimal design of automotive tires whose mechanical response, durability and dynamical forces/moments generated in the contact patch strongly depends on the outer profile of the structure. Shape design sensitivity analysis (DSA) is still a challenge for tire industry researchers due to the high level of nonlinearity in material, geometry and contact stresses [2]. In this case, finite element analysis obtains an optimal design at affordable computational cost.

Global optimization algorithms that combine different types of response surface approximations are more efficient than local search methods that may get trapped in local minima. The Kriging method is more robust than standard response surface fitting because it does not require assuming the form of the response surface a priori. The gradient-based Kriging (GBK) method [3] can further reduce the number of samples. However, the issue of disposing of a stable and accurate sensitivity formulation becomes even more critical in this case [3, 4]. Another way to reduce the computational cost of shape optimization is to use sample points determined by means of the Latin hypercube design minimizing pairwise correlations and maximizing inter-site distances of sample points [4, 5].

The material derivative approach (MDA) (see the textbook by Choi and Kim [6, 7]) has been adopted in different engineering areas by [8-20]. The adjoint variable method (AVM) [8, 21-34] allows improving computational efficiency of sensitivity evaluations. However, the theoretical derivation of AVM for shape DSA of highly nonlinear structures as tires is a challenging task in view of the complexity of kinematic relationships. Here, a new adjoint shape DSA formulation is proposed for nonlinear tire structures with contact forces. The new formulation entails two basic aspects: (i) derivation of design velocity using a geometrical mapping approach (GMA); (ii) formulation of shape DSA for nonlinear tire structures starting from the parameterized tire profile. The final goal is to perform accurate and stable DSA at low computational cost.

The idea behind GMA is that shape variations are regarded as a mapping characterized by the shape variation velocity field. Using the virtual work principle, sensitivity equations of state variables (i.e., displacements) with respect to the shape variation are formulated. The AVM for sensitivity analysis of average shear strain in the tire belt area is hence formulated. The tire profile is parameterized using parameterized arcs, which makes boundary design velocities explicitly calculable according to its design variables. A passenger radial tire is chosen as case study: remarkably, AVM is about 80% faster than direct differentiation method (DDM) in this case. This provides a useful framework for shape sensitivity analysis and optimization of highly nonlinear structures.

The paper is structured as follows. Section 2 describes design variables and objective function of the tire shape optimization problem. Section 3 presents the geometrical mapping approach that will be used for deriving sensitivity formulation. Section 4 illustrates how tire shape is parameterized and presents the definition of shape design velocity. Section 5 explains how to formulate sensitivity of displacements and displacement gradients with respect to shape variables. Sensitivity derivation for the objective function considered in this study is presented in Sections 6 and 7, respectively for the DDM and AVM. Applications to a passenger radial tire contour optimization are discussed in Section 8. Finally, Section 9 summarizes the main findings of this study.

2. Shape Variables and Objective Function

This article addresses nonlinear structures with contact forces, with tires being the application example. The tire shape optimization problem in this study included the five design variables shown in Figure 1: radius of the first crown arc R_1 , radius of the second crown arc R_2 , radius of the shoulder arc R_3 , radius of the sidewall-shoulder arc R_4 , meridian coordinate of the intersection point of the first and second arcs W_1 , tread contact width W_2 . Since R_4 is not an independent variable as its value can be known from W_2 , the radius related to the shape of the inner boundary and parameters remain constant in the optimization.

Figure 1: Design variables in the shape optimization of a tire (units/mm)

The objective function to be minimized is the average effective shear strain which develops in the belt end area near the tire-ground contact region. The corresponding area in the meridian plane is shown in Figure 2, and the 3D finite element model of tire is obtained from this 2D model by rotating around the center of tire. The circle of 360 degrees is divided into 50 sections on average and the footprint length of tire tread in the circumferential direction comes to 6 sections. Therefore the circumferential angle is taken as $2\pi' 6/50$ which is shown in Figure 4. The objective function (also called performance function) can be expressed as

$${}^t_\tau \psi^1 = \frac{1}{V_b} \int_{{}_\tau V_b} {}^t_\tau \bar{\epsilon}(\mathbf{x}) d_\tau V \quad (1)$$

Where the left subscript τ denotes the initial configuration corresponding to the shape parameter τ_α ($\alpha=1, 2 \dots 5$) mentioned in Figure 1, the left superscript t denotes the load parameter, ${}_\tau V_b$ is the total volume of the design area, and ${}^t_\tau \bar{\epsilon}(\mathbf{x})$ [35] is the effective shear strain:

$${}^t_\tau \bar{\epsilon} = \sqrt{\frac{2}{3} {}^t_\tau e_{ij} {}^t_\tau e_{ij}} = \sqrt{\frac{2}{3} \left({}^t_\tau \epsilon_{ij} - \frac{1}{3} {}^t_\tau \epsilon_{kk} \delta_{ij} \right) \left({}^t_\tau \epsilon_{ij} - \frac{1}{3} {}^t_\tau \epsilon_{ll} \delta_{ij} \right)} \quad (2)$$

Figure 2: FE model of the meridian plane of the tire and definition of the cost function in belt end area

Figure 2 shows the FE mesh adopted for the meridian plane of the tire. The model is comprised of different rubber materials and rebar elements as reinforcements. The FE model, homogeneously divided into 50 sections in the circumferential direction, includes in most part trilinear hexahedral elements (HEX8). The level of mesh refinement was such to have mesh independent solutions at reasonably low computational cost. The structure is subject to two independent load cases: (i) inflation pressure; (ii) contact with rigid road surface under the load carried.

3. Geometrical Mapping Approach

In order to evaluate the sensitivity of cost function with respect to shape variations, strain sensitivities must be determined. For that purpose, sensitivity of displacements with respect to changes in shape must be obtained first. In this study, the GMA was utilized because the MDA may be misleading as the real material derivative is not suitable for formulating the variation in shape design. The partial derivative with respect to the shape parameter does not have any physical meaning, and the gradient is the derivative with respect to the Euler coordinates rather than to the Lagrange coordinates that appear in the displacement gradient of the solids. The GMA is presented with clear physical meaning, wherein the shape variation is regarded as a mapping characterized by the shape variation velocity field, or a fictitious deformation of the continuum (i.e. not a real displacement) and there are no strains or stresses corresponding to such fictitious displacement fields. The shape variation is formulated as a geometrical mapping, characterized by the shape variation velocity or **shape design velocity** field ${}_0V_{i\alpha}$. The **mapping of coordinates** can be written as

$${}_0x_i = {}_0x_i + {}_0V_{i\alpha}\tau_\alpha = {}_0x_i + {}_0u_i \quad i = 1, \dots, m; \alpha = 1, \dots, k \quad (3)$$

Where ${}_0x_i, {}_0x_i$ denote the Lagrange coordinates for the original shape and the shape design corresponding to parameters τ_α respectively, m is the number of independent degrees of freedom

(DOFs), k denotes the number of design parameters, and ${}^0_\tau u_i$ can be regarded as a virtual displacement, which is not a real displacement of the continuum. In addition, there are no strains or stresses corresponding to such a virtual displacement field.

For a given loading condition, the relationship between the displacements of the corresponding mapping points can be expressed as

$${}^t_\tau u_i = {}^t_0 u_i + {}^t_0 u_i^{\circ\alpha} \tau_\alpha \quad (4)$$

Where the right superscript $\circ\alpha$ denotes $\partial/\partial\tau_\alpha$, which means the change rate of the variable for the corresponding mapping point during the shape variation, and ${}^t_0 u_i^{\circ\alpha}$ at $\tau_\alpha = 0$ is the **sensitivity of displacements**.

The relationship between the displacement gradients of the corresponding mapping points can be expressed as

$${}^t_\tau u_{i,j} = {}^t_0 u_{i,j} + {}^t_0 (u_{i,j})^{\circ\alpha} \tau_\alpha \quad (5)$$

Where ${}^t_0 (u_{i,j})^{\circ\alpha}$ denotes the sensitivity of displacement gradients, subscript j indicates the derivative of the displacement u_i with respect to the j^{th} coordinate, which can be formulated as:

$${}^t_0 (u_{i,j})^{\circ\alpha} @ \frac{\nabla}{\nabla \tau_\alpha} ({}^t_\tau u_{i,j}) \Big|_{\tau=0} = {}^t_0 u_i^{\circ\alpha}, j - {}^t_0 u_{i,k} {}^t_0 V_{k\alpha}, j \quad (6)$$

In which $\frac{d}{d\tau} ({}^t_\tau u_{i,j})$ consists of two parts: one is due to the difference of ${}^t_\tau u_i$ and ${}^t_0 u_i$, namely ${}^t_0 u_i^{\circ\alpha}$; the other one comes from the difference of ${}^t_\tau (u_{i,j})$ and ${}^t_0 (u_{i,j})$ in the case of identical ${}^t_\tau u_i$ and ${}^t_0 u_i$.

4. Tire shape parameterization and shape design velocity calculation

An important concept in the derivation of sensitivity is the shape design velocity. For the shape design, the **boundary shape design velocity** can be directly obtained from the definition used to map the coordinates (Eq. (3)):

$${}_0 V_{i\alpha} = \frac{\nabla}{\nabla \tau_\alpha} ({}_0 x_i - {}^t_\tau x_i) \quad (7)$$

where the shape velocity field $V_{i\alpha}$ represents the variation relative of the i^{th} degree of freedom defining the shape with respect to the α_{th} design parameter (variable). Provided $({}^0_\tau x_i - {}_0x_i)$ can be formulated explicitly using the design variables τ_α . Otherwise, the finite difference approach can be followed,

$${}_0V_{i\alpha} = ({}^0_\tau x_i (\Delta \tau_\alpha) - {}_0x_i) / \Delta \tau_\alpha \quad (8)$$

where $\Delta \tau_\alpha$ is a small increment of a design variable.

As mentioned in Section 2, we use five design variables, i.e., R_1 , R_2 , R_3 , R_4 , W_1 , and W_2 , as shown in Figure 1, to describe the outer profile for a series of 50-70 typical passenger car radial tires (PCR). Moreover, note that the specified data in Figure 1 for each parameter are simply an example, and R_4 can be determined by W_2 . The controlling equations of all the arcs can explicitly be described as follows:

$$\begin{aligned} (x_1 - x_{o1})^2 + (y_1 - y_{o1})^2 &= R_1^2, \quad x_{o1} = 0, \quad y_{o1} = D - R_1 \\ (W_1 - x_{o1})^2 + (y_2 - y_{o1})^2 &= R_1^2, \quad (W_1 - x_{o2})^2 + (y_2 - y_{o2})^2 = R_2^2 \\ (W_2 - x_{o2})^2 + (\bar{y} - y_{o2})^2 &= R_2^2, \quad (W_2 - x_{o4})^2 + (\bar{y} - y_{o4})^2 = R_4^2, \\ y_{o4} &= D - H, \quad x_{o4} = B - R_4 \\ (x_3 - x_{o2})^2 + (y_3 - y_{o2})^2 &= R_2^2, \quad (x_3 - x_{o3})^2 + (y_3 - y_{o3})^2 = R_3^2, \\ (x_4 - x_{o3})^2 + (y_4 - y_{o3})^2 &= R_3^2, \quad (x_4 - x_{o4})^2 + (y_4 - y_{o4})^2 = R_4^2 \\ \frac{W_1 - x_{o2}}{y_2 - y_{o2}} &= \frac{W_1 - x_{o1}}{y_2 - y_{o1}}; \quad \frac{x_3 - x_{o2}}{y_3 - y_{o2}} = \frac{x_3 - x_{o3}}{y_3 - y_{o3}}; \quad \frac{x_4 - x_{o3}}{y_4 - y_{o3}} = \frac{x_4 - x_{o4}}{y_4 - y_{o4}} \end{aligned} \quad (9)$$

where B is the tire width; D is the diameter; x_{oi}, y_{oi} are the coordinates of the i^{th} arc center; x_i, y_i are outer surface point coordinates belonging to the i^{th} arc.

This means that, giving the five design variables, the tire width B and the diameter D , all 4 arcs as well as the intersection points can be calculated; therefore, the tire shape can be determined.

The design velocity calculation is a key to the shape optimization problem, which is closely related to shape parameterization. A shape change can be described by a mapping function T expressed by

$$x_\tau \equiv T(x, \tau), \quad \Omega_\tau \equiv T(\Omega, \tau) \quad (10)$$

where Ω represents the whole shape domain.

The design velocity, physically, can be understood as the partial derivative of the shape coordinates with the shape parameter. If one imagines this partial derivative as “the design velocity”, the shape parameter τ can be taken as the “time”, therefore the **design velocity** can be defined as

$$V(x_\tau, \tau) \equiv \frac{dx_\tau}{d\tau} = \frac{dT(x, \tau)}{d\tau} = \frac{\partial T(x, \tau)}{\partial \tau} \quad (11)$$

It can be shown that the design velocity is the key parameter for mesh updating and to evaluate sensitivity of objective function with respect to shape variables.

Because all the arcs can be explicitly formulated using Eq. (9), the tire contour can be parameterized using arcs. That is:

$$\begin{bmatrix} \mathbf{p}_x(u) & \mathbf{p}_y(u) \end{bmatrix} = \mathbf{F}\mathbf{B}(\mathbf{R}) \quad (12)$$

Where u are the parametric coordinates (i.e. the length of each arc); $\mathbf{F}$ and $\mathbf{B}$ are coefficient matrices, and $\mathbf{R}$ is the vector of design variables.

The real meaning of Eq. (12) is that one can explicitly calculate all outer surface point coordinates, from which the design velocity can be calculated.

Then, the design velocity of each mesh point can be determined by

$$\mathbf{V} = \mathbf{F} \frac{\partial \mathbf{B}}{\partial \mathbf{R}} \quad (13)$$

After the 2D design velocity on the meridian plane is obtained, the 3D design velocities can be calculated by

$$V_x = \bar{V}_x, V_y = \bar{V}_y * \cos(\theta), V_z = -\bar{V}_y * \sin(\theta) \quad (14)$$

Figure 3: Sketch of the tire for 3D design velocities

The boundary design velocity can be easily calculated with Eq. (13). Various approaches have been presented in the literature [8, 9], such as the boundary displacement method and the fictitious load method, for determining the **domain shape design velocity field** based on the boundary design velocity. The resulting domain velocity field is almost distributed across the domain, and the computation is time-consuming in general. Conversely, in the present approach, the domain velocity field is localized in one layer of finite elements, which is directly related to the design modified boundary. For HEX8 elements, the domain velocity field of the elements is directly determined by the

boundary velocity field, and no additional computing cost is required. Furthermore, undesirable mesh distortions for composite structures can be naturally avoided.

5. Sensitivity of displacements and displacement gradients

The system of equations for the sensitivity of displacements can be derived from the virtual work principle. For the shape-varied structure in the shape optimization problem, the virtual work equation can be formulated as

$$\begin{aligned} \dot{\mathbf{0}}_{\tau V} {}^t S_{ij} \delta_{\tau}^t \varepsilon_{ij} d_{\tau} V &= \dot{\mathbf{0}}_{\tau S^t} {}^t t_k \delta_0 u_k d_{\tau} S + \dot{\mathbf{0}}_{\tau V} \tau \rho {}^t f_k \delta_0 u_k d_{\tau} V + {}^t C_c^{\text{LM}} \\ {}^t C_c^{\text{LM}} &= \dot{\mathbf{0}}_{\tau S_c^{\text{st}}} \left({}^t \lambda_n \delta_0 u_n + ({}^t u_n + {}^0 g_n) \delta_0 \lambda_n \right) d_{\tau} S \end{aligned} \quad (15)$$

In this equation ${}^t C_c^{\text{LM}}$ is the contact boundary term in Lagrange multiplier approach, where S_c^{st} denotes the actually contacted part of the possibly in contact boundary, ${}_0 \lambda_n$, ${}^t \lambda_n$ denote the Lagrange multipliers while $\mathbf{g}_n$ denotes the normal gap, $\mathbf{u}_i$ ($i = 1, 2, 3$) denotes the displacement, the left subscript τ denotes the initial configuration corresponding to the shape parameter τ_{α} , and the left superscript t denotes the loading condition. The entire boundary is divided into a traction-free boundary ${}^t S^f$, a traction-given boundary ${}^t S^t$, and the contacted boundary ${}^t S^{\text{st}}$. The traction-free boundary, including the part of the tire boundary which may come in contact with the load, is excluded from the virtual work equation. It should be noted that the division of the boundary is load and shape dependent. Here, for simplicity, frictionless contact is considered; thus, only a normal Lagrange multiplier is used.

For the shape optimization problem, the configuration corresponding to $\tau = 0$ is taken as reference. Hence, Eq. (15) can be rewritten as

$$\begin{aligned} \dot{\mathbf{0}}_{\tau V} {}^t S_{ij} \delta_{\tau}^t \varepsilon_{ij} \Big|_0^{\tau} \mathbf{J} d_0 V &= \dot{\mathbf{0}}_{\tau S^t} {}^t t_k \delta_{\tau}^t u_k (d_{\tau} S / d_0 S) d_0 S + \dot{\mathbf{0}}_{\tau V} \tau \rho {}^t f_k \delta_{\tau}^t u_k \Big|_0^{\tau} \mathbf{J} d_0 V \\ &+ \dot{\mathbf{0}}_{\tau S_c^{\text{st}}} \left({}^t \lambda_n \delta_0 u_n + ({}^t u_n + {}^0 g_n) \delta_0 \lambda_n \right) (d_{\tau} S / d_0 S) d_0 S \end{aligned} \quad (16)$$

Recalling Eq. (5), the strain field can be expressed as follows:

$$\begin{aligned} {}^t \varepsilon_{ij} &= ({}^t u_{i,j} + {}^t u_{j,i} + {}^t u_{k,i} {}^t u_{k,j}) / 2 = \frac{1}{2} \left\{ ({}_0^t u_{i,j} + {}_0^t u_{j,i} + {}_0^t u_{k,i} {}_0^t u_{k,j}) + \right. \\ &\left. \left({}_0^t (u_{i,j})^{\circ\alpha} + {}_0^t (u_{j,i})^{\circ\alpha} + {}_0^t (u_{k,i})^{\circ\alpha} {}_0^t u_{k,j} + {}_0^t u_{k,i} {}_0^t (u_{k,j})^{\circ\alpha} \right) \tau_{\alpha} + \left({}_0^t (u_{k,i})^{\circ\alpha} {}_0^t (u_{k,j})^{\circ\beta} \right) \tau_{\alpha} \tau_{\beta} \right\} \end{aligned} \quad (17)$$

where τ_{α} and τ_{β} denote the α_{ih} and β_{ih} shape design variables. Note that the strain expressed in Eq. (17) is defined according to the nonlinear theory.

For the virtual displacement $\delta_{\tau} u_i$, which is arbitrary and geometrically admissible for the design shape vector $\bar{\tau}$ including all shape variables, the corresponding virtual strain can be expressed as follows:

$$\begin{aligned}
\delta_\tau \varepsilon_{ij} &= \frac{1}{2} \left\{ (\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{k,i} {}^t_0 u_{k,j} + {}^t_0 u_{k,i} \delta_0 u_{k,j}) + \right. \\
&\quad \left. + \left(\delta_0 (u_{i,j})^{\circ\alpha} + \delta_0 (u_{j,i})^{\circ\alpha} + \delta_0 (u_{k,i})^{\circ\alpha} {}^t_0 u_{k,j} + {}^t_0 (u_{k,i})^{\circ\alpha} \delta_0 u_{k,j} \right. \right. \\
&\quad \left. \left. + \delta_0 u_{k,i} {}^t_0 (u_{k,j})^{\circ\alpha} + {}^t_0 u_{k,i} \delta_0 (u_{k,j})^{\circ\alpha} \right) \tau_\alpha \right\}
\end{aligned} \tag{18}$$

where the small higher order terms are neglected. The full derivation of Eq. (18) is given in Appendix 3.

For the second Piola-Kirchhoff stress tensor, it can be written as

$$\begin{aligned}
{}^t_\tau S_{ij} &= {}^t_0 S_{ij} + {}^t_0 S^{\circ\alpha}_{ij} \tau_\alpha \\
{}^t_0 S^{\circ\alpha}_{ij} \tau_\alpha &\approx {}^t_0 D_{Tijk} \left({}^t_0 (u_{k,l})^{\circ\alpha} + {}^t_0 (u_{l,k})^{\circ\alpha} + {}^t_0 (u_{m,k})^{\circ\alpha} {}^t_0 u_{m,l} + {}^t_0 u_{m,k} {}^t_0 (u_{m,l})^{\circ\alpha} \right) \tau_\alpha / 2
\end{aligned} \tag{19}$$

where the left term represent the stress increment, which is obtained from the product between the fourth order tangent constitutive tensor $D_{Tijkl}, i, j, k, l = 1, 2, 3$ and the displacement gradient sensitivity tensor.

As far as it concerns the other terms in Eq. (16), it can be mentioned that

$${}_{\tau}\rho = {}_0\rho \quad (20)$$

which implies keeping the same mass density during shape changes, for a generic element e :

$$\begin{aligned} d_{\tau}V_e &= \left| {}^{\tau}\mathbf{J} \right| d_0V_e \quad d_{\tau}V_e = \left| {}^{\tau}\mathbf{J} \right| d_0V_e = \left| {}^{\tau}\mathbf{J} \right|_0 \mathbf{J} | d\xi_{1e} d\xi_{2e} d\xi_{3e} \\ d_{\tau}V_e &= \left| {}^{\tau}\mathbf{J} \right| d_0V_e = \left| {}^{\tau}\mathbf{J} \right|_0 \mathbf{J} | d\xi_{1e} d\xi_{2e} d\xi_{3e} \end{aligned} \quad (21)$$

Introducing the above equations into the equation of virtual work corresponding to the shape variation - Eq. (16) - it follows:

$$\begin{aligned}
& \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\mathring{\mathbf{0}}_{0V_e} S_{ij} + {}^t_0 D_{ijkl} \left({}^t_0(u_{k,l})^{\circ\alpha} + {}^t_0(u_{l,k})^{\circ\alpha} + {}^t_0(u_{m,k})^{\circ\alpha} {}^t_0 u_{m,l} + {}^t_0 u_{m,k} {}^t_0(u_{m,l})^{\circ\alpha} \right) \tau_\alpha / 2 \right) \\
& \times \left(\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{k,i} {}^t_0 u_{k,j} + {}^t_0 u_{k,i} \delta_0 u_{k,j} \right) / 2 + \left(\delta_0(u_{i,j})^{\circ\alpha} + \delta_0(u_{j,i})^{\circ\alpha} \right. \\
& \left. + \delta_0(u_{k,i})^{\circ\alpha} {}^t_0 u_{k,j} + {}^t_0(u_{k,i})^{\circ\alpha} \delta_0 u_{k,j} + \delta_0 u_{k,i} {}^t_0(u_{k,j})^{\circ\alpha} + {}^t_0 u_{k,i} \delta_0(u_{k,j})^{\circ\alpha} \right) \tau_\alpha / 2 \left(1 + {}^t_0 V_{i\alpha,i} \tau_\alpha \right) d_0 V_e \quad (22) \\
& = \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_e^t} {}^t_0 t_k (1 + {}^t_0 S_e^{\circ\alpha} \tau_\alpha) \delta_0 u_k d_0 S_e + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} {}^t_0 \rho_0 f_k (1 + {}^t_0 V_{i\alpha,i} \tau_\alpha) \delta_0 u_k d_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_e^{\text{st}}} \left\{ ({}^t_0 \lambda_n + {}^t_0 \lambda_n^{\circ\alpha} \tau_\alpha) \delta_0 u_n (1 + {}^t_0 S_e^{\circ\beta} \tau_\beta) + ({}^t_0 u_n + {}^t_0 u_n^{\circ\alpha} \tau_\alpha + {}^0_0 g_n) \delta_0 \lambda_n (1 + {}^t_0 S_e^{\circ\beta} \tau_\beta) \right\} d_0 S_e
\end{aligned}$$

Note that S_e denotes the element surface. Developing this equation with respect to the shape variation factor τ_α , after having neglected the small higher order terms related to the stress field ${}^t_0 S_{ij}$ and the loads ${}^t_0 t_k$, ${}^t_0 \rho_0 f_k$, which are in equilibrium in the original shape, and introducing Eq. (6) for the sensitivity of the displacement gradient, Eq. (22) can be rewritten as

$$\begin{aligned}
& \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{i,j} + \delta_0 u_{j,i} \right) / 2 \mathring{\mathbf{D}}_{Tijkl} \left(\delta_0^t u_{k,l}^{\circ\alpha} + {}^t_0 u_{l,k}^{\circ\alpha} \right) / 2 \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{i,j} + \delta_0 u_{j,i} \right) / 2 \mathring{\mathbf{D}}_{Tijkl} \left(\delta_0^t u_{m,k}^{\circ\alpha} + {}^t_0 u_{m,l}^{\circ\alpha} + {}^t_0 u_{m,l}^{\circ\alpha} + {}^t_0 u_{m,k}^{\circ\alpha} \right) / 2 \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{n,i} + {}^t_0 u_{n,j} + \delta_0 u_{n,j} + {}^t_0 u_{n,i} \right) / 2 \mathring{\mathbf{D}}_{Tijkl} \left(\delta_0^t u_{k,l}^{\circ\alpha} + {}^t_0 u_{l,k}^{\circ\alpha} \right) / 2 \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{n,i} + {}^t_0 u_{n,j} + \delta_0 u_{n,j} + {}^t_0 u_{n,i} \right) / 2 \mathring{\mathbf{D}}_{Tijkl} \left(\delta_0^t u_{m,k}^{\circ\alpha} + {}^t_0 u_{m,l}^{\circ\alpha} + {}^t_0 u_{m,l}^{\circ\alpha} + {}^t_0 u_{m,k}^{\circ\alpha} \right) / 2 \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} (\delta_0 u_{n,i}) {}^t_0 S_{ij} ({}^t_0 u_{n,j}^{\circ\alpha}) \mathring{\mathbf{d}}_0 V_e \\
& = \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_e^t} (\delta_0 u_k) (\chi_0^t t_k) {}^t_0 S_e^{\circ\alpha} \mathring{\mathbf{d}}_0 S_e + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} (\delta_0 u_k) ({}^t_0 \rho_0^t f_k) {}^t_0 V_{ia,i} \mathring{\mathbf{d}}_0 V_e \\
& - \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{n,i} + {}^t_0 u_{n,j} + \delta_0 u_{n,j} + {}^t_0 u_{n,i} \right) / 2 \mathring{\mathbf{S}}_{ij} {}^t_0 V_{k\alpha,k} \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left((\delta_0 u_{n,i} + {}^t_0 u_{n,k} + {}^t_0 V_{k\alpha,j} + \delta_0 u_{n,j} + {}^t_0 u_{n,k} + {}^t_0 V_{k\alpha,i}) / 2 \right) {}^t_0 S_{ij} \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left((\delta_0 u_{i,k} + {}^t_0 V_{k\alpha,j} + \delta_0 u_{j,k} + {}^t_0 V_{k\alpha,i}) / 2 \right) {}^t_0 S_{ij} \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left((\delta_0 u_{n,k} + {}^t_0 V_{k\alpha,i} + {}^t_0 u_{n,j} + \delta_0 u_{n,k} + {}^t_0 V_{k\alpha,j} + {}^t_0 u_{n,i}) / 2 \right) {}^t_0 S_{ij} \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left((\delta_0 u_{i,j} + \delta_0 u_{j,i}) / 2 \right) {}^t_0 D_{Tijkl} \left(({}^t_0 u_{k,m} + {}^t_0 V_{m\alpha,l} + {}^t_0 u_{l,m} + {}^t_0 V_{m\alpha,k}) / 2 \right) \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left((\delta_0 u_{i,j} + \delta_0 u_{j,i}) / 2 \right) {}^t_0 D_{Tijkl} \left(({}^t_0 u_{m,k} + {}^t_0 u_{m,n} + {}^t_0 V_{n\alpha,l} + {}^t_0 u_{m,l} + {}^t_0 u_{m,n} + {}^t_0 V_{n\alpha,k}) / 2 \right) \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left((\delta_0 u_{n,i} + {}^t_0 u_{n,j} + \delta_0 u_{n,j} + {}^t_0 u_{n,i}) / 2 \right) {}^t_0 D_{Tijkl} \left(({}^t_0 u_{k,m} + {}^t_0 V_{m\alpha,l} + {}^t_0 u_{l,m} + {}^t_0 V_{m\alpha,k}) / 2 \right) \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left((\delta_0 u_{n,i} + {}^t_0 u_{n,j} + \delta_0 u_{n,j} + {}^t_0 u_{n,i}) / 2 \right) {}^t_0 D_{Tijkl} \left(({}^t_0 u_{m,k} + {}^t_0 u_{m,n} + {}^t_0 V_{n\alpha,l} + {}^t_0 u_{m,l} + {}^t_0 u_{m,n} + {}^t_0 V_{n\alpha,k}) / 2 \right) \mathring{\mathbf{d}}_0 V_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_e^{\text{st}}} (\delta_0 u_n^t \lambda_n^{\circ\alpha} + \delta_0 \lambda_n^0 g_n^{\circ\alpha} + \delta_0 \lambda_n^t u_n^{\circ\alpha} + (\delta_0 u_n^t \lambda_n)_0 S_e^{\circ\alpha}) \mathring{\mathbf{d}}_0 S_e
\end{aligned} \tag{23}$$

Finally, the equations for evaluating sensitivity of displacements can be formulated in matrix form as follows:

$$\begin{aligned}
({}_0 \mathbf{K}_L + {}^t_0 \mathbf{K}_N + {}^t_0 \mathbf{K}_G) {}^t_0 \mathbf{u}^{\circ\alpha} &= {}^t_0 \mathbf{F}^{\circ\alpha} + {}^t_0 \mathbf{F}_C^{\circ\alpha} - {}^t_0 \mathbf{C}_C {}^t_0 \boldsymbol{\lambda}^{\circ\alpha} \\
{}^t_0 \mathbf{G}_C {}^t_0 \mathbf{u}^{\circ\alpha} &= \mathbf{g}^{\circ\alpha}
\end{aligned} \tag{24}$$

where $\mathbf{G}_C$ is the kinematic matrix and $\mathbf{C}_C$ denotes the contact matrix with regard to Lagrange multiplier, ${}_0 \mathbf{K}_L$, ${}^t_0 \mathbf{K}_N$, ${}^t_0 \mathbf{K}_G$ are identical to those in the conventional nonlinear finite element analysis for the original design shape. That is:

$${}_0\mathbf{K}_L = \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V} {}_0\mathbf{B}_L^T {}_0^t\mathbf{D} {}_0\mathbf{B}_L d_0V \quad (25)$$

$${}_0^t\mathbf{K}_N = \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V} ({}_0\mathbf{B}_L^T {}_0^t\mathbf{D} {}_0^t\mathbf{B}_N + {}_0^t\mathbf{B}_N^T {}_0^t\mathbf{D} {}_0\mathbf{B}_L + {}_0^t\mathbf{B}_N^T {}_0^t\mathbf{D} {}_0^t\mathbf{B}_N) d_0V \quad (26)$$

$${}_0^t\mathbf{K}_G = \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V} {}_0\mathbf{B}_G^T {}_0^t\mathbf{S} {}_0\mathbf{B}_G d_0V \quad (27)$$

The load vector in Eq. (24) can be formulated as follows:

$${}_0^t\mathbf{F}^{\circ\alpha} = {}_0^t\mathbf{Q}^{\circ\alpha} + {}_0^t\mathbf{F}_F^{\circ\alpha} \quad (28)$$

Where ${}_0^t\mathbf{Q}^{\circ\alpha}$, ${}_0^t\mathbf{F}_F^{\circ\alpha}$ denote the change rates of the real loads and the fictitious loads, respectively.

For the shape-modified design, the system of equations in the nonlinear finite element analysis can be written as

$$({}_0\mathbf{K}_L + {}_0\mathbf{K}_L^{\circ\alpha}\tau_\alpha + {}_0^t\mathbf{K}_N + {}_0^t\mathbf{K}_N^{\circ\alpha}\tau_\alpha + {}_0^t\mathbf{K}_G + {}_0^t\mathbf{K}_G^{\circ\alpha}\tau_\alpha)({}_0^t\mathbf{u} + {}_0^t\mathbf{u}^{\circ\alpha}\tau_\alpha) = {}_0^t\mathbf{F} + {}_0^t\mathbf{F}_R^{\circ\alpha}\tau_\alpha \quad (29)$$

In the sensitivity analysis, the system of equations to be solved is

$$({}_0\mathbf{K}_L + {}_0^t\mathbf{K}_N + {}_0^t\mathbf{K}_G){}_0^t\mathbf{u}^{\circ\alpha} = {}_0^t\mathbf{F}_R^{\circ\alpha} - ({}_0\mathbf{K}_L^{\circ\alpha} + {}_0^t\mathbf{K}_N^{\circ\alpha} + {}_0^t\mathbf{K}_G^{\circ\alpha}){}_0^t\mathbf{u} = {}_0^t\mathbf{F}_R^{\circ\alpha} + {}_0^t\mathbf{F}_F^{\circ\alpha} = {}_0^t\mathbf{F}^{\circ\alpha} \quad (30)$$

where the fictitious load ${}_0^t\mathbf{F}_F^{\circ\alpha}$ can be formulated as

$$\begin{aligned} {}_0^t\mathbf{F}_F^{\circ\alpha} = & - \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} ({}_0\mathbf{B}_L^T + {}_0^t\mathbf{B}_N^T) {}_0^t\mathbf{S} {}_0V_{i\alpha} \cdot_i d_0V_e + \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} {}_0\mathbf{B}_{GV\alpha}^T {}_0^t\mathbf{S} d_0V_e \\ & + \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} {}_0\mathbf{B}_{LV\alpha}^T {}_0^t\mathbf{S} d_0V_e + \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} {}_0\mathbf{B}_{NV\alpha}^T {}_0^t\mathbf{S} d_0V_e \\ & + \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} {}_0\mathbf{B}_L^T {}_0^t\mathbf{D}\mathbf{E}_{LV\alpha} d_0V_e + \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} {}_0\mathbf{B}_L^T {}_0^t\mathbf{D}\mathbf{E}_{NV\alpha} d_0V_e \\ & + \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} {}_0\mathbf{B}_N^T {}_0^t\mathbf{D}\mathbf{E}_{LV\alpha} d_0V_e + \mathring{\mathbf{a}}_e \mathring{\mathbf{\partial}}_{{}_0V_e} {}_0\mathbf{B}_N^T {}_0^t\mathbf{D}\mathbf{E}_{NV\alpha} d_0V_e \end{aligned} \quad (31)$$

where all the terms can be derived from Eq. (23). In Eqs. (25-27), stiffness matrix ${}_0^t\mathbf{K}_N$, ${}_0^t\mathbf{K}_N$, ${}_0^t\mathbf{K}_N$ including strain matrix $\mathbf{B}_L, \mathbf{B}_N, \mathbf{B}_G$ preserve their classical meaning. However, the strain matrix parameters $\mathbf{B}_{LV\alpha}, \mathbf{B}_{NV\alpha}, \mathbf{E}_{LV\alpha}, \mathbf{E}_{NV\alpha}$ related to design velocity in Eq. (31) are different from those defined in conventional finite element analysis as explained in Appendix 4. The complete derivation is a routine process and omitted here for the sake of brevity.

On the other hand, the change rates of the real loads can be calculated as

$${}^t_0 \mathbf{F}_R^{\circ\alpha} \circ {}^t_0 \mathbf{Q}^{\circ\alpha} = \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} N^T ({}_0 \rho {}^t_0 \mathbf{f}) {}_0 V_{i\alpha}, {}_i d_0 V_e + \mathring{\mathbf{a}}_e \mathring{\partial}_{0S_e} N^T {}^t_0 \mathbf{S}_e^{\circ\alpha} d_0 S_e \quad (32)$$

Sensitivity of displacements can be determined from the linear system(30).

6. Determination of Cost Function Sensitivities with the Direct Differentiation Method

The sensitivities of cost function defined by Eq. (1) can be computed once sensitivities of displacements and the displacement gradients are known. The average strain functional (1) defined for the tire shape optimization problem can be expressed as

$${}^t_0 \psi = \int_{0V} f({}^t_0 u_i({}_0 \mathbf{x}), {}^t_0 u_{i,j}({}_0 \mathbf{x})) d_0 V \quad (33)$$

If we assume that density remains unchanged during the shape variation, the cost function can be formulated as

$$\begin{aligned} {}^t_\tau \psi &= \int_{\tau V} f({}^t_\tau u_i({}_\tau \mathbf{x}), {}^t_\tau u_{i,j}({}_\tau \mathbf{x})) d_\tau V \\ &= \int_{0V} f({}^t_0 u_i({}_0 \mathbf{x}) + {}^t_0 u_i^{\circ\alpha}({}_0 \mathbf{x}) \tau_\alpha, {}^t_0 u_{i,j}({}_0 \mathbf{x}) + {}^t_0 (u_{i,j})^{\circ\alpha} \tau_\alpha) \left[1 + ({}_0 |\mathbf{J}|^{\circ\beta} / {}_0 |\mathbf{J}|) \tau_\beta \right] d_0 V \\ &\approx {}^t_0 \psi + {}^t_0 \psi^{\circ\alpha} \tau_\alpha \end{aligned} \quad (34)$$

where

$$\begin{aligned} {}^t_0 \psi^{\circ\alpha} &= \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} \left(\frac{\partial f}{\partial u_i} \frac{\partial}{\partial x} N^m(\xi) {}^t_0 u_i^{m\circ\alpha} + \frac{\partial f}{\partial u_{i,j}} \frac{\partial}{\partial x} (N^m,{}_j) {}^t_0 u_i^{m\circ\alpha} \right) d_0 V_e \\ &\quad - \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} \left(\frac{\partial f}{\partial u_{i,j}} \frac{\partial}{\partial x} (N^m,{}_k) {}^t_0 u_i^{m\circ\alpha} \frac{\partial}{\partial x} (N^n,{}_j) {}^t_0 V_{k\alpha}^n \right) d_0 V_e \\ &\quad + \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} f({}^t_0 u_i({}_0 \mathbf{x}), {}^t_0 u_{i,j}({}_0 \mathbf{x})) \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} (N^m,{}_i) {}_0 V_{i\alpha}^m d_0 V_e \end{aligned} \quad (35)$$

Introducing

$$\mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} \left(\frac{\partial f}{\partial u_i} \frac{\partial}{\partial x} N^m(\xi) {}^t_0 u_i^{m\circ\alpha} + \frac{\partial f}{\partial u_{i,j}} \frac{\partial}{\partial x} (N^m,{}_j) {}^t_0 u_i^{m\circ\alpha} \right) d_0 V_e @ \frac{\partial \psi}{\partial {}^t_0 \mathbf{u}} {}^t_0 \mathbf{u}^{\circ\alpha} \quad (36)$$

$$\begin{aligned} &- \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} \left(\frac{\partial f}{\partial u_{i,j}} \frac{\partial}{\partial x} (N^m,{}_k) {}^t_0 u_i^{m\circ\alpha} \frac{\partial}{\partial x} (N^n,{}_j) {}^t_0 V_{k\alpha}^n \right) d_0 V_e \\ &+ \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} f({}^t_0 u_i({}_0 \mathbf{x}), {}^t_0 u_{i,j}({}_0 \mathbf{x})) \mathring{\mathbf{a}}_e \mathring{\partial}_{0V_e} (N^m,{}_i) {}_0 V_{i\alpha}^m d_0 V_e @ {}^t_0 \bar{\psi}^{\circ\alpha} \end{aligned} \quad (37)$$

The sensitivity of cost function can hence be expressed as

$${}^t_0\psi^{o\alpha} = \frac{\frac{\partial {}^t_0\psi}{\partial {}^t_0\mathbf{u}}}{{}^t_0\mathbf{u}} {}^t_0\mathbf{u}^{o\alpha} + {}^t_0\bar{\psi}^{o\alpha} \quad (38)$$

Displacement sensitivities ${}^t_0\mathbf{u}^{o\alpha}$ are obtained by solving the Eq. (30). Determination of cost function sensitivities with Eq. (38) and (30) is called direct sensitivity calculation (DDM). Since this method is computationally expensive as it requires solving the system (30), the AVM was developed in this study.

7. Determination of Cost Function Sensitivities with the Adjoint Variable Method

The system (24) that allows sensitivity of displacements with respect to shape variables to be computed can be rewritten as follows

$$\begin{bmatrix} \frac{\partial {}^t_0\mathbf{K}_L}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_N}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_G}{\partial \mathbf{e}} \\ \frac{\partial {}^t_0\mathbf{G}_C}{\partial \mathbf{e}} \end{bmatrix} \begin{bmatrix} {}^t_0\mathbf{C}_C \frac{\partial \mathbf{u}}{\partial \mathbf{e}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} {}^t_0\mathbf{u}^{o\alpha} \\ {}^t_0\lambda^{o\alpha} \end{bmatrix} = \begin{bmatrix} {}^t_0\mathbf{F}^{o\alpha} + {}^t_0\mathbf{F}_C^{o\alpha} \\ \mathbf{g}^{o\alpha} \end{bmatrix} \quad (39)$$

The sensitivity of cost function can be determined with the following expression

$${}^t_0\psi^{o\alpha} = \frac{\frac{\partial {}^t_0\psi}{\partial \mathbf{e}}}{\frac{\partial {}^t_0\psi}{\partial \mathbf{u}}} \frac{\frac{\partial {}^t_0\psi}{\partial \mathbf{e}}}{\frac{\partial {}^t_0\psi}{\partial \mathbf{u}}} \begin{bmatrix} \frac{\partial {}^t_0\mathbf{K}_L}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_N}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_G}{\partial \mathbf{e}} \\ \frac{\partial {}^t_0\mathbf{G}_C}{\partial \mathbf{e}} \end{bmatrix} \begin{bmatrix} {}^t_0\mathbf{C}_C \frac{\partial \mathbf{u}}{\partial \mathbf{e}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} {}^t_0\mathbf{F}^{o\alpha} + {}^t_0\mathbf{F}_C^{o\alpha} \\ \mathbf{g}^{o\alpha} \end{bmatrix} + {}^t_0\bar{\psi}^{o\alpha} \quad (40)$$

where $\frac{\partial {}^t_0\psi}{\partial \lambda} = \mathbf{0}$ because the cost function does not directly depend on the traction at the contact surface.

Using the adjoint approach, let us put

$$\frac{\frac{\partial {}^t_0\psi}{\partial \mathbf{e}}}{\frac{\partial {}^t_0\psi}{\partial \mathbf{u}}} \frac{\frac{\partial {}^t_0\psi}{\partial \mathbf{e}}}{\frac{\partial {}^t_0\psi}{\partial \mathbf{u}}} \begin{bmatrix} \frac{\partial {}^t_0\mathbf{K}_L}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_N}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_G}{\partial \mathbf{e}} \\ \frac{\partial {}^t_0\mathbf{G}_C}{\partial \mathbf{e}} \end{bmatrix} \begin{bmatrix} {}^t_0\mathbf{C}_C \frac{\partial \mathbf{u}}{\partial \mathbf{e}} \\ \mathbf{0} \end{bmatrix} = \frac{\partial {}^t_0\psi}{\partial \mathbf{e}} \mathbf{A}^T \boldsymbol{\alpha} \quad (41)$$

The adjoint equation system can hence be formulated as

$$\begin{bmatrix} \frac{\partial {}^t_0\mathbf{K}_L}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_N}{\partial \mathbf{e}} + \frac{\partial {}^t_0\mathbf{K}_G}{\partial \mathbf{e}} \\ \frac{\partial {}^t_0\mathbf{G}_C}{\partial \mathbf{e}} \end{bmatrix} \begin{bmatrix} {}^t_0\mathbf{C}_C \frac{\partial \mathbf{u}}{\partial \mathbf{e}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{A}^T \boldsymbol{\alpha} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \frac{\partial {}^t_0\psi}{\partial \mathbf{e}} \frac{\partial {}^t_0\mathbf{F}^{o\alpha}}{\partial \mathbf{e}} + \frac{\partial {}^t_0\psi}{\partial \mathbf{e}} \frac{\partial {}^t_0\mathbf{F}_C^{o\alpha}}{\partial \mathbf{e}} \\ \frac{\partial {}^t_0\psi}{\partial \mathbf{e}} \frac{\partial \mathbf{g}^{o\alpha}}{\partial \mathbf{e}} \end{bmatrix} \quad (42)$$

The matrix of coefficients of the adjoint equation system is the transpose of the original system of equations because the matrix of coefficients is not symmetric. Then,

$$\begin{bmatrix} \frac{\partial \psi}{\partial \mathbf{u}} \\ \frac{\partial \psi}{\partial \mathbf{p}} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \mathbf{u}} \mathbf{K}_L + \frac{\partial}{\partial \mathbf{u}} \mathbf{K}_N + \frac{\partial}{\partial \mathbf{u}} \mathbf{K}_G \\ \frac{\partial}{\partial \mathbf{p}} \mathbf{C}_C^T \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \mathbf{u}} \mathbf{G}_C^T \\ \frac{\partial}{\partial \mathbf{p}} \mathbf{u} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial \mathbf{u}} \mathbf{u} \\ \frac{\partial}{\partial \mathbf{p}} \mathbf{u} \\ \frac{\partial}{\partial \mathbf{p}} \mathbf{u} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \mathbf{u}} \mathbf{u} \\ \frac{\partial}{\partial \mathbf{p}} \mathbf{u} \\ \frac{\partial}{\partial \mathbf{p}} \mathbf{u} \end{bmatrix}^T \quad (43)$$

After the adjoint variables $\frac{\partial \psi}{\partial \mathbf{u}}^T$ $\frac{\partial \psi}{\partial \mathbf{p}}^T$ are solved from Eq. (43), and then introducing Eq. (41) into Eq. (40), the sensitivity of cost function is hence expressed as:

$$\frac{\partial \psi}{\partial \mathbf{u}}^T = \mathbf{A}^T \left\{ \frac{\partial}{\partial \mathbf{u}} \mathbf{F}^{\circ\alpha} + \frac{\partial}{\partial \mathbf{u}} \mathbf{F}_C^{\circ\alpha} \right\} + \mathbf{a}^T \left\{ \mathbf{g}^{\circ\alpha} \right\} + \frac{\partial \psi}{\partial \mathbf{p}}^T \quad (44)$$

For example, for the average strain to be determined in the belt end area of the tire - Eq. (1) - the cost function is

$$\begin{aligned} \frac{\partial \psi}{\partial \mathbf{u}} &= \frac{1}{V_b} \int_{V_b} \frac{\partial}{\partial \mathbf{u}} \bar{\epsilon}(\mathbf{x}) dV \\ \frac{\partial \bar{\epsilon}}{\partial \mathbf{u}} &= \sqrt{\frac{2}{3} \frac{\partial}{\partial \mathbf{u}} \epsilon_{ij} \frac{\partial}{\partial \mathbf{u}} \epsilon_{ij}} = \sqrt{\frac{2}{3} \left(\frac{\partial}{\partial \mathbf{u}} \epsilon_{ij} - \frac{1}{3} \frac{\partial}{\partial \mathbf{u}} \epsilon_{kk} \delta_{ij} \right) \left(\frac{\partial}{\partial \mathbf{u}} \epsilon_{ij} - \frac{1}{3} \frac{\partial}{\partial \mathbf{u}} \epsilon_{ll} \delta_{ij} \right)} \end{aligned} \quad (45)$$

Sensitivities with respect to shape variables can be determined as follows

$$\left(\frac{\partial \psi}{\partial \mathbf{p}} \right)^{\circ\alpha} = \mathbf{A}^T \left\{ \frac{\partial}{\partial \mathbf{p}} \mathbf{F}^{\circ\alpha} + \frac{\partial}{\partial \mathbf{p}} \mathbf{F}_C^{\circ\alpha} \right\} + \mathbf{a}^T \left\{ \mathbf{g}^{\circ\alpha} \right\} + \left(\frac{\partial \psi}{\partial \mathbf{p}} \right)^{\circ\alpha} \quad (46)$$

If the design velocity is zero in a given region of design space, cost function will be insensitive to shape variations. In the adjoint system (43), the right side term is called adjoint load. In the present case, it can be computed as follows

$$\begin{aligned} \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T &= \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \dots \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \dots \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \\ \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T &= \frac{1}{V_b} \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \\ &+ \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \\ &- \frac{2}{3} \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial \psi}{\partial \mathbf{u}} \right)^T \end{aligned} \quad (47)$$

$$\begin{aligned}
& \dot{\bar{\mathbf{u}}}_{-1}^1 \dot{\bar{\mathbf{u}}}_{-1}^1 \dot{\bar{\mathbf{u}}}_{-1}^1 \{\dots\} d\xi_1 d\xi_2 d\xi_3 = \mathring{\mathbf{a}}_{\alpha=1}^8 \frac{1}{3} ({}^t_0 \bar{\boldsymbol{\varepsilon}}(\mathbf{x}))^{-1} |{}_0 J| W_{\alpha} \mathring{\mathbf{c}}_{ij}^t \varepsilon_{kj} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{kj} \frac{\ddot{\mathbf{a}}_{ij}^8}{\bar{\theta}} \mathring{\mathbf{a}}_{0(N^p, j)}^8 \delta_{ik} \\
& + \mathring{\mathbf{c}}_{ij}^t \varepsilon_{ik} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ik} \frac{\ddot{\mathbf{a}}_{ij}^8}{\bar{\theta}} \mathring{\mathbf{a}}_{0(N^p, i)}^8 \delta_{jk} + \mathring{\mathbf{c}}_{ij}^t \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_{ij}^8}{\bar{\theta}} \mathring{\mathbf{a}}_{0(N^p, i)}^8 \delta_{kl} \mathring{\mathbf{a}}_{0(N^q, j)}^8 u_k^q \\
& + \mathring{\mathbf{c}}_{ij}^t \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_{ij}^8}{\bar{\theta}} \mathring{\mathbf{a}}_{0(N^p, i)}^8 u_k^p \mathring{\mathbf{a}}_{0(N^q, j)}^8 \delta_{kl} - \frac{2}{3} \mathring{\mathbf{c}}_{ij}^t \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_{ij}^8}{\bar{\theta}} \mathring{\mathbf{a}}_{0(N^p, k)}^8 \delta_{kl} \delta_{ij} \\
& - \frac{2}{3} \mathring{\mathbf{c}}_{ij}^t \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_{ij}^8}{\bar{\theta}} \mathring{\mathbf{a}}_{0(N^p, l)}^8 u_k^p \mathring{\mathbf{a}}_{0(N^q, l)}^8 \delta_{km} \delta_{ij}
\end{aligned} \tag{48}$$

where the subscript β of the displacement u is the global number of the node displacement component for all nodes related to the cost function, which is denoted as $\beta(I, k)$ for the I -th node k -th degree of freedom, and takes the value $\beta=3(I-1)+k$; n is the number of element related to the cost function, the summation on all related boundary elements, number 1 to n_b , the summation performed only on the elements connected to the I -th node; q or p is the local number of the element node, the condition $(n, p)=I$ under summation symbol means that, if the summation conducted only for the node in element n , coincides with the global I -th node; N^p or N^q denote the shape functions of the p -th or q -th element node.

In order to simplify Eq. (48), it can be written

$$\begin{aligned}
\{\dots\} = & 2\cancel{\mathcal{C}_0^t} \varepsilon_{kj} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{kj} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^p, j) + \cancel{\mathcal{C}_0^t} \varepsilon_{kj} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{kj} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^p, j) \\
& + 2\cancel{\mathcal{C}_0^t} \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^p, i) \mathbf{a}_0^8 (N^q, j) {}^t_0 u_k^q \\
& + \cancel{\mathcal{C}_0^t} \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^q, j) {}^t_0 u_k^q \mathbf{a}_0^8 (N^p, i) - \frac{2}{3} \cancel{\mathcal{C}_0^t} \varepsilon_{ii} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ii} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^p, k) \delta_{ij} \\
& - \frac{2}{3} \cancel{\mathcal{C}_0^t} \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^q, i) {}^t_0 u_k^q \mathbf{a}_0^8 (N^p, i) \delta_{ij} \\
= & 2\cancel{\mathcal{C}_0^t} \varepsilon_{kj} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{kj} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^p, j) + 2\cancel{\mathcal{C}_0^t} \varepsilon_{ij} - \frac{1}{3} {}^t_0 \varepsilon_{hh} \delta_{ij} \frac{\ddot{\mathbf{a}}_0^8}{\theta} (N^p, i) \mathbf{a}_0^8 (N^q, j) {}^t_0 u_k^q
\end{aligned} \tag{49}$$

The terms indicated in red are eliminated.

In summary, the AVM requires to calculate the adjoint load and solve Eq. (43). Therefore, its computational cost is much lower than in the case of DDM if the number of optimization variables is larger than the number of objective functions.

8. Determination of Sensitivities in the Shape Optimization of a Passenger Car Radial Tire

The sensitivity analysis technique described in this study was integrated with a finite element program available to authors. Computational tasks are distributed as follows:

Sensitivity calculations were carried out for a 205/50R16 passenger radial car tire using both DDM and AVM. It was found that the two methods give identical results. However, the pure computational time of AVM (determined by subtracting the computation time required by the contact problem solver from the total CPU time required by the whole process) is only about 1/5 of the CPU time required by DDM for evaluating sensitivities (see Table 3). DDM and AVM require almost the same time to solve the contact problem. However, because of the advantage of AVM mentioned in Section 7, the pure computational time of present approach can reduce to $1/n$ of DDM, where n is the number of design variables. Further discussion on this aspect is provided in Appendix 2. Sensitivity of cost function was evaluated with respect to the five shape variables included in the optimization problem. For that purpose, eight levels were evenly selected by Latin Hypercube Design method [3] in the range of $W_1 \in [40.6, 50.6]$, $W_2 \in [90.3, 95.3]$, $R_1 \in [600, 1200]$, $R_2 \in [248, 600]$, and $R_3 \in [40, 53]$ covering the whole design range for all variables as shown in Table 1. According to orthogonal-maximin Latin Hypercube Design algorithm [4, 5], a number of samples greater than 1.5 times of the number of design variables is enough. Hence, the eight samples selected in the present analysis can efficiently cover the whole design range for five design variables.

The 2D and 3D models of the selected 205/55R16 passenger car radial tire are shown in Figure 2 and 4, respectively. Table 2 presents the results of the sensitivity analysis carried out for eight design points generated with the design of experiments technique. $\psi_A^\alpha = \mathbf{A}_0^{Tt} \mathbf{F}^{\alpha\alpha}$, $\psi_B^\alpha = \mathbf{A}_0^{Tt} \mathbf{F}_C^{\alpha\alpha}$, $\psi_C^\alpha = \boldsymbol{\alpha}^T \{\mathbf{g}^{\alpha\alpha}\}$, ψ_1 is the objective function, $\psi_{AVM}^\alpha = \psi_A^\alpha + \psi_B^\alpha - \psi_C^\alpha$ is the sensitivity value determined with the adjoint variable method, ψ_{FDM}^α is the sensitivity value determined with the finite differentiation method, and ψ_{DDM}^α is the sensitivity value determined with the direct differentiation method. It can be seen that AVM and DDM obtained identical values sensitivities while the FDM results show a slight error that may be caused by the fact that, unlike AVM and DDM, FDM computes sensitivity by directly changing design variables. In this study, the interval for design variable change was set as one thousand of the original variable value.

Figure 4: 3D finite element model of 205/65R16 tire

9. Summary and Conclusions

This paper described an efficient method for performing sensitivity analysis in shape optimization problems of complex nonlinear structures such as automotive tires. Since computations for each design sample are time consuming, reducing the number of design samples is very important. In shape optimization problems, two issues must be addressed: (i) sensitivity calculation and (ii) disposing of an efficient optimizer. A possible strategy may be to combine AVM and GBK. While the latter was already studied in the literature [4], this research developed a novel AVM including a new adjoint shape design sensitivity formulation for nonlinear structures subject to contact forces. Sensitivity equations of displacements with respect to shape variations were formulated using the virtual work principle. Then, sensitivity analysis of average shear strains in the tire belt area was carried out. It was found that the proposed approach is more accurate than traditional finite difference sensitivity computations and much faster than the direct differentiation method as shown in Table 3. In view of this, the present approach has full potentiality for becoming a useful framework for shape optimization and sensitivity analysis of highly nonlinear structures including contact forces.

Table 1. Partitions of design space defined for the 205/55R16 tire shape optimization problem

	W_1	W_2	R_1	R_2	R_3
Level 1	40.6	90.3	600	248	40
Level 2	42.02857	91.01429	685.7143	298.2857	41.85714
Level 3	43.45714	91.72857	771.4286	348.5714	43.71429
Level 4	44.88571	92.44286	857.1429	398.8571	45.57143
Level 5	46.31429	93.15714	942.8571	449.1429	47.42857
Level 6	47.74286	93.87143	1028.571	499.4286	49.28571
Level 7	49.17143	94.58571	1114.286	549.7143	51.14286
Level 8	50.6	95.3	1200	600	53

Table 2. Results of sensitivity analysis carried out for the 205/55R16 tire shape optimization problem

DOE sample	Design variables	Value of design variables	ψ_A^α	ψ_B^α	ψ_C^α	ψ_1	ψ_{AVM}^α	ψ_{FDM}^α	ψ_{DDM}^α
#1	1	4.06E+01	-1.46E-05	-4.61E-07	-9.23E-05	4.99E-02	7.72E-05	7.44E-05	7.72E-05
	2	9.46E+01	9.89E-07	-2.63E-07	7.55E-07	4.99E-02	-2.77E-08	-1.40E-06	-2.77E-08
	3	1.03E+03	-6.21E-07	-1.30E-07	-3.09E-06	4.99E-02	2.34E-06	2.33E-06	2.34E-06
	4	3.49E+02	-6.98E-09	7.80E-08	-3.92E-06	4.99E-02	3.99E-06	3.22E-06	3.99E-06
	5	5.30E+01	1.64E-04	-1.21E-04	4.54E-04	4.99E-02	-4.10E-04	-1.18E-04	-4.10E-04
#2	1	4.20E+01	-3.10E-06	-4.92E-07	-5.82E-05	4.84E-02	5.46E-05	5.89E-05	5.46E-05
	2	9.32E+01	-6.96E-05	2.99E-05	1.01E-04	4.84E-02	-1.41E-04	-2.69E-04	-1.40E-04
	3	6.00E+02	-1.31E-06	-6.50E-07	-1.71E-05	4.84E-02	1.52E-05	1.72E-05	1.52E-05
	4	4.49E+02	1.37E-09	1.88E-07	-8.12E-06	4.84E-02	8.31E-06	8.33E-06	8.31E-06
	5	4.00E+01	2.95E-04	-7.20E-05	-1.67E-04	4.84E-02	3.89E-04	2.88E-04	3.89E-04
#3	1	4.35E+01	-6.03E-06	-1.02E-06	-3.87E-05	5.04E-02	3.17E-05	3.27E-05	3.17E-05
	2	9.03E+01	-6.86E-07	4.84E-06	-1.11E-05	5.04E-02	1.53E-05	-2.03E-05	1.53E-05

	3	8.57E+02	-9.31E-07	-3.19E-07	-4.92E-06	5.04E-02	3.67E-06	4.04E-06	3.67E-06
	4	4.99E+02	-7.11E-08	-1.04E-08	-1.76E-06	5.04E-02	1.68E-06	1.45E-06	1.68E-06
	5	4.93E+01	1.93E-04	-1.06E-04	3.77E-04	5.04E-02	-2.90E-04	-2.00E-04	-2.90E-04
#4	1	4.49E+01	-8.71E-06	-1.47E-06	-8.98E-05	5.25E-02	7.96E-05	8.71E-05	7.96E-05
	2	9.17E+01	-1.25E-05	2.14E-05	-3.54E-05	5.25E-02	4.43E-05	-7.87E-05	4.43E-05
	3	1.20E+03	-3.92E-07	-1.72E-07	-2.91E-06	5.25E-02	2.35E-06	2.80E-06	2.35E-06
	4	3.99E+02	-5.07E-08	9.13E-08	-3.62E-06	5.25E-02	3.66E-06	3.43E-06	3.66E-06
	5	4.37E+01	2.60E-04	-7.96E-05	2.56E-04	5.25E-02	-7.64E-05	-1.98E-04	-7.64E-05
#5	1	4.63E+01	-1.72E-05	2.33E-06	-1.63E-04	5.07E-02	1.49E-04	1.54E-04	1.49E-04
	2	9.10E+01	1.33E-05	1.23E-05	-1.23E-05	5.07E-02	3.79E-05	-7.19E-05	3.79E-05
	3	9.43E+02	-6.96E-07	-1.58E-07	-4.98E-06	5.07E-02	4.13E-06	4.62E-06	4.13E-06
	4	2.48E+02	-2.36E-07	5.45E-07	-9.33E-06	5.07E-02	9.64E-06	8.56E-06	9.64E-06
	5	4.56E+01	2.33E-04	-6.59E-05	1.76E-04	5.07E-02	-8.56E-06	-8.98E-05	-8.56E-06
#6	1	4.77E+01	-1.50E-06	-4.92E-08	-1.31E-05	4.94E-02	1.16E-05	1.10E-05	1.16E-05
	2	9.24E+01	1.29E-06	3.02E-07	-6.11E-07	4.94E-02	2.20E-06	-6.40E-06	2.20E-06
	3	6.86E+02	-1.18E-06	-3.81E-07	-8.19E-06	4.94E-02	6.62E-06	6.57E-06	6.62E-06
	4	5.50E+02	2.73E-08	1.76E-10	-8.26E-07	4.94E-02	8.54E-07	7.06E-06	8.54E-07
	5	5.11E+01	1.64E-04	-1.11E-04	4.10E-04	4.94E-02	-3.56E-04	-1.22E-04	-3.56E-04
#7	1	4.92E+01	-9.96E-06	1.57E-06	-9.11E-05	5.00E-02	8.28E-05	8.06E-05	8.28E-05
	2	9.53E+01	-6.76E-06	5.05E-06	-7.20E-06	5.00E-02	5.48E-06	-3.18E-05	5.48E-06
	3	7.71E+02	-1.04E-06	-2.59E-07	-7.60E-06	5.00E-02	6.30E-06	6.56E-06	6.30E-06
	4	2.98E+02	-5.85E-08	2.06E-07	-4.08E-06	5.00E-02	4.22E-06	3.62E-06	4.22E-06
	5	4.74E+01	1.95E-04	-8.98E-05	2.68E-04	5.00E-02	-1.63E-04	-8.01E-05	-1.63E-04
#8	1	5.06E+01	-1.43E-06	-1.19E-07	-3.19E-05	5.31E-02	3.03E-05	3.23E-05	3.03E-05
	2	9.39E+01	-2.26E-05	2.39E-05	-4.11E-05	5.31E-02	4.24E-05	-8.34E-05	4.24E-05
	3	1.11E+03	-3.49E-07	-2.18E-07	-3.71E-06	5.31E-02	3.14E-06	3.71E-06	3.14E-06
	4	6.00E+02	6.00E-08	3.78E-08	-9.46E-07	5.31E-02	1.04E-06	9.50E-07	1.04E-06
	5	4.19E+01	2.86E-04	-8.34E-05	2.74E-04	5.31E-02	-7.14E-05	-2.07E-04	-7.14E-05

Table 3 Comparison of computing time of DDM and AVM (CPU time/seconds)

DOE sample	DDM			AVM			% reduction of CPU time passing from DDM to AVM
	Total time	Contact solving time	DDM algorithm time	Total time	Contact solving time	AVM algorithm time	
#1	430.50	341.05	89.45	359.22	339.64	19.58	78.11%
#2	436.67	346.59	90.08	364.97	345.80	19.17	78.72%
#3	444.28	355.19	89.09	373.56	354.27	19.30	78.34%
#4	440.23	346.27	93.97	364.17	344.89	19.28	79.48%
#5	436.19	346.05	90.14	363.75	344.52	19.23	78.66%
#6	440.42	348.83	91.59	366.13	347.11	19.02	79.24%
#7	440.59	351.38	89.22	369.03	349.72	19.31	78.35%
#8	437.98	348.38	89.61	366.42	346.70	19.72	78.00%

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Appendix 1

Notation

${}^t_\tau \psi^1$	The objective function
τ	The initial configuration corresponding to the shape parameter
t	The load parameter
e_{ij}	The principle strains
δ_{ij}	The Kronecker delta
${}_\tau V_b$	The total volume of the design area
${}^t_\tau \bar{\varepsilon}(\mathbf{x})$	The effective shear strain
${}_0x_i, {}^\circ_\tau x_i$	The Lagrange coordinates for the original shape and the shape design corresponding to parameters τ_α
m	The number of independent degrees of freedom (DOFs)
k	The number of design parameters
${}^\circ_\tau u_i$	Virtual displacement
${}^t_0 u_i^{\circ\alpha}$	The sensitivity of the displacements when $\tau_\alpha = 0$
${}^t_0 (u_{i,j})^{\circ\alpha}$	The sensitivity of displacement gradients
${}_0 V_{i\alpha}$	Shape design velocity field
τ_α	The design variables
$\Delta \tau_\alpha$	A small increment of a design variable
B, D	The tire width and diameter
Ω	The whole shape domain
x_{oi}, y_{oi}	The coordinates of the i^{th} arc center
x_i, y_i	The outer surface point coordinates belonging to the i^{th} arc
T	Mapping function
u	The parametric coordinates
$\mathbf{F}, \mathbf{B}$	The coefficient matrices
$\mathbf{R}$	The vector of design variables
${}^t_\tau C_c^{LM}$	The contact boundary term in Lagrange multiplier approach
${}_0 \lambda_n, {}^\circ_\tau \lambda_n$	The Lagrange multipliers
${}^\circ_\tau S^f, {}^\circ_\tau S^t, {}^\circ_\tau S^{st}$	The traction-free boundary, the traction-given boundary, and the contacted boundary
${}^t_0 S_{ij}$	The stress field
${}^t_0 t_k, {}^\circ_0 \rho_0 f_k$	The loads which are in equilibrium in the original shape
${}_0 \mathbf{K}_L, {}^\circ_0 \mathbf{K}_N, {}^\circ_0 \mathbf{K}_G$	Identical to those in the conventional nonlinear finite element analysis for the original design shape
${}^t_0 \mathbf{Q}^{\circ\alpha}, {}^\circ_0 \mathbf{F}_F^{\circ\alpha}$	The change rates of the real loads and the fictitious loads

${}^t_0 \mathbf{F}_F^{\circ\alpha}$	The change rate of the fictitious load
${}^t_0 \mathbf{F}_R^{\circ\alpha}$	The change rate of the real loads
${}^t_0 \mathbf{u}^{\circ\alpha}$	Displacement sensitivities
ψ_{AVM}^{α}	The sensitivity value determined with the adjoint variable method
ψ_{FDM}^{α}	The sensitivity value determined with the finite differentiation method
ψ_{DDM}^{α}	The sensitivity value determined with the direct differentiation method

Appendix 2: Calibration case

In order to generalize the findings presented in this study, FDM, DDM, and AVM are applied to the square rod problem schematized in Figure A2.1. The top of rod is subject to uniform pressure while the bottom is put in frictionless contact with a stationary rigid surface. The rod is divided into 10 cubic HEX8 elements with length of 10 mm (see Figure A2.1). For the sake of simplicity, the side length l_c of the first element is defined as design variable, closely related to nodal forces of top element. We assume that all four sides displace inwards or outwards together in a length d (see Figure A2.1). The objective function is the effective shear strain -Eq. (2)- developed in the ninth element. The shape design velocity is 0.5, i.e. the coordinate x_c change rate with respect to the l_c design variable. One can calculate the design sensitivities ψ_{FDM}^{α} , ψ_{DDM}^{α} , ψ_{AVM}^{α} and the corresponding CPU times listed in Table A2.1.

Similar to Table 2, AVM and DDM get identical sensitivity values while FDM shows a small error. For this simple case, AVM is only 22% faster than DDM. Hence, the advantage of using AVM becomes much more significant for large scale problems.

Table A2.1 Comparison of FDM, DDM, and AVM results for the square rod problem

	FDM	DDM	AVM
CPU time (sec)	2.80	3.25	2.55
Sensitivity of objective function	2.66E-02	2.61E-02	2.61E-02

Figure A2.1 3D finite element model and boundary conditions of the square rod.

Appendix 3: Derivation of strain field sensitivity equation Eq. (18)

According to the nonlinear theory, the strain field can be expressed as follows:

$${}^t_0 \mathcal{E}_{ij} = ({}^t_0 u_{i,j} + {}^t_0 u_{j,i} + {}^t_0 u_{k,i} {}^t_0 u_{k,j})/2 \quad (\text{A3.1})$$

$$\begin{aligned} {}^t_\tau \mathcal{E}_{ij} &= ({}^t_\tau u_{i,j} + {}^t_\tau u_{j,i} + {}^t_\tau u_{k,i} {}^t_\tau u_{k,j})/2 = ({}^t_0 u_{i,j} + {}^t_0 u_{j,i} + {}^t_0 u_{k,i} {}^t_0 u_{k,j})/2 \\ &+ \left({}^t_0 (u_{i,j})^{\circ\alpha} + {}^t_0 (u_{j,i})^{\circ\alpha} + {}^t_0 (u_{k,i})^{\circ\alpha} {}^t_0 u_{k,j} + {}^t_0 u_{k,i} {}^t_0 (u_{k,j})^{\circ\alpha} \right) \tau_\alpha / 2 \\ &+ \left({}^t_0 (u_{k,i})^{\circ\alpha} {}^t_0 (u_{k,j})^{\circ\beta} \right) \tau_\alpha \tau_\beta / 2 \end{aligned} \quad (\text{A3.2})$$

where ${}^t_\tau u_{i,j} = {}^t_0 u_{i,j} + {}^t_0 (u_{i,j})^{\circ\alpha} \tau_\alpha$, and ${}^t_0 (u_{i,j})^{\circ\alpha} = \frac{\partial}{\partial \tau} \frac{d}{d\tau} (u_{i,j}) \Big|_{\tau=0}$, and in which $\frac{d}{d\tau} \frac{d}{d\tau} (u_{i,j}) \Big|_{\tau=0}$

consists of two parts: one is due to the difference of ${}^t_\tau u_i$ and ${}^t_0 u_i$, namely ${}^t_0 u_i^{\circ\alpha}$, another one comes from the difference of ${}^t_\tau (u_{i,j})$ and ${}^t_0 (u_{i,j})$ in the case of identical ${}^t_\tau u_i$ and ${}^t_0 u_i$.

For the virtual displacement $\delta_\tau u_i$, which is arbitrary and geometrically admissible for the design shape vector $\bar{\tau}$ including all shape variables, the corresponding virtual strain can be expressed as follows

$$\begin{aligned} \delta_\tau \mathcal{E}_{ij} &= \delta_\tau (u_{i,j}) + \delta_\tau (u_{j,i}) + \delta_\tau (u_{k,i}) {}^t_\tau (u_{k,j}) + {}^t_\tau (u_{k,i}) \delta_\tau (u_{k,j}) \\ &= (\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{k,i} {}^t_0 u_{k,j} + {}^t_0 u_{k,i} \delta_0 u_{k,j})/2 \\ &+ \left((\delta_0 (u_{i,j})^{\circ\alpha} + \delta_0 (u_{j,i})^{\circ\alpha} + \delta_0 (u_{k,i})^{\circ\alpha} {}^t_0 u_{k,j} + {}^t_0 (u_{k,i})^{\circ\alpha} \delta_0 u_{k,j} + \delta_0 u_{k,i} {}^t_0 (u_{k,j})^{\circ\alpha} \right. \\ &\quad \left. + {}^t_0 u_{k,i} \delta_0 (u_{k,j})^{\circ\alpha} \right) \tau_\alpha / 2 + \left(\delta_0 (u_{k,i})^{\circ\alpha} {}^t_0 (u_{k,j})^{\circ\beta} + {}^t_0 (u_{k,i})^{\circ\alpha} \delta_0 (u_{k,j})^{\circ\beta} \right) \tau_\alpha \tau_\beta / 2 \end{aligned} \quad (\text{A3.3})$$

By omitting the small higher order term $(\delta_0(u_{k,i})^{\circ\alpha} {}^t_0(u_{k,j})^{\circ\beta} + {}^t_0(u_{k,i})^{\circ\alpha} \delta_0(u_{k,j})^{\circ\beta})\tau_\alpha\tau_\beta/2$

which implies that the shape variation is small terms, the strain sensitivities can be rewritten as

$$\begin{aligned} \delta_\tau \varepsilon_{ij} = & (\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{k,i} {}^t_0 u_{k,j} + {}^t_0 u_{k,i} \delta_0 u_{k,j})/2 \\ & + \left((\delta_0(u_{i,j})^{\circ\alpha} + \delta_0(u_{j,i})^{\circ\alpha} + \delta_0(u_{k,i})^{\circ\alpha} {}^t_0 u_{k,j} + {}^t_0(u_{k,i})^{\circ\alpha} \delta_0 u_{k,j} \right. \\ & \left. + \delta_0 u_{k,i} {}^t_0(u_{k,j})^{\circ\alpha} + {}^t_0 u_{k,i} \delta_0(u_{k,j})^{\circ\alpha} \right) \tau_\alpha / 2 \end{aligned} \quad (\text{A3.4})$$

Appendix 4: Derivation of displacement gradient sensitivity Eq. (23)

Introducing Eqs (18)-(20) into the equation of virtual work corresponding to the shape variation, it follows

$$\begin{aligned} \dot{\mathbf{0}}_{0V} {}^t_\tau S_{ij} \delta'_\tau \varepsilon_{ij} \Big|_0^{\tau} \mathbf{J} \Big|_0 d_0 V = & \dot{\mathbf{0}}_{0S^t} {}^t_\tau t_k \delta'_\tau u_k (d_\tau S / d_0 S) d_0 S + \dot{\mathbf{0}}_{0V} \rho {}^t_\tau f_k \delta'_\tau u_k \Big|_0^{\tau} \mathbf{J} \Big|_0 d_0 V \\ & + \dot{\mathbf{0}}_{0S^c} ({}^t_\tau \lambda_n \delta_0 u_n + {}^t_\tau \lambda_s \delta_0 u_s + {}^t_\tau \lambda_t \delta_0 u_t) (d_\tau S / d_0 S) d_0 S \\ & + \dot{\mathbf{0}}_{0S^c} ({}^t_\tau u_n + {}^0_0 g_n) \delta_0 \lambda_n + ({}^t_\tau u_s - {}^t_0 u_s) \delta_0 \lambda_s + ({}^t_\tau u_t - {}^t_0 u_t) \delta_0 \lambda_t \Big|_0^{\tau} (d_\tau S / d_0 S) d_0 S \\ & + \dot{\mathbf{0}}_{0S^c} ({}^t_\tau \lambda_n \delta_0 u_n + {}^t_\tau t_s \delta_0 u_s + {}^t_\tau t_t \delta_0 u_t) (d_\tau S / d_0 S) d_0 S \\ & + \dot{\mathbf{0}}_{0S^c} ({}^t_\tau u_n + {}^0_0 g_n) \delta_0 \lambda_n \Big|_0^{\tau} (d_\tau S / d_0 S) d_0 S \end{aligned} \quad (\text{A4.1})$$

The above equation can be rewritten as

$$\begin{aligned} & \dot{\mathbf{a}}_e \dot{\mathbf{0}}_{0V_e} \dot{\mathbf{0}}_{0S^c} {}^t_\tau S_{ij} + {}^t_0 D_{ijkl} \left(({}^t_0(u_{k,l})^{\circ\alpha} + {}^t_0(u_{l,k})^{\circ\alpha} + {}^t_0(u_{m,k})^{\circ\alpha} {}^t_0 u_{m,l} + {}^t_0 u_{m,k} {}^t_0(u_{m,l})^{\circ\alpha} \right) \tau_\alpha / 2 \Big|_0^{\tau} \\ & \times (\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{k,i} {}^t_0 u_{k,j} + {}^t_0 u_{k,i} \delta_0 u_{k,j}) / 2 + \left((\delta_0(u_{i,j})^{\circ\alpha} + \delta_0(u_{j,i})^{\circ\alpha} \right. \\ & \left. + \delta_0(u_{k,i})^{\circ\alpha} {}^t_0 u_{k,j} + {}^t_0(u_{k,i})^{\circ\alpha} \delta_0 u_{k,j} + \delta_0 u_{k,i} {}^t_0(u_{k,j})^{\circ\alpha} + {}^t_0 u_{k,i} \delta_0(u_{k,j})^{\circ\alpha} \right) \tau_\alpha / 2 \Big|_0^{\tau} (1 + {}^0_0 V_{i\alpha} \tau_\alpha) d_0 V_e \\ = & \dot{\mathbf{a}}_e \dot{\mathbf{0}}_{0S^c} {}^t_\tau t_k (1 + {}^0_0 S^{\circ\alpha}_e \tau_\alpha) \delta_0 u_k d_0 S_e + \dot{\mathbf{a}}_e \dot{\mathbf{0}}_{0V_e} \rho {}^t_\tau f_k (1 + {}^0_0 V_{i\alpha} \tau_\alpha) \delta_0 u_k d_0 V_e \\ & + \dot{\mathbf{a}}_e \dot{\mathbf{0}}_{0S^c} ({}^t_0 \lambda_n + {}^t_0 \lambda_n^{\circ\alpha} \tau_\alpha) \delta_0 u_n (1 + {}^0_0 S^{\circ\beta}_e \tau_\beta) d_0 S_e \\ & + \dot{\mathbf{a}}_e \dot{\mathbf{0}}_{0S^c} ({}^t_0 u_n + {}^t_0 u_n^{\circ\alpha} \tau_\alpha + {}^0_0 g_n) \delta_0 \lambda_n (1 + {}^0_0 S^{\circ\beta}_e \tau_\beta) d_0 S_e \end{aligned} \quad (\text{A4.2})$$

In order to simplify the present derivation, only frictionless contact is considered here. In Eq. (A4.2), S_c^{st} denotes the contact element. Developing equation (A4.2) with respect to the shape variation factor τ_α , we have

$$\begin{aligned}
& \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{n,i} {}^t_0 u_{n,j} + \delta_0 u_{n,j} {}^t_0 u_{n,i} \right) / 2 \mathring{\mathbf{u}}_0 S_{ij} d_0 V_e \\
& + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{n,i} {}^t_0 u_{n,j} + \delta_0 u_{n,j} {}^t_0 u_{n,i} \right) / 2 {}^t_0 D_{ijkl} \left(({}^t_0 (u_{k,l})^\circ{}^\alpha + {}^t_0 (u_{l,k})^\circ{}^\alpha \right. \\
& \quad \left. + {}^t_0 u_{m,k} {}^t_0 (u_{m,l})^\circ{}^\alpha + {}^t_0 u_{m,l} {}^t_0 (u_{m,k})^\circ{}^\alpha \right) / 2 \mathring{\mathbf{u}}_0 d_0 V_e \\
& + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 (u_{i,j})^\circ{}^\alpha + \delta_0 (u_{j,i})^\circ{}^\alpha + \delta_0 (u_{n,i}) {}^t_0 (u_{n,j})^\circ{}^\alpha + \delta_0 (u_{n,j}) {}^t_0 (u_{n,i})^\circ{}^\alpha \right. \\
& \quad \left. + \delta_0 (u_{n,i})^\circ{}^\alpha {}^t_0 (u_{n,j}) + \delta_0 (u_{n,j})^\circ{}^\alpha {}^t_0 (u_{n,i}) \right) / 2 \mathring{\mathbf{u}}_0 S_{ij} d_0 V_e \\
& + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{n,i} {}^t_0 u_{n,j} + \delta_0 u_{n,j} {}^t_0 u_{n,i} \right) / 2 {}^t_0 S_{ij} {}^t_0 V_{k\alpha}, \mathring{\mathbf{u}}_0 V_e \\
& + \tau_\alpha \tau_\beta \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 u_{i,j} + \delta_0 u_{n,i} {}^t_0 u_{n,j} \right) {}^t_0 D_{ijkl} \left(({}^t_0 (u_{k,l})^\circ{}^\alpha + {}^t_0 u_{m,k} {}^t_0 (u_{m,l})^\circ{}^\alpha \right) \\
& \quad + \left(\delta_0 (u_{i,j})^\circ{}^\alpha + \delta_0 (u_{n,i}) {}^t_0 (u_{n,j})^\circ{}^\alpha + \delta_0 (u_{n,i})^\circ{}^\alpha {}^t_0 (u_{n,j}) \right) {}^t_0 S_{ij} \mathring{\mathbf{u}}_0 V_{s\beta}, d_0 V_e \\
& + \tau_\alpha \tau_\beta \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 (u_{i,j})^\circ{}^\alpha + \delta_0 (u_{n,i}) {}^t_0 (u_{n,j})^\circ{}^\alpha + \delta_0 (u_{n,i})^\circ{}^\alpha {}^t_0 (u_{n,j}) \right) {}^t_0 D_{ijkl} \left(({}^t_0 (u_{k,l})^\circ{}^\beta + {}^t_0 u_{m,k} {}^t_0 (u_{m,l})^\circ{}^\beta \right) \mathring{\mathbf{u}}_0 d_0 V_e \\
& + \tau_\alpha \tau_\beta \tau_\gamma \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \left(\delta_0 (u_{i,j})^\circ{}^\alpha + \delta_0 (u_{n,i}) {}^t_0 (u_{n,j})^\circ{}^\alpha + \delta_0 (u_{n,i})^\circ{}^\alpha {}^t_0 (u_{n,j}) \right) {}^t_0 D_{ijkl} \left(({}^t_0 u_{k,l}^\circ{}^\beta + {}^t_0 u_{m,k} {}^t_0 u_{m,l}^\circ{}^\beta \right) \mathring{\mathbf{u}}_0 V_{s\gamma}, d_0 V_e \\
& = \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 u_{k,l} {}^t_0 t_k d_0 S_e + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \delta_0 u_{k,l} {}^t_0 \rho_{0f,k} d_0 V_e + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 u_{n,l} {}^t_0 \lambda_n d_0 S_e \\
& \quad + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 \lambda_n ({}^t_0 u_n + {}^0_0 g_n) d_0 S_e \\
& + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_e} \delta_0 u_{k,l} {}^t_0 t_k {}^t_0 S_e^\circ{}^\alpha d_0 S_e + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0V_e} \delta_0 u_{k,l} {}^t_0 \rho_{0f,k} {}^t_0 V_{s\alpha}, d_0 V_e + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 u_{n,l} {}^t_0 \lambda_n {}^t_0 S_e^\circ{}^\alpha d_0 S_e \\
& + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 \lambda_n {}^t_0 u_n {}^t_0 S_e^\circ{}^\alpha d_0 S_e + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 u_{n,l} {}^t_0 \lambda_n {}^t_0 S_e^\circ{}^\alpha d_0 S_e \\
& + \tau_\alpha \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 \lambda_n ({}^t_0 u_n + {}^0_0 g_n) {}^t_0 S_e^\circ{}^\alpha d_0 S_e + \tau_\alpha \tau_\beta \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \delta_0 u_{n,l} {}^t_0 \lambda_n {}^t_0 S_e^\circ{}^\beta d_0 S_e \\
& + \mathring{\mathbf{a}}_e \cdot \mathring{\mathbf{0}}_{0S_c^{st}} \tau_\alpha \tau_\beta \delta_0 \lambda_n {}^t_0 u_n {}^t_0 S_e^\circ{}^\beta d_0 S_e
\end{aligned} \tag{A4.4}$$

Neglecting small higher order terms, the terms related to the stress field ${}^t_0 S_{ij}$ and the loads ${}^t_0 t_k, {}^t_0 \rho_{0f,k}$ which are in equilibrium in the original shape, Eq. (A4.4) can be rewritten as

$$\begin{aligned}
& \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \left((\delta_0 u_{i,j} + \delta_0 u_{j,i})/2 \right)_0^t D_{ijkl} \left(\left({}^t_0(u_{k,l})^{\circ\alpha} + {}^t_0(u_{l,k})^{\circ\alpha} \right)/2 \right) \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \left((\delta_0 u_{i,j} + \delta_0 u_{j,i})/2 \right)_0^t D_{ijkl} \left(\left({}^t_0 u_{m,k} {}^t_0(u_{m,l})^{\circ\alpha} + {}^t_0 u_{m,l} {}^t_0(u_{m,k})^{\circ\alpha} \right)/2 \right) \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \left((\delta_0 u_{n,i} {}^t_0 u_{n,j} + \delta_0 u_{n,j} {}^t_0 u_{n,i})/2 \right)_0^t D_{ijkl} \left(\left({}^t_0(u_{k,l})^{\circ\alpha} + {}^t_0(u_{l,k})^{\circ\alpha} \right)/2 \right) \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \left((\delta_0 u_{n,i} {}^t_0 u_{n,j} + \delta_0 u_{n,j} {}^t_0 u_{n,i})/2 \right)_0^t D_{ijkl} \left(\left({}^t_0 u_{m,k} {}^t_0(u_{m,l})^{\circ\alpha} + {}^t_0 u_{m,l} {}^t_0(u_{m,k})^{\circ\alpha} \right)/2 \right) \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \left(\delta_0 u_{k,i} {}^t_0 S_{ij} {}^t_0(u_{k,j})^{\circ\alpha} \right) \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& = \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} (\delta_0 u_k) ({}^t_0 t_k)_0 S_{\mathbf{e}}^{\circ\alpha} \mathring{\mathbf{d}}_0 S_{\mathbf{e}} + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} (\delta_0 u_k) ({}_0 \rho {}^t_0 f_k)_0 V_{i\alpha, i} \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& - \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \left((\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{n,i} {}^t_0 u_{n,j} + \delta_0 u_{n,j} {}^t_0 u_{n,i})/2 \right)_0^t S_{kl} ({}_0 V_{k\alpha, k}) \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& - \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \left((\delta_0 (u_{i,j})^{\circ\alpha} + \delta_0 (u_{j,i})^{\circ\alpha} + \delta_0 (u_{n,i})^{\circ\alpha} {}^t_0 (u_{n,j}) + \delta_0 (u_{n,j})^{\circ\alpha} {}^t_0 (u_{n,i})) / 2 \right)_0^t S_{ij} \mathring{\mathbf{d}}_0 V_{\mathbf{e}} \\
& + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \delta_0 u_n {}^t_0 \lambda_n^{\circ\alpha} \mathring{\mathbf{d}}_0 S_{\mathbf{e}} + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \delta_0 \lambda_n ({}^t_0 u_n^{\circ\alpha} + {}_0 g_n^{\circ\alpha}) \mathring{\mathbf{d}}_0 S_{\mathbf{e}} \\
& + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \delta_0 u_n {}^t_0 \lambda_n S_{\mathbf{e}}^{\circ\alpha} \mathring{\mathbf{d}}_0 S_{\mathbf{e}} + \mathring{\mathbf{a}}_{\mathbf{e}} \mathring{\mathbf{d}}_{\mathbf{e}} \mathring{\mathbf{e}}_{\mathbf{e}} \delta_0 \lambda_n ({}^t_0 u_n + {}_0 g_n) S_{\mathbf{e}}^{\circ\alpha} \mathring{\mathbf{d}}_0 S_{\mathbf{e}}
\end{aligned} \tag{A4.5}$$

When the structure is modelled using HEX8 element, one can obtain the following weak form control equations

$$N^m = \frac{1}{8} (1 + \xi_{1'}^m \xi_{1'}) (1 + \xi_{2'}^m \xi_{2'}) (1 + \xi_{3'}^m \xi_{3'}) \tag{A4.6}$$

$$\begin{aligned}
{}_0 x_j(\xi) &= \sum_{m=1}^8 N^m(\xi_{1'}, \xi_{2'}, \xi_{3'}) {}_0 x_j^m \\
{}^t_0 u_j(\xi) &= \sum_{m=1}^8 N^m(\xi_{1'}, \xi_{2'}, \xi_{3'}) {}^t_0 u_j^m
\end{aligned} \tag{A4.7}$$

$$\begin{aligned}
& {}^t_0(u_{i,j})^{\circ\alpha} @ \frac{\mathbb{I}}{\mathbb{I} \tau_{\alpha}} ({}^t_0 u_{i,j}) \Big|_{\tau=0} = {}^t_0 u_{i,j}^{\circ\alpha} - {}^t_0 u_{i,k} {}_0 V_{k\alpha, j} \\
& = \frac{{}_0 J_{k\ell}^*}{|{}_0 J|} \mathring{\mathbf{a}}_{\mathbf{e}} \frac{\mathbb{I} N^m}{\mathbb{I} \xi_{k\ell}} {}^t_0 u_i^{m\circ\alpha} - \frac{\mathring{\mathbf{a}}_{\mathbf{e}} J_{k\ell}^*}{|{}_0 J|} \mathring{\mathbf{a}}_{\mathbf{e}} \frac{\mathbb{I} N^m}{\mathbb{I} \xi_{k\ell}} {}^t_0 u_i^{m\frac{\circ\alpha}{\frac{\partial}{\partial \xi_{k\ell}}}} - \frac{\mathring{\mathbf{a}}_{\mathbf{e}} J_{k\ell}^*}{|{}_0 J|} \mathring{\mathbf{a}}_{\mathbf{e}} \frac{\mathbb{I} N^m}{\mathbb{I} \xi_{k\ell}} \frac{\partial}{\partial \xi_{k\ell}} \frac{{}_0 J_{k\ell}^*}{|{}_0 J|} \mathring{\mathbf{a}}_{\mathbf{e}} \frac{\mathbb{I} N^m}{\mathbb{I} \xi_{k\ell}} {}_0 V_{k\alpha, j}^{\frac{\partial}{\partial \xi_{k\ell}}}
\end{aligned} \tag{A4.8}$$

$$\begin{aligned}
\delta_0 u_{i,j} &= \frac{{}_0 J_{k\ell}^*}{|{}_0 \mathbf{J}|} \mathring{\mathbf{a}}^8 \frac{\mathbb{I} N^m}{\mathbb{I} \xi_{k\ell}} \delta_0 u_i^m \\
\delta_0 (u_{i,j})^{\circ\alpha} &= \frac{\mathring{\xi}}{\mathring{\xi}} \frac{\mathbb{I}}{\tau_\alpha} \delta_\tau (u_{i,j}) \Big|_{\mathring{\mathbf{u}}=0}^{\mathring{\mathbf{u}}} = - \delta_0 u_{i,k} {}_0 V_{k\alpha,j} \\
&= - \frac{\mathring{\xi}}{\mathring{\xi}} \frac{{}_0 J_{k\ell}^*}{|{}_0 \mathbf{J}|} \mathring{\mathbf{a}}^8 \frac{\mathbb{I} N^m}{\mathbb{I} \xi_{k\ell}} \delta_0 u_i^m \frac{\mathring{\xi}}{\mathring{\xi}} \frac{{}_0 J_{k\ell}^*}{|{}_0 \mathbf{J}|} \mathring{\mathbf{a}}^8 \frac{\mathbb{I} N^n}{\mathbb{I} \xi_{k\ell}} {}_0 V_{k\alpha}^n \frac{\mathring{\xi}}{\mathring{\xi}}
\end{aligned} \tag{A4.9}$$

Where

$${}_0 \mathbf{J}(\xi_{1'}, \xi_{2'}, \xi_{3'}) = \begin{bmatrix} \frac{\partial_0 x_1}{\partial \xi_{1'}} & \frac{\partial_0 x_2}{\partial \xi_{1'}} & \frac{\partial_0 x_3}{\partial \xi_{1'}} \\ \frac{\partial_0 x_1}{\partial \xi_{2'}} & \frac{\partial_0 x_2}{\partial \xi_{2'}} & \frac{\partial_0 x_3}{\partial \xi_{2'}} \\ \frac{\partial_0 x_1}{\partial \xi_{3'}} & \frac{\partial_0 x_2}{\partial \xi_{3'}} & \frac{\partial_0 x_3}{\partial \xi_{3'}} \end{bmatrix} = \begin{bmatrix} {}_0 \mathbf{J}_{11'} & {}_0 \mathbf{J}_{21'} & {}_0 \mathbf{J}_{31'} \\ {}_0 \mathbf{J}_{12'} & {}_0 \mathbf{J}_{22'} & {}_0 \mathbf{J}_{32'} \\ {}_0 \mathbf{J}_{13'} & {}_0 \mathbf{J}_{23'} & {}_0 \mathbf{J}_{33'} \end{bmatrix} \tag{A4.10}$$

Using index symbol, it can be expressed compactly as

$${}_0 J_{ji} = \sum_{m=1}^8 \frac{\xi_{j'0}^m x_i^m}{8} \prod_{\substack{n'=1 \\ n' \neq j'}}^3 (1 + \xi_{n'}^m \xi_{n'}) \tag{A4.11}$$

where indices $i, j, n, \dots$, except those related to $\mathring{\mathbf{a}}$ and $\tilde{\mathbf{O}}$, obey the rules of index symbol including the summation convention, $e_{i\ell\ell\ell}$ is the permutation symbol. The determinant of the Jacobian matrix can be expressed compactly as

$$\begin{aligned}
|{}_0 \mathbf{J}| &= e_{i'j'k'} J_{i'1} J_{j'2} J_{k'3} \\
&= e_{i'j'k'} \left(\sum_{m=1}^8 \frac{\xi_{i'0}^m x_1^m}{8} \prod_{\substack{n'=1 \\ n' \neq i'}}^3 (1 + \xi_{n'}^m \xi_{n'}) \right) \left(\sum_{p=1}^8 \frac{\xi_{j'0}^p x_2^p}{8} \prod_{\substack{s'=1 \\ s' \neq j'}}^3 (1 + \xi_{s'}^p \xi_{s'}) \right) \left(\sum_{q=1}^8 \frac{\xi_{k'0}^q x_3^q}{8} \prod_{\substack{t'=1 \\ t' \neq k'}}^3 (1 + \xi_{t'}^q \xi_{t'}) \right) \tag{A4.12}
\end{aligned}$$

For the shape variation, the geometrical coordinates for the initial configuration corresponding to the shape variation parameter τ_α can be formulated as

$${}_0 x_i = {}_0 x_i + {}_0 V_{i\alpha} \tau_\alpha \tag{A4.13}$$

$${}_0 x_i(\xi) = \mathring{\mathbf{a}}^8 N^m(\xi) {}_0 x_i^m + \mathring{\mathbf{a}}^8 N^m(\xi) {}_0 V_{i\alpha}^m \tau_\alpha \tag{A4.14}$$

The change rate of the geometrical coordinates of the mapping points with respect to the shape variation parameter τ_a for a generic point and nodal points of an element can be formulated as

$${}_0x_i^{\circ\alpha}(\xi) = \mathring{a} \sum_{m=1}^8 N^m(\xi) {}_0V_{i\alpha}^m \quad {}_0x_i^{m\circ\alpha} = {}_0V_{i\alpha}^m \quad (\text{A4.15})$$

Considering Eqs. (A4.6) through (A4.15), the virtual work equation (4.18) and its finite element form can further be rewritten as

[illegible]

Finally, sensitivity equations for state variables can be put in the matrix form of Eq. (24).