1 An integral validation technique of RANS turbulence 2 models

3 Ian Pond^a, Alireza Ebadi^b, Yves Dubief^{a,*}, Christopher M. White^b

4 ^a*School of Engineering, University of Vermont, Burlington VT 05405 USA*

5 ^b*Department of Mechanical Engineering, University of New Hampshire, Durham NH*
6 *03824 USA*

7 **Abstract**

8 An integral technique to validate Reynolds-averaged Navier-Stokes (RANS)
9 turbulence models is presented. The technique has the advantage of provid-
10 ing a direct connection between wall fluxes and mean flow dynamics, thus
11 providing the necessary means to evaluate if a model correctly predicts the
12 flow physics. In turn, the technique provides needed information critical
13 to the improved development of turbulence models. To assess the value of
14 the technique, it is used to evaluate the performance of two low-Reynolds-
15 number turbulence models against DNS of reciprocating channel flow with
16 heat transfer. The evaluation demonstrates that the integral technique is an
17 improved validation technique compared to standard validation techniques.
18 *Keywords:* RANS, model validation, reciprocating flow, turbulent wall flux

*Corresponding author

Email addresses: `ian.b.pond@gmail.com` (Ian Pond), `alireza.ebadi@unh.edu`
(Alireza Ebadi), `yves.dubief@uvm.edu` (Yves Dubief), `chris.white@unh.edu`
(Christopher M. White)

1. Introduction

Reynolds-averaged Navier-Stokes (RANS) numerical simulations are used extensively to solve application relevant fluid transport problems across a broad range of disciplines and industries [1–5]. The widespread application of RANS simulations is primarily due to both ease of use and low computational cost, making RANS simulations ideal for system design and optimization, or investigative studies when experimental measurements are not feasible, among other practical utilizations. One shortcoming of RANS simulations is that the most often used linear eddy viscosity models (EVM) are largely incapable of accurately simulating complex flows [6–10]. Owing to these deficiencies, the base EVM (k - ε and k - ω) have been reformulated to seek improved performance [1, 11–16]. The purportedly improved variants (some being nonlinear) of the base EVM are typically validated following what we term the “standard validation technique” where the user compares averaged variable profiles (e.g., mean velocity or Reynolds shear stress) or averaged wall fluxes (e.g., wall shear stress or wall heat flux) against baseline standards. Model performance is assessed based on the “agreement” between model computed metrics and baseline standard metrics [17–21].

The standard validation technique is generally sufficient for model validation in simple canonical flows, like steady boundary layers or free shear flows, since much is known about these flows and there is a wealth of data for baseline comparison [19, 22–24]. In complex flows, commonly found in real-world applications, the standard validation technique is often deficient since typically little is known about the underlying flow physics to justify model validation based on a qualitative comparison of a few select metrics.

44 In the present study, we formulate an integral validation technique that is
 45 well-suited to validate and verify RANS turbulence models in complex flows.
 46 The metrics for the integral validation technique are attained by thrice in-
 47 tegrating the RANS momentum and scalar temperature transport equations
 48 to derive expressions for the wall shear stress and wall heat flux, respectively,
 49 in terms of integrated mean flow variables. The strength of the technique is
 50 that a direct connection is provided between mean flow dynamics and wall
 51 fluxes that is embedded in the validation metrics. Consequently, the integral
 52 validation technique provides an improved means, compared to the standard
 53 validation technique, to better evaluate if a RANS turbulence model accu-
 54 rately captures the underlying flow dynamics.

55 Model validation using the integral technique is demonstrated for recip-
 56 roating channel flow. This particular flow type was chosen since it is an
 57 unsteady, non-equilibrium flow that is simple enough for direct numerical
 58 simulation (DNS) but a very challenging flow for RANS modeling. Two
 59 low-Reynolds-number RANS turbulence models are validated against DNS
 60 first using the standard validation technique and second using the integral
 61 validation technique. The designation low-Reynolds-number model means
 62 that no wall-functions are employed and the near-wall region is solved di-
 63 rectly. For the standard validation technique, phase-averaged wall fluxes and
 64 wall-normal profiles of mean and fluctuating variables and their correlations
 65 are compared to the DNS results. Similarly, for the integral validation tech-
 66 nique, the phase-averaged contributing terms to the wall shear stress and
 67 the wall heat flux are compared to the DNS results. Lastly, the two valida-
 68 tion techniques are compared, and the benefits of the integral technique are

69 described.

70 **2. Reciprocating Channel Flow**

71 Reciprocating channel flow is a periodic flow that oscillates at a fixed
72 angular frequency ω with a cycle-averaged zero mean velocity. The cross-
73 sectional mean flow over a half-period first accelerates to maximum velocity,
74 next decelerates to zero velocity, then reverses direction. The process is then
75 repeated indefinitely. The flow at a fixed point in space may either lead
76 or lag the mean flow dependent on wall-normal position and flow forcing.
77 The important similarity variables are the Womersley number $\alpha = h\sqrt{\omega/\nu}$,
78 where h is the channel half-height, $\omega = 2\pi/T$ with T the period of oscilla-
79 tion, and ν is the fluid kinematic viscosity; and the peak Reynolds number
80 $Re_p = 2U_m h/\nu$, where U_m is the maximum amplitude of the cross-sectional
81 average velocity. The square of the Womersley number, α^2 , characterizes the
82 ratio of the diffusion time scale h^2/ν to the oscillation time scale ω^{-1} . The
83 ratio $Re_p/\alpha = \sqrt{2}Re_s$, where $Re_s = U_m l_s/\nu$ is the Stokes' Reynolds number
84 and $l_s = \sqrt{2\nu/\omega}$ is the Stokes' length. At a critical value of Re_s (≈ 750), a
85 reciprocating channel flow will transition from a self-sustaining transitional
86 flow to an intermittently turbulent flow [25]. In an intermittently turbulent
87 reciprocating channel flow, turbulence is first observed during the early de-
88 celerating phases of the cycle, while in the early accelerating phases of the
89 cycle, turbulence is damped and the flow degenerates towards relaminariza-
90 tion. With further increase in Re_s (i.e., by a decrease in α and/or an increase
91 in Re_p), the flow will become turbulent throughout an increasing portion of
92 its cycle [26, 27].

93 In the present study, reciprocating channel flow is used as a test-case
 94 flow to evaluate the performance of RANS turbulence models using both the
 95 standard validation technique and our integral validation technique. The
 96 baseline standard used for validation is DNS of reciprocating channel flow.
 97 This particular test-case flow was chosen since it is a challenging flow for
 98 RANS modeling owing to two primary reasons: (1) for $\alpha \gtrsim 0.1$, there is
 99 a phase difference between the stress and strain fields, with the phase dif-
 100 ference depending on α [28], and (2) the flow is inherently transitional (as
 101 described above), in particular at application relevant Re_s . The difficulty
 102 with (1) is that most EVM models assume that the stress and strain fields
 103 are in phase. The difficulty with (2) is that transitional flows and laminar-
 104 to-turbulent transition are not reliably predicted using conventional RANS
 105 modeling approaches, and thus require special treatment such as using low-
 106 Reynolds-number models, transitional models, and empirical correlations.
 107 [29, 30].

108 **3. Numerical Simulation of Reciprocating Channel Flow**

109 All simulations employ a cartesian domain defined by the orthonormal
 110 vector base $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ where boldface denotes a vector, and
 111 x , y and z are the streamwise, spanwise, and wall-normal directions, respec-
 112 tively. Throughout the manuscript, the index notation is used for complete
 113 equations and the coordinate (x, y, z) notation is used for plots. The com-
 114 ponents of the velocity vector $u_i = \mathbf{u}$ are u , v , and w in the x , y and z
 115 directions, respectively. The flow is incompressible,

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (1)$$

116 and driven by a cosinusoidal pressure gradient of amplitude $P_{x,0}$ at a pre-
 117 scribed frequency ω , given by the last term on the right hand side of the
 118 following momentum transport equation:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{dp}{dx_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{\rho} P_{x,0} \cos(\omega t) \delta_{i1}, \quad (2)$$

119 where ρ is the fluid density, p is pressure, and δ_{ij} is the Kronecker delta. The
 120 computational domain is periodic in the streamwise (x) and spanwise (y)
 121 directions. The wall boundary conditions are no-slip and no-through. The
 122 design of the forcing term yields a net-zero flow rate over a full period.

123 Temperature is incorporated in the simulation as a passive scalar with a
 124 transport equation:

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = \frac{\nu}{Pr} \frac{\partial^2 \theta}{\partial x_i^2}, \quad (3)$$

125 where θ is temperature and $Pr \equiv \nu/(\lambda/\rho Cp)$ is the Prandtl number, where
 126 λ is the fluid thermal conductivity and Cp is the fluid specific heat. In the
 127 present study $Pr = 0.7$. The temperature dynamics in the flow is driven by
 128 isothermal boundary conditions at the walls. The bottom wall is set to a
 129 non-dimensional temperature of 1 and the top wall to 0.

130 The study of Di Liberto and Ciofalo [26] was used to identify two Reynolds
 131 numbers: $Re_s = 648$ and $R_s = 1019$, with Womersley numbers of 20.47 and
 132 17.72, respectively. The former Re_s is at the edge of the interface between
 133 a disturbed laminar flow and an intermittently turbulent flow, whereas the
 134 latter is well within the intermittently turbulent regime and exhibits fully
 135 turbulent behaviors over a portion of the cycle. The statistics described in the
 136 present paper are phase-averaged over 10 periods. Statistical convergence is
 137 achieved after 10 periods and 15 periods for the DNS and RANS simulations,

138 respectively. A period is divided into 32 phases: $1\pi/16 \leq \phi \leq 32\pi/16$,
 139 where ϕ denotes the phase angle. Wall-normal distributions of velocity and
 140 temperature statistics are collected at each phase.

141 3.1. Direct Numerical Simulation (DNS)

142 DNS simulations are carried out in a domain of dimensions $L_x = 10h$,
 143 $L_y = 5h$ and $L_z = 2h$ in the streamwise, spanwise and wall-normal directions,
 144 respectively. These dimensions are identical to the computational domain
 145 used in [26], although the present resolution is higher with most simulations
 146 using 128^3 cells. Statistics collected on a higher resolution of 256^3 did not
 147 show appreciable deviation from the lower resolution simulations.

148 The DNS code is a well-validated finite difference code [31] using non-
 149 dissipative second-order schemes for spatial derivatives on staggered grid.
 150 The fractional step method enforces incompressibility. The time advance-
 151 ment is 3rd-order Runge-Kutta for the advection terms in all directions and
 152 the diffusive terms in the streamwise and spanwise directions. To relax the
 153 viscous stability constraint, the wall-normal diffusive term is advanced with
 154 a 2nd-order Crank-Nicolson scheme. All simulations are performed with a
 155 dynamic time step which satisfies the Courant-Friedrichs-Lewy condition,
 156 $\Delta t u_i / \Delta x_i \leq 0.9$, and a viscous stability condition, $\nu \Delta t / \min(\Delta x, \Delta y) < 0.2$.
 157 Both conditions are satisfied well below their maximum values for the nu-
 158 merical methods used here.

159 3.2. Reynolds Average Navier-Stokes (RANS)

160 The RANS simulations were carried out with OpenFOAM version 2.3.0
 161 using the PISO incompressible-transient flow solver in a 2D computational

domain with streamwise periodic boundary condition. The present study compares two low-Reynolds-number turbulence models. The low-Reynolds-number models require the first grid point to be within a viscous unit, i.e. at $z^+ \leq 1$, where the superscript $+$ denotes normalization by friction velocity $u_\tau = \sqrt{\tau_w/\rho}$ and kinematic viscosity ν , where τ_w is the shear stress at the wall. Computational grids were created to satisfy this condition for the maximum phase-averaged wall shear stress. A grid refinement study determined that the solution is grid independent for $N_z = 256$ for both Re_s .

In the RANS simulations Eqs (1-3) are phase-averaged and each variable is decomposed into its phase-average and its fluctuations: $a_i = A_i + a'_i$, where a denotes a generic variable, a capitalized variable denotes its phase average, and a prime denotes its fluctuations. The resulting governing equations correspond to:

$$\frac{\partial U_i}{\partial x_j} = 0, \quad (4)$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \left(\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial(\rho \overline{u'_i u'_j})}{\partial x_j} \right), \quad (5)$$

$$\frac{\partial \Theta}{\partial t} + U_i \frac{\partial \Theta}{\partial x_i} = \frac{\nu}{Pr} \frac{\partial^2 \Theta}{\partial x_i^2} - \frac{\partial(\overline{u'_i \theta'})}{\partial x_i}, \quad (6)$$

where μ is the dynamic viscosity and an overbar denotes a correlation. Equations (5) and (6) include two new terms which result from the averaging process, the turbulent or Reynolds stress $(-\rho \overline{u'_i u'_j})$ and the turbulent heat flux $(-\overline{u'_i \theta'})$, both require models for closure.

179 3.3. Low-Reynolds-number turbulence models

180 The two low-Reynolds-number models chosen for validation suffer from
181 both modeling difficulties described in §2. In particular, the EVM models do
182 not account for the phase difference between the stress and strain fields and,
183 with the exception of resolving the wall-region, no transitional flow modeling
184 techniques were employed. Moreover, Reynolds analogy is used to compute
185 the scalar temperature field. While there are more sophisticated turbulence
186 models that are better suited to model reciprocating channel flow [29, 32, 33],
187 in particular with heat transfer [34], the models investigated here were pur-
188 posefully chosen so that they do not reliably predict the flow behaviors in
189 reciprocating channel flow. Hence, the value of the integral validation tech-
190 nique can be better demonstrated in comparison to the standard validation
191 technique.

192 The specific models are the widely used Launder-Sharma (LS) k - ε [15]
193 model and the v^2 - f model [35, 36]. Both models use the transport of aver-
194 age turbulent kinetic energy (k) and the rate of dissipation of the average
195 turbulent kinetic energy (ε). The transport equations for both low-Reynolds-
196 number models are presented below. Similar to base EVM presentation, the
197 subscripted C 's denote a model constant and the subscripted σ 's denote a
198 turbulent Prandtl number, where the subscript serves as an identifier that
199 typically contains an associated variable. The LS model has the following
200 transport equations and turbulent viscosity (ν_T):

$$\begin{aligned} \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = & \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] + \\ & \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} - (\tilde{\varepsilon} + D), \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial \tilde{\varepsilon}}{\partial t} + U_j \frac{\partial \tilde{\varepsilon}}{\partial x_j} = & \frac{\partial}{\partial x_j} \left(\frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \tilde{\varepsilon}}{\partial x_j} \right) + C_{\varepsilon 1} \frac{\tilde{\varepsilon}}{k} \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \\ & - C_{\varepsilon 2} f_2 \frac{\tilde{\varepsilon}^2}{k} + E, \end{aligned} \quad (8)$$

$$\nu_T = C_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}}, \quad (9)$$

$$\varepsilon = \tilde{\varepsilon} + D, \quad D = 2\nu \left(\frac{\partial \sqrt{k}}{\partial z} \right)^2 \approx 2\nu \left(\nabla \sqrt{k} \right)^2, \quad (10)$$

$$f_\mu = \exp \left(\frac{-3.4}{(1 + Re_T/50)^2} \right), \quad f_2 = 1 - 0.3 \exp(-Re_T^2), \quad (11)$$

$$E = 2\nu\nu_T \left(\frac{\partial^2 U}{\partial z^2} \right)^2, \quad Re_T = \frac{k^2}{\nu \tilde{\varepsilon}}. \quad (12)$$

$$\sigma_k = 1, \quad \sigma_\varepsilon = 1.3, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad C_\mu = 0.09.$$

201 The v^2 - f model (effectively the w^2 - f model in the present coordinate
 202 system) uses the standard (k) and (ε) equations of a high-Reynolds-number
 203 model as well as the transport of the wall normal fluctuating velocity and an
 204 elliptic relaxation function (f):

$$\begin{aligned} \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = & \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \\ & + \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} - \varepsilon, \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = & \frac{\partial}{\partial x_j} \left(\frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_j} \right) \\ & + C_{\varepsilon 1} \frac{\varepsilon}{k} \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}, \end{aligned} \quad (14)$$

$$\frac{\partial \overline{w'^2}}{\partial t} + U_j \frac{\partial \overline{w'^2}}{\partial x_j} = kf - \frac{\overline{w'^2}}{k} \varepsilon + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_T}{\sigma_{w'^2}} \right) \frac{\partial \overline{w'^2}}{\partial x_j} \right], \quad (15)$$

$$L^2 \nabla^2 f - f = \frac{C_1 - 1}{\Lambda} \left(\frac{\overline{w'^2}}{k} - \frac{2}{3} \right) - C_2 \frac{P_k}{\varepsilon}, \quad (16)$$

$$\nu_T = C_\mu \overline{w'^2} \Lambda, \quad P_k = \nu_T C_\mu k^2 / \epsilon \quad (17)$$

$$L = C_L \max \left[\frac{k^{3/2}}{\varepsilon}, C_\eta \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} \right], \quad \Lambda = \max \left[\frac{k}{\varepsilon}, C_T \left(\frac{\nu}{\varepsilon} \right)^{1/2} \right]. \quad (18)$$

$\sigma_k = 1,$	$\sigma_\varepsilon = 1.3,$	$C_{\varepsilon 1} = 1.44,$	$C_{\varepsilon 2} = 1.9,$	$\sigma_{w'^2} = 1,$	$C_1 = 1.4$
$C_2 = 0.3,$	$C_\mu = 0.22,$	$C_L = 0.25,$	$C_\eta = 85,$	$C_T = 6.$	

205 As is seen in equations (8) and (9), the LS model requires the use of
206 damping functions listed in equation (11). These functions are necessary to

207 accurately damp the turbulent quantities as they approach the wall otherwise
 208 unrealistic values would be predicted. The v^2 - f model on the other hand does
 209 not require the use of damping functions, which is an important characteristic
 210 of the model. In its formulation, the use of the fluctuating wall normal
 211 velocity $\overline{w'^2}$ provides the correct scaling for the damping of turbulence near
 212 the wall, hence the use of damping functions is removed. For a more in depth
 213 description of the v^2 - f model see [35, 37].

214 3.4. Turbulent Heat Flux Closure

215 In the present work, the turbulent heat flux ($-\overline{u'_i\theta}$) is closed using the sim-
 216 ple eddy diffusivity approach much like that of the eddy viscosity approach
 217 for the turbulent stress:

$$-\overline{u'_i\theta} = \frac{\nu_T}{Pr_T} \frac{\partial \Theta}{\partial x_i}. \quad (19)$$

218 A main component of this closure is the Reynolds analogy, which assumes
 219 that the turbulent momentum flux is proportional to the turbulent heat flux.
 220 The Reynolds analogy is incorporated in the RANS simulations by the use of
 221 a turbulent Prandtl number ($Pr_T = \nu_T/\alpha_T$), where α_T is the turbulent ther-
 222 mal diffusivity. For the RANS simulations of reciprocating channel flow with
 223 heat transfer $Pr_T=0.9$, which is a common assumption in many engineer-
 224 ing applications and a reasonable assumption for the isothermal boundary
 225 conditions of the present study.

Figure 1: Wall-normal profiles of the phase-averaged streamwise velocity difference between the RANS turbulence models and the DNS normalized by static friction velocity for $Re_s = 648$ (panel 1 and 2) and $Re_s = 1019$ (panel 3 and 4) for accelerating phases (panel 1 and 3) and decelerating phases (panel 2 and 4). Phase increases from bottom-to-top in a panel and the corresponding phase is written above the curves. The line styles correspond to: $---$ v^2-f , $---$ LS .

226 4. Standard Validation Technique

227 Phase-averaged wall-normal profiles and wall fluxes computed from the
 228 RANS simulations are first compared to the DNS profiles following the stan-
 229 dard validation technique. For ease of interpretation, and when appropriate,
 230 data is presented only for a half-cycle (owing to anti-symmetry) and sepa-
 231 rated between the accelerating phases and decelerating phases of the cycle.

232 The phase-averaged streamwise velocity difference profiles between the RANS
 233 turbulence models and the DNS ($U_{RANS} - U_{DNS}$) normalized by static friction
 234 velocity ($u_{\tau_0} \equiv \sqrt{\tau_0/\rho}$ where $\tau_0 = h|dP/dx|_{max} = hP_{x,0}$ is the static shear
 235 stress and $P_{x,0}$ is the amplitude of the cosinusoidal pressure gradient) for
 236 $Re_s=648$ and $Re_s=1019$ are shown in Fig. 1. In general, and not unexpected,
 237 is that the performance of the RANS turbulence models vary with phase and
 238 Re_s . The largest differences are observed from $6\pi/16 \leq \phi \leq 11\pi/16$ when
 239 the cross-sectional averaged flow is transitioning from an accelerating flow
 240 to a decelerating flow. In general, the two turbulence models differ between
 241 each other in the near-wall region at several phases in the cycle, in particular
 242 during the decelerating phases for $Re_s = 1019$.

243 The phase-averaged Reynolds stress difference profiles ($\overline{u'w'}_{RANS} - \overline{u'w'}_{DNS}$)
 244 normalized by $u_{\tau_0}^2$ for $Re_s = 648$ and $Re_s = 1019$ are shown in Fig. 2. For
 245 both Re_s , the largest differences between the DNS and the RANS turbulence
 246 models occur in the near-wall region from $6\pi/16 \leq \phi \leq 11\pi/16$. In addi-
 247 tion, the differences for $Re_s = 1019$ are greater than for $Re_s = 648$. The
 248 two RANS models are better correlated with each other compared to their
 249 correlation for mean velocity.

250 The normalized wall shear stress, τ_w/τ_0 , computed from the DNS and
 251 RANS simulations are plotted in Fig. 3 as a function of ϕ for $Re_s = 648$ (left
 252 panel) and $Re_s = 1019$ (right panel). Here τ_w is computed from the velocity
 253 gradient at the wall using a finite difference method. For $Re_s = 1019$, the
 254 RANS models capture the rapid rise in τ_w but the peak magnitude is lower
 255 and the rise occurs at a phase-lead compared to the DNS. Since it is during
 256 these phase angles ($6\pi/16 \lesssim \phi \lesssim 11\pi/16$) that the flow transitions to a

Figure 2: Reynolds shear stress difference profiles between the RANS turbulence models and the DNS normalized by $u_{\tau 0}^2$. Panel layout and line styles are the same as in Fig. 1.

257 turbulent flow, the phase-lead suggests that transition occurs at an earlier
 258 phase angle compared to the DNS. A similar rise in τ_w predicted by the
 259 RANS models for $Re_s = 648$ but not observed in the DNS is consistent with
 260 the idea that transition is predicted too early by the RANS models.

261 The analog plots to those shown in Figs. 1–3 with respect to the thermal
 262 field are shown in Figs. 4–6. The phase-averaged mean temperature difference
 263 profiles between the RANS turbulence models and the DNS ($\Theta_{RANS} - \Theta_{DNS}$)
 264 normalized by the difference between the centerline and wall mean tem-
 265 perature ($\Theta_{cl} - \Theta_w$) are shown in Fig. 4. Surprisingly, the RANS models

Figure 3: Phase-averaged normalized wall shear stress plotted as function of ϕ for $Re_s = 648$ (left panel) and $Re_s = 1019$ (right panel). The black solid line denotes the DNS. The line styles for the RANS turbulence models are the same as in Fig. 1.

show larger differences for $Re_s = 648$ compared to $Re_s = 1019$. The phase-averaged turbulent heat flux difference profiles $(\overline{w'\theta'}_{RANS} - \overline{w'\theta'}_{DNS})$ normalized by $u_{\tau 0}(\Theta_{cl} - \Theta_w)$ are shown in Fig. 5. Independent of Re_s , the RANS models shows the largest differences in the near-wall region from $6\pi/16 \leq \phi \leq 11\pi/16$, similar to the Reynolds shear stress differences.

The phase-averaged wall heat flux, non-dimensionalized in the form of a Nusselt (Nu) number:

Figure 4: Wall-normal profiles of the phase-averaged temperature difference between the RANS turbulence models and the DNS normalized by the difference between the phase-averaged centerline temperature and wall temperature. Panel layout and line styles are the same as in Fig. 1.

$$Nu = \frac{q_w''}{\lambda(\Theta_{cl} - \Theta_w)/h}, \quad (20)$$

273 plotted as a function of ϕ computed from the DNS and RANS simulations
 274 at $Re_s = 648$ (left panel) and $Re_s = 1019$ (right panel) are shown in Fig. 6.
 275 The RANS models do a reasonable job of reproducing the DNS period and
 276 magnitude of Nu . At $Re_s=1019$, both low-Reynolds-number models show
 277 an overshoot spike near the end of the decelerating portion of the cycle,

Figure 5: Turbulent heat flux difference profiles between the RANS turbulence models and the DNS normalized by $u_{\tau 0}(\Theta_{cl} - \Theta_w)$. Panel layout and line styles are the same as in Fig. 1.

although the spike for the LS model is much larger than that for the v^2-f model. Lastly, similar to τ_w both low-Reynolds-number models phase lead the DNS during the later phases of the accelerating portion of the cycle.

The wall flux predictive capabilities of the RANS turbulence models are quantified by integrating the magnitude difference between the phase-averaged wall flux values computed from the DNS and the RANS simulations over a full-cycle normalized by the cycle-averaged magnitude of the wall-flux:

Figure 6: Phase-averaged normalized heat flux (i.e., Nu) plotted as function of ϕ for $Re_s = 648$ (left panel) and $Re_s = 1019$ (right panel). The black solid line denotes the DNS. The line styles for the RANS turbulence models are the same as in Fig. 1.

$$\Delta \widehat{\tau}_w = \frac{1}{2\pi \langle |\tau_w| \rangle} \int_0^{2\pi} |\tau_{wRANS} - \tau_{wDNS}| d\phi, \quad (21)$$

$$\Delta \widehat{Nu} = \frac{1}{2\pi \langle Nu \rangle} \int_0^{2\pi} |Nu_{RANS} - Nu_{DNS}| d\phi, \quad (22)$$

285 where $|\cdot|$ denotes an absolute magnitude and $\langle \cdot \rangle$ denotes a cycle-average.
 286 The values of $\Delta \widehat{\tau}_w$ and $\Delta \widehat{Nu}$ for the two RANS turbulence models at both
 287 Re_s are tabulated in Table 1. The tabulated results are generally consistent

Table 1: Tabulated values of $\Delta\widehat{\tau}_w$ and $\Delta\widehat{Nu}$ given by Eq. 21 and Eq. 22, respectively, for the two RANS turbulence models.

Model	$\Delta\widehat{\tau}_w \times 100$		$\Delta\widehat{Nu} \times 100$	
	$Re_s = 648$	$Re_s = 1019$	$Re_s = 648$	$Re_s = 1019$
<i>LS</i>	16.7	18.7	19.1	23.3
<i>v²-f</i>	20.3	20.0	17.8	12.9

with a qualitative evaluation of Figs. 3 and 6. Specifically, the $\Delta\widehat{\tau}_w$ and $\Delta\widehat{Nu}$ metric shows that both RANS models predict the wall fluxes to within approximately 20% of the DNS. The metric also shows that the *v²-f* model performs the best at $Re_s = 1019$. The apparent poor performance of the *LS* model in predicting Nu , however, is primarily owed to the large spike at $\phi = 15\pi/16$. It follows that the general conclusion from this standard validation technique is that *v²-f* and *LS* models in conjunction with Reynolds analogy do a reasonable job of modeling the flow physics of reciprocating channel flow with heat transfer at both $Re_s = 648$ and $Re_s = 1019$. Moreover, model improvements can be achieved by incorporating transitional modeling strategies that delay transition.

5. Integral Validation Technique

The wall shear stress and wall heat flux are not local quantities but depend on the integrated effects of the flow dynamics above the wall. Consequently, a comparison of the wall flux computed from a RANS simulation to that computed from a benchmark dataset does not provide sufficient information on how well the turbulence model predicts the flow dynamics. In

305 particular, while such a comparison can identify model failures, it cannot
 306 identify false-positives owed to the cancellation of model errors. Moreover,
 307 such a comparison cannot be used to provide physics-based guidance on how
 308 to improve a model. The integral validation technique presented here pro-
 309 vides a direct connection between wall fluxes and mean flow dynamics, thus
 310 providing the necessary means to evaluate if a model correctly predicts the
 311 flow physics. In turn, providing needed information critical to the improved
 312 development of turbulence models.

313 The integral validation technique is derived following the work of Fuka-
 314 gata *et al.* [38] and the complementary works of Mehdi *et al.* [39, 40] and
 315 Ebadi *et al.* [41]. Integral expressions for the wall shear stress and wall heat
 316 flux are obtained by thrice integrating the momentum and scalar temperature
 317 transport equation in the wall-normal direction, respectively. The integral
 318 expressions for τ_w and q_w'' provided here are given in their most general form¹.
 319 In addition, following Mehdi *et al.* and Ebadi *et al.*, the streamwise gradient
 320 terms and unsteady term have been replaced by mathematically equivalent
 321 wall-normal gradient terms.

322 For a two-dimensional, wall-bounded turbulent flow of an incompressible
 323 fluid (here temperature is modeled as a passive scalar), neglecting heat trans-
 324 fer from viscous dissipation, the wall shear stress and wall heat flux can be
 325 expressed in integral form as

¹Alternative forms of the expressions are given in Appendix A

$$\tau_w = \underbrace{\frac{2\mu}{z_t^2} \int_0^{z_t} U dz}_I - \underbrace{\frac{2\rho}{z_t^2} \int_0^{z_t} (z_t - z) \overline{u'w'} dz}_{II} - \underbrace{\frac{\rho}{z_t^2} \int_0^{z_t} (z_t - z)^2 \frac{\partial}{\partial z} \left(\nu \frac{\partial U}{\partial z} - \overline{u'w'} \right) dz}_{III}, \quad (23)$$

326 and

$$q_w'' = \underbrace{\frac{2\lambda}{z_t^2} \int_0^{z_t} (\Theta_w - \Theta) dz}_{I^*} + \underbrace{\frac{2\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z) \overline{w'\theta'} dz}_{II^*} + \underbrace{\frac{\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z)^2 \frac{\partial}{\partial z} \left(\frac{\lambda}{\rho C_p} \frac{\partial \Theta}{\partial z} - \overline{w'\theta'} \right) dz}_{III^*}, \quad (24)$$

327 where z_t is an arbitrary height in the wall-normal direction. The equivalent
 328 integral relations for the RANS turbulence models are obtained by replacing
 329 the turbulent flux in terms II, II^*, III, III^* by the RANS modeling equiv-
 330 alent:

$$-\overline{u'w'} = \nu_T \frac{\partial U}{\partial z} \text{ and } -\overline{w'\theta'} = \frac{\nu_T}{Pr_T} \frac{\partial \Theta}{\partial z}.$$

331 Equations 23 and 24 each consist of three contributing terms to the wall
 332 flux: I, I^* is the contribution from the mean velocity or temperature pro-
 333 file, respectively; II, II^* is the contribution from the turbulent flux; and
 334 III, III^* is the contribution from the gradient of the total (i.e., molecular

335 plus turbulent) flux, or equivalently the substituted terms (e.g., in recipro-
 336 cating channel flow, term III is effectively the sum of the unsteady velocity
 337 and pressure gradient terms and term III^* is the unsteady temperature term,
 338 see Appendix A). The $(z_t - z)$ or $(z_t - z)^2$ weightings place more emphasis on
 339 the near-wall values and, consequently, as $z_t \rightarrow h$ (or, alternatively, bound-
 340 ary layer thickness δ), term I, I^* decreases while terms II, II^* and III, III^*
 341 increase. Importantly, the decomposition provides two major advantages:
 342 (a) it provides a means to connect the wall flux to the mean flow dynamics
 343 and (b) the expressions are amenable to experimental validation of turbu-
 344 lence models requiring only measurements of wall-normal profiles of mean
 345 variables (velocity or temperature) and turbulent flux at one streamwise lo-
 346 cation up to an arbitrary height z_t [39–41].

347 The phase-averaged contributions to τ_w/τ_0 from the three terms in Eq. 23
 348 when integrated to the channel half-height (i.e., $z_t = h$) are plotted in Fig. 7
 349 for $Re_s = 648$ (left panel) and $Re_s = 1019$ (right panel). First inspecting
 350 the DNS data, consistent between the two periods is that the contribution
 351 from term I is small (note the scale is one order-of-magnitude smaller than
 352 the others), representing approximately 10% or less of the total wall-shear
 353 stress at all phases. Relative to the sign of τ_w , term I follows with a slight
 354 phase-lag. This phase lag is smaller for $Re_s = 1019$ compared to $Re_s = 648$.
 355 For $Re_s = 648$, term II is of opposite sign of τ_w during the first quarter-
 356 period (i.e., accelerating phases) and the same sign during the second quarter-
 357 period (i.e., decelerating phases). For $Re_s = 1019$, the sign change occurs
 358 at $\phi \approx 6\pi/16$. Term III , however, changes sign relative to τ_w at a phase =
 359 $12\pi/16$ for both Re_s . Observe that at the phase angle where $\tau_w = 0$ (i.e., flow

Figure 7: Contribution of term I (2nd row), II (third row) and III (fourth row) in the right hand side of Eq. 23 plotted as a function of ϕ for $Re_s = 648$ (left panel) and $Re_s = 1019$ (right panel). The top row is τ_w/τ_0 shown for reference. Note that at any ϕ the sum of I, II, III equals τ_w/τ_0 . Panel layout and line styles are the same as in Fig. 3.

360 reversal at the wall), the contributing terms to the wall-shear stress are all
 361 finite but sum to zero. Also note that since term II and III are the dominant
 362 terms, the reliable modeling of the turbulent flux term, in particular in the
 363 near-wall region, is critical to yield an accurate prediction of the wall shear
 364 stress.

365 The RANS models show the largest differences compared to the DNS
 366 for $Re_s = 1019$. Term II is slightly overpredicted during the accelerating
 367 phases of the cycle and underpredicted during the early phases of deceleration

Figure 8: Contribution of term I^* (2nd row), II^* (third row) and III^* (fourth row) in the right hand side of Eq. 24 plotted as a function of ϕ for $Re_s = 648$ (left panel) and $Re_s = 1019$ (right panel). The top row is Nu shown for reference, where $\Delta\Theta = \Theta_{cl} - \Theta_w$. Note that at any ϕ the sum of I^*, II^*, III^* equals Nu . Panel layout and line styles are the same as in Fig. 6.

368 $(9\pi/16 \leq \phi \leq 11\pi/16)$. This indicates that the RANS modeled turbulent
369 flux term is overpredicted at the phase angles when the flow is transitional but
370 underpredicted when the flow is turbulent. Term III is overpredicted during
371 the late phases of acceleration $(6\pi/16 \leq \phi \leq 8\pi/16)$. Owing to substitution
372 (see Appendix A) and given that the pressure gradient is prescribed, the
373 underprediction of term III indicates that the unsteady term $(\partial U/\partial t)$ is not
374 reliably predicted. The error in predicting the unsteady term is of course a

375 consequence of not reliably modeling the turbulent flux term since the two
 376 terms are coupled by the momentum equation. The sum of the differences in
 377 term *II* and *III* is observed in the predicted τ_w , where the RANS turbulence
 378 models capture the rapid rise in τ_w , but the peak magnitude is lower and
 379 the rise occurs at a phase-lead compared to the DNS. It follows that the
 380 phase-lead is due to an overprediction of the contribution from the unsteady
 381 term (term *III*) during the late stages of the accelerating portion of the
 382 cycle, and the lower peak magnitude is due to an underprediction of the
 383 contribution from the turbulent flux term (term *II*) during the early phases of
 384 the deceleration portion of the cycle when the flow transitions to a turbulent
 385 flow.

386 The phase-averaged contributions to Nu from the three terms in Eq. 24
 387 when integrated to the channel half-height (i.e., $z_t = h$) are plotted in Fig. 8
 388 for $Re_s = 648$ (left panel) and $Re_s = 648$ (right panel). The DNS data shows,
 389 as expected, that the period of each term is half the forcing period (unlike the
 390 terms for τ_w which have a period equal to the forcing period). The dominant
 391 contribution is term II^* , the contribution from term I^* is small, and term
 392 II^* and III^* are π radians out of phase.

393 Compared to τ_w , the RANS turbulence models perform far worse in pre-
 394 dicting the contributing terms to Nu . Surprisingly, the RANS model better
 395 predict term I^* at the higher Re_s . For $Re_s = 1019$, both RANS models have
 396 the correct phase for term *II* but significantly under estimate its contribu-
 397 tion during the early phases of deceleration ($9\pi/16 \leq \phi \leq 11\pi/16$) when the
 398 flow transitions to turbulence. Conversely, during the early phases of accel-
 399 eration when the flow is transitional, the RANS models slightly overpredict

400 the contribution from term II . It follows that the turbulent flux term in the
 401 RANS models is significantly underestimated when the flow transitions to
 402 turbulence. The largest differences are observed in term III where at both
 403 Re_s the contribution from the RANS models are π radians out of phase and
 404 from $6\pi/16 \leq \phi \leq 16\pi/16$ of opposite sign compared to the DNS. Since in
 405 reciprocating channel flow term III^* is effectively the unsteady temperature
 406 term (i.e., $\partial\Theta/\partial t$), these results suggest that the unsteady temperature term
 407 is not reliably predicted.

408 The error in predicting the contribution from the unsteady term is a
 409 consequence of not reliably modeling the turbulent flux term since the two
 410 are coupled by the heat equation. Specifically, from $7\pi/16 \leq \phi \leq 16\pi/16$ the
 411 RANS modeled wall heat flux is comparable to the DNS computed wall heat
 412 flux. The contribution from the RANS modeled turbulent heat flux (term
 413 II^*) is, however, significantly underestimated. Consequently, to balance the
 414 diffusion term, the contribution from the unsteady temperature term must
 415 be positive (of opposite sign of the DNS). In effect, the unsteady temperature
 416 profile computed from the RANS simulation must behave very differently (in
 417 shape, sign, and magnitude) compared to the DNS (this has been verified
 418 but not shown here). Importantly, the differences in terms II^* and III^*
 419 computed from the RANS models serendipitously cancel to yield a reasonable
 420 prediction of the wall heat flux compared to the DNS.

421 A similar metric as used for the standard validation technique is used
 422 here to quantify the differences between the phase-averaged RANS predicted
 423 integral terms in Eqs. 23 and 24 compared to the DNS:

$$\Delta \hat{i} = \frac{1}{2\pi \langle |i| \rangle} \int_0^{2\pi} |i_{RANS} - i_{DNS}| d\phi, \quad (25)$$

424 where $i = \{I, I^*, II, II^*, III, III^*\}$. The values of $\Delta \hat{I}, \Delta \hat{I}^*$; $\Delta \hat{II}, \Delta \hat{II}^*$; and
 425 $\Delta \hat{III}, \Delta \hat{III}^*$ are tabulated in Table 2, 3, and 4, respectively. The tabu-
 426 lated results are consistent with the qualitative evaluation of Figs. 7 and 8
 427 and their accompanying descriptions. Specifically, term I is generally well-
 428 predicted. Term I^* is better predicted at the higher Re_s . Note that the
 429 good prediction of term I, I^* does not necessarily mean that the U and Θ
 430 profiles are well predicted, it only means that the integral contributions to
 431 the wall-flux from U and Θ are well-predicted. The error in term II, II^* are
 432 comparable between the two RANS models and on the order of 20-30% indi-
 433 cating that the turbulent models are not accurately modeling the turbulent
 434 flux terms. Term III is generally reasonably predicted by both RANS mod-
 435 els with an error of ≈ 10 -15% Surprisingly, despite the reasonable prediction
 436 of term III , term III^* is poorly predicted by both RANS models with an
 437 error greater than 100%. Since term III^* is effectively the unsteady temper-
 438 ature term, this indicates that the RANS models do not reliably model the
 439 unsteady temperature term in the heat equation. Understanding the root
 440 cause of these failures is a necessary first step needed to improve the exist-
 441 ing models or seek out RANS models better suited to model reciprocating
 442 channel flow with heat transfer.

443 In summary, while a validation based on a comparison of Nu alone would
 444 suggest that the LS and v^2 - f models perform reasonably well, the integral
 445 validation technique demonstrates that the RANS turbulence models do not
 446 accurately capture the underlying flow physics. In fact, the reasonable pre-

Table 2: Tabulated values of $\Delta\widehat{I}$ and $\Delta\widehat{I}^*$ given by Eq. 25 for the two RANS turbulence models.

	$\Delta\widehat{I} \times 100$		$\Delta\widehat{I}^* \times 100$	
Model	$Re_s = 648$	$Re_s = 1019$	$Re_s = 648$	$Re_s = 1019$
<i>LS</i>	1.6	2.5	9.2	0.8
v^2 -f	2.1	2.8	11.0	0.7

Table 3: Tabulated values of $\Delta\widehat{II}$ and $\Delta\widehat{II}^*$ given by Eq. 25 for the two RANS turbulence models.

	$\Delta\widehat{II} \times 100$		$\Delta\widehat{II}^* \times 100$	
Model	$Re_s = 648$	$Re_s = 1019$	$Re_s = 648$	$Re_s = 1019$
<i>LS</i>	28.1	25.7	17.0	28.6
v^2 -f	35.6	22.1	28.7	28.8

Table 4: Tabulated values of $\Delta\widehat{III}$ and $\Delta\widehat{III}^*$ given by Eq. 25 for the two RANS turbulence models.

	$\Delta\widehat{III} \times 100$		$\Delta\widehat{III}^* \times 100$	
Model	$Re_s = 648$	$Re_s = 1019$	$Re_s = 648$	$Re_s = 1019$
<i>LS</i>	11.8	12.6	125.2	122.8
v^2 -f	15.8	15.7	126.3	115.3

447 diction of Nu is only owed to the cancellation of errors. The technique
448 identified that the contribution from the turbulent flux terms (II, II^*) are
449 overpredicted at the phase angles when the flow is transitional but underpre-

dicted when the flow is turbulent. In addition, the technique also identified that the phase-lead of the RANS models compared to the DNS during the rapid rise in τ_w is due to the overprediction of the contribution from the unsteady term. Importantly, it follows that the evolution of the unsteady term in the momentum equation is likely an important parameter to study to better understand transition to turbulence in reciprocating channel flow. The technique also identified that the unsteady temperature term is poorly modeled from the late-stages of the acceleration portion of the cycle through the decelerating portion of the cycle. Lastly, the reasonable success in predicting the integral terms for τ_w and the failure in predicting the terms for Nu is strong evidence that at least one issue in the RANS modeling lies in the assumption that the turbulent Prandtl number is constant.

6. Reynolds Analogy

The fact that the RANS turbulence models better predict the contributing integral terms to τ_w compared to q_w'' is suggestive that the turbulent heat flux closure is not properly modeled. Specifically, since Reynolds analogy is employed in the RANS numerical simulations, it suggests that Reynolds analogy may not hold in reciprocating channel flow. The potential breakdown of Reynolds analogy in reciprocating channel flow is, therefore, briefly explored here.

The Reynolds analogy relies on the assumption that there exists a turbulent Prandtl number Pr_T , which relates the momentum scales to the thermal scales. In standard modeling practice, Pr_T is assumed to be a constant across the flow domain. This assumption is relatively valid in steady equilibrium

474 flow but is known to breakdown in non-equilibrium flow [42–44]. Import-
 475 tantly, to assess the validity of Reynolds analogy, DNS allows for the Pr_T to
 476 be computed directly:

$$Pr_T = \frac{\nu_T}{\alpha_T} = \frac{\overline{u'w'}(\partial\Theta/\partial z)}{\overline{w'\theta'}(\partial U/\partial z)}. \quad (26)$$

477 It follows that examination of Pr_T profiles allows for a direct evaluation of
 478 the validity of Reynolds analogy. For example, Fig. 9 shows wall-normal
 479 profiles of Pr_T in steady-state turbulent channel flow at two $Re_\tau = u_\tau h/\nu$.
 480 Evident is that Pr_T is approximately constant across the channel height.

Figure 9: Pr_T profiles as a function of the distance from the wall in a steady channel flow, with temperature as a passive scalar. Figure courtesy of Samir Sid.

Figure 10: Pr_T profiles as a function of distance from the wall for $Re_s = 648$ (left panel) and $Re_s = 1019$ (right panel) at a few select phases in the cycle. The horizontal dashed lines correspond to $Pr_T=1$.

481 Wall-normal profiles of Pr_T computed from the DNS of reciprocating
482 channel flow are shown in Fig. 10 for $Re_s = 648$ (left panel) and $Re_s = 1019$
483 (right panel) at a few select phases in the cycle. Apparent is that Pr_T
484 is highly variable both across the channel half-height as well as between
485 different phases. With the exception of a few decelerating phases at $Re_s =$
486 1019, the Pr_T does not approximate a constant across the flow domain. In
487 particular, the Pr_T profiles are complex and of generally similar shape for
488 many of the phases. For both Re_s and during the accelerating phases of the
489 cycle, Pr_T decreases with increasing z in the inner-region of the flow with a

490 phase-dependent z -position zero crossing followed by a rapid decrease/rapid
 491 increase (with a second zero crossing). Pr_T then approaches either a constant
 492 value, decays to zero, or has some complex behavior as the channel centerline
 493 is approached. The first zero crossing is where $\overline{u'w'}$ changes sign and the rapid
 494 sign change occurs at the zero crossing of $\partial U/\partial z$. It is apparent from Fig. 10
 495 that Reynolds analogy is a flawed assumption in reciprocating channel flow.

496 The underlying failure of Reynolds analogy is most likely caused by the
 497 combined effects of the flow being intermittently turbulent (i.e., transitional
 498 like at most phases) and forced by an unsteady pressure gradient. Regarding
 499 the latter effect, as described by Bradshaw and Bradshaw and Huang [43, 45],
 500 Reynolds analogy may breakdown in flows with imposed pressure gradients
 501 because the velocity field depends on the pressure field while the temperature
 502 field does not (explicitly) depend on the pressure field. It follows that the
 503 inability of the RANS turbulence models to accurately predict the contribut-
 504 ing integral terms to Nu is most likely due to a combination of deficiencies in
 505 the turbulence models employed and the use of Reynolds analogy. It follows
 506 that reliable prediction of the flow behaviors in reciprocating channel flow
 507 with heat transfer requires an independent numerical solution of the energy
 508 equation or a more sophisticated representation of Pr_T .

509 7. Discussion

510 DNS of reciprocating channel flow with heat transfer was used to evalu-
 511 ate the performance of two low-Reynolds-number RANS turbulence models.
 512 Two validation techniques were used in the evaluation: (1) the so-called stan-
 513 dard validation technique, where results from RANS simulations of averaged

514 variable profiles and wall fluxes are compared to the DNS dataset, and (2)
 515 an integral validation technique, where the contributing terms to the wall
 516 shear stress and wall heat flux are compared between the RANS results and
 517 the DNS dataset. The contributing terms to the wall shear stress and wall
 518 heat flux used in the integral validation technique were obtained by thrice
 519 integrating the momentum and scalar temperature transport equation, re-
 520 spectively.

521 The standard validation technique indicated that the RANS models rea-
 522 sonably predicted the transport of momentum and thermal energy in recip-
 523 roating channel flow. Specifically, the mean variable profiles predicted by
 524 the RANS simulations qualitatively agreed reasonably well with the DNS
 525 profiles. Moreover, using the wall flux of momentum and wall flux of heat
 526 as validation metrics, both RANS models predicted the wall-shear stress and
 527 wall-heat flux to within approximately 20% of the DNS results as quantified
 528 by Eqs. 21 and 22, respectively. The largest differences observed between
 529 the RANS models and the DNS were from $6\pi/16 \leq \phi \leq 11\pi/16$, when the
 530 flow transitions from an accelerating flow to a decelerating flow. Since it is
 531 during these phases that the flow transitions to a turbulent flow, it can be
 532 concluded that the RANS turbulence models do not accurately capture the
 533 transition. With respect to the wall fluxes, this modeling deficiency mani-
 534 fested itself in a phase-lead of the RANS modeled wall fluxes compared to
 535 the DNS computed wall fluxes.

536 The results of the integral validation technique, however, provided a very
 537 different assessment of the performance of the RANS turbulence models.
 538 The most striking result is that while the RANS models performed reason-

ably well in predicting the contributing terms to the wall shear stress, the models performed rather poorly in predicting the contributing terms to the wall heat flux. Specifically, the contributions from the turbulent heat flux (term II^*) were underpredicted by as much as approximately 40% during the decelerating phases of the cycle. In addition, the contribution from the gradient of the total heat flux (term III^* , which is effectively the contribution from the unsteady temperature), was of opposite sign and π radians out-of-phase compared to the DNS. Importantly, the differences in term II^* and III^* serendipitously canceled each other yielding a reasonable prediction of the wall heat flux compared to the DNS. Consequently, despite a reasonable agreement in predicting the wall heat flux, the low-Reynolds-number models do not accurately model the transport of thermal energy in reciprocating channel flow. This important conclusion likely would have been missed using the standard validation technique. In addition, the integral validation technique identified the root causes of the modeling failures, namely (a) underprediction of the contribution of turbulent flux term during the decelerating portion of the cycle, (b) inability to reliably predict the unsteady velocity and temperature fields, (c) the breakdown of Reynolds analogy in reciprocating channel flow. This information is critical to the improved development of turbulence models.

559

560 **Appendix A. Alternative forms of the integral expressions.**

561 The integral expressions for τ_w and q_w'' are derived following the work of
562 Fukagata *et al.* [38] and the complementary works of Mehdi *et al.* [39, 40]

563 and Ebadi *et al.* [41]. The derivation involves thrice integrating the RANS
 564 momentum and scalar temperature transport equations in the wall-normal
 565 direction.

566 The most general form of the expressions are

$$\begin{aligned} \tau_w = & \frac{2\mu}{z_t^2} \int_0^{z_t} U dz - \frac{2\rho}{z_t^2} \int_0^{z_t} (z_t - z) \overline{u'w'} dz \\ & - \frac{\rho}{z_t^2} \int_0^{z_t} (z_t - z)^2 I_x dz - \frac{\rho}{z_t^2} \int_0^{z_t} (z_t - z)^2 \frac{\partial U}{\partial t} dz - \frac{1}{\rho} \frac{\partial P}{\partial x} \frac{z_t^3}{6}. \end{aligned} \quad (\text{A.1})$$

567 and

$$\begin{aligned} q_w'' = & \frac{2\lambda}{z_t^2} \int_0^{z_t} (\Theta_w - \Theta) dz + \frac{2\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z) \overline{w'\theta'} dz \\ & + \frac{\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z)^2 G dz + \frac{\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z)^2 \frac{\partial \Theta}{\partial t} dz, \end{aligned} \quad (\text{A.2})$$

568 where $I_x = U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z} - \frac{\partial}{\partial x} (\nu \frac{\partial U}{\partial x} - \overline{u'^2})$, $G = U \frac{\partial \Theta}{\partial x} + W \frac{\partial \Theta}{\partial z} - \frac{\partial}{\partial x} (\frac{\lambda}{\rho c_p} \frac{\partial \Theta}{\partial x} - \overline{u'\theta'})$,

569 and z_t is an arbitrary height in the wall-normal direction.

570 Combining the last three terms in Eq.A.1 and the last two terms in Eq.A.2

571 yields

$$\begin{aligned} \tau_w = & \frac{2\mu}{z_t^2} \int_0^{z_t} U dz - \frac{2\rho}{z_t^2} \int_0^{z_t} (z_t - z) \overline{u'w'} dz \\ & - \frac{\rho}{z_t^2} \int_0^{z_t} (z_t - z)^2 \left(I_x + \frac{\partial U}{\partial t} + \frac{1}{\rho} \frac{\partial P}{\partial x} \right) dz, \end{aligned} \quad (\text{A.3})$$

572 and

$$q_w'' = \frac{2\lambda}{z_t^2} \int_0^{z_t} (\Theta_w - \Theta) dz + \frac{2\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z) \overline{w'\theta'} dz + \frac{\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z)^2 \left(G + \frac{\partial \Theta}{\partial t} \right) dz. \quad (\text{A.4})$$

573 Replacing I_x and G_x with their mathematical equivalents [39, 41], respec-
 574 tively: $I_x = \frac{\partial}{\partial z}(\nu \frac{\partial U}{\partial z} - \overline{u'w'}) - \frac{\partial U}{\partial t} - \frac{1}{\rho} \frac{\partial P}{\partial x}$ and $G = \frac{\partial}{\partial z}(\frac{\lambda}{\rho C_p} \frac{\partial \Theta}{\partial z} - \overline{w'\theta'} - \frac{\partial \Theta}{\partial t})$,
 575 results in the following expressions:

$$\tau_w = \frac{2\mu}{z_t^2} \int_0^{z_t} U dz - \frac{2\rho}{z_t^2} \int_0^{z_t} (z_t - z) \overline{u'w'} dz - \frac{\rho}{z_t^2} \int_0^{z_t} (z_t - z)^2 \frac{\partial}{\partial z} \left(\nu \frac{\partial U}{\partial z} - \overline{u'w'} \right) dz, \quad (\text{A.5})$$

576 and

$$q_w'' = \frac{2\lambda}{z_t^2} \int_0^{z_t} (\Theta_w - \Theta) dz + \frac{2\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z) \overline{w'\theta'} dz + \frac{\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z)^2 \frac{\partial}{\partial z} \left(\frac{\lambda}{\rho C_p} \frac{\partial \Theta}{\partial z} - \overline{w'\theta'} \right) dz. \quad (\text{A.6})$$

577 In fully-developed channel flow $I_x = 0$ and $G = 0$, expressions A.3 and A.4
 578 then reduce to

$$\tau_w = \frac{2\mu}{z_t^2} \int_0^{z_t} U dz - \frac{2\rho}{z_t^2} \int_0^{z_t} (z_t - z) \overline{u'w'} dz - \frac{\rho}{z_t^2} \int_0^{z_t} (z_t - z)^2 \left(\frac{\partial U}{\partial t} + \frac{1}{\rho} \frac{\partial P}{\partial x} \right) dz, \quad (\text{A.7})$$

579 and

$$q_w'' = \frac{2\lambda}{z_t^2} \int_0^{z_t} (\Theta_w - \Theta) dz + \frac{2\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z) \overline{w'\theta'} dz + \frac{\rho c_p}{z_t^2} \int_0^{z_t} (z_t - z)^2 \frac{\partial \Theta}{\partial t} dz. \quad (\text{A.8})$$

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