Modeling of Nonlocal Damage-Plasticity in Beams using Isogeometric Analysis

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Abstract

A nonlocal damage-plastic Euler-Bernoulli beam model is proposed based on resultant damage-plasticity together with implicit approximations of the nonlocal integral operators. Isogeometric concept is used to approximate the higher order gradient terms, and numerical algorithms and consistent tangents are derived. Test cases are presented to show the important features of the proposed nonlocal beam element. Numerical results show that the proposed nonlocal beam is successfully able to preserve the objectivity of the results and a meaningful convergence is achieved with decreasing mesh size. In addition, various softening responses associated with localized damage mechanisms – including snap-back and snap-through responses – are obtained.

Keywords: Nonlocal damage; Resultant plasticity; Isogeometric analysis; Beam finite elements; Implicit gradient approximations.

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1. Introduction

Successful simulation of structural system response under extreme events requires appropriate models that incorporate the physics of associated damage and failure with high fidelity. To this end, macromodels that utilize a combination of beams and discrete finite elements are well suited for the overall global response simulation [1-3]. A key constituent of these macromodels is the description of inelastic zones wherein energy dissipation occurs and the associated length-scale effects [4-7], and appropriate models of dissipative damage mechanisms are required to obtain physically reasonable results. In the case of beams, damage due to dissipative mechanisms is typically confined to critical locations termed “inelastic-hinge” regions, and the response is usually accompanied by localization of damage at these locations [8, 9]. When undergoing damage localization, significant internal force redistribution may occur under extreme loads. Therefore, it is imperative to simulate with high fidelity the physics of localized damage mechanisms to capture the associated internal force redistribution and the resulting system behavior.

In the past, various inelastic hinge models have been proposed for capturing the plastic response and the associated damage and softening behavior in beam elements. Examples include lumped models [8, 10, 11], distributed models [12-14], and nonlocal models [15-17], among others. The introduction of softening behavior in inelastic hinges, which is necessary to model damage and capture the load redistribution, make these models length-scale dependent, and further regularization is required to obtain results that are finite element mesh size independent [6]. For instance, in lumped hinge models, the constitutive behavior of an inelastic hinge is tied to a fixed hinge length [10, 11]; however, a main drawback of this approach is that the plasticity and damage can only evolve at the hinge locations that have to be specified a priori. In the distributed

hinge models, on the other hand, the inelasticity can occur at any location in the beam, and in this case, regularization is typically carried out by making the constitutive response dependent on characteristic mesh size [2, 12]. Another approach for addressing the softening effect is to consider nonlocal behavior with the inclusion of nonlocal terms in the constitutive models [15-18]. Moreover, two different methodologies are typically employed in such nonlocal theories [19-21]: (a) Integral based nonlocal theories – in this case, nonlocal variables are calculated as the weighted average value of the corresponding local variables and depends on a prescribed length-scale. Although an attractive approach from a theoretical point of view, these nonlocal constitutive models are difficult to implement and are numerically inefficient [22]; (b) Gradient based nonlocal theories – in this case, implicit or explicit gradient approximations of the nonlocal integral operator are used to model the nonlocal effects. Most of the studies on gradient-based models have focused on second-order gradient approximations, since only C^0 -continuity of the finite element interpolation functions is required in this case. However, other studies have shown that the inclusion of higher order gradient terms is necessary to approximate the integral nonlocal operators [22, 23]. Furthermore, finite element interpolation functions of higher order continuity are needed to meet the increased continuity requirements associated with higher order gradient approximations.

In this study, a displacement based nonlocal damage-plastic Euler-Bernoulli beam model is proposed for response simulation of nonlinear beams and frames that can be used for modeling of plasticity and the associated nonlocal damage resulting in a softening response. A distributed resultant nonlocal damage-plasticity model is introduced following the work of Engelen et al. [20] and Hjelmstad and Taciroglu [14], together with implicit approximations of the nonlocal gradient operators. Finite elements are formulated using the isogeometric concept introduced by Hughes et

al. [24] and the consistent tangent modulus for the resulting nonlocal beam element is derived. Isogeometric analysis (IGA) uses B-splines for interpolation of the geometry and the solution spaces, and offers an alternative methodology for constructing higher order interpolation functions that can be effectively used for approximation of the gradient terms in the implicit nonlocal approximations. IGA offers many advantages over conventional finite element bases including exact representation of geometry, higher continuity of the basis functions and variation diminishing properties [24, 25], among others. Since the beam has relatively simple geometry, the IGA's advantage of preserving the exact geometry is less important in this case. However, the higher order continuity and smoothness of the B-spline interpolations is exploited for constructing smooth finite element interpolation functions for nonlocal beam elements. Various test cases are presented to show the application and the important features of the proposed nonlocal inelastic beam element. The paper is organized as follows: Section 2 provides the theoretical description of the proposed nonlocal Euler-Bernoulli beam. Isogeometric analysis is reviewed in Section 3, and the isogeometric finite element beam formulation is presented in Section 4. Various numerical examples including convergence studies, comparison of different order of gradient models, parametric studies and length-scale effects are shown in Section 5. Finally, the important conclusions of this study are given in Section 6.

2. Nonlocal Euler-Bernoulli beam

2.1 Kinematics and kinetics

A planar geometrically nonlinear Euler-Bernoulli beam is considered with small-strain moderate rotation assumption as shown in Figure 1. The displacement field, $\mathbf{u}(x) = \{u(x), v(x)\}$, comprises of the horizontal displacement ($u(x)$) and the vertical displacement ($v(x)$) of the points along the centroidal axis of the beam. In Figure 1, $q(x)$ and $\tau(x)$ are the vertical and

axial traction per unit length, and $P(x)$, $V(x)$ and $M(x)$ are the resultant axial force, shear force and bending moment at a cross-section, respectively. With the assumption of moderate rotations, the strain is expressed as

$$\varepsilon_{mr}(x) = \varepsilon_a(x) - y\kappa(x) \quad (1)$$

with

$$\varepsilon_a(x) = \frac{du}{dx} + \frac{1}{2} \left(\frac{dv}{dx} \right)^2 \quad (2)$$

$$\kappa(x) = \frac{d^2v}{dx^2} \quad (3)$$

where $\varepsilon_a(x)$ is the axial strain that includes the nonlinear moderate rotation term $\frac{1}{2} \left(\frac{dv}{dx} \right)^2$ and $\kappa(x)$ is the curvature. Equilibrium equations for the Euler-Bernoulli beam are expressed as [14]

$$\frac{dP}{dx} = -\tau(x) \quad (4)$$

$$\frac{d^2M}{dx^2} - \frac{d}{dx} \left(P \frac{dv}{dx} \right) = q(x) \quad (5)$$

Figure 1. Planar Euler-Bernoulli beam.

96 2.2 Constitutive model

97 A nonlocal resultant damage-plasticity model is proposed to describe the loss of strength and
 98 stiffness due to damage during an inelastic process under the applied loads. To this end, the strain
 99 resultants, $\boldsymbol{\varepsilon} \equiv [\varepsilon_a \quad \kappa]^T$ and stress resultants, $\boldsymbol{s} \equiv [P \quad M]^T$, are introduced. With the
 100 assumption of additive decomposition, the strain is decomposed into elastic ($\boldsymbol{\varepsilon}^e$) and plastic ($\boldsymbol{\varepsilon}^p$)
 101 components as follows

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$$

with (6)

$$\boldsymbol{\varepsilon}^e = \begin{bmatrix} \varepsilon_a^e \\ \kappa^e \end{bmatrix}, \quad \boldsymbol{\varepsilon}^p = \begin{bmatrix} \varepsilon_a^p \\ \kappa^p \end{bmatrix}$$

102 Stress resultants are related to the elastic resultant strain through the elastic moduli ($\boldsymbol{D}$) as

$$\boldsymbol{s} = \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \boldsymbol{D}\boldsymbol{\varepsilon}^e$$

$$\boldsymbol{D} = \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix}$$

(7)

103 where E is the Young's modulus, A is the area of cross-section and I is the second moment of
 104 area. To finish the model description the evolution of resultant plastic strain ($\boldsymbol{\varepsilon}^p$), which is an
 105 internal state variable, has to be specified. To this end, a resultant nonlocal damage-plasticity
 106 model is introduced, wherein the elastic and plastic states are described using a yield function
 107 defined in terms of stress resultants and internal state variables. The yield function is formulated
 108 as [14, 20]

$$\phi(\mathbf{s}, \alpha, \bar{\alpha}) = \sqrt{\mathbf{s}^T \mathbf{Q} \mathbf{s}} - (1 - \omega^d(\bar{\alpha}))(1 + \zeta^p(\alpha))$$

$$\mathbf{Q}^{-1} = \begin{bmatrix} P_y^2 & 0 \\ 0 & M_y^2 \end{bmatrix} \quad (8)$$

109 where α and $\bar{\alpha}$ are the local and nonlocal effective plastic strains, $\omega^d = \omega^d(\bar{\alpha})$ is the damage
 110 parameter that is assumed to be a function of nonlocal effective plastic strain ($\bar{\alpha}$), $\zeta^p = \zeta^p(\alpha)$ is
 111 the hardening parameter defined in terms of local effective plastic strain (α), and P_y and M_y are
 112 the yield axial force and the yield moment, respectively. Thus, two state variables – α and $\bar{\alpha}$ –
 113 are introduced in the above formulation, and the damage (ω^d) and the hardening (ζ^p) variables
 114 are prescribed as functions of the state variables. Next, the rates of internal variable are given as

$$\dot{\alpha} \stackrel{\text{def}}{=} \sqrt{\dot{\boldsymbol{\varepsilon}}^p{}^T \mathbf{Q}^{-1} \dot{\boldsymbol{\varepsilon}}^p} = \gamma \quad (9)$$

$$\dot{\boldsymbol{\varepsilon}}^p \stackrel{\text{def}}{=} \gamma \frac{\partial \phi}{\partial \mathbf{s}} = \gamma \frac{\mathbf{Q} \mathbf{s}}{\sqrt{\mathbf{s}^T \mathbf{Q} \mathbf{s}}} \quad (10)$$

115 where $\boldsymbol{\varepsilon}^p$ is the plastic strain resultant defined in Eq. (6) and γ is the plastic multiplier [14].
 116 Finally, the Kuhn-Tucker complementarity conditions (Eq. (11)) and plastic consistency
 117 conditions (Eq. (12)) are introduced to describe the plastic loading and unloading process

$$\gamma \geq 0, \quad \phi \leq 0, \quad \gamma \phi = 0 \quad (11)$$

$$\gamma \dot{\phi} = 0 \quad (12)$$

118 A combination of linear and exponential hardening is adopted to describe the plastic hardening
 119 (Eq. (13))

$$\zeta^p(\alpha) = k_1 \alpha + k_2(1 - \exp(-\delta \alpha)) \quad (13)$$

where k_1 , k_2 and δ are the plastic hardening model parameters. Moreover, for the nonlocal damage variable $\omega^d = \omega^d(\bar{\alpha})$, two different models – a linear model (Eq. (14)) and an exponential model (Eq. (15)) – are considered

Linear model:

$$\omega^d(\bar{\alpha}) = \begin{cases} \frac{\bar{\alpha}}{\bar{\alpha}_c} & \text{if } \bar{\alpha} \leq \bar{\alpha}_c \\ 1 & \text{if } \bar{\alpha} > \bar{\alpha}_c \end{cases} \quad (14)$$

Exponential model:

$$\omega^d(\bar{\alpha}) = 1 - \exp(-\beta \bar{\alpha}) \quad (15)$$

where $\bar{\alpha}_c$ and β are the corresponding damage model parameters.

2.3 Nonlocal damage

The use of the local measure of plastic strain in a damage function, i.e. $\omega^d = \omega^d(\alpha)$, will lead to the localization of damage, with the extent of localization depending on the finite element mesh size. Moreover, as the mesh-size is refined, there is no convergence to a physically meaningful solution. In the limit, when the mesh size approaches zero, the deformation is localized within a zone of zero measure with zero energy dissipation [26]. One of the methods to fix this pathology is to use nonlocal theory [19, 26-28], wherein nonlocal fields are defined using spatial averaging of the corresponding local fields. Damage models are then formulated using the nonlocal fields [29]. The dimension of spatial averaging domain is dependent on a material parameter called length-scale or internal length. The local theory can be recovered by setting the length-scale to zero. For the resultant damage-plasticity model described above, the nonlocal effective plastic strain $\bar{\alpha}(x)$ field corresponding to the local effective plastic strain ($\alpha(x)$) is introduced, and is defined as

$$\bar{\alpha}(x) = \frac{\int_{\Omega} \Psi(\hat{x}; x) \alpha(\hat{x}) d\hat{x}}{\int_{\Omega} \Psi(\hat{x}; x) d\hat{x}} \quad (16)$$

$$\Psi(\hat{x}; x) = \exp\left(-\frac{|x - \hat{x}|^2}{2l_c^2}\right) \quad (17)$$

137 where $\Psi(\hat{x}; x)$ is the weighting function, l_c is the internal length, x and $\hat{x}$ are the coordinates
 138 on the centroid axis, and $\Omega = [0, L]$ denotes the domain of integration. In this study, a bell-shape
 139 Gaussian function is chosen as the weighting function [22, 23], however, other choices are also
 140 possible [20, 28, 30]. Figure 2 shows an example of a local field $\alpha(x)$ and the corresponding
 141 nonlocal fields $\bar{\alpha}(x)$ for different values of l_c .

142

143 Figure 2. Nonlocal regularization ($\bar{\alpha}(x)$) of a local field ($\alpha(x)$) for different length scales ($\Omega =$
 144 $[0,100]$).

145

146 For computational purposes, the above nonlocal formulation involves calculating the integral to
 147 determine $\bar{\alpha}(x)$ at each integration point (Eq. (16)), which is intractable and computationally
 148 expensive. In order to circumvent this issue, the integral nonlocal formulation (Eq. (16)) can be
 149 replaced by gradient approximations by expressing $\alpha(\hat{x})$ near x using the Taylor series
 150 expansion as

$$\alpha(\hat{x}) = \alpha(x) + \left. \frac{\partial \alpha}{\partial \hat{x}} \right|_{\hat{x}=x} (\hat{x} - x) + \frac{1}{2!} \left. \frac{\partial^2 \alpha}{\partial \hat{x}^2} \right|_{\hat{x}=x} (\hat{x} - x)^2 + \frac{1}{3!} \left. \frac{\partial^3 \alpha}{\partial \hat{x}^3} \right|_{\hat{x}=x} (\hat{x} - x)^3 + \dots \quad (18)$$

The explicit gradient approximation of the integral nonlocal formulation (Eq. (16)) is then obtained by substituting Eq. (18) into Eq. (16) and letting the domain Ω of integration to be infinity. This is a reasonable assumption as the weighting function $\Psi(\hat{x}; x)$ decreases rapidly to zero when $|\hat{x} - x| \gg l_c$, and the internal length scale (l_c) is usually small when compared to the domain Ω . The resulting explicit gradient approximation can be expressed as

$$\bar{\alpha}(x) = \alpha(x) + \frac{1}{2} l_c^2 \frac{\partial^2 \alpha(x)}{\partial x^2} + \frac{1}{8} l_c^4 \frac{\partial^4 \alpha(x)}{\partial x^4} + \frac{1}{48} l_c^6 \frac{\partial^6 \alpha(x)}{\partial x^6} + \frac{1}{384} l_c^8 \frac{\partial^8 \alpha(x)}{\partial x^8} + \dots \quad (19)$$

Though easy to implement, the explicit gradient approximation (Eq. (19)) has its own drawbacks such as weakness of the nonlocality, inability to model complete damage, and the associated numerical instabilities [20]. To further improve the numerical performance, implicit gradient approximation is frequently used, which is obtained by expressing higher order derivatives $\frac{\partial^k \bar{\alpha}(x)}{\partial x^k}$ in terms of $\alpha(x)$ through Eq. (19) and then adding various order derivatives of $\bar{\alpha}(x)$ to cancel out higher order derivatives of $\alpha(x)$. The result is expressed as

$$\bar{\alpha}(x) - \frac{1}{2} l_c^2 \frac{\partial^2 \bar{\alpha}(x)}{\partial x^2} + \frac{1}{8} l_c^4 \frac{\partial^4 \bar{\alpha}(x)}{\partial x^4} - \frac{1}{48} l_c^6 \frac{\partial^6 \bar{\alpha}(x)}{\partial x^6} + \frac{1}{384} l_c^8 \frac{\partial^8 \bar{\alpha}(x)}{\partial x^8} + \dots = \alpha(x) \quad (20)$$

When compared to the explicit gradient approximation, the implicit gradient approximation (Eq. (20)) is shown to be numerically robust and stable [20, 21]. This implicit gradient approximation is considered in this study, with various degrees of approximations by using different number of terms in the series in Eq. (20).

3. Isogeometric analysis

3.1 B-splines

168 A B-spline curve of order p on x - y plane is represented by a linear combination of B-spline
 169 basis functions as follows

$$\mathbf{C}(\xi) = \sum_{i=1}^n \mathbf{C}_i N_{i,p}(\xi) \quad (21)$$

170 where $\xi_1 \leq \xi \leq \xi_{n+p+1}$, $\mathbf{C}(\xi) = [C_x(\xi), C_y(\xi)]$ is the position vector that defines coordinates
 171 of point on the B-spline curve, $\mathbf{C}_i = [C_{ix}, C_{iy}]$ is the coordinate vector of the i^{th} control point
 172 and $N_{i,p}(\xi)$ is the i^{th} B-spline basis function of order p . A B-spline patch $\mathcal{P}$ is defined by a
 173 knot vector $\mathbf{\xi} = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ with $n + p + 1$ knots in which the knots are arranged in
 174 a non-decreasing order, i.e. $\xi_i \leq \xi_{i+1}$. Thus, a B-spline curve is parametrized by a scalar ξ on an
 175 interval $\mathcal{P} = [\xi_1, \xi_{n+p+1}]$ that defines a patch. The basis functions $N_{i,p}(\xi)$ can be recursively
 176 obtained as follows [31-33]

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}, \quad p = 0 \quad (22)$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), \quad p \geq 1$$

177 An example of a B-spline curve is shown in Figure 3. Figure 3(a) shows the B-spline basis
 178 functions $\{N_{i,p}(\xi)\}_{i=1}^9$ based on an open non-uniform knot vector, where *open* implies that the
 179 multiplicity of the first and the last knot values is equal to $p + 1$, and *non-uniform* implies that
 180 the interior knots are non-uniformly distributed over the patch. Figure 3 (b) shows the B-spline
 181 curve constructed by using the control points $\mathbf{C}_i$ ($i = 1, \dots, 9$) and the basis $\{N_{i,p}(\xi)\}_{i=1}^9$.
 182 Continuity of the B-spline curve within each non-zero knot span is C^∞ , while the continuity
 183 across an interior knot is C^{p-k} , where k is the multiplicity of the interior knot. When the
 184 multiplicity of an interior knot is equal to the order of basis ($k = p$), the corresponding control

point is interpolated by the curve; see the control point $\mathbf{C}_5$ in Figure 3 (b), for example. The starting and ending control points are interpolated when an open knot vector is used [31]. The B-spline curve shown in Figure 3(b) is constructed by using one patch; however, multiple patches can also be used to build more complex curves. A B-spline surface in 2-D and B-spline solid in 3-D parameter space can be constructed in the similar way with basis as the tensor products of the univariate B-spline basis functions [24]. For modeling beams, which is the focus of this study, the B-spline bases defined in 1-D parameter space are only considered.

(a) Basis functions (defined in parameter space) (b) B-spline curve with control points

Figure 3. A B-spline curve with control points $\{\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_9\}$ and cubic basis functions based on an open non-uniform knot vector $\mathbf{\Xi} = \{\xi_1, \xi_2, \dots, \xi_{13}\} = \{0, 0, 0, 0, 1, 2, 2, 2, 3, 4, 4, 4, 4\}$.

3.2 Isogeometric analysis (IGA)

In isogeometric analysis (IGA) both the geometry and the unknown fields of interest are interpolated using the B-spline bases. In IGA, an “element” is defined as a non-zero knot span for the numerical integration purposes, since the bases functions are C^∞ polynomials within each

non-zero knot span; for example, there are four elements in the patch defined by knot vector $\mathcal{E}$ in Figure 3. To effectively use the architecture of existing finite element codes, Borden et al. [34] proposed the Bézier extraction operator to construct the basis functions. This method provides an element-wise structure to IGA that allows the implementation of IGA into existing FEA codes without changing subroutines such as assembly, data storing and processing algorithms. In this study, Bézier extraction operators are used to construct the B-spline shape functions and the shape functions derivatives; further details about the Bézier extraction can be found in Borden et al. [34].

In IGA, refinements such as p -refinement and h -refinement can be achieved through degree elevation or knot insertion [24, 31]. Besides the benefit of the exact representation of geometry, the B-spline bases have high continuity that can be further augmented by k -refinement [24]. Moreover, B-splines are variation diminishing in the neighborhood of discontinuous data and the Gibbs effect observed with Lagrangian bases is not present since the control points are not necessarily interpolated [25]. However, due to this property the application of boundary conditions is not straightforward, and special techniques such as Lagrange multiplier method [35], the penalty method [36] and Nitsche's method [37, 38], are needed to enforce the boundary conditions; further details on IGA can be found in Cottrell et al. [25].

4. IGA Implementation

4.1 Weak forms

The governing equations for the nonlinear Euler-Bernoulli beam can be summarized as

$$\frac{dP}{dx} = -\tau(x) \quad (23)$$

$$\frac{d^2M}{dx^2} - \frac{d}{dx} \left(P \frac{dv}{dx} \right) = q(x) \quad (24)$$

$$\bar{\alpha}(x) - \frac{1}{2} l_c^2 \frac{\partial^2 \bar{\alpha}(x)}{\partial x^2} + \frac{1}{8} l_c^4 \frac{\partial^4 \bar{\alpha}(x)}{\partial x^4} - \frac{1}{48} l_c^6 \frac{\partial^6 \bar{\alpha}(x)}{\partial x^6} = \alpha(x) \quad (25)$$

220 Note that in the above equations sixth-order implicit gradient approximation is considered for the
 221 nonlocal plastic equivalent strain ($\bar{\alpha}$) to show the implementation process; however, higher order
 222 gradient terms can be included in a similar fashion, if needed. The boundary conditions for the
 223 nonlocal plastic strain is assumed as

$$\frac{\partial^r \bar{\alpha}}{\partial x^r} = 0 \quad r \in \{1, 2, \dots, 5\} \quad (26)$$

224 These boundary condition are shown to be consistent with the nonlocal integral formulation
 225 [22]. Multiplying by the appropriate weighting functions and integrating over the domain Ω
 226 using integration by parts, the corresponding weak forms are given by

$$\int_{\Omega} \frac{dw_u}{dx} P dx = \int_{\Omega} w_u \tau(x) dx + w_u P|_{\Gamma} \quad (27)$$

$$\int_{\Omega} \frac{d^2 w_v}{dx^2} M dx + \int_{\Omega} \frac{dw_v}{dx} P \frac{dv}{dx} dx = \int_{\Omega} w_v q(x) dx + \frac{dw_v}{dx} M \Big|_{\Gamma} - w_v \left(\frac{dM}{dx} - P \frac{dv}{dx} \right) \Big|_{\Gamma} \quad (28)$$

$$\int_{\Omega} \left(w_{\bar{\alpha}} \bar{\alpha} + \frac{1}{2} l_c^2 \frac{dw_{\bar{\alpha}}}{dx} \frac{d\bar{\alpha}}{dx} + \frac{1}{8} l_c^4 \frac{\partial^2 w_{\bar{\alpha}}}{\partial x^2} \frac{\partial^2 \bar{\alpha}}{\partial x^2} + \frac{1}{48} l_c^6 \frac{\partial^3 w_{\bar{\alpha}}}{\partial x^3} \frac{\partial^3 \bar{\alpha}}{\partial x^3} - w_{\bar{\alpha}} \alpha(x) \right) dx = 0 \quad (29)$$

227 where w_u , w_v and $w_{\bar{\alpha}}$ are the weighting functions and $\Gamma = \{0, L\}$ denotes the boundary points.
 228 In this weak form, the continuity requirements on the fields are as follows: $u(x) \in C^0$, $v(x) \in$
 229 C^1 and $\bar{\alpha}(x) \in C^2$ and similarly $w_u(x) \in C^0$, $w_v \in C^1$ and $w_{\bar{\alpha}}(x) \in C^2$. Note that due to the
 230 increased continuity requirements higher order shape functions will be needed to interpolate the
 231 nonlocal field $\bar{\alpha}(x)$ if the standard Hermite interpolation functions are employed.

4.2 Finite element discretization using B-spline

Galerkin method is used for deriving the finite element approximations, wherein both the unknown fields (u , v and $\bar{\alpha}$) and the corresponding weighting functions (w_u , w_v and $w_{\bar{\alpha}}$) are approximated using the same interpolation functions. Furthermore, the isoparametric concept is invoked implying that the solution space, weighting function space and geometry space are spanned by the same B-spline basis functions. Since the highest continuity requirement is C^2 , $p \geq 3$ in the B-spline basis is needed, assuming the multiplicity of all interior knots to be one.

Figure 4. Discretization in geometry space for planar beam (knot vector $\Xi = \{0,0,0,0,0.25,0.5,0.75,1,1,1,1\}$).

Consider a beam with n control points such that each control point has three degrees of freedom: axial displacement $u(x)$, transverse displacement $v(x)$, and nonlocal effective plastic strain $\bar{\alpha}(x)$. If the order of basis is taken as p then the knot vector can be written as $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$. In IGA terminology, the beam is considered as a patch while the non-zero knot span $[\xi_i, \xi_{i+1}]$ is considered as an element over which the numerical integration is carried out. Within

one element $\Omega^e \in [\xi_i, \xi_{i+1}]$ ($\xi_i \neq \xi_{i+1}$), the number of support control points and the
 corresponding interpolation functions is $p + 1$ due to the local support property of the B-spline
 basis. This means that only $p + 1$ interpolation functions are non-zero within an element, and
 these $p + 1$ interpolation functions can be used to span the geometry and solution spaces within
 the element. An example of isogeometric interpolation using B-splines for a beam is shown in
 Figure 4. In this case the beam is discretized using 4 Bézier elements with 3rd order B-spline basis
 ($p = 3$). The control points are located such that the mapping from the parametric space to
 physical space is linear which implies that the Jacobian of the mapping is constant (i.e. $dx/d\xi =$
 constant) [25]. Figure 4(a) shows the B-spline basis functions within each Bézier element. The
 local support property of B-spline basis ensures only $p + 1 = 4$ basis functions are nonzero in
 an element. Using the local support property, the geometry (x^h) and various fields (u^h , v^h and
 $\bar{\alpha}^h$) in the e^{th} Bézier element are interpolated as

$$x^h(\xi) = \mathbf{N}_x^e \mathbf{x}^e \quad (30)$$

$$u^h(\xi) = \mathbf{N}_u^e \mathbf{u}^e, \quad v^h(\xi) = \mathbf{N}_v^e \mathbf{v}^e, \quad \bar{\alpha}^h(\xi) = \mathbf{N}_{\bar{\alpha}}^e \bar{\alpha}^e \quad (31)$$

$$\mathbf{w}_u^h(\xi) = \mathbf{N}_u^e \mathbf{w}_u^e, \quad \mathbf{w}_v^h(\xi) = \mathbf{N}_v^e \mathbf{w}_v^e, \quad \mathbf{w}_{\bar{\alpha}}^h(\xi) = \mathbf{N}_{\bar{\alpha}}^e \mathbf{w}_{\bar{\alpha}}^e \quad (32)$$

$$\mathbf{N}_x^e = \mathbf{N}_u^e = \mathbf{N}_v^e = \mathbf{N}_{\bar{\alpha}}^e = [N_{e,p}(\xi), N_{e+1,p}(\xi), \dots, N_{e+p,p}(\xi)] \quad (33)$$

where $\mathbf{x}^e$, $\mathbf{u}^e$, $\mathbf{v}^e$ and $\bar{\alpha}^e$ are the x -coordinates, horizontal and vertical displacements and the
 nonlocal plastic strain vectors at the control points from e to $e + p$; $\mathbf{w}_u^e$, $\mathbf{w}_v^e$ and $\mathbf{w}_{\bar{\alpha}}^e$ are the
 corresponding arbitrary weights at the control points; and $\mathbf{N}_u$, $\mathbf{N}_v$, and $\mathbf{N}_{\bar{\alpha}}$ are the B-spline
 basis functions. Note that the same order of interpolation is used for various fields and this is in
 contrast with standard finite elements wherein different order of interpolations are usually
 employed for different fields. Using these interpolations in the weak forms (Eq. (27), Eq. (28) and

Eq. (29)) together with the standard finite element assembly operations, the global discrete system can be expressed in the residual form as

$$\begin{aligned}\mathbf{R}(\mathbf{d}) &= \mathbf{F}_{int}(\mathbf{d}) - \mathbf{F}_{ext} = \mathbf{0} \\ \mathbf{F}_{int}(\mathbf{d}) &= \mathcal{A} \sum_{e=1}^{n_{ele}} \mathbf{F}_{int}^e \\ \mathbf{F}_{ext} &= \mathcal{A} \sum_{e=1}^{n_{ele}} \mathbf{F}_{ext}^e + \bar{\mathbf{F}}_{ext}\end{aligned}\tag{34}$$

where $\mathbf{R}(\mathbf{d})$ is the global residual vector, $\mathbf{F}_{int}$ is the internal force vector, $\mathbf{F}_{ext}$ is the external force vector, $\mathbf{F}_{int}^e$ is the internal force vector for a Bézier element, $\mathbf{F}_{ext}^e$ is the external force vector for a Bézier element due to distributed loads, $\bar{\mathbf{F}}_{ext}$ is the external force contribution of the point loads, $\mathcal{A}$ is the standard finite element assembly operator, and n_{ele} is the total number of Bézier elements in the domain. The internal and external force vectors are given by

$$\mathbf{F}_{int}^e = \begin{bmatrix} \mathbf{F}_{int,u}^e \\ \mathbf{F}_{int,v}^e \\ \mathbf{F}_{int,\bar{\alpha}}^e \end{bmatrix}, \quad \mathbf{F}_{ext}^e = \begin{bmatrix} \mathbf{F}_{ext,\tau}^e \\ \mathbf{F}_{ext,q}^e \\ \mathbf{0} \end{bmatrix}\tag{35}$$

$$\mathbf{F}_{int,u}^e = \int_{\Omega^e} \mathbf{B}_u^{eT} P dx, \quad P = EA(\varepsilon_a - \varepsilon_a^p)\tag{36}$$

$$\mathbf{F}_{int,v}^e = \int_{\Omega^e} \mathbf{B}_{vv}^{eT} M dx + \int_{\Omega^e} \mathbf{B}_v^{eT} P (\mathbf{B}_v^e \mathbf{v}^e) dx, \quad M = EI(\kappa - \kappa^p)\tag{37}$$

$$\begin{aligned}\mathbf{F}_{int,\bar{\alpha}}^e &= \int_{\Omega^e} \left(\mathbf{N}_{\bar{\alpha}}^{eT} \mathbf{N}_{\bar{\alpha}}^e \bar{\alpha}^e + \frac{1}{2} l_c^2 \mathbf{B}_{\bar{\alpha}}^{eT} \mathbf{B}_{\bar{\alpha}}^e \bar{\alpha}^e + \frac{1}{8} l_c^4 \mathbf{B}_{\bar{\alpha}\bar{\alpha}}^{eT} \mathbf{B}_{\bar{\alpha}\bar{\alpha}}^e \bar{\alpha}^e + \frac{1}{48} l_c^6 \mathbf{B}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}}^{eT} \mathbf{B}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}}^e \bar{\alpha}^e \right. \\ &\quad \left. - \mathbf{N}_{\bar{\alpha}}^{eT} \alpha \right) dx\end{aligned}\tag{38}$$

$$\mathbf{F}_{ext,\tau}^e = \int_{\Omega^e} \mathbf{N}_u^{eT} \tau(x) dx, \quad \mathbf{F}_{ext,q}^e = \int_{\Omega^e} \mathbf{N}_v^{eT} q(x) dx\tag{39}$$

The shape function derivatives – $\mathbf{B}_u^e$, $\mathbf{B}_v^e$, $\mathbf{B}_{vv}^e$, $\mathbf{B}_{\bar{\alpha}}^e$, $\mathbf{B}_{\bar{\alpha}\bar{\alpha}}^e$, $\mathbf{B}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}}^e$ – are given by

$$\mathbf{B}_u^e = \frac{d\mathbf{N}_u^e}{dx}, \quad \mathbf{B}_v^e = \frac{d\mathbf{N}_v^e}{dx}, \quad \mathbf{B}_{vv}^e = \frac{d^2\mathbf{N}_v^e}{dx^2}\tag{40}$$

$$\mathbf{B}_{\bar{\alpha}}^e = \frac{d\mathbf{N}_{\bar{\alpha}}^e}{dx}, \quad \mathbf{B}_{\bar{\alpha}\bar{\alpha}}^e = \frac{d^2\mathbf{N}_{\bar{\alpha}}^e}{dx^2}, \quad \mathbf{B}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}}^e = \frac{d^3\mathbf{N}_{\bar{\alpha}}^e}{dx^3}\tag{41}$$

and are obtained by chain rule (Eq. (42))

$$\frac{d(\blacksquare)}{dx} = \frac{d(\blacksquare)}{d\xi} \frac{1}{dx/d\xi} \quad (42)$$

275 Finally, the global nonlinear system (Eq. (34)) is solved using the Newton-Raphson method [39],

276 and to this end, the global tangent stiffness is needed. The global tangent stiffness is obtained as

$$\mathbf{K}_T = \sum_{e=1}^{n_{ele}} \mathcal{A} \mathbf{K}_T^e \quad (43)$$

277 where $\mathbf{K}_T^e$ is the tangent stiffness matrix of the e^{th} Bézier element given by

$$\mathbf{K}_T^e = \begin{bmatrix} \frac{d\mathbf{F}_{int,u}^e}{d\mathbf{u}^e} & \frac{d\mathbf{F}_{int,u}^e}{d\mathbf{v}^e} & \frac{d\mathbf{F}_{int,u}^e}{d\bar{\alpha}^e} \\ \frac{d\mathbf{F}_{int,v}^e}{d\mathbf{u}^e} & \frac{d\mathbf{F}_{int,v}^e}{d\mathbf{v}^e} & \frac{d\mathbf{F}_{int,v}^e}{d\bar{\alpha}^e} \\ \frac{d\mathbf{F}_{int,\bar{\alpha}}^e}{d\mathbf{u}^e} & \frac{d\mathbf{F}_{int,\bar{\alpha}}^e}{d\mathbf{v}^e} & \frac{d\mathbf{F}_{int,\bar{\alpha}}^e}{d\bar{\alpha}^e} \end{bmatrix} \quad (44)$$

278 The derivatives of $\mathbf{F}_{int,u}^e$ are obtained as

$$\frac{d\mathbf{F}_{int,u}^e}{d\mathbf{u}^e} = \int_{\Omega^e} \mathbf{B}_u^{eT} \frac{dP}{d\varepsilon_a} \mathbf{B}_u^e dx \quad (45)$$

$$\frac{d\mathbf{F}_{int,u}^e}{d\mathbf{v}^e} = \int_{\Omega^e} \mathbf{B}_u^{eT} \left(\frac{dP}{d\varepsilon_a} (\mathbf{B}_v^e \mathbf{v}^e) \mathbf{B}_v^e + \frac{dP}{d\kappa} \mathbf{B}_{vv}^e \right) dx \quad (46)$$

$$\frac{d\mathbf{F}_{int,u}^e}{d\bar{\alpha}^e} = \int_{\Omega^e} \mathbf{B}_u^{eT} \frac{dP}{d\bar{\alpha}} \mathbf{N}_{\bar{\alpha}}^e dx \quad (47)$$

279 whereas, the derivatives of $\mathbf{F}_{int,v}^e$ are given by

$$\frac{d\mathbf{F}_{int,v}^e}{d\mathbf{u}^e} = \int_{\Omega^e} \mathbf{B}_{vv}^{eT} \frac{dM}{d\varepsilon_a} \mathbf{B}_u^e dx + \int_{\Omega^e} \mathbf{B}_v^{eT} \left((\mathbf{B}_v^e \mathbf{v}^e) \frac{dP}{d\varepsilon_a} \right) \mathbf{B}_u^e dx \quad (48)$$

$$\begin{aligned} \frac{d\mathbf{F}_{int,v}^e}{d\mathbf{v}^e} = & \int_{\Omega^e} \mathbf{B}_{vv}^{eT} \left(\frac{dM}{d\varepsilon_a} (\mathbf{B}_v^e \mathbf{v}^e) \mathbf{B}_v^e + \frac{dM}{d\kappa} \mathbf{B}_{vv}^e \right) dx \\ & + \int_{\Omega^e} \mathbf{B}_v^{eT} \left[\left(\frac{dP}{d\varepsilon_a} (\mathbf{B}_v^e \mathbf{v}^e)^2 \mathbf{B}_v^e + \frac{dP}{d\kappa} (\mathbf{B}_v^e \mathbf{v}^e) \mathbf{B}_{vv}^e \right) + P \mathbf{B}_v^e \right] dx \end{aligned} \quad (49)$$

$$\frac{d\mathbf{F}_{int,v}^e}{d\bar{\alpha}^e} = \int_{\Omega^e} \mathbf{B}_{vv}^{eT} \frac{dM}{d\bar{\alpha}} \mathbf{N}_{\bar{\alpha}}^e dx + \int_{\Omega^e} \mathbf{B}_v^{eT} (\mathbf{B}_v^e \mathbf{v}^e) \frac{dP}{d\bar{\alpha}} \mathbf{N}_{\bar{\alpha}}^e dx \quad (50)$$

280 Finally, the derivatives of $\mathbf{F}_{int,\bar{\alpha}}^e$ are given by

$$\frac{d\mathbf{F}_{int,\bar{\alpha}}^e}{d\mathbf{u}^e} = - \int_{\Omega^e} \mathbf{N}_{\bar{\alpha}}^{eT} \frac{d\alpha}{d\varepsilon_a} \mathbf{B}_u^e dx \quad (51)$$

$$\frac{d\mathbf{F}_{int,\bar{\alpha}}^e}{d\mathbf{v}^e} = - \int_{\Omega^e} \mathbf{N}_{\bar{\alpha}}^{eT} \left(\frac{d\alpha}{d\varepsilon_a} (\mathbf{B}_v^e \mathbf{v}^e) \mathbf{B}_v^e + \frac{d\alpha}{d\kappa} \mathbf{B}_{vv}^e \right) dx \quad (52)$$

$$\begin{aligned} \frac{d\mathbf{F}_{int,\bar{\alpha}}^e}{d\bar{\alpha}^e} = \int_{\Omega^e} & \left(\mathbf{N}_{\bar{\alpha}}^{eT} \mathbf{N}_{\bar{\alpha}}^e + \frac{1}{2} l_c^2 \mathbf{B}_{\bar{\alpha}}^{eT} \mathbf{B}_{\bar{\alpha}}^e + \frac{1}{8} l_c^4 \mathbf{B}_{\bar{\alpha}\bar{\alpha}}^{eT} \mathbf{B}_{\bar{\alpha}\bar{\alpha}}^e + \frac{1}{48} l_c^6 \mathbf{B}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}}^{eT} \mathbf{B}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}}^e \right. \\ & \left. - \mathbf{N}_{\bar{\alpha}}^{eT} \frac{d\alpha}{d\bar{\alpha}} \mathbf{N}_{\bar{\alpha}}^e \right) dx \end{aligned} \quad (53)$$

Numerical implementation of the constitutive model and calculation of the consistent algorithmic derivatives $dP/d\varepsilon_a$, $dP/d\kappa$, $dP/d\bar{\alpha}$, $dM/d\varepsilon_a$, $dM/d\kappa$, $dM/d\bar{\alpha}$, $d\alpha/d\varepsilon_a$, $d\alpha/d\kappa$ and $d\alpha/d\bar{\alpha}$, needed for computing the tangent stiffness matrix, are given in the Appendix A.

5. Numerical simulations

5.1 Example 1: Planar pinned-pinned beam

Figure 5. Pinned-pinned beam under point load.

In the first example, a pinned-pinned beam under point load at mid-span shown in Figure 5 is considered. The beam has a W12×65 cross-section with depth $h = 0.308$ m, area $A = 0.0123$ m², area moment of inertia $I = 2.219 \times 10^{-4}$ m⁴ and a span of $L = 6$ m. The Young's modulus is $E = 200$ GPa, and the yield axial force and the yield moment are $P_y = 4243.5$ kN and $M_y = 552$ kNm, respectively. All the analyses are carried out using a Matlab[®] based finite element program CPSSL-FEA developed at Notre Dame. Moreover, the arc-length method, with cylindrical

arc-length constraint [40], is used to obtain the softening and snap back responses associated with the damage.

Nonlocal implicit gradient approximations

In the first case, the length-scale is taken as $l_c = h = 0.308$ m and various orders of implicit gradient approximations of the nonlocal operator (Eq. (20)) are investigated. In addition, the following model parameters are considered: $k_1 = 0.02$ and $k_2 = \delta = 0$ in the plastic hardening function (Eq. (13)) and $\bar{\alpha}_c = 40$ in the linear damage model (Eq. (14)). Figure 6 shows the force-displacement response curves for second, fourth and sixth order implicit gradient approximations. Different number of cubic Bézier elements (i.e. $p = 3$) are used to obtain convergence, and the number of elements used are given in Figure 6. These results indicate that for second-order approximation 120 elements are needed to obtain adequate convergence (Figure 6a), while for fourth-order (Figure 6b) and sixth-order approximations (Figure 6c), 90 elements are adequate to represent the solution. Thus, with higher order approximations fewer numbers of elements are needed to achieve adequate convergence. This phenomenon was also reported by Verhoosel et al. [22], where it was shown that the higher order approximations generate smoother results which lead to better convergence. In Figure 6(d), the converged results of second, fourth and sixth order gradient approximations are compared to show the difference brought by adding higher order gradient terms. These results also indicate that sixth-order formulation gives better approximation to initial nonlocal formulation, as expected.

Figure 6. Mesh convergence studies using cubic B-splines: (a) second-order, (b) fourth-order and (c) sixth-order gradient approximations, and (d) comparison of second, fourth and sixth order approximations.

Figure 7 shows the membrane strain (Figure 7a), the curvature strain (Figure 7b) and the damage parameter (Figure 7c) along the beam span at the last stage (point ④ in Figure 6(d)), and the evolution of $\sqrt{\mathbf{s}^T \mathbf{Q} \mathbf{s}}$ at the midpoint together with the yield surface during the entire loading path (Figure 7d). The results are shown for sixth-order implicit gradient approximation with 120 cubic Bézier elements. After the snap-back point is reached, the difference between total strain and plastic strain approaches zero, implying that $\boldsymbol{\varepsilon}^e \rightarrow \mathbf{0}$ and $\mathbf{s} \rightarrow \mathbf{0}$. The damage is localized at

the center of the beam as expected and complete damage (i.e. $\omega^d \rightarrow 1$) occurs at the mid-point of the beam at the last stage (Figure 7c). Figure 7d shows the evolution of yield surface, wherein it can be observed that the yield surface first expands before the damage starts. As the damage progresses, the yield surface contracts and the load finally drops.

Figure 7. Sixth-order gradient approximation: (a) membrane strain, (b) curvature strain, (c) damage parameter and (d) axial force vs. moment at the mid-point of the beam.

Figure 8. Stress resultants, effective plastic strain and yield function components at three different stages.

Figure 8 shows the distribution of stress resultants, local and nonlocal effective plastic strains and yield function in the beam at three different stages of loading for sixth-order implicit gradient approximation. In Figure 8(a) to (f) some oscillations can be observed in the stress resultants, and these oscillations typically increase during the last stages. The oscillations occur at the locations where the plasticity and damage is evolving and these oscillations are purely a numerical artifact; similar oscillations have been observed in the case of nonlocal damage models in past studies [22, 41, 42]. In addition, since both the total and plastic strain are bounded, the oscillations are also bounded. For engineering purposes, these oscillations can be removed through post-processing as suggested by Simone et al. [41] and this is adopted for the rest of the numerical examples. Finally, from Figure 8(g), Figure 8(h) and Figure 8(i) two different inelastic zones can be identified. The zone-PD where both $\alpha(x)$ and $\bar{\alpha}(x)$ are nonzero, and in this zone there is both the plastic hardening and the damage softening effect. The zone-D where only $\bar{\alpha}(x)$ is nonzero while $\alpha(x)$ is zero and in this zone only the damage softening effect is present. In Figure 8(j), Figure 8(k) and Figure 8(l), the two parts of the yield potential $-\sqrt{\mathbf{s}^T \mathbf{Q} \mathbf{s}}$ and $(1 - \omega^d(\bar{\alpha}))(1 + \zeta^p(\alpha))$ are plotted separately for each stage and the two zones can be again identified in these figures.

Local damage case

In this case, a local damage model is considered with the following parameters: $k_1 = 0.02$, $k_2 = \delta = 0$ in the hardening function (Eq. (13)) and $\bar{\alpha}_c = 40$ in the linear damage model (Eq. (14)). The length-scale $l_c = 0$ is used implying that $\bar{\alpha}(x) = \alpha(x)$, and a local elasto-plastic damage constitutive model is recovered. Figure 9 gives a mesh convergence study for sixth-order implicit gradient approximation. This result clearly shows that no convergence to a meaningful

solution exists upon mesh refinement in this case, thus signifying the importance of the nonlocal model.

Figure 9. Mesh convergence studies for local damage model.

Influence of model parameters

Figure 10 shows the influence of the damage evolution rules on the global response. Linear and exponential rules are considered with different model parameters. Sixth-order gradient approximation with length-scale $l_c = h$ and plastic hardening parameters $k_1 = 0.02$, $k_2 = 0$, and $\delta = 0$ are used with 60 cubic Bézier elements in IGA. In the linear damage evolution case (Figure 10a), the parameter $\bar{\alpha}_c$ controlled the ductility of the response: higher $\bar{\alpha}_c$ leading to higher ductility. In the exponential damage evolution case (Figure 10b), the parameter β controls both the ductility and the strength, with lower β leading to higher strength and ductility.

Figure 11 shows the influence of the plastic hardening model parameters on the global response with linear and exponential damage models. Sixth-order gradient approximation with length-scale $l_c = h$ is used with 60 cubic Bézier elements in IGA. Clearly, different hardening and softening responses curves can be obtained by choosing different models and parameters. In practice, final

selection of these parameters will depend on the experimental calibration of these parameters. Finally, the influence of the length-scale parameter is shown in Figure 12. Sixth-order gradient approximation with plastic hardening parameters $k_1 = 0.02$, $k_2 = 0$ and $\delta = 0$ and the linear damage model with $\bar{\alpha}_c = 40$ are used with 60 cubic Bzier elements in IGA. It can be observed that with a larger length-scale parameter l_c , the damage spreads to a wider region (Figure 12a) and this results in increased ductility (Figure 12b).

Figure 10. Influence of damage model and parameters $\bar{\alpha}_c$ and β : (a) linear model and (b) exponential model.

(c)

(d)

Figure 11. Influence of plastic hardening function and parameters: (a) linear damage ($\bar{\alpha}_c = 40$) with linear hardening, (b) linear damage ($\bar{\alpha}_c = 40$) with exponential hardening ($k_1 = 0$, $k_2 = 1$), (c) exponential damage ($\beta = 0.03$) with linear hardening ($k_2 = \delta = 0$) and (d) exponential damage ($\beta = 0.03$) with exponential hardening ($k_1 = 0$, $k_2 = 1$).

(a)

(b)

Figure 12. Influence of length-scale: (a) nonlocal effective plastic strain and (b) force-displacement response.

5.2 Example 2: Multi-span beam

Figure 13. Multi-span beam: (a) geometry and loading pattern and (b) discretization on patch level in IGA.

In the second example, a multi-span beam shown in Figure 13(a) is considered. The beam has a W6×20 cross-section with depth $h = 0.1575$ m, area $A = 0.0038$ m², moment of inertia $I = 1.7232 \times 10^{-5}$ m⁴. The Young's modulus is $E = 200$ GPa, and yield axial force and yield moment are $P_y = 1306.515$ kN and $M_y = 84.046$ kNm, respectively. In this example, linear damage evolution law with $\bar{\alpha}_c = 30$ and plastic hardening with $k_1 = 0.02$, $k_2 = 0$ and $\delta = 0$ are adopted. The length-scale parameter is taken as the depth of the beam, i.e. $l_c = h$. Also, the proportional loading is applied with $|F| = |q| = \lambda$. The continuous spans are modeled with three IGA patches, as shown in Figure 13(b). By default the continuity of field variables across the IGA patches is C^0 ; however, in the case of multi-span beam higher order continuity is required for the vertical displacement ($v(x)$) and nonlocal damage ($\bar{\alpha}(x)$) fields. In particular, $v'(x)$ should be continuous across the patch boundary in this case of Euler-Bernoulli beam. Moreover, for the nonlocal damage field, $\bar{\alpha}''(x)$ should be continuous across the patch boundary for the sixth-order gradient formulation (Eq. (29)). This continuity requirement can be relaxed, however,

418 as there is no physical requirement for this continuity. To further verify this assertion, two cases
 419 are considered: in the first case, continuity of both $v'(x)$ and $\bar{\alpha}''(x)$ is enforced across the
 420 patch boundaries, while in the second case only the continuity of $v'(x)$ is enforced and $\bar{\alpha}(x)$ is
 421 assumed to have C^0 -continuity across the patch boundaries. The continuity across the patch
 422 boundaries is enforced using the Lagrange multiplier method.

423
 424 Figure 14. Mesh convergence studies for sixth-order formulation with C^2 -continuous $\bar{\alpha}$ using
 425 cubic B-splines: (a) vertical displacement at point A, (b) vertical displacement at point B and (c)
 426 vertical displacement at point C.

Figure 15. Mesh convergence studies for sixth-order formulation with C^0 -continuous $\bar{\alpha}$ using cubic B-splines: (a) vertical displacement at point A, (b) vertical displacement at point B and (c) vertical displacement at point C.

Figure 14 and Figure 15 show the mesh convergence study with sixth-order gradient approximation for C^2 and C^0 -continuous $\bar{\alpha}(x)$, respectively, across the patch boundaries. It can be observed that 350 cubic Bézier elements in three patches are adequate to obtain satisfactory convergence. In Figure 16, the converged solutions of both C^2 and C^0 -continuous $\bar{\alpha}(x)$ are compared, and this result indicates that the continuity enforcement has little influence on the global load-displacement response in this case.

Figure 16. Comparison of the force-displacement response with C^2 and C^0 -continuous $\bar{\alpha}(x)$: (a) vertical displacement at point A, (b) vertical displacement at point B, and (c) vertical displacement at point C.

Furthermore, four critical points are marked on the response curves, denoting four stages of the response for which the moment, axial force, nonlocal effective plastic strain distribution and damage parameter are shown in Figure 17. From Figure 17(a), Figure 17(c), Figure 17(e) and Figure 17(g) it can be observed that the moment and axial force responses based on two different assumptions, C^0 -continuity and C^2 -continuity for $\bar{\alpha}(x)$ field across adjacent spans, are quite similar. However, due to the continuity enforcement, the nonlocal effective plastic strain distribution is different at the boundary of two patches, i.e. between two different spans as shown in Figure 17(b), Figure 17(d), Figure 17(f) and Figure 17(h). Clearly, when C^2 -continuity of

451 $\bar{\alpha}(x)$ is enforced, a smooth $\bar{\alpha}(x)$ curve at the interface of two patches can be observed, whereas
 452 without any enforcement there is C^0 -continuity across the patch boundaries, as expected.

453

(a)

(b)

Response at Stage ①

(c)

(d)

Response at Stage ②

454

Response at Stage ③

Response at Stage ④

Figure 17. Response quantities at integrations point at different stages marked in Figure 16 (*left column*: axial force and moment; *right column*: nonlocal effective plastic strain and damage).

5.3 Example 3: Axially loaded beam

Figure 18. Simply supported beam under axial loading.

In the third example, a simply supported beam with imperfection under axial loading is considered (Figure 18). The beam has the same material property and cross-section as that considered in Example 1. Imperfection is included as an initial eccentricity in the middle of the beam, and three magnitudes of imperfections – $\Delta = L/100$, $L/200$ and $L/400$ – are considered. In this example, exponential damage evolutionary law with $\beta = 0.03$, and exponential hardening rule with $k_1 = 0$, $k_2 = 1$ and $\delta = 0.02$ are used. Furthermore, based on the convergence study in Example 1, an IGA mesh with 120 cubic Bézier elements are employed. The beam is modeled with two IGA patches, and the continuity of both $v'(x)$ and $\bar{\alpha}''(x)$ is enforced at midspan.

Figure 19. Sixth-order gradient approximation: (a) load-displacement curve and (b) axial force vs. moment at the mid-point of the beam.

Figure 19(a) shows the load-displacement curves for various magnitudes of imperfections. The elastic Euler critical load for this case is $\lambda_{cr} = \pi^2 EI/L^2 = 1.216 \times 10^4$ kN. Since nonlocal elasto-plastic material is considered, the limit load for the imperfect beam is lower than the elastic Euler critical load. In addition, with increasing imperfections, the limit load is reduced, as expected. Figure 19(b) shows the evolution of plastic potential term $\sqrt{s^T Q s}$ (Eq. (8)) at the

midpoint together with the yield surfaces during loading path. As evident from the yield surface, in this case the axial load dominated the inelastic response, as opposed to Example 1 (see Figure 7d)) wherein the moment controls the response.

5.4 Example 4: Portal frame

Figure 20. Planar portal frame: (a) geometry and loading and (b) discretization at patch level.

In the last example, a clamped portal frame under vertical and horizontal loads as shown in Figure 20(a) is considered. A similar frame was studied by Cocchetti and Maier [8] but with a different softening plastic-hinge model without incorporating damage or nonlocal effects. The beam and columns have the same cross-section and material property as the multi-span beam in Example 2. The length of columns is $H = 3$ m and the span of beam is $L = 4$ m. In this example, linear damage evolution law with $\bar{\alpha}_c = 30$ and plastic hardening rule with $k_1 = 0.02$, $k_2 = 0$ and $\delta = 0$ is used. Two different length-scale parameters are chosen for two different test cases. Furthermore, three patches are used in isogeometric analysis as shown in Figure 20(b). The nonlocal effective plastic strain $\bar{\alpha}(x)$ is considered to be C^0 -continuous across the patch boundaries, while the continuity of the rotations at the joints (patch-boundaries) is enforced using the Lagrange multiplier method.

Figure 21. Mesh convergence studies: (a) second-order, (b) fourth-order, (c) sixth-order gradient approximations, and (d) comparison of converged second, fourth and sixth order approximations.

Figure 21(a), Figure 21(b) and Figure 21(c) show the mesh convergence results for second, fourth and sixth order gradient approximations. As shown, satisfactory convergence is achieved with 300 cubic Bézier elements in three patches. Moreover, the converged fourth and sixth order approximations give similar response as shown in Figure 21(d), implying the convergence of sixth-order gradient approximation to the nonlocal integral formulation. Considering the accuracy as well as computation cost, the sixth-order gradient approximation with 200 cubic Bézier elements in three patches is chosen for further analysis.

(a)

(b)

Figure 22. Force-displacement response curves: (a) horizontal displacement (u_A) and (b) vertical displacement (v_B). Solid circle ● indicates the plastic hinge region.

In the first test case, the length-scale parameter is taken as the depth of the beam, i.e. $l_c = h$.

Figure 22(a) and Figure 22(b) show the force-displacement response curve for horizontal displacement at A (u_A) and vertical displacement at B (v_B), respectively, for this case. There are four points of interest on this response curve: at point ① first two plastic hinges are formed; at point ② the third plastic hinge is formed at right column base; at point ③ the fourth plastic hinge is formed at left column base. The strength of the frame is then reduced and the load drops before the final failure by damage localization occurs at point ④. These points are marked in Figure 22 together with the locations of the plastic hinge region. The moment and axial force diagrams together with the nonlocal effective plastic strain and damage distribution is shown in Figure 24 at each stage. The position x in Figure 24 is defined according to the axis illustrated in Figure 23. Together with the evolution of the inelastic regions during the loading history, the redistribution of moments and axial forces in the frame can be observed from these figures. In particular, at point ④ the moment at the bottom of the right column approaches zero (Figure 24g), where the damage first localizes (i.e. $\omega^d \rightarrow 1$).

Figure 23. Axis illustration for plotting response quantities.

In the second test case, to illustrate the influence of length-scale parameter on the response, the length-scale parameter is increased from $l_c = h$ to $l_c = 1.5h$, and the response of the frame is again evaluated. The results for this case are presented in Figure 25. In this case, it can be observed that another plastic hinge is formed under the vertical load at point ④ on the response curve. This plastic hinge is not present in the preceding example with $l_c = h$. The axial force and moment response together with the nonlocal plastic strain and damage distribution is presented in Figure 26 at each critical point. It can be observed that up to point ③, the responses are quite similar to that in Figure 24. However, after this point, due to the initialization of another plastic hinge, the redistribution of internal force begins to differ from that in Figure 24. It can be observed that the final failure due to damage localization occurs at the right end of the beam and the upper end of the right column, where the moment is reduced to zero as shown in Figure 26(i). These results clearly indicate the important influence of length-scale parameter (l_c) on the global response including force redistribution and damage localization.

Response at Stage ①

(b) Response at Stage ②

Response at Stage ③

544

Response at Stage ④

545

546 Figure 24. Response quantities at integrations point at different stages marked in Figure 22 (*left*
547 *column*: axial force and moment; *right column*: nonlocal effective plastic strain and damage).
548

549

550 Figure 25. Force-displacement response curves: (a) horizontal displacement (u_A) and (b) vertical
551 displacement (v_B). Solid circle $\bullet$ indicates the plastic hinge region.
552

(a)

(b)

Response at Stage ①

(c)

(d)

Response at Stage ②

(e)

(f)

Response at Stage ③

553

554

(g)

Response at Stage ④

(h)

(i)

Response at Stage ⑤

(j)

Figure 26. Response quantities at integrations point at different stages marked in Figure 25 (*left column*: axial force and moment; *right column*: nonlocal effective plastic strain and damage).

6. Conclusions

A displacement based nonlocal damage-plastic Euler-Bernoulli beam model is proposed using a distributed resultant plasticity approach together with implicit gradient approximation of the nonlocal operator. IGA is used for the finite element formulation of the beam model due to the smoothness provided by the B-splines, and the fields are approximated using the same interpolation functions. Various plastic hardening rules and damage functions are incorporated in

the nonlocal damage-plastic constitutive model, and the numerical algorithm and the consistent tangents are derived for implementation of the beam model. Numerical results using different test cases show that the nonlocal theory is successfully able to preserve the objectivity of the results, and the finite element results converge with decreasing mesh size. In addition, various hardening and softening responses can be obtained by appropriately varying the model parameters. It is also shown that the length-scale plays an important role and can alter the response including internal force redistribution and the formation of plastic hinges. Furthermore, in the presented displacement based approach the strain-driven format of the global finite element formulation is preserved, and both snap-back and snap-through responses are obtained without any numerical difficulties.

Although the presented beam model is able to resolve the inelastic and damage zones with high fidelity, it is important to note that the models presented in this study are computationally expensive when compared to traditional beam elements with lumped plasticity. However, with the increase in computational capacity over the last decade these analyses can be conveniently carried out on desktop computers. Future studies can consider including shear deformations, which can allow for more complex failure modes including axial force, shear force and moment interaction, and nonlocal formulation of 3D beam elements with anisotropic damage models. Furthermore, since failure often occurs at large displacements, it may be necessary to develop beam models that allow for large displacements. Further work is under progress to address these important issues.

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Appendix A: Operator splitting algorithm

Numerical implementation of the nonlocal damage-plastic beam discussed in Section 4 is presented in this Appendix. In the context of strain-driven finite element formulation, for the given data at an integration point: $\boldsymbol{\varepsilon}_n^p = \begin{bmatrix} \varepsilon_a^p \\ \kappa^p \end{bmatrix}_n$ and α_n at step n , and $\bar{\alpha}_{n+1}$ and $\boldsymbol{\varepsilon}_{n+1} = \begin{bmatrix} \varepsilon_a \\ \kappa \end{bmatrix}_{n+1}$ at step $n+1$, the goal is to find the unknown variables: $\boldsymbol{\varepsilon}_{n+1}^p = \begin{bmatrix} \varepsilon_a^p \\ \kappa^p \end{bmatrix}_{n+1}$, $\mathbf{s}_{n+1} = \begin{bmatrix} P_{n+1} \\ M_{n+1} \end{bmatrix}$, α_{n+1} together with the derivatives $\frac{dP_{n+1}}{d\varepsilon_{a_{n+1}}}$, $\frac{dP_{n+1}}{d\kappa_{n+1}}$, $\frac{dP_{n+1}}{d\bar{\alpha}_{n+1}}$, $\frac{dM_{n+1}}{d\varepsilon_{a_{n+1}}}$, $\frac{dM_{n+1}}{d\kappa_{n+1}}$, $\frac{dM_{n+1}}{d\bar{\alpha}_{n+1}}$, $\frac{d\alpha_{n+1}}{d\varepsilon_{a_{n+1}}}$, $\frac{d\alpha_{n+1}}{d\kappa_{n+1}}$ and $\frac{d\alpha_{n+1}}{d\bar{\alpha}_{n+1}}$ at step $n+1$. The consistent evaluation of these derivatives will ensure the quadratic convergence of the global Newton-Raphson solver. An operator splitting method with trial elastic step followed by plastic step (if needed) is employed for solving the constitutive model.

Step 1: Elastic step

In this step plastic flow is assumed to be zero, i.e. $\gamma = 0$ and the trial state is defined as follows

$$\boldsymbol{\varepsilon}_{n+1}^{p^{tr}} = \boldsymbol{\varepsilon}_n^p, \quad \alpha_{n+1}^{tr} = \alpha_n \quad (\text{A1})$$

$$\mathbf{s}_{n+1}^{tr} = \begin{bmatrix} P_{n+1}^{tr} \\ M_{n+1}^{tr} \end{bmatrix} = \mathbf{D}(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_{n+1}^{p^{tr}}) = \begin{bmatrix} EA(\varepsilon_{a_{n+1}} - \varepsilon_{a_n}^p) \\ EI(\kappa_{n+1} - \kappa_n^p) \end{bmatrix} \quad (\text{A2})$$

$$\phi_{n+1}^{tr}(\mathbf{s}_{n+1}^{tr}, \alpha_{n+1}^{tr}, \bar{\alpha}_{n+1}) = \sqrt{\mathbf{s}_{n+1}^{trT} \mathbf{Q} \mathbf{s}_{n+1}^{tr}} - (1 - \omega^d(\bar{\alpha}_{n+1}))(1 + \zeta^p(\alpha_{n+1}^{tr})) \quad (\text{A3})$$

If $\phi_{n+1}^{tr} \leq 0$ then the current step is indeed an elastic step and the following elastic updates are made

$$\boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_{n+1}^{p^{tr}}, \quad \alpha_{n+1} = \alpha_{n+1}^{tr} \quad \text{and} \quad \mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{tr} \quad (\text{A4})$$

$$\begin{bmatrix} \frac{dP_{n+1}}{d\varepsilon_{a_{n+1}}} & \frac{dP_{n+1}}{d\kappa_{n+1}} & \frac{dP_{n+1}}{d\bar{\alpha}_{n+1}} \\ \frac{dM_{n+1}}{d\varepsilon_{a_{n+1}}} & \frac{dM_{n+1}}{d\kappa_{n+1}} & \frac{dM_{n+1}}{d\bar{\alpha}_{n+1}} \\ \frac{d\alpha_{n+1}}{d\varepsilon_{a_{n+1}}} & \frac{d\alpha_{n+1}}{d\kappa_{n+1}} & \frac{d\alpha_{n+1}}{d\bar{\alpha}_{n+1}} \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A5})$$

602 Else, if $\phi_{n+1}^{tr} > 0$ then there is plastic flow in this step and the algorithm proceeds to Step 2.

603 **Step 2: Plastic step**

604 In this step, plastic flow is nonzero ($\gamma > 0$). Using backward Euler method the flow rules in Eq.

605 (9) and Eq. (10) are discretized as

$$\alpha_{n+1} = \alpha_n + \Delta\gamma \quad (\text{A6})$$

$$\boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \frac{\Delta\gamma}{q_{n+1}} \mathbf{Q} \mathbf{s}_{n+1} \quad (\text{A7})$$

$$q_{n+1} \stackrel{\text{def}}{=} \sqrt{\mathbf{s}_{n+1}^T \mathbf{Q} \mathbf{s}_{n+1}}$$

606 The resultant force updates can be written as

$$\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{tr} - \frac{\Delta\gamma}{q_{n+1}} \mathbf{D} \mathbf{Q} \mathbf{s}_{n+1} \quad (\text{A8})$$

607 Next, the consistency condition $\phi_{n+1} = 0$ when combined with Eq. (A8), yield a system of

608 three nonlinear equations $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ (Eq. (A9)), with $\mathbf{z} = [P_{n+1}, M_{n+1}, \Delta\gamma]^T$ and this system is

609 solved using the Newton-Raphson method for the three unknown variables P_{n+1} , M_{n+1} and $\Delta\gamma$.

610 The initial guess for the Newton-Raphson process is taken as: $\mathbf{z}^{(0)} = [P_{n+1}^{tr}, M_{n+1}^{tr}, 0]^T$. The

611 Jacobian matrix of this nonlinear system, which is required for the Newton-Raphson updates, is

612 denoted by $\mathbf{A}$ and is given in Eq. (A10).

$$F_1(P_{n+1}, M_{n+1}, \Delta\gamma) = P_{n+1} - P_{n+1}^{tr} + \frac{EA\Delta\gamma}{q_{n+1}} \frac{P_{n+1}}{P_y^2} = 0$$

$$F_2(P_{n+1}, M_{n+1}, \Delta\gamma) = M_{n+1} - M_{n+1}^{tr} + \frac{EI\Delta\gamma}{q_{n+1}} \frac{M_{n+1}}{M_y^2} = 0 \quad (A9)$$

$$F_3(P_{n+1}, M_{n+1}, \Delta\gamma) = q_{n+1} - (1 - \omega^d(\bar{\alpha}_{n+1}))(1 + \zeta^p(\alpha_{n+1})) = 0$$

$$\mathbf{A} = \frac{d\mathbf{F}}{d\mathbf{z}}$$

$$A_{11} = 1 - \left(\frac{1}{q_{n+1}^2} \frac{P_{n+1}^2}{P_y^2} - 1 \right) \frac{EA\Delta\gamma}{q_{n+1}} \frac{1}{P_y^2}, A_{12} = -\frac{EA\Delta\gamma}{q_{n+1}^3} \frac{P_{n+1}}{P_y^2} \frac{M_{n+1}}{M_y^2}, A_{13} = \frac{EA}{q_{n+1}} \frac{P_{n+1}}{P_y^2}$$

$$A_{21} = -\frac{EI\Delta\gamma}{q_{n+1}^3} \frac{P_{n+1}}{P_y^2} \frac{M_{n+1}}{M_y^2}, A_{22} = 1 - \left(\frac{1}{q_{n+1}^2} \frac{M_{n+1}^2}{M_y^2} - 1 \right) \frac{EI\Delta\gamma}{q_{n+1}} \frac{1}{M_y^2}, A_{23} = \frac{EI}{q_{n+1}} \frac{M_{n+1}}{M_y^2} \quad (A10)$$

$$A_{31} = \frac{1}{q_{n+1}} \frac{P_{n+1}}{P_y^2}, A_{32} = \frac{1}{q_{n+1}} \frac{M_{n+1}}{M_y^2}, A_{33} = -[1 - \omega^d(\bar{\alpha}_{n+1})]K_{n+1}^p$$

$$K_{n+1}^p = \left. \frac{d\zeta^p(\alpha)}{d\alpha} \right|_{\alpha=\alpha_{n+1}} \quad \text{and} \quad K_{n+1}^d = \left. \frac{d\omega^d(\bar{\alpha})}{d\bar{\alpha}} \right|_{\bar{\alpha}=\bar{\alpha}_{n+1}}$$

613 To obtain the consistent tangents Eqs. (A9) are expressed in differential form as follows

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} dP_{n+1} \\ dM_{n+1} \\ d\alpha_{n+1} \end{bmatrix} = \begin{bmatrix} EA d\varepsilon_{a_{n+1}} \\ EI d\kappa_{n+1} \\ -[1 + \zeta^p(\alpha_{n+1})]K_{n+1}^d d\bar{\alpha} \end{bmatrix} \quad (A11)$$

614 Finally, using Eq. (A11) the consistent tangents are obtained as

$$\begin{bmatrix} \frac{dP_{n+1}}{d\varepsilon_{a_{n+1}}} & \frac{dP_{n+1}}{d\kappa_{n+1}} & \frac{dP_{n+1}}{d\bar{\alpha}_{n+1}} \\ \frac{dM_{n+1}}{d\varepsilon_{a_{n+1}}} & \frac{dM_{n+1}}{d\kappa_{n+1}} & \frac{dM_{n+1}}{d\bar{\alpha}_{n+1}} \\ \frac{d\alpha_{n+1}}{d\varepsilon_{a_{n+1}}} & \frac{d\alpha_{n+1}}{d\kappa_{n+1}} & \frac{d\alpha_{n+1}}{d\bar{\alpha}_{n+1}} \end{bmatrix} = \begin{bmatrix} EAB_{11} & EIB_{12} & -[1 + \zeta^p(\alpha_{n+1})]K_{n+1}^d B_{13} \\ EAB_{21} & EIB_{22} & -[1 + \zeta^p(\alpha_{n+1})]K_{n+1}^d B_{23} \\ EAB_{31} & EIB_{32} & -[1 + \zeta^p(\alpha_{n+1})]K_{n+1}^d B_{33} \end{bmatrix} \quad (A12)$$

615 where $\mathbf{B}_{ij}$ are the ij components of $\mathbf{A}^{-1}$.

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