Modeling and Simulations for Fluid and Rotating Structure Interactions

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Abstract

In this paper, we study a dynamic fluid-structure interaction (FSI) model for an elastic structure immersed and spinning in the fluid. To describe the motion of a rotating elastic structure, we develop a linear constitutive model, that is suitable for the application of the arbitrary Lagrangian-Eulerian (ALE) method in FSI simulations. Additionally, a new ALE mapping method is designed to generate the moving fluid mesh while the deformable structure spins in a non-axisymmetric fluid channel. The structure velocity is adopted as the principle unknown to form a monolithic saddle-point system together with fluid velocity and pressure. Using the mixed finite element method and Newton's linearization, we discretize the nonlinear saddle-point system, and prove that the discrete saddle-point system is well-posed. The developed methodology is applied to a self-defined elastic structure and a realistic hydro-turbine under a prescribed angular velocity. Numerical validation is also conducted to demonstrate the accuracy of the models and the numerical methods.

Keywords: Fluid-rotating structure interaction, arbitrary Lagrangian Eulerian (ALE) method, monolithic algorithm, mixed finite element method, linear elasticity, master-slave relations.

1. Introduction

The fluid-structure interaction (FSI) problem remains one of the most challenging in computational mechanics and computational fluid dynamics. Researchers have conducted various studies on certain types of FSI problems (such as the fluid-rigid body interaction [1, 2, 3, 4, 5], fluids with a non-rotational structure [6, 7, 8, 9, 10, 11], and FSI with a fixed fluid domain [12, 13, 14, 15]). However, there is a dearth of practical models on FSI problems involving a structure that rotates and deforms, i.e., an elastic rotor. The development of mathematical models and numerical methodologies is critical in practice for large-scale advanced FSI simulations involving an elastic rotor, in order to guide the design, evaluation, and prediction of various applications, such as hydro-turbines, jet engines, and artificial heart pumps. Therefore, it is of great significance to develop an efficient and accurate mathematical model and numerical method to handle fluid-structure interactions involving a rotational and deformable structural motion.

However, numerical simulations of fluids interacting with a rotating structure can be challenging. The difficulties associated with simulating the rotational structure in FSI stem from the fact that the simulations rely on the coupling of two distinct descriptions: the Lagrangian description for the solid and the Eulerian

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coordinate for the fluid. The arbitrary Lagrangian Eulerian (ALE) method [6, 16, 17, 18, 19] copes with this difficulty by adapting the fluid mesh to accommodate the deformations of the solid on the interface. When the ALE method is used, the meshes of the fluid and structure conform on the interface if the Lagrangian structure mesh is moved to the Eulerian one following ALE mapping. This is important as the degrees of freedom on the interface are naturally shared by both the fluid and the structure, which facilitates the implementation of the discretization proposed herein. However, the ALE method has a severe drawback; i.e., when the structure has a large displacement or deformation, it is very likely that ALE mapping will distort the fluid mesh. Even the most advanced and most finely tuned ALE-based scheme cannot perform well without re-meshing. And, if re-meshing is used to produce the fluid mesh, then the number of mesh nodes and/or elements over different time levels can no longer be guaranteed to be the same. Thus, interpolations of variables between every two adjacent time steps are unavoidable, resulting in a time-consuming and even unstable geometrical process, especially in high-dimension cases. In fact, it is difficult to directly apply the ALE method to fluid-rotating structure interaction problems.

Many approaches have been proposed to deal with rotational structures in FSI problems. Several of these approaches model the wind turbine rotor [20, 21, 22, 23, 24] by coupling the finite element method (FEM) for fluid dynamics, isogeometric analysis (IGA) for structural mechanics, and the non-conforming discretization on the interface between fluid and structure. In order to apply ALE, these approaches introduce an artificial cylindrical buffer zone to enclose the rotor; let the mesh of the sliding cylindrical interface weakly enforce the continuity of the solution fields; and introduce extra unknowns and/or penalties to reinforce the continuity on the interface via, for example, the Lagrange multiplier or the discontinuous Galerkin (DG) method. The shear-slip method [25, 26, 20] was introduced to locally reconnect the mesh in order to maintain its the quality when the structure is undergoing translation or rotation. There are also some interesting ALE remeshing techniques that are able to deal with large structure displacement. For example, the fixed-mesh ALE method [27] and the universal mesh method [28] locally rebuild the mesh in order to accommodate the arbitrary motion of the interface.

In this paper, we develop a new ALE method in order to produce a body-fitted moving fluid mesh that conforms with the rotational and deformable structure mesh on the interface. And, to make our ALE method work for an elastic rotor immersed in a fluid, we first derive a linear structure equation that involves the rotational matrix from the nonlinear structure model based on decomposing the structure displacement into two components: rotation and deformation. In addition, we define an artificial cylindrical buffer zone in the fluid domain to enclose the elastic rotor and rotate together on the same axis of rotation with the same angular velocity. If the fluid channel is non-axisymmetric, we need to determine the relative motion information between the rotational fluid subdomain (the cylindrical buffer zone) and the fixed fluid subdomain (the rest of the fluid domain) by matching the grid on the sliding interface and defining them in our new ALE mapping. Finally, we develop a very stable and easily attainable ALE method designed to generate a rotational and deformable fluid mesh that matches the structure mesh on the interface.

Next, with the velocity instead of the displacement as the principle unknown of the structure, we define a monolithic saddle-point system for the studied FSI problem, and further a monolithic algorithm to solve the coupled fluid and structure equations. Moreover, we are able to prove the well-posedness of the discrete monolithic linear saddle-point system resulting from mixed finite element discretization and Newton's method. Numerical experiments are carried out for a self-defined elastic rotor and a realistic hydro-turbine to illustrate that our developed structure model and ALE-based monolithic method are efficient and stable for the FSI problem involving an elastic rotor. A numerical validation is also conducted to demonstrate the consistency of our rotational structure model with respect to the different structure parameters.

The paper is organized as follows. In Section 2, we define the general governing equations, interface conditions, and boundary conditions of the FSI problem, and their weak formulations along with the ALE techniques, and then we introduce the monolithic weak formulation of FSI. In Section 3, we develop an approximate formulation of the constitutive equation for the rotating structure. Then, in Section 4, we define a new ALE mapping for an elastic rotor immersed in the fluid, and we define a monolithic numerical discretization in Section 5, where we also analyze the well-posedness of the resulting discrete saddle-point system. In Section 6, we describe our monolithic algorithm in detail. Then, in Section 7, we show some nu-

merical experiments and conduct the numerical validations. We draw our conclusions and outline directions for future research in Section 8.

2. Modeling the fluid-structure interaction (FSI) problem

In this section, we briefly introduce a general fluid-structure interaction problem, including its strong and weak forms, as well as the ALE method. For details on different types of FSI problems, we refer to [29, 30].

Figure 1: A schematic domain of FSI involving an elastic rotor Ω_s in 2D.

As shown in Fig. 1, we consider an elastic rotor $\Omega_s(t)$ that is immersed in the fluid domain $\Omega_f(t)$ and spins around its rigid axis of rotation displayed as an inner circle Γ_{in} . $\Omega_f(t) \cap \Omega_s(t) = \emptyset$. Denote the whole domain as $\bar{\Omega}(t) = \bar{\Omega}_f(t) \cup \bar{\Omega}_s(t) \subset \mathbb{R}^d$ ($d = 2, 3$), and the interface of the fluid and structure as $\Gamma(t) = \partial\Omega_s(t) \cap \partial\Omega_f(t)$. We use ∂_t to denote the time derivative $\frac{\partial}{\partial t}$ in the rest of this paper.

For any position $\hat{\mathbf{x}} \in \hat{\Omega} = \Omega(0)$, we denote its position at time t by $\mathbf{x}(\hat{\mathbf{x}}, t) \in \Omega(t)$. $\mathbf{x}(\hat{\mathbf{x}}, t)$ is usually referred to as the flow map. We can also use $\mathbf{x}(\hat{\mathbf{x}}, t)$ to denote the *flow map*: $\hat{\mathbf{x}} \mapsto \mathbf{x}(\hat{\mathbf{x}}, t)$. Then, for any given $t > 0$, $\mathbf{x}(\cdot, t)$ is a diffeomorphism from $\Omega(0)$ to $\Omega(t)$. For $(\hat{\mathbf{x}}, t) \in \hat{\Omega} \times [0, T]$, we introduce the following variables in *Lagrangian coordinates*: the displacement $\hat{\mathbf{u}}(\hat{\mathbf{x}}, t) = \mathbf{x}(\hat{\mathbf{x}}, t) - \hat{\mathbf{x}}$, and the velocity $\hat{\mathbf{v}}(\hat{\mathbf{x}}, t) = \partial_t \hat{\mathbf{u}}(\hat{\mathbf{x}}, t)$. Using the relationship $\mathbf{x} = \mathbf{x}(\hat{\mathbf{x}}, t)$, we also introduce the velocity in *Eulerian coordinates*: $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}}(\hat{\mathbf{x}}, t)$. The symmetric part of the gradient of velocity is denoted by $\boldsymbol{\epsilon}(\mathbf{v}) = (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)/2$.

For the sake of brevity, in the rest of this paper we omit the temporal variable t in notations defined in Eulerian coordinates, i.e., $\phi = \phi(t)$, and use $\hat{\phi}$ to denote variables defined in Lagrangian coordinates, i.e., $\hat{\phi} = \phi(0)$.

Fluid motion. The fluid is assumed to be Newtonian and incompressible. It is described by the following conservations of momentum and mass, i.e., Navier-Stokes equations, in terms of the fluid velocity $\mathbf{v}_f$ and pressure p_f in the Eulerian description:

$$\begin{cases} \rho_f(\partial_t \mathbf{v}_f + \mathbf{v}_f \cdot \nabla \mathbf{v}_f) = \nabla \cdot \boldsymbol{\sigma}_f + \mathbf{g}_f, & \text{in } \Omega_f, \\ \nabla \cdot \mathbf{v}_f = 0, & \text{in } \Omega_f, \end{cases} \quad (1)$$

where

$$\boldsymbol{\sigma}_f = -p_f \mathbf{I} + 2\mu_f \boldsymbol{\epsilon}(\mathbf{v}_f),$$

ρ_f denotes the density of the fluid, μ_f the dynamic viscosity of the fluid, and $\mathbf{g}_f$ the external body force of the fluid.

Structure motion.

Structure motion. The structure motion is described by the following elastodynamics equation with St.Venant-Kirchhoff (STVK) material [31] in Lagrangian coordinates

$$\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_s = \nabla \cdot \mathbf{P}_s + \hat{\mathbf{g}}_s, \quad \text{in } \hat{\Omega}_s, \quad (2)$$

where $\hat{\rho}_s$ denotes the density of the structure in Lagrangian coordinates, and $\mathbf{g}_s$ denotes the external body force of the structure. Moreover, $\mathbf{P}_s = 2\mu_s \mathbf{F}\mathcal{E} + \lambda_s (\text{tr}\mathcal{E}) \mathbf{F}$ is the first Piola-Kirchhoff stress tensor for STVK materials, $\mathbf{F}(\hat{\mathbf{x}}, t) = \partial \mathbf{x} / \partial \hat{\mathbf{x}} = \mathbf{I} + \nabla_{\hat{\mathbf{x}}} \hat{\mathbf{u}}_s$ is the deformation gradient tensor, $\mathcal{E} = (\mathbf{F}^T \mathbf{F} - \mathbf{I})/2$ is the Green-Lagrange finite strain tensor, and μ_s and λ_s are the Lamé constants.

It is evident that (2) is a nonlinear equation. In Section 3, we develop a linearized model, which is an approximation to (2) under the assumption of a small deformation, specifically for a deformable structure undergoing a large rotation.

Interface conditions. Suppose no-slip boundary conditions hold on the interface of the fluid and structure, i.e., both velocity and normal stress are continuous across interface Γ . The interface conditions on Γ can be defined as follows in the Eulerian description:

$$\mathbf{v}_f = \mathbf{v}_s = \partial_t \mathbf{u}_s, \quad \text{on } \Gamma, \quad (3)$$

$$\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_s \mathbf{n}_s = 0, \quad \text{on } \Gamma, \quad (4)$$

where the unit normal vector to the interface Γ is $\mathbf{n}_f$ from the fluid and $\mathbf{n}_s$ from the structure.

Boundary conditions. To fully define the boundary conditions with the exception of the interface conditions, we split the outer boundary of the fluid domain into two types: $\partial\Omega_f \setminus \Gamma = \Gamma_{D,f} \cup \Gamma_{N,f}$ and $\Gamma_{D,f} \cap \Gamma_{N,f} = \emptyset$, where $\Gamma_{D,f}$ represents a part of the outer fluid boundary on which the Dirichlet boundary condition is imposed, and $\Gamma_{N,f}$ represents the rest of the outer fluid boundary on which Neumann boundary condition is defined. Similarly, we split the outer boundary of the structure into $\partial\hat{\Omega}_s \setminus \hat{\Gamma} = \hat{\Gamma}_{D,s} \cup \hat{\Gamma}_{N,s}$ and $\hat{\Gamma}_{D,s} \cap \hat{\Gamma}_{N,s} = \emptyset$. Note that the above notations we use here to define the outer boundary are for general FSI cases, their usages are beyond the specific FSI problem shown in Fig. 1.

The Dirichlet and Neumann boundary conditions for the fluid and structure are as follows:

$$\mathbf{v}_f = \mathbf{v}_{D,f}, \quad \text{on } \Gamma_{D,f}, \quad (5)$$

$$\hat{\mathbf{u}}_s = \hat{\mathbf{u}}_{D,s}, \quad \text{on } \hat{\Gamma}_{D,s}, \quad (6)$$

$$\boldsymbol{\sigma}_f \mathbf{n}_f = \mathbf{g}_{N,f}, \quad \text{on } \Gamma_{N,f}, \quad (7)$$

$$\boldsymbol{\sigma}_s \mathbf{n}_s = \hat{\mathbf{g}}_{N,s}, \quad \text{on } \hat{\Gamma}_{N,s}, \quad (8)$$

where $\mathbf{v}_{D,f}(\mathbf{x}, t)$, $\hat{\mathbf{u}}_{D,s}(\hat{\mathbf{x}}, t)$, $\mathbf{g}_{N,f}(\mathbf{x}, t)$, and $\hat{\mathbf{g}}_{N,s}(\hat{\mathbf{x}}, t)$ are all prescribed. For example, the Dirichlet boundary conditions for the fluid can be the inflow boundary condition at the inlet and the no-slip boundary condition on the wall. Dirichlet boundary conditions for the structure can be used when it is clamped to the boundary. With $\mathbf{g}_{N,f} = 0$, the Neumann boundary condition is adopted for fluid as the stress-free boundary condition at the outlet of the flow channel. The Neumann boundary condition is used for the structure in FSI when the fluid is enclosed in the structure and an external force imposes on the structure outer boundary.

For the case studied in this paper, the structure is immersed in the fluid. Therefore, the Neumann boundary condition is not applied to the structure, i.e., $\hat{\Gamma}_{N,s} = \emptyset$. And, $\hat{\Gamma}_{D,s} = \Gamma_{in}$, on which $\hat{\mathbf{u}}_{D,s}$ is given if a rotation is prescribed. In the rest of this paper, we use the structure velocity, $\hat{\mathbf{v}}_s$, as the principle unknown instead of $\hat{\mathbf{u}}_s$, noting that $\hat{\mathbf{v}}_s = \partial_t \hat{\mathbf{u}}_s$. Then, the corresponding Dirichlet boundary condition is changed to

$$\hat{\mathbf{v}}_s = \partial_t \hat{\mathbf{u}}_{D,s}, \quad \text{on } \hat{\Gamma}_{D,s}. \quad (9)$$

Initial conditions. The following initial conditions hold for fluid and structure equations defined on the time interval $[0, T]$, with $T > 0$:

$$\begin{aligned} \mathbf{v}_f(0) &= \mathbf{v}_f^0(\mathbf{x}) && \text{in } \Omega_f \\ \hat{\mathbf{u}}_s(0) &= \hat{\mathbf{u}}_s^0(\hat{\mathbf{x}}), \quad \hat{\mathbf{v}}_s(0) = \hat{\mathbf{v}}_s^0(\hat{\mathbf{x}}) && \text{in } \hat{\Omega}_s, \end{aligned} \quad (10)$$

where $\hat{\mathbf{u}}_s^0(\hat{\mathbf{x}})$, $\hat{\mathbf{v}}_s^0(\hat{\mathbf{x}})$, and $\mathbf{v}_f^0(\mathbf{x})$ are all prescribed.

Now we derive weak formulations of the FSI problem. First, for any given domain $\Omega \subset \mathbb{R}^d$, we define the H^1 space of the vector functions as

$$\mathbf{H}^1(\Omega) = (H^1(\Omega))^d.$$

Moreover, we define

$$\begin{aligned} \mathbf{H}_0^1(\Omega_f) &:= \{\mathbf{u} \in \mathbf{H}^1(\Omega_f) \mid \mathbf{u} = 0, \text{ on } \partial\Omega \cap \partial\Omega_f\}, \\ \mathbf{H}_0^1(\hat{\Omega}_s) &:= \{\mathbf{u} \in \mathbf{H}^1(\hat{\Omega}_s) \mid \mathbf{u} = 0, \text{ on } \partial\hat{\Omega} \cap \partial\hat{\Omega}_s\}, \\ \mathbf{H}_0^1(\hat{\Omega}_f) &:= \{\mathbf{u} \in \mathbf{H}^1(\hat{\Omega}_f) \mid \mathbf{u} = 0, \text{ on } \partial\hat{\Omega} \cap \partial\hat{\Omega}_f\}. \end{aligned}$$

Then, we define the following Sobolev spaces:

$$\begin{aligned} \mathbb{V} &:= \{(\mathbf{v}_f, \hat{\mathbf{v}}_s) \in \mathbf{H}_0^1(\Omega_f) \times \mathbf{H}_0^1(\hat{\Omega}_s) \text{ such that } \mathbf{v}_f \circ \mathbf{x}(\hat{\mathbf{x}}, t) = \hat{\mathbf{v}}_s, \text{ on } \hat{\Gamma}\}, \\ \mathbb{Q} &:= L^2(\Omega_f). \end{aligned} \quad (11)$$

Thus, we are able to define the weak formulations of FSI as follows. Find $\mathbf{v}_f$, p , and $\hat{\mathbf{u}}_s$ satisfying (5), (6), and (10), such that for any given $t > 0$, the following equations hold for any $(\phi, \hat{\phi}) \in \mathbb{V}$ and $q \in \mathbb{Q}$:

$$\left\{ \begin{aligned} &(\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_s, \hat{\phi})_{\hat{\Omega}_s} + (\rho_f D_t \mathbf{v}_f, \phi)_{\Omega_f} + (\mathbf{P}_s(\hat{\mathbf{u}}_s), \nabla \hat{\phi})_{\hat{\Omega}_s} + (\boldsymbol{\sigma}_f, \boldsymbol{\epsilon}(\phi))_{\Omega_f} \\ &\quad = (\hat{\mathbf{g}}_s, \hat{\phi}) + (\mathbf{g}_f, \phi), \\ &(\nabla \cdot \mathbf{v}_f, q)_{\Omega_f} = 0, \\ &\mathbf{v}_f \circ \mathbf{x}(\hat{\mathbf{x}}, t) = \partial_t \hat{\mathbf{u}}_s, \quad \text{on } \hat{\Gamma}, \end{aligned} \right. \quad (12)$$

where the pair $(\cdot, \cdot)_\Omega$ stands for the L^2 -inner product on a domain Ω and where $\nabla \phi = \nabla_{\mathbf{x}} \phi$, $\nabla \hat{\phi} = \nabla_{\hat{\mathbf{x}}} \hat{\phi}$.

The above weak formulation shows that on the continuous level, the continuity of velocity of fluid and structure across the interface is enforced in the definition of the Sobolev space $\mathbb{V}$. However, on the discrete level, we only need one set of degrees of freedom of velocity to be shared by the fluid and structure on the interface. To that end, first of all, a conforming mesh across the interface is needed, which is guaranteed by the ALE method. Secondly, we need to merge the two sets of degrees of freedom of velocity arising from both the fluid and the structure on the interface in order to keep only one set, which can be achieved using a well-known technique of ‘‘Master-Slave Relation’’ [32, 33]. With this technique, one of the connecting mesh boundaries is designated as master and the other as the slave, and the nodes on the slave boundary are constrained to lie on the master boundary. Thus, both master and slave nodes at the same position on the common mesh boundaries (interfaces) share the same degrees of freedom (DOFs). There are no further redundant DOFs to eliminate and no need to use the Lagrange multiplier to enforce the continuity of velocity. We discuss this technique further in combination with the ALE method in the next section.

Due to the interaction between fluid and structure, we need to solve the Navier-Stokes equations on a moving fluid domain. Although always an option for a moving domain, re-meshing introduces extra computational cost and requires interpolation to transfer data between time steps. Another frequently used approach is the arbitrary Lagrangian-Eulerian (ALE) method, with which the moving fluid mesh sticks to the moving structure mesh and is updated by maintaining the same number of mesh nodes. Thus, no interpolation occurs. The essential idea of the ALE method is that of formulating and solving the fluid

problem on a deforming mesh, which deforms with the structure at the interface such that the fluid mesh is smoothed within the fluid domain. Introduced for use with finite element discretizations of incompressible fluids in [16, 34], the ALE method provides an approach to finding the fluid mesh that can fit the moving fluid domain Ω_f . This mapping is a diffeomorphism on the continuous level, and we use piecewise polynomials to approximate it on the discrete level. In the rest of this paper, we assume that the mesh motion is piecewise linear in order to ensure that the updated mesh by ALE mapping remains an undistorted triangular mesh.

We denote the image of $\hat{\Omega}_f$ under the piecewise linear map $\mathbf{x}_{h,f}$ by Ω_f^n . Ω_f^n is triangulated by a moving mesh with respect to time, denoted by $T_h(\Omega_f^n)$. Note that Ω_f^n is assumed to be a polygonal domain in 2D, and a polyhedral domain in 3D. There are two main ingredients in the ALE approach:

1. Define with respect to time how the grid is moving such that it matches the structure displacement at the fluid-structure interface.
2. Define how the material derivatives are discretized on the moving grid.

Given the structure trajectory $\mathbf{x}(\hat{\mathbf{x}}, t)$ defined on $\hat{\Gamma}$, the moving grid can be described by a diffeomorphism $\mathcal{A}(\cdot, t) : \hat{\Omega}_f \mapsto \Omega_f$ that satisfies

$$\begin{cases} \mathcal{A}(\hat{\mathbf{x}}, t) = \hat{\mathbf{x}}, & \text{on } \partial\hat{\Omega}_f \cap \partial\hat{\Omega}, \\ \mathcal{A}(\hat{\mathbf{x}}, t) = \mathbf{x}(\hat{\mathbf{x}}, t), & \text{on } \hat{\Gamma}. \end{cases} \quad (13)$$

ALE mappings satisfying (13) are by no means unique. In the interior of $\hat{\Omega}_f$, the ALE mapping can be “arbitrary”. One popular approach to uniquely determine $\mathcal{A}$ is to solve a partial differential equation:

$$\mathcal{L}\mathcal{A} = 0, \quad \text{in } \hat{\Omega}_f. \quad (14)$$

A popular choice for the operator $\mathcal{L}$ is the Laplacian $\mathcal{L} = -\Delta$. For more choices to formulate the ALE mapping equation, we refer to [34, 35, 36, 30] and references therein.

With the ALE technique, we can reformulate the fluid momentum equation on a moving fluid domain as follows:

$$\rho_f \partial_t^A \mathbf{v}_f + \rho_f ((\mathbf{v}_f - \mathbf{w}) \cdot \nabla) \mathbf{v}_f = \nabla \cdot \boldsymbol{\sigma}_f + \mathbf{g}_f, \quad (15)$$

where $\mathbf{w} = \partial_t \mathcal{A} \circ \mathcal{A}^{-1}$ denotes the velocity of the fluid mesh and

$$\partial_t^A \mathbf{v}_f|_{(\mathbf{x}, t)} := \partial_t \mathbf{v}_f + (\mathbf{w} \cdot \nabla) \mathbf{v}_f$$

is referred to as the ALE material derivative.

In particular, we define the discretization of the ALE material derivative on the moving grid as follows:

$$\partial_t^A \mathbf{v}_f \approx \frac{\mathbf{v}_f(\mathbf{x}, t_n) - \mathbf{v}_f(\mathcal{A}_{n-1} \circ \mathcal{A}_n^{-1}(\mathbf{x}), t_{n-1})}{\Delta t}, \quad (16)$$

where $\mathcal{A}_{n-1} \circ \mathcal{A}_n^{-1}(\mathbf{x})$ is the location of $\mathbf{x}$ at $t = t_{n-1}$ before $\mathbf{x}$ is moved by ALE mesh motion. For example, if at $t = t_{n-1}$, $\mathcal{A}_{n-1} \circ \mathcal{A}_n^{-1}(\mathbf{x})$ is a mesh node, the ALE mesh motion moves it to $\mathbf{x}$ at $t = t_n$, which is a mesh node at $t = t_n$. Thus, interpolation between the meshes of different time steps is avoided.

In order to precisely fulfill the kinematic interface condition (3) and apply the Master-Slave Relation, we need to reformulate the structure equation in terms of the structure velocity $\hat{\mathbf{v}}_s$, instead of its displacement $\hat{\mathbf{u}}_s$ in (12). We use the following relation of $\hat{\mathbf{v}}_s$ and $\hat{\mathbf{u}}_s$:

$$\hat{\mathbf{v}}_s = \partial_t \hat{\mathbf{u}}_s, \quad \text{or} \quad \hat{\mathbf{u}}_s = \hat{\mathbf{u}}_s^0 + \int_0^t \hat{\mathbf{v}}_s(\tau) d\tau \quad (17)$$

to reformulate (12) and rewrite the kinematic condition (3) in the reference configuration as

$$\hat{\mathbf{v}}_s(\hat{\mathbf{x}}) = \mathbf{v}_f(\mathcal{A}(\hat{\mathbf{x}}, t)), \quad \text{on } \hat{\Gamma}.$$

Thus, the Master-Slave Relation can be efficiently applied as both the master and slave degrees of freedom (DOFs) are equal to each other at the same mesh node on the interface. In addition, we introduce an equivalent ALE mapping for the displacement of the fluid mesh $\mathcal{A}_{\mathbf{u}}(\hat{\mathbf{x}}, t) = \mathcal{A}(\hat{\mathbf{x}}, t) - \hat{\mathbf{x}}(\hat{\mathbf{x}}, t)$, satisfying

$$-\Delta \mathcal{A}_{\mathbf{u}} = 0, \quad \text{in } \hat{\Omega}_f \quad (18)$$

$$\mathcal{A}_{\mathbf{u}} = 0, \quad \text{on } \partial\hat{\Omega}_f \cap \partial\hat{\Omega}, \quad (19)$$

$$\mathcal{A}_{\mathbf{u}} = \hat{\mathbf{u}}_s, \quad \text{on } \hat{\Gamma}. \quad (20)$$

The resulting new weak formulation is, thus, defined as follows with the fluid velocity $\mathbf{v}_f$, the fluid pressure p , the structure velocity $\hat{\mathbf{v}}_s$, and the ALE mapping $\mathcal{A}_{\mathbf{u}}$ as unknowns:

Find $\mathbf{v}_f$, p , $\hat{\mathbf{v}}_s$, and $\mathcal{A}_{\mathbf{u}}$ satisfying (3), (5), (9), and (10), such that for any given $t > 0$, the following system of equations holds for $\forall(\hat{\phi}, \phi) \in \mathbb{V}$, $q \in \mathbb{Q}$, and $\xi \in \mathbf{H}_0^1(\hat{\Omega}_f)$:

$$\left\{ \begin{array}{l} (\hat{\rho}_s \partial_t \hat{\mathbf{v}}_s, \hat{\phi})_{\hat{\Omega}_s} + (\mathbf{P}_s(\hat{\mathbf{u}}_s^0 + \int_0^t \hat{\mathbf{v}}_s(\tau) d\tau), \nabla \hat{\phi})_{\hat{\Omega}_s} + (\rho_f \partial_t^A \mathbf{v}_f, \phi)_{\Omega_f} \\ \quad + (\rho_f (\mathbf{v}_f - \mathbf{w}) \cdot \nabla \mathbf{v}_f, \phi)_{\Omega_f} + (\boldsymbol{\sigma}_f, \boldsymbol{\epsilon}(\phi))_{\Omega_f} = (\hat{\mathbf{g}}_s, \hat{\phi}) + (\mathbf{g}_f, \phi), \\ \quad (\nabla \cdot \mathbf{v}_f, q)_{\Omega_f} = 0, \\ \quad (\nabla \mathcal{A}_{\mathbf{u}}, \nabla \xi)_{\hat{\Omega}_f} = 0. \\ \quad \mathcal{A}_{\mathbf{u}} = \hat{\mathbf{u}}_s^0 + \int_0^t \hat{\mathbf{v}}_s(\tau) d\tau, \text{ on } \hat{\Gamma}. \end{array} \right. \quad (21)$$

To simplify the notation, in the following sections, we suppose $\mathbf{g}_f = \hat{\mathbf{g}}_s = 0$ by assuming that no external body force is acting on the fluid or the structure.

3. Remodeling the rotational elastic structure

In this section, we develop a new model for an elastic rotor with large rotation and small deformation, which is simultaneously suitable for using the ALE method for FSI problems that involve both the rotation and deformation in the structure.

From (2), we know that the equation of an elastic structure that follows the constitutive law of STVK material and is spinning around the axis of rotation, Γ_{in} , is specifically defined as

$$\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_s = \nabla \cdot \mathbf{P}_s = \nabla \cdot (\lambda_s (\text{tr} \mathcal{E}) \mathbf{F} + 2\mu_s \mathbf{F} \mathcal{E}) \quad (22)$$

with the boundary condition

$$\hat{\mathbf{u}}_s = \mathbf{x} - \hat{\mathbf{x}} = (R(\theta) - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0), \quad \text{on } \Gamma_{in}, \quad (23)$$

where $R(\theta)$ is the rotational matrix that depends on the angle of rotation $\theta(t)$. For instance, let us specify a case in which the axis of rotation is the z -axis, then

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (24)$$

For the sake of brevity, in the rest of the paper, we just use R to denote the rotational matrix.

Here we consider a hyperelastic structure such as the SVTK material for which the deformation can be small while a large rotation occurs. The key idea is that we can decompose the structure motion into a rotational and a deformation part [37, 38, 23]. Suppose $\hat{\mathbf{x}}_0$ is a point on the axis of rotation. We can decompose the trajectory of each material particle as follows by using the rotational matrix R arising from

the boundary condition (23),

$$\mathbf{x} - \hat{\mathbf{x}}_0 = R(\hat{\mathbf{x}} + \hat{\mathbf{u}}_d - \hat{\mathbf{x}}_0), \quad (25)$$

where $\hat{\mathbf{u}}_d$ is the local deformation displacement in the reference configuration. Note that $\hat{\mathbf{u}}_d$ does not contain any information about the rotation, such that $\hat{\mathbf{u}}_d = 0$ on Γ_{in} . Hence, the total structure displacement, $\hat{\mathbf{u}}_s$, can be decomposed as

$$\hat{\mathbf{u}}_s = \mathbf{x} - \hat{\mathbf{x}} = (R - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0) + R\hat{\mathbf{u}}_d = \hat{\mathbf{u}}_\theta + R\hat{\mathbf{u}}_d, \quad (26)$$

where $\hat{\mathbf{u}}_\theta = (R - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)$ is referred to as the rotational part of the structure displacement.

From (25), we obtain the deformation gradient tensor $\mathbf{F} = R(\mathbf{I} + \mathbf{H})$, where $\mathbf{H} = \nabla \hat{\mathbf{u}}_d$. Then, the Green-Lagrangian finite strain tensor $\mathcal{E} = (\mathbf{F}^T \mathbf{F} - \mathbf{I})/2 = (\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H})/2$. Thus, the first Piola-Kirchhoff stress in (22), $\mathbf{P}_s = \mathbf{F}(\lambda_s \text{tr} \mathcal{E} \mathbf{I} + 2\mu_s \mathcal{E})$, leads to a function of $\mathbf{H}$ as follows:

$$\mathbf{P}_s(\mathbf{H}) = R(\mathbf{I} + \mathbf{H}) \left(\frac{\lambda_s}{2} \text{tr}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}) \mathbf{I} + \mu_s (\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}) \right).$$

As we are only considering a small deformation, i.e., $\mathbf{H} \approx 0$, we then conduct a linear approximation for the above equation by Taylor expansion around the non-deformable configuration $\mathbf{H} = 0$, resulting in

$$\mathbf{P}_s \approx \frac{1}{2} \lambda_s R((\text{tr} \mathbf{H}) \mathbf{I} + (\text{tr} \mathbf{H}^T) \mathbf{I}) + \mu_s R(\mathbf{H} + \mathbf{H}^T) = R(\lambda_s \text{tr}(\epsilon(\hat{\mathbf{u}}_d)) \mathbf{I} + 2\mu_s \epsilon(\hat{\mathbf{u}}_d)), \quad (27)$$

where $\epsilon(\hat{\mathbf{u}}_d) = (\nabla \hat{\mathbf{u}}_d + (\nabla \hat{\mathbf{u}}_d)^T)/2$ is the linear approximation of $\mathcal{E}(\hat{\mathbf{u}}_d)$. Therefore, we obtain the linear approximation for the stress tensor in the Lagrangian description,

$$\hat{\boldsymbol{\sigma}}_s = \mathbf{P}_s \approx R(\lambda_s \text{tr}(\epsilon(\hat{\mathbf{u}}_d)) \mathbf{I} + 2\mu_s \epsilon(\hat{\mathbf{u}}_d)).$$

We can equivalently rewrite $\hat{\boldsymbol{\sigma}}_s = R\mathbf{D}\epsilon(\hat{\mathbf{u}}_d)$, where $\mathbf{D}_{ijkl} = 2\mu_s \delta_{ik} \delta_{jl} + \lambda_s \delta_{ij} \delta_{kl}$ is a fourth-order tensor and $\mathbf{D}\epsilon(\hat{\mathbf{u}}_d)$ is the contraction of $\mathbf{D}$ and $\epsilon(\hat{\mathbf{u}}_d)$, i.e., $(\mathbf{D}\epsilon(\hat{\mathbf{u}}_d))_{ij} = \sum_{kl} \mathbf{D}_{ijkl} (\epsilon(\hat{\mathbf{u}}_d))_{kl}$.

Thus, (22) is approximated by

$$\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_s = \nabla \cdot (R(\lambda_s \text{tr}(\epsilon(\hat{\mathbf{u}}_d)) \mathbf{I} + 2\mu_s \epsilon(\hat{\mathbf{u}}_d))) = \nabla \cdot (R\mathbf{D}\epsilon(\hat{\mathbf{u}}_d)), \quad (28)$$

which can be treated as a linear elasticity model of the elastic rotor, noting that the right-hand side of (28) is linear with respect to $\hat{\mathbf{u}}_d$.

Next, we reformulate (28) in terms of a unique principle unknown $\hat{\mathbf{u}}_s$. Due to (26), we have $R\hat{\mathbf{u}}_d = \hat{\mathbf{u}}_s - \hat{\mathbf{u}}_\theta$; thus, $\epsilon(\hat{\mathbf{u}}_d) = \epsilon(R^T \hat{\mathbf{u}}_s) - \epsilon(R^T \hat{\mathbf{u}}_\theta)$. Note that $\hat{\mathbf{u}}_\theta = (R - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)$, then $\mathbf{D}\epsilon(R^T \hat{\mathbf{u}}_\theta) = \mathbf{D}\epsilon((\mathbf{I} - R^T)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0))$. Therefore, we obtain the following equation for the elastic structure in terms of $\hat{\mathbf{u}}_s$ only:

$$\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_s - \nabla \cdot (R\mathbf{D}\epsilon(R^T \hat{\mathbf{u}}_s)) = -\nabla \cdot (R\mathbf{D}\epsilon((R^T - \mathbf{I})(\hat{\mathbf{x}}_0 - \hat{\mathbf{x}}))). \quad (29)$$

It is assumed that the structure only rotates on the axis of rotation, i.e., the inner boundary Γ_{in} . Then, with a given rotational matrix R , the admissible solution set for $\hat{\mathbf{u}}_s$ is defined as follows:

$$\mathbf{H}_R^1(\hat{\Omega}_s) = \{\hat{\mathbf{u}} \in \mathbf{H}^1(\hat{\Omega}_s) | \hat{\mathbf{u}}(\hat{\mathbf{x}}) = (R - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0), \hat{\mathbf{x}} \in \Gamma_{in}\}.$$

By (26), $\hat{\mathbf{u}} \in \mathbf{H}_R^1(\hat{\Omega}_s)$ can be decomposed into the rotational part $\hat{\mathbf{u}}_\theta = (R - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)$ and the deformation part $\hat{\mathbf{u}}_d \in \mathbf{H}_0^1(\hat{\Omega}_s)$ such that

$$\hat{\mathbf{u}} = R\hat{\mathbf{u}}_d + (R - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0). \quad (30)$$

It is obvious that given any $\hat{\mathbf{u}} \in \mathbf{H}_R^1(\hat{\Omega}_s)$, there exists a unique $\hat{\mathbf{u}}_d \in \mathbf{H}_0^1(\hat{\Omega}_s)$ such that (30) holds.

Assume that R is given for $t \in [0, T]$. Then, the weak formulation of (29) can be defined as follows:

$\forall t \in [0, T]$, find $\hat{\mathbf{u}}_s(t) \in \mathbf{H}_R^1(\hat{\Omega}_s)$ such that $\forall \hat{\phi} \in (H_0^1(\hat{\Omega}_s))^d$

$$(\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_s, \hat{\phi})_{\hat{\Omega}_s} + (\mathbf{D}\epsilon(R^T \hat{\mathbf{u}}_s), \epsilon(R^T \hat{\phi}))_{\hat{\Omega}_s} = (\mathbf{D}\epsilon((\mathbf{I} - R^T)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)), \epsilon(R^T \hat{\phi}))_{\hat{\Omega}_s} + \int_{\hat{\Gamma}} \mathbf{g}_{inf} \hat{\phi} ds, \quad (31)$$

where the stiffness term on the left-hand side of (31) is symmetric positive definite. Here $\mathbf{g}_{inf}$ on the right-hand side is the interface force from the fluid, which will be canceled in the weak form of FSI due to the continuity of normal stress on the interface of the fluid and structure. We conclude that the weak formulation (31) can also be derived based on the linear elasticity equations on the rotational configuration, as demonstrated in the rest of this section.

Let $\hat{\Omega}$ be the reference configuration of elasticity. Suppose the material undergoes a large rotation along with a small deformation. At time t , the rotational matrix is given as R . Without loss of generality, let $\hat{\mathbf{x}}_0 = 0$. Define

$$\hat{\Omega}_R := \{\hat{\mathbf{x}}_R = R\hat{\mathbf{x}}, \hat{\mathbf{x}} \in \hat{\Omega}\},$$

which is the rotational configuration. It is known that when formulating the equation in $\hat{\Omega}_R$, we can just use linear elasticity. When the equation is formulated on $\hat{\Omega}$, however, the linearization is more complicated. Now we want to show that we can obtain the same equation by formulating it in different configurations.

Recall the decomposition of displacements in $\hat{\Omega}$:

$$\hat{\mathbf{u}}_s = (R - \mathbf{I})\hat{\mathbf{x}} + R\hat{\mathbf{u}}_d = \hat{\mathbf{u}}_\theta + \hat{\mathbf{u}}_R,$$

where $\hat{\mathbf{u}}_R := R\hat{\mathbf{u}}_d$ is the unknown of the linear elasticity equation defined in $\hat{\Omega}_R$. Note that we use the subscript R for the variables defined in $\hat{\Omega}_R$, and no subscript for variables defined in $\hat{\Omega}$. To simplify the notation, we do not distinguish between $f(\hat{\mathbf{x}})$ and $f(R\hat{\mathbf{x}})$.

First, we look at the stress term in the weak form,

$$\begin{aligned} & (2\mu_s \epsilon_R(\hat{\mathbf{u}}_R) + \lambda_s \text{tr}(\epsilon_R(\hat{\mathbf{u}}_R))\mathbf{I}, \nabla_R \phi)_{\hat{\Omega}_R} \\ &= (\mu_s (\nabla_R \hat{\mathbf{u}}_R + (\nabla_R \hat{\mathbf{u}}_R)^T) + \lambda_s \text{tr}(\nabla_R \hat{\mathbf{u}}_R + (\nabla_R \hat{\mathbf{u}}_R)^T)\mathbf{I}/2, \nabla_R \phi)_{\hat{\Omega}_R} \\ &= (\mu_s (\nabla \hat{\mathbf{u}}_R R^T + (\nabla \hat{\mathbf{u}}_R R^T)^T) + \lambda_s \text{tr}(\nabla \hat{\mathbf{u}}_R R^T + (\nabla \hat{\mathbf{u}}_R R^T)^T)\mathbf{I}/2, \nabla \phi R^T)_{\hat{\Omega}} \\ &= (\mu_s (\nabla \hat{\mathbf{u}}_R + R(\nabla \hat{\mathbf{u}}_R)^T R) + \lambda_s \text{tr}(R^T \nabla \hat{\mathbf{u}}_R + (R^T \nabla \hat{\mathbf{u}}_R)^T)R/2, \nabla \phi)_{\hat{\Omega}} \\ &= (R[\mu_s (\nabla(R^T \hat{\mathbf{u}}_R) + (\nabla(R^T \hat{\mathbf{u}}_R))^T) + \lambda_s \text{tr}(\nabla(R^T \hat{\mathbf{u}}_R) + (\nabla(R^T \hat{\mathbf{u}}_R))^T)\mathbf{I}/2], \nabla \phi)_{\hat{\Omega}} \\ &= (R[2\mu\epsilon(\hat{\mathbf{u}}_d) + \lambda_s \text{tr}(\epsilon(\hat{\mathbf{u}}_d))\mathbf{I}], \nabla \phi)_{\hat{\Omega}}. \end{aligned}$$

This stress term is the same as that shown in (28). Meanwhile, we have the following inertia term in the rotational configuration

$$(\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_R, \phi)_{\Omega_R}.$$

The rotational configuration is a non-inertial reference frame, for the sake of force balance, the term $(\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_\theta, \phi)_{\Omega_R}$, representing the centrifugal force, is added to the equation. Thus, we can recover the inertia term in (31) given that

$$(\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_R, \phi)_{\Omega_R} + (\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_\theta, \phi)_{\Omega_R} = (\hat{\rho}_s \partial_{tt} \hat{\mathbf{u}}_s, \phi)_{\hat{\Omega}}.$$

We have, therefore, shown the equivalence with the derivation of equation (31).

After updating the structure equation with the linear elasticity involving the rotation, we have the following new weak formulation of the FSI system with a rotational elastic structure. Find $\mathbf{v}_f$, p , $\hat{\mathbf{v}}_s$, and $\mathcal{A}_u$, satisfying (3), (5), (9), and (10), such that for any given $t > 0$, the following equations hold for

$\forall(\hat{\phi}, \phi) \in \mathbb{V}, q \in \mathbb{Q}, \xi \in \mathbf{H}_0^1(\hat{\Omega}_f)$:

$$\left\{ \begin{array}{l} (\hat{\rho}_s \partial_t \hat{\mathbf{v}}_s, \hat{\phi})_{\hat{\Omega}_s} + (\mathbf{D} \epsilon(R^T(\hat{\mathbf{u}}_s^0 + \int_0^t \hat{\mathbf{v}}_s(\tau) d\tau)), \epsilon(R^T \hat{\phi}))_{\hat{\Omega}_s} + (\rho_f \partial_t^A \mathbf{v}_f, \phi)_{\Omega_f} \\ \quad + (\rho_f (\mathbf{v}_f - \mathbf{w}) \cdot \nabla \mathbf{v}_f, \phi)_{\Omega_f} + (\boldsymbol{\sigma}_f, \boldsymbol{\epsilon}(\phi))_{\Omega_f} = (\mathbf{D} \epsilon((\mathbf{I} - R^T)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)), \epsilon(R^T \hat{\phi}))_{\hat{\Omega}_s}, \\ \quad (\nabla \cdot \mathbf{v}_f, q)_{\Omega_f} = 0, \\ \quad (\nabla \mathcal{A}_u, \nabla \xi)_{\hat{\Omega}_f} = 0. \\ \quad \mathcal{A}_u = \hat{\mathbf{u}}_s^0 + \int_0^t \hat{\mathbf{v}}_s(\tau) d\tau, \text{ on } \hat{\Gamma}. \end{array} \right. \quad (32)$$

Remark 1. Note that the decomposition of structure displacement into rotation and deformation has been applied to FSI in [23], where the structure is modeled as nonlinear composite shell. In [23], this decomposition is used to modified the generalized-alpha time integration to produce exact structural motion in case of pure rotation. Moreover, in [23], the deformation displacement is used to obtain the deflection component of the mesh motion, where the rotation component of the mesh motion is handled exactly.

4. The modified ALE method for an elastic rotor immersed in a fluid

In this section, we develop a modified ALE method to deal with an elastic rotor immersed in a non-axisymmetric fluid domain. Roughly speaking, we introduce an artificial cylindrical buffer zone Ω_{rf} in the fluid domain in which the elastic rotor Ω_s is enclosed and rotates together on the same axis of rotation and with the same angular velocity. And, we refer to the remaining region in Ω_f as the fixed fluid subdomain, denoted by Ω_{sf} , thus $\Omega_f = \Omega_{sf} \cup \Omega_{rf}$. Denote the interface between Ω_{rf} and Ω_{sf} by Γ_{rs} , i.e., $\Gamma_{rs} = \partial\Omega_{rf} \cap \partial\Omega_{sf}$. On the other hand, we measure the relative motion of each pair of mesh nodes on Γ_{rs} and add the difference of motion to the mesh nodes on $\partial\Omega_{rf}$. Thus, the mesh in $\bar{\Omega}_{rf}$ needs to be updated all the time in the numerical computation in order to conform with the mesh in Ω_s through the interface of Ω_{rf} and Ω_s , Γ , and with the mesh in Ω_{sf} through the interface Γ_{rs} , as shown in Fig. 2.

Figure 2: The cylindrical buffer zone Ω_{rf} that embraces an elastic rotor Ω_s .

For the sake of brevity, we use a 2D example to show how to use the ALE method to generate the fluid mesh in the fluid domain Ω_f . Unlike the traditional ALE approach, the modified ALE method only moves the mesh nodes in the rotational fluid buffer zone Ω_{rf} . We handle the fluid mesh motion in a way that is similar to our handling of the structure: that is, we decompose the displacement into the rotational part and the deformation part: $\hat{\mathbf{u}}_s = \hat{\mathbf{u}}_\theta + R\hat{\mathbf{u}}_d$. In particular, we decompose $\mathcal{A}_u$, the displacement of the rotational fluid mesh defined in $\hat{\Omega}_{rf}$, into two parts: the rotational part $\hat{\mathbf{u}}_\theta$ and the deformation part $\mathcal{A}_D$:

$$\mathcal{A}_u = \hat{\mathbf{u}}_\theta + \mathcal{A}_D,$$

where $\hat{\mathbf{u}}_\theta = (R - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)$ is the rotational displacement of the mesh determined by the given rotational matrix, R , defined on the axis of rotation Γ_{in} . In addition, we also need the rotational fluid mesh to conform with the rotational and deformable structure mesh from inside and with the fixed fluid mesh from outside. The deformation displacement, $\mathcal{A}_D$, serves this purpose by moving the mesh according to the structure deformation displacement, $\hat{\mathbf{u}}_s - \hat{\mathbf{u}}_\theta$, on the interface $\hat{\Gamma}$, and by locally moving the mesh nodes on the interface $\hat{\Gamma}_{rs} := \partial\hat{\Omega}_{rf} \cap \partial\hat{\Omega}_{sf}$ to match with the fixed fluid mesh. Specifically, we let $\mathcal{A}_D$ satisfy the following modified ALE mapping:

$$\begin{cases} -\Delta \mathcal{A}_D = 0, & \text{in } \hat{\Omega}_{rf}, \\ \mathcal{A}_D = \hat{\mathbf{u}}_s - \hat{\mathbf{u}}_\theta, & \text{on } \hat{\Gamma}, \\ \mathcal{A}_D = \hat{\mathbf{u}}_m, & \text{on } \hat{\Gamma}_{rs}, \end{cases} \quad (33)$$

where $\hat{\mathbf{u}}_m$ needs to be defined such that each sliding mesh node on $\hat{\Gamma}_{rs}$ from the side of $\hat{\Omega}_{rf}$ matches with a fixed mesh node on $\hat{\Gamma}_{rs}$ from the other side of $\hat{\Omega}_{sf}$ by locally moving for such an amount of displacement $\hat{\mathbf{u}}_m$ on $\hat{\Gamma}_{rs}$.

Now, in Fig. 3, we illustrate how to find such a small displacement $\hat{\mathbf{u}}_m$ in the 2D case. The interface is locally shown as a straight line for the purpose of illustration. In practice, the curved interface is approximated by piecewise line segments in 2D or piecewise triangular slices in 3D. The technique shown here is applicable to general cases with piecewise line segments/triangular slices.

Figure 3: Re-matching grid points on the interface Γ_{rs}

Let $\{N_i^r\}_{i=0}^{m-1}$ be coordinates of boundary nodes from $\partial\Omega_{rf}$, and let $\{N_i^{st}\}_{i=0}^{m-1}$ be the counterpart from $\partial\Omega_{sf}$. We assume that, after Ω_{rf} is rotated, N_0^r falls in between N_{K-1}^{st} and N_K^{st} . Then, we can choose the following mapping between the nodes:

$$N_i^r \mapsto N_{[K+i]_m}^{st}.$$

Here, $[m]_n$ denotes the modulo operation. $\hat{\mathbf{u}}_m$ can be correspondingly defined on $\{N_i^r\}$:

$$\hat{\mathbf{u}}_m(x) = N_{[K+i]_m}^{st} - N_i^r, \quad \text{if } x = N_i^r.$$

On $\hat{\Gamma} \setminus \{N_i^r\}$, $\hat{\mathbf{u}}_m$ is defined by linear interpolation of the nodal values. Each boundary node from $\partial\Omega_{rf}$ has a nearby node from $\partial\Omega_{sf}$ to match so that the magnitude of $\hat{\mathbf{u}}_m$ is bounded by the mesh diameter. Therefore, this local shifting of boundary nodes will not cause problems for ALE mapping. Note that it is also possible to match N_i^r with $N_{[K+i-1]_m}^{st}$. The choice depends on practical considerations such as mesh quality.

Via this approach, the ALE mapping preserves the mesh quality. Further, under the mapping, the rotational fluid mesh still conforms with the fixed fluid mesh in Ω_{sf} through Γ_{rs} and with the rotational structure mesh in Ω_s through Γ .

Remark 2. *The modified ALE method we propose here moves the mesh in the entire rotational fluid domain Ω_{rf} to accommodate the displacement of mesh through the interfaces. This approach could be viewed as a shear-slip method [25, 26] followed by a smoothing via solving the ALE equations (33), which further improves the mesh quality.*

Note that there are alternatives to the shear-slip type method. For example, the sliding-interface formulation [39] does not resolve the incompatibility of the discretization near the interface and enforce the continuity of solution fields weakly.

Remark 3. *We cannot simply match each of the nodes from $\partial\Omega_{rf}$ with its nearest node from $\partial\Omega_{sf}$ as this might cause hanging nodes and degenerate elements. We need to move all the interface nodes clockwise uniformly, or counterclockwise uniformly.*

Remark 4. *In 3D, the treatment needed for the rotational fluid mesh is much more sophisticated than in 2D. In order to let structure rotate, the buffer zone is chosen as a cylinder when viewed from outside. At the top and bottom of this cylindrical fluid domain, a natural analogue of the 2D case is to generate a mesh that is radially symmetric about the axis of rotation. Therefore, this mesh can match with the meshes in the fixed fluid domain in a similar way to that shown in Figure 3. However, the radially symmetric meshes result in higher aspect ratios for the meshes near the axis of rotation than for those away from the axis of rotation (Figure 10). Without the radial symmetry, it may be difficult to let the meshes on the interface match. This dilemma is addressed in [26].*

5. Numerical discretizations

In this section, we introduce full discretizations for the weak forms of the FSI model. We discretize the temporal derivative with the characteristic finite difference scheme (16) based on the moving mesh and on the time interval $[0, T]$, which is divided into

$$0 = t_0 < t_1 < \dots < t_N = T.$$

In the space dimension, we adopt the mixed finite element method to discretize the saddle-point problem (32), where we denote $\mathbb{V}_h \subset \mathbb{V}$, $\mathbb{Q}_h \subset \mathbb{Q}$ and $\mathbb{W}_h \subset H_0^1(\hat{\Omega}_f)$ for the finite element spaces of velocity, pressure, and the fluid mesh displacement $\mathcal{A}_u$, respectively.

5.1. A monolithic mixed finite element approximation

Suppose that all the necessary solution data from the last time step, i.e. $\mathbf{v}_f^{n-1}$, $\hat{\mathbf{v}}_s^{n-1}$, $\hat{\mathbf{u}}_s^{n-1}$, w^{n-1} , and $\mathcal{A}^{n-1}$, are known. And, suppose that the mesh on the last time step $\mathbb{T}_h^{n-1}$ is given as $\mathbb{T}_h^{n-1} = \mathbb{T}_{sf,h} \cup \mathbb{T}_{rf,h} \cup \hat{\mathbb{T}}_{s,h}$, where $\hat{\mathbb{T}}_{s,h}$ is the Lagrangian structure mesh in $\hat{\Omega}_s$ which is always fixed; $\mathbb{T}_{sf,h}$ is the mesh in the fixed fluid subdomain Ω_{sf} , which is also always fixed; and $\mathbb{T}_{rf,h}$ is the mesh in the rotational fluid domain Ω_{rf} , which needs to be computed all the time. In addition, we discretize (17) in the time interval $[t_n, t_{n+1}]$ with the trapezoidal quadrature rule as

$$\hat{\mathbf{u}}_s^n = \hat{\mathbf{u}}_s^{n-1} + \frac{\Delta t}{2}(\hat{\mathbf{v}}_s^n + \hat{\mathbf{v}}_s^{n-1}). \quad (34)$$

Then, a monolithic mixed finite element discretization within a fixed-point iteration for both the fluid and the structure equations can be defined as follows: Let $\mathbf{w}^{n,0} = \mathbf{w}^{n-1}$, $\mathbb{T}_{f,h}^{n,0} = \mathbb{T}_{f,h}^{n-1}$, find $\mathbf{v}_f^{n,j}$, $\hat{\mathbf{v}}_s^{n,j}$, $p^{n,j}$, $\mathcal{A}_u^{n,j}$ ($j =$

1, 2, $\dots$ until convergence), satisfying (3), (5), (9), (10), (19), and (20). And, let $\forall(\phi, \hat{\phi}) \in \mathbb{V}_h, q \in \mathbb{Q}_h, \xi \in \mathbb{W}_h$, such that

$$\left\{ \begin{array}{l} \left(\rho_f \frac{\mathbf{v}_f^{n,j} - \mathbf{v}_f^{n-1}}{\Delta t}, \phi \right)_{\Omega_f^{n,j-1}} + \left(\rho_f (\mathbf{v}_f^{n,j} - \mathbf{w}^{n,j-1}) \cdot \nabla \mathbf{v}_f^{n,j}, \phi \right)_{\Omega_f^{n,j-1}} + \left(\mu_f \epsilon(\mathbf{v}_f^{n,j}), \epsilon(\phi) \right)_{\Omega_f^{n,j-1}} \\ - (p^{n,j}, \nabla \cdot \phi)_{\Omega_f^{n,j-1}} + \left(\rho_s \frac{\hat{\mathbf{v}}_s^{n,j} - \hat{\mathbf{v}}_s^{n-1}}{\Delta t}, \hat{\phi} \right)_{\hat{\Omega}_s} + \frac{\Delta t}{2} \left(\mathbf{D}\epsilon((R^n)^T \hat{\mathbf{v}}_s^{n,j}), \epsilon((R^n)^T \hat{\phi}) \right)_{\hat{\Omega}_s} = \\ - \frac{\Delta t}{2} \left(\mathbf{D}\epsilon((R^n)^T \hat{\mathbf{v}}_s^{n-1}), \epsilon((R^n)^T \hat{\phi}) \right)_{\hat{\Omega}_s} - \left(\mathbf{D}\epsilon((R^n)^T \hat{\mathbf{u}}_s^{n-1}), \epsilon((R^n)^T \hat{\phi}) \right)_{\hat{\Omega}_s} \\ + \left(\mathbf{D}\epsilon((I - (R^n)^T)(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)), \epsilon((R^n)^T \hat{\phi}) \right)_{\hat{\Omega}_s}, \\ \left(\nabla \cdot \mathbf{v}_f^{n,j}, q \right)_{\Omega_f^{n,j-1}} = 0, \\ \left(\nabla \mathcal{A}_{\mathbf{u}}^{n,j}, \nabla \xi \right)_{\hat{\Omega}_f} = 0, \end{array} \right. \quad (35)$$

where R^n is the rotational matrix at time t_n given on the axis of rotation Γ_{in} , and

$$\mathbf{w}^{n,j} = \frac{\mathcal{A}_{\mathbf{u}}^{n,j} - \mathcal{A}_{\mathbf{u}}^{n-1}}{\Delta t}, \quad \mathbb{T}_{f,h}^{n,j} = \mathbb{T}_{f,h}^{n-1} + \mathcal{A}_{\mathbf{u}}^{n,j}.$$

To obtain a higher-quality mesh from the ALE mapping, the practical experiences show that a linear anisotropic elasticity equation can produce more high-quality shape-regular mesh than harmonic mapping (18) can, as we can increase the stiffness of small elements to prevent them from becoming distorted [40].

The only nonlinear term in (35) is the convection term $(\rho_f \mathbf{v}_f^{n,j} \cdot \nabla \mathbf{v}_f^{n,j}, \phi)$ in the first equation, which can be linearized by using Newton's method as

$$\begin{aligned} (\rho_f \mathbf{v}_f^{n,j} \cdot \nabla \mathbf{v}_f^{n,j}, \phi) &\leftarrow (\rho_f \mathbf{v}_f^{n,j,k-1} \cdot \nabla \mathbf{v}_f^{n,j,k}, \phi) + (\rho_f \mathbf{v}_f^{n,j,k} \cdot \nabla \mathbf{v}_f^{n,j,k-1}, \phi) \\ &- (\rho_f \mathbf{v}_f^{n,j,k-1} \cdot \nabla \mathbf{v}_f^{n,j,k-1}, \phi), \text{ for } k = 1, 2, \dots \text{ until convergence.} \end{aligned}$$

If we use P_1 - P_1 -type mixed finite elements to discretize (35), then in order to obtain a stable solution of velocity and pressure, the pressure stabilization term, $\delta h^2 (\nabla p^{n,j}, \nabla q)$, originally derived from the Galerkin/least-square scheme [41, 42, 43, 44], needs to be added to the second equation of (35), i.e., the mass equation. The stabilization parameter δh^2 can be chosen as $\frac{\delta_0 h^2}{\mu_f}$, and $0 < \delta_0 < 1$ is an appropriately tuned parameter. In comparison with other stable mixed elements such as the P_2 - P_1 (Taylor-Hood) element, the P_1 - P_1 element with pressure stabilization can save considerable computational cost with acceptable accuracy, especially for 3D FSI problems.

In addition, if the Reynolds number of the fluid is large, then the convection term in the momentum equation of fluid is dominating and the streamline-upwind/Petrov-Galerkin scheme [45] can be used to achieve a stable and convergent solution. That is, extra terms as follows needs to be added to the momentum equation:

$$\sum_{\tau \in T_h} \frac{\delta_{SUPG} h}{\|\mathbf{v}_f^{n,j} - \mathbf{w}^{n,j-1}\|_{\infty}} \left(R(\mathbf{v}_f^{n,j}, \mathbf{w}^{n,j-1}, p^{n,j}), (\mathbf{v}_f^{n,j} - \mathbf{w}^{n,j-1}) \cdot \nabla \phi \right)_{\tau}$$

with the residual $R(\mathbf{v}_f^{n,j}, \mathbf{w}^{n,j-1}, p^{n,j})$ of the strong form of fluid equations in FSI and an appropriately tuned parameter δ_{SUPG} . The summation is over each element τ of the triangulation T_h . For more details on $SUPG$ and related methods, we refer to [45, 46, 44].

5.2. Well-posedness of the discrete linear system

If Stokes-stable finite element pairs are used to discretize velocity and pressure, then robust block preconditioners can be developed following the approach in [11]. We consider the linearization of (35) and use the notation convention from [11], where $\mathbf{v} = (\mathbf{v}_f, \hat{\mathbf{v}}_s)$. That is, the velocity has two components: the fluid velocity in Eulerian coordinates and the structure velocity in Lagrangian coordinates. We assume that the

discretized function spaces $\mathbb{V}_h$ and $\mathbb{Q}_h$ are Stokes stable. For brevity, we also assume that the fixed fluid subdomain vanishes; i.e., the entire fluid domain constitutes the rotational fluid domain. The technique shown here can be applied to general cases without any essential difficulty.

We consider the linearization of the fluid-structure equations of (35). Note that the ALE mesh motion is not included in the equations under consideration. Define

$$\begin{aligned} a(\mathbf{v}, \phi) &= \frac{1}{\Delta t} (\rho_f \mathbf{v}_f, \phi_f)_{\Omega_f} + \frac{1}{\Delta t} (\hat{\rho}_s \hat{\mathbf{v}}_s, \hat{\phi}_s)_{\hat{\Omega}_s} + (\mu_f \epsilon(\mathbf{v}_f), \epsilon(\phi_f))_{\Omega_f} \\ &\quad + \Delta t (D\epsilon(R^T \hat{\mathbf{v}}_s), \epsilon(R^T \hat{\phi}_s))_{\hat{\Omega}_s}, \\ c(\mathbf{v}, \phi) &= \int_{\Omega_f} \rho_f (\eta \cdot \nabla) \mathbf{v}_f \cdot \phi_f + \int_{\Omega_f} \rho_f (\mathbf{v}_f \cdot \nabla) \mathbf{z} \cdot \phi_f, \end{aligned}$$

and

$$b(\mathbf{v}, p) = (\nabla \cdot \mathbf{v}_f, p)_{\Omega_f},$$

where $a(\cdot, \cdot)$ denotes the symmetric terms on the velocity field, and $c(\cdot, \cdot)$ denotes the nonsymmetric terms due to linearization; and η and $\mathbf{z}$ are given functions based on previous iteration steps. Then, the momentum and continuity equations in (35) can be formulated as follows:

Find $\mathbf{v} \in \mathbb{V}_h$ and $p \in \mathbb{Q}_h$ such that

$$\begin{cases} \tilde{a}(\mathbf{v}, \phi) + b(\phi, p) &= (\tilde{\mathbf{g}}, \phi), \quad \forall \phi \in \mathbb{V}_h, \\ b(\mathbf{v}, q) &= 0, \quad \forall q \in \mathbb{Q}_h, \end{cases} \quad (36)$$

where $\tilde{a}(\mathbf{v}, \phi) = a(\mathbf{v}, \phi) + c(\mathbf{v}, \phi)$ and $\tilde{\mathbf{g}}$ denotes the summation of all the known terms after linearization in the first equation of (35). Note that we rescale the pressure by multiplying -1 such that the problem can be formulated as a symmetric saddle point problem. This variational problem is similar to those studied in [11]. The only difference is the elastic stress term, where R is added due to the rotation of the structure. However, well-posedness can still be proved in a similar way.

It is well-known that (36) is well-posed if the following conditions can be verified [47]:

•

$$\tilde{a}(\cdot, \cdot) \text{ is bounded and coercive in } \mathbb{Z}_h := \{\mathbf{v} \in \mathbb{V}_h | (\nabla \cdot \mathbf{v}, q) = 0, \forall q \in \mathbb{Q}_h\}, \quad (37)$$

•

$b(\cdot, \cdot)$ is bounded and satisfies the inf-sup condition

$$\inf_{p \in \mathbb{Q}_h} \sup_{\mathbf{v} \in \mathbb{V}_h} \frac{b(\mathbf{v}, p)}{\|\mathbf{v}\|_V \|p\|_Q} \geq \beta > 0. \quad (38)$$

Here, $\|\cdot\|_V$ and $\|\cdot\|_Q$ are given norms of $\mathbb{V}_h$ and $\mathbb{Q}_h$, respectively. Similar to [11], we define

$$\begin{aligned} \|\mathbf{v}\|_V^2 &:= a(\mathbf{v}, \mathbf{v}) + r \|\mathcal{P}_h \nabla \cdot \mathbf{v}_f\|_{0, \Omega_f}^2, \quad \forall \mathbf{v} \in \mathbb{V}_h, \\ \|q\|_Q^2 &:= r^{-1} \|q\|_0^2, \quad \forall q \in \mathbb{Q}_h, \end{aligned}$$

where $\mathcal{P}_h$ represents the L^2 projection from $\mathbb{Q}$ to $\mathbb{Q}_h$, and $r = \max\{1, \mu_f, \rho_f \Delta t^{-1}, \hat{\rho}_s \Delta t^{-1}, \Delta t \mu_s, \Delta t \lambda_s\}$.

First, we have the following lemma from [11]. Let $\mathbb{V}_{h,f}$ be the discrete fluid velocity space, and let $\mathbf{x}(\hat{\mathbf{x}}, t)$ denote the fluid mesh motion. In fact, $\mathbf{x}(\hat{\mathbf{x}}, t)$ is the total mesh displacement of the fluid computed by ALE mapping: $\mathcal{A}_u = \hat{\mathbf{u}}_\theta + \mathcal{A}_D$.

Lemma 1. Assume that $\mathbf{x}(\hat{\mathbf{x}}, t)$ is continuous and satisfies

$$\mathbf{x}(\hat{\mathbf{x}}, t)|_\tau \in \mathcal{P}_1, \quad \forall \tau \in T_h(\hat{\Omega}_s) \quad \text{and} \quad \inf_{\hat{\mathbf{x}} \in \hat{\Omega}_s} \det(\nabla_{\hat{\mathbf{x}}} \mathbf{x}(\hat{\mathbf{x}}, t)) > 0,$$

and that the finite element pair $(\mathbb{V}_{h,f}, \mathbb{Q}_h)$ for the fluid variables satisfies that

$$\inf_{q \in \mathbb{Q}_h} \sup_{\mathbf{v}_f \in \mathbb{V}_{h,f}} \frac{(\nabla \cdot \mathbf{v}_f, q)_{\Omega_f}}{\|\mathbf{v}_f\|_1 \|q\|_0} \geq \beta > 0. \quad (39)$$

Then, the following inf-sup condition holds:

$$\inf_{q \in \mathbb{Q}_h} \sup_{\mathbf{v} \in \mathbb{V}_h} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1 \|q\|_0} \geq \frac{\beta}{\alpha_0^{d/2+1} \alpha_1}, \quad (40)$$

where

$$\alpha_0 = \max \left\{ \sup_{\hat{\mathbf{x}} \in \Gamma} \|(\nabla_{\hat{\mathbf{x}}} \mathbf{x}(\hat{\mathbf{x}}, t))\|_2, 1 \right\}, \quad \alpha_1 = \max \left\{ \sup_{\hat{\mathbf{x}} \in \Gamma} \{ \det(\nabla_{\hat{\mathbf{x}}} \mathbf{x}(\hat{\mathbf{x}}, t))^{-1} \}, 1 \right\}. \quad (41)$$

The following theorem shows that (36) is well-posed under certain conditions.

Theorem 1. Assume the following conditions hold:

- The time step size Δt is small enough such that the following inequality holds:

$$C_K (\Delta t \rho_f / \mu_f)^{1/2} \|\eta\|_\infty + \Delta t \|\nabla \mathbf{z}\|_\infty \leq c_0 < 1, \quad (42)$$

where $0 < c_0 < 1$ is a constant. Here, C_K is a constant such that $\|\nabla \mathbf{u}_f\| \leq C_K \|\epsilon(\mathbf{u}_f)\|$ holds for $\forall \mathbf{u}_f$. Moreover, $\|\eta\|_\infty = \text{ess sup}_{\mathbf{x} \in \Omega_f} \|\eta(\mathbf{x})\|_2$, $\|\nabla \mathbf{z}\|_\infty = \text{ess sup}_{\mathbf{x} \in \Omega_f} \|\nabla \mathbf{z}(\mathbf{x})\|_2$ such that

- The assumptions in Lemma 1 hold.
- At a given time step t^n , α_0, α_1 and c_0 are independent of material and discretization parameters.

Then, under the norms $\|\cdot\|_V$ and $\|\cdot\|_Q$ the variational problem (36) at the given time step t^n is uniformly well-posed with respect to the material and discretization parameters.

Proof. We prove this theorem by verifying Brezzi's conditions (37) and (38). First, we verify the boundedness of $\tilde{a}(\cdot, \cdot)$ and its coercivity in $\mathbb{Z}_h$. By definition, we first have

$$\begin{aligned} a(\mathbf{v}, \phi) &\leq \|\mathbf{v}\|_V \|\phi\|_V, & \forall \mathbf{v}, \phi \in \mathbb{V}_h, \\ a(\mathbf{v}, \mathbf{v}) &\geq \|\mathbf{v}\|_V^2, & \forall \mathbf{v} \in \mathbb{V}_h. \end{aligned}$$

Based on the first assumption (42), $c(\cdot, \cdot)$ is a small perturbation, as shown by the following estimates:

$$\begin{aligned} \left| \int_{\Omega_f} \rho_f (\eta \cdot \nabla) \mathbf{u}_f \cdot \mathbf{v}_f \right| &\leq C_K \left(\frac{\Delta t \rho_f}{\mu_f} \right)^{1/2} \|\eta\|_\infty \|\mathbf{u}\|_V \|\mathbf{v}\|_V, \\ \left| \int_{\Omega_f} \rho_f (\mathbf{u}_f \cdot \nabla) \mathbf{z} \cdot \mathbf{v}_f \right| &\leq \Delta t \|\nabla \mathbf{z}\|_\infty \|\mathbf{u}\|_V \|\mathbf{v}\|_V, \\ |c(\mathbf{u}, \mathbf{v})| &\leq \left(C_K (\Delta t \rho_f / \mu_f)^{1/2} \|\eta\|_\infty + \Delta t \|\nabla \mathbf{z}\|_\infty \right) \|\mathbf{u}\|_V \|\mathbf{v}\|_V \leq c_0 \|\mathbf{u}\|_V \|\mathbf{v}\|_V. \end{aligned} \quad (43)$$

Then, we have the boundedness and coercivity of $\tilde{a}(\mathbf{v}, \phi) = a(\mathbf{v}, \phi) + c(\mathbf{v}, \phi)$:

$$\begin{aligned} \tilde{a}(\mathbf{v}, \mathbf{v}) &\geq (1 - c_0) \|\mathbf{v}\|_V^2, & \forall \mathbf{v} \in \mathbb{Z}_h, \\ \tilde{a}(\mathbf{v}, \phi) &\leq (1 + c_0) \|\mathbf{v}\|_V \|\phi\|_V, & \forall \mathbf{v}, \phi \in \mathbb{V}_h. \end{aligned} \quad (44)$$

The boundedness of $b(\cdot, \cdot)$ is shown by

$$b(\mathbf{v}, q) \leq \|\mathcal{P}_h \nabla \cdot \mathbf{v}\|_{0, \Omega_f} \|q\|_0 \leq r^{1/2} \|\mathcal{P}_h \nabla \cdot \mathbf{v}\|_{0, \Omega_f} r^{-1/2} \|q\|_0 \leq \|\mathbf{v}\|_V \|q\|_Q. \quad (45)$$

Therefore, we only need to prove the inf-sup condition of $b(\cdot, \cdot)$.

As we have

$$\|\epsilon(R^T \hat{\mathbf{v}}_s)\|^2 \leq C \|\nabla(R^T \hat{\mathbf{v}}_s)\|^2 = C \|\nabla \hat{\mathbf{v}}_s\|^2,$$

the following inequality holds:

$$\|\mathbf{v}\|_V \leq Cr^{1/2} \|\mathbf{v}\|_{1,\Omega}, \quad \forall \mathbf{v} \in \mathbb{V}_h. \quad (46)$$

Lemma 1 indicates that

$$\inf_{q \in \mathbb{Q}_h} \sup_{\mathbf{v} \in \mathbb{V}_h} \frac{(\nabla \cdot \mathbf{v}, q)_{\Omega_f}}{\|\mathbf{v}\|_V \|q\|_Q} \geq \frac{C}{\alpha_0^{d/2+1} \alpha_1}.$$

As α_0 and α_1 are independent of material and discretization parameters, the inf-sup constant is uniformly bounded below. Therefore, we have shown that (36) is uniformly well-posed with respect to the material and discretization parameters. $\square$

Remark 5. *Based on the well-posedness shown above, we can also derive robust block preconditioners for the coupled fluid-structure equations as in [11]. However, we do not elaborate on this topic in this paper, as the mixed finite element we use for the numerical experiments in Section 7 is the P_1 - P_1 type with pressure stabilization in order to save what would otherwise be huge computational cost for 3D. A consideration of robust block preconditioners for P_1 - P_1 discretization of FSI with pressure stabilization will be included in our future research. In our numerical experiments, we adopt some commonly used block preconditioners for saddle-point problems, more details of which will be shown in the next section. For solving FSI discretized by P_1 - P_1 FEM, we refer to some recent works [48, 49].*

6. Algorithm description

In this section, we describe a monolithic algorithm involving a relaxed fixed-point iteration for the FSI simulation at the current n -th time step. Roughly speaking, we conduct a fixed-point iteration between the coupled system of fluid-structure equations and the ALE mapping equation, where the fluid-structure coupling system is solved together, i.e., in the monolithic fashion. Basically, we solve the coupled equations of fluid and structure on a known mesh $\mathbb{T}_{f,h} \cup \hat{\mathbb{T}}_{s,h}$ by a nonlinear iteration until convergence, where, $\mathbb{T}_{f,h}$ is the fluid mesh updated from the previous iteration step and $\mathbb{T}_{s,h}$ is the fixed structure mesh. Then, we update the fluid mesh $\mathbb{T}_{f,h}$ by solving the ALE mapping based on the new solution of the structure velocity on the interface Γ , and so on. This fixed-point iteration is continued until the fluid mesh converges. Then, we go to the next time step. Details are shown in Algorithm 1, where the rotation matrix R , depending on the time only, is given if the angular velocity is prescribed on the axis of rotation for the active rotation case. In the case of passive rotation, R is unknown and depends on both space and time; therefore, it needs to be updated at each iteration step. We will consider the passive rotation case in our future research.

Algorithm 1 ALE method for FSI involving an elastic rotor

At the n -th time step, let $\mathbf{w}^{n,0} = \mathbf{w}^{n-1}$, $\mathbb{T}_{f,h}^{n,0} = \mathbb{T}_{f,h}^{n-1}$.

for $j = 1, 2, \dots$ **until** *convergence* **do**

1. Solve (35) for $(\mathbf{v}_f^{n,j}, p^{n,j}, \hat{\mathbf{v}}_s^{n,j})$.
2. Compute $\hat{\mathbf{u}}_{s,*}^{n,j}$ with (34).
3. Compute $\hat{\mathbf{u}}_\theta^{n,j} = (R^n - \mathbf{I})(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0)$.
4. Find $\hat{\mathbf{u}}_m^{n,j}$ on $\hat{\Gamma}_{rs}$ by the searching scheme shown in Section 4.
5. Update the displacements on the interface:

$$\begin{aligned}\hat{\mathbf{u}}_{s,*}^{n,j} &= (1 - \omega)\hat{\mathbf{u}}_{s,*}^{n,j-1} + \omega\hat{\mathbf{u}}_s^{n,j}, & \text{on } \hat{\Gamma}, \\ \hat{\mathbf{u}}_\theta^{n,j} &= (1 - \omega)\hat{\mathbf{u}}_\theta^{n,j-1} + \omega\hat{\mathbf{u}}_\theta^{n,j}, & \text{on } \hat{\Gamma},\end{aligned}$$

where $\omega \in (0, 1]$ is the relaxation number.

6. Solve the ALE mapping equation (33) for $\mathcal{A}_D^{n,j}$ with $\hat{\mathbf{u}}_{s,*}^{n,j} - \hat{\mathbf{u}}_{\theta,*}^{n,j}$ and $\hat{\mathbf{u}}_m^{n,j}$ as the Dirichlet boundary conditions on $\hat{\Gamma}$ and $\hat{\Gamma}_{rs}$, respectively.
 7. Calculate the total fluid mesh displacement $\hat{\mathbf{u}}_f^{n,j}$ with $\hat{\mathbf{u}}_f^{n,j} = \hat{\mathbf{u}}_{\theta,*}^{n,j} + \mathcal{A}_D^{n,j}$, with which the rotational fluid mesh, $\mathbb{T}_{rf,h}^{n,j}$, is updated as $\mathbb{T}_{rf,h}^{n,j} = \mathbb{T}_{rf,h}^{n-1} + \hat{\mathbf{u}}_f^{n,j}$.
 8. Update the fluid mesh velocity $\mathbf{w}^{n,j}$ by $\mathbf{w}^{n,j} = \frac{\hat{\mathbf{u}}_f^{n,j} - \hat{\mathbf{u}}_f^{n-1}}{\Delta t}$.
 9. Check if the iteration converges. If not, let $j \leftarrow j + 1$, and continue the iteration. If yes, let $n \leftarrow n + 1$, and go to the next time step.
-

The most time-consuming aspect of Algorithm 1 is solving (35). Therefore, an efficient and robust linear solver is crucial to speed up the simulation. In the rest of this section, we briefly introduce the preconditioning technique we used in our solver. For the convenience of notation, we denote the linear system of the coupled fluid-structure system as

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad (47)$$

where v and p are the vectors corresponding to velocity and pressure, respectively. The block matrices A and B arise from the corresponding discretizations of the bilinear forms $\tilde{a}(\cdot, \cdot)$ and $b(\cdot, \cdot)$, respectively. The nonzero block C is from the pressure stabilization if the P_1 - P_1 mixed element is used, otherwise $C = \emptyset$.

We use the flexible GMRes as the iterative solver for the coupled system with a block triangular preconditioner P , which is an approximation of

$$\begin{pmatrix} A & 0 \\ B & -S \end{pmatrix}^{-1},$$

where $S := C + B \text{diag}(A)^{-1} B^T$ is an approximation of the Schur complement $C + BA^{-1}B^T$. We define the action of $\mathbb{P}$ as follows. Given $\begin{pmatrix} f \\ g \end{pmatrix}$, we obtain $\begin{pmatrix} v \\ p \end{pmatrix} = \mathbb{P} \begin{pmatrix} f \\ g \end{pmatrix}$ by the following procedure:

1. Solve $Av = f$ using the algebraic multigrid preconditioned GMRes. In particular, we use the unsmoothed aggregation AMG with ILU and Gauss-Seidel smoothers.
2. Solve $Sp = -g + Bv$ using the algebraic multigrid preconditioned GMRes. Here, we use the unsmoothed aggregation AMG with the Gauss-Seidel smoother.

There is an extensive literature on fast solvers for the monolithic FSI simulation. We refer to [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 11] for details. In particular, when stable mixed finite element pairs are used, we can develop robust block preconditioners based on the well-posedness shown in Theorem 1. (See [11].)

Remark 6. Because of the simultaneous solution of both fluid and structure in the monolithic method, the interface conditions are naturally enforced. Therefore, the resulting algorithm is much more stable than partitioned algorithms. In particular, if the densities of fluid and structure become comparable or if the domain has a slender shape, the partitioned method turns out to be unconditionally unstable with an explicit scheme, and an oscillating iteration with questionable convergence when an implicit scheme is used. This phenomenon is also referred to as the added-mass effect [60], an effect that monolithic algorithms can avoid [61, 62, 63, 64].

7. Numerical experiments

To verify the correctness and efficiency of our proposed model and numerical methods, we first introduce a simplified two-dimensional hydro-turbine with the shape of a cross immersed in the laminar flow that is fully developed in a straight open channel. As shown in Fig. 4, the cross lies close to the inlet and stays in the same plane with the flow channel, the axis of rotation of the cross is orthogonal to this plane. Therefore, the side of the cross faces the incoming flow, rendering the cross to rotate about its axis of rotation as well as to deform if the fluid impact is strong enough. Further, we define a circular fluid subdomain $\Omega_{rf} \subset \Omega_f$ as

Figure 4: The 2D self-defined elastic rotor: $\Omega = \Omega_s + \Omega_f$ and $\Omega_f = \Omega_{sf} + \Omega_{rf}$, and the computational mesh in which two interfaces, $\Gamma_{rs} = \partial\Omega_{rf} \cap \partial\Omega_{sf}$ (red) and $\Gamma = \partial\Omega_{rf} \cap \partial\Omega_s$ (blue), exist between three subdomains.

Table 1: Geometrical, physical, and operating parameters

<i>Parameters/properties</i>	<i>Symbol</i>	<i>Value</i>	<i>Unit</i>
<i>Modeling domain dimensions</i>			
Channel length	L_{channel}	0.5	m
Channel width (height)	W_{channel}	0.2	m
Cross length	L_{cross}	0.1	m
Cross width	W_{cross}	0.015	m
Cross thickness	T_{cross}	0.05	m
<i>Physical and transport parameters</i>			
Dynamic viscosity of fluid	μ_f	1.0	kg/(m·s)
Density of fluid	ρ_f	1000	kg/m ³
Density of structure	ρ_s	1280	kg/m ³
Young's modulus	E	2.5×10^6	Pa
Poisson's ratio	ν	0.384	
<i>Operating parameters</i>			
Maximum incoming velocity	$\mathbf{u}_{\text{in}}$	1.5	m/s
Prescribed angular velocity	ω	1.0	rad/s
Tolerance of nonlinear iteration	ε_{tol}	10^{-6}	

the buffer zone to enclose the cross Ω_s and rotate together on the same axis of rotation. Let $\Omega_{sf} := \Omega_f \setminus \Omega_{rf}$ denote the rest of the fluid domain that is fixed. Moreover, all meshes are conforming through the interface

between Ω_{sf} and Ω_{rf} , Γ_{rs} , and the interface between Ω_{rf} and Ω_s , Γ , we are then able to apply the Master-Slave relation to the pairs of grid points on interfaces. We employ the approach shown in Fig. 3 to re-match each point pair on Γ_{rs} when they rotate away from each other in the computation. Therefore, the entire fluid mesh retains conformity all the time, as illustrated in Fig. 5, where, the moving quasi-uniform triangulation inside the buffer zone Ω_{rf} , $\mathcal{T}_h(\Omega_{rf})$, is computed by means of the ALE equation (33). Due to the setup of boundary conditions in (33), $\mathcal{T}_h(\Omega_{rf})$ remains conforming with the structure mesh $\mathcal{T}_h(\Omega_s)$ through Γ and with the fixed fluid mesh $\mathcal{T}_h(\Omega_{sf})$ through Γ_{rs} at all times while rotating together with $\mathcal{T}_h(\Omega_s)$ on the same axis of rotation, Γ_{in} .

Figure 5: A moving mesh inside the buffer zone Ω_{rf} (blue region) with time that always remains conforming with the rotating structure mesh inside Ω_s (yellow region) and the fixed fluid mesh inside Ω_{sf} (red region).

Figure 6: A self-defined 2D elastic rotor: the evolution of magnitude of velocity with time.

The geometrical and physical parameters of this model problem are given in Table 1. The hydro-turbine rotor is simulated under the prescribed angular velocity of 1 rad/s at the axis of rotation of rotor Γ_{in} and the prescribed steady incoming flow, which is described as a parabolic velocity function at the inlet with a

maximum value of 1.5 m/s. The incoming flow at the inlet and the angular velocity at Γ_{in} are responsible for the Dirichlet boundary conditions of the fluid velocity and structure velocity, respectively. In addition, the no-slip boundary condition is applied to the wall and the do-nothing boundary condition [65] is applied to the fluid velocity at the outlet.

Then, by means of Algorithm 1, the P_1 - P_1 mixed element with pressure stabilization to discretize the momentum and mass equations, and the linear finite element to discretize the ALE equation, we are able to attain the following numerical results illustrated in Fig. 6 that is based upon the initial mesh shown in Fig. 4 with the mesh size $h \leq 0.05\text{m}$ and the time step size $\Delta t = 0.01\text{s}$. Fig. 6 shows the evolution of cross and its surrounding fluid flow in 2.5s during which the cross, that is impacted by the incoming flow from the left, keeps spinning forward (in clockwise direction) around its axis of rotation at the prescribed angular velocity. On the other hand, due to the interaction of fluid and cross, the cross's motion impacts the fluid flow as well, producing dramatic changes of flow field in fine scale surrounding the cross and in large scale along the channel.

To validate a convergent approximation of the obtained numerical results, we compute the above set up 2D example on a series of meshes with a decreasing mesh size h (or an increasing number of mesh nodes) and a decreasing time step size Δt which basically satisfies $\Delta t = O(h)$, and measure the maximum error of structure displacement on the interface Γ at each mesh level by comparing with its numerically “true” solution that is computed with much smaller h and Δt at the same time point. The maximum error curves of structure displacement on Γ versus time and mesh levels are plotted in Fig. 7, where, after the initial period of preparation time 0.5s, the approximation error decreases along with an increasing number of mesh node (due to a decreasing h) at the same point of time, and, the approximation remains convergent for the rest period of time with respect to h and Δt . Once the structure displacement is accurately approximated on the interface, considering that the interface condition plays such a crucial role to correctly simulate the interaction mechanism between fluid and structure, we may be able to use the approximation property of structure displacement on the interface to estimate the convergence of finite element solution in the entire domain to some extent.

Figure 7: Maximum error curves of structure displacement (in the magnitude of 10^{-5}) on the interface Γ versus time and different mesh levels with increasing number of nodes.

In the following, we numerically study the influence of the size of buffer zone, Ω_{rf} , on the mesh quality, further, on the accuracy of finite element solutions. Mesh quality is defined as the minimum radius ratio (inradius over circumradius) of all mesh cells, it is well known that a good mesh quality, which represents the shape of all mesh cells are regular, certainly improves the accuracy of finite element approximation. In the above 2D example, the size of circular buffer zone Ω_{rf} is determined by its radius R_b . We adopt different radii of Ω_{rf} to conduct a series of computations for the above 2D example, and compute the mesh quality at

each time step for different meshes $\mathcal{T}_h(\Omega_{r,f})$ which are generated by ALE mapping, together with the fixed fluid mesh $\mathcal{T}_h(\Omega_{s,f})$. Fig. 8 illustrates the relationship of the mesh quality versus time and different radii R_b of $\Omega_{r,f}$, where, if $R_b = 0.1$ or larger, mesh quality curves show a relatively horizontal line with time, i.e., the mesh quality barely changes with time, achieving its steady state. And, if R_b is as large as 0.19, we notice that the mesh quality is lower than that of $R_b = 0.1$, and even lower than that of $R_b = 0.08$ when its mesh quality reaches the steady state. This is because the boundary of buffer zone is closer to the outer boundary of flow channel and make the fixed fluid mesh's quality worse within the thin area between the buffer zone and the outer boundary. On the other hand, if R_b increases, then the number of mesh node inside the buffer zone increases too, resulting in a higher computational cost of ALE mapping as well. If R_b is as small as 0.06 – 0.08, Fig. 8 shows a dramatic drop on mesh qualities before their respective steady states are reached, that is because a smaller R_b leads to a small distance between the side boundaries of buffer zone and structure, within that thin area the ALE mapping can reduce the quality of mesh cells. Therefore, we generally use the optimal value $R_b = 0.1$, which produces the best mesh quality and bears a less expensive computational cost, in other numerical computations for the self-defined 2D example.

Figure 8: Mesh quality curves versus time and different radii (R_b) of buffer zone $\Omega_{r,f}$.

Next, we numerically study a self-defined three-dimensional hydro-turbine with the shape of a curving cross immersed in the laminar flow fully developed in a straight open channel. As shown in Fig. 9, the cross lies close to the inlet and rotates about its rigid axis of rotation in the plane. The axis of rotation is parallel to the flow direction, making the cross faces the incoming flow and may deform to some extent due to the fluid impact. Similarly, we define a cylindrical fluid subdomain $\Omega_{r,f} \subset \Omega_f$ as the buffer zone to enclose the

Figure 9: The 3D self-defined elastic rotor: $\Omega = \Omega_s + \Omega_f$ and $\Omega_f = \Omega_{s,f} + \Omega_{r,f}$ and the computational mesh of the half domain, where two interfaces, $\Gamma_{rs} = \partial\Omega_{r,f} \cap \partial\Omega_{s,f}$ and $\Gamma = \partial\Omega_{r,f} \cap \partial\Omega_s$, exist between three subdomains.

elastic rotor Ω_s and rotate together on the same axis of rotation, and to be embraced by the fixed fluid subdomain Ω_{sf} . We apply the Master-Slave relation to the pairs of grid points on interfaces Γ_{sf} and Γ along with a certain triangulation pattern in three-dimensional case. This triangulation pattern on the interface Γ_{rs} is defined in a specific way in 3D such that it is easy to search and reconnect each pair of the grid point when the slave point on one side ($\partial\Omega_{rf}$) rotates away from its master point on the other side ($\partial\Omega_{sf}$). For instance, we need to choose an axisymmetric domain as the rotational fluid domain Ω_{rf} . Therefore, the cross-section perpendicular to the central axis of Ω_{rf} is always a disk on which the grid pattern can be triangulated as shown in Fig. 10 in order to easily search the paired grid points on Γ_{rs} .

Figure 10: The triangulation pattern on the interface of the rotational fluid domain Ω_{rf} and the fixed fluid subdomain Ω_{sf} .

Thereafter, we employ the same numerical approach as we do for the 2D example, but in three dimension, for the above set up 3D fluid-rotating structure interaction problem, then we obtain the following numerical results shown in Figs. 11 and 12 with the time step size $\Delta t = 0.01s$ and the mesh size $h < 0.02m$. Fig. 11

Figure 11: A self-defined 3D elastic rotor: the evolution of magnitude of velocity with time in 1/4 fluid domain.

illustrates the evolution of the magnitude of the velocity field with time marching in a quarter part of the

fluid domain along the flow direction, while the incoming fluid keeps flowing in the inlet with the prescribed maximum velocity magnitude, 1.5 m/s, and is detoured by the elastic rotor, which continues spinning around its axis of rotation in the plane perpendicular to the flow direction at the prescribed angular velocity. With these given steady velocities, we can observe that the entire fluid field achieves a relatively steady state after 2s, that the fluid flow is significantly disturbed near the spinning turbine, and that this disturbance diminishes with increasing distance from the turbine. The magnitude of the velocity field forms a nearly vacuum region behind the turbine in which the velocity magnitude is much smaller than elsewhere due to the lower pressure there, and the portion of this region that is nearest to the turbine is periodically twisted by the spin. Fig. 12 shows both the streamline field and the velocity vector field at 19s. This figure shows the disturbance status of the fluid right after it flows over the spinning turbine: the streamline is twisted away from the mainstream near the turbine due to its spin, and then merges back again when the fluid is relatively far away from the turbine, and continues to flow straight to the outlet.

Note that the magnitude of the Reynolds number used in the above 2D and 3D examples is just 100 or so, resulting in a truly laminar flow. Our focus in the methodology study is on simulating the dynamical interaction between an elastic rotor and the fluid, such that the fluid with a large Reynolds number or even turbulence flow is not considered herein. In our future research, we will consider a turbulence model for fluids with a large Reynolds number. However, it should be noted that the ALE approach and the monolithic algorithm developed in this paper are valid for simulating the interaction between the turbulent flow and a hydro-turbine.

Figure 12: A self-defined elastic rotor: magnitude of velocity field on the central cross section along flow direction (left); streamline (middle) and velocity (right) fields at 19s.

It is not easy to observe the deformation of the elastic rotor, as its stiffness is measured by a relatively large Young's modulus (2.5 MPa) in this example. To show the deformation of a spinning rotor due to the fluid's impact, we need to look at some extreme circumstances, e.g., a flexible and/or a thinner rotor with a smaller Young's modulus, in order to observe a dramatically large deformation on the hydro-turbine blades while they are rotating. Fig. 13 illustrates the desired numerical results, where the self-defined flexible and thinner blades with a relatively smaller Young's modulus deform dramatically during rotation.

Now, we conduct a consistency check for the developed structure model. We carry out a series of numerical computations for the fluid-turbine interaction with respect to an increasing Young's modulus of the turbine: for instance, from $E = 2.5 \times 10^4$ Pa all the way to $E = 2.5 \times 10^9$ Pa, which means the turbine tends to become stiffer and stiffer such that it becomes a rigid body. Because a rigid body does not deform, the deformation displacement at any place on the turbine can be expected to approach zero along with an increasing Young's modulus. To that end, we just need to check the displacement component in the flow direction, as there are no rotational displacement components in that direction. By the configuration, we set the turbine to spin only in the plane that is perpendicular to the flow direction. In addition, as the thinnest part of the turbine, the blade tip becomes more deformed during the interaction with the fluid than the other parts of the turbine do. Thus, by checking the variation tendency of the blade tip displacement in the flow direction along with the increasing Young's modulus, we are able to validate our developed structure model in terms of asymptotic stiffness. Fig. 14 shows a semi-logarithmic plot of the variation of blade-tip displacement in the flow direction along with time and the increasing Young's modulus, where the displacement component

Figure 13: A more flexible and thinner elastic rotor, which deforms while its blades rotate: the evolution of magnitude of the velocity field with time on the central cross-section along the flow direction.

is logarithmically scaled in order to distinctly illustrate that the blade-tip displacement approaches zero while the Young's modulus becomes much larger at any instantaneous time. Thus, we demonstrate that the developed structure model is consistently correct and the developed numerical method is convergent with respect to a crucial physical parameter of structure that may vary, asymptotically.

Figure 14: The variation tendency of blade-tip displacement in the flow direction at the central line of the rotor thickness along with the increasing Young's modulus and time, where the ordinate is scaled logarithmically.

Finally, we apply our validated monolithic ALE method to a practical fluid-structure interaction problem involving a realistic 3D hydro-turbine, which is still under construction for a new hydro plant. As illustrated in Figs. 15 and 16, a realistic turbine bears five blades curved in a 3D fashion and is immersed in the fluid domain in the shape of circular truncated cone. Thus, we take advantage of the shape of the fluid flow channel and make the entire fluid domain rotational, such that there is no fixed fluid subdomain in this example. In this way, we simply solve the ALE mapping equation in the entire fluid domain to obtain the entire fluid mesh with the zero-boundary condition on the outer boundary of $\hat{\Omega}_f$. Under the same prescribed incoming velocity and angular velocity as given for the previous self-defined rotor, we obtain similar illustrations for

the numerical results shown in Figs. 17 and 18. Fig. 17 displays the development of velocity magnitude with time marching in a quarter part of the fluid domain along the flow direction, and Fig. 18 shows the velocity field in the streamline as well as the vector version.

Figure 15: A realistic 3D turbine.

Figure 16: The realistic turbine immersed in a fluid domain in the shape of a circular truncated cone: $\Omega = \Omega_s + \Omega_f$ and $\Omega_f = \Omega_{rf}$; there is no fixed fluid subdomain Ω_{sf} ; and the only interface is $\Gamma = \partial\Omega_f \cap \partial\Omega_s$.

Remark 7. *In some ways, our current model and numerical method also work for the passive rotational case, i.e., for which there is not a prescribed angular velocity on the axis of rotation. In this case, we do not know the structure velocity on the cylindrical axis of rotation Γ_{in} . In order to make the initially static turbine spin up while the fluid flow starts to impact, we can shrink the cylindrical axis of rotation as a point axis (or a line axis in 3D) on its barycenter, and fix the structure displacement and then velocity as zero on the point/line axis. Thus, if the external fluid force acts on the turbine, the turbine is not carried away by the fluid flow but is likely to rotate about the point/line axis. This configuration is inaccurate from a mathematical point of view but works numerically to some extent. We are, therefore, still able to apply our developed model and numerical methods to the FSI problem involving a passively rotational hydro-turbine and to observe the following: the elastic turbine starts to spin up from its initially static position and accelerates its rotation until a steady rotational state is reached; i.e., if no other external torque is exerted on the turbine, the fluid impact force acting on the turbine attains a balance with the resistance force arising from the fluid itself due to the viscosity.*

However, we do not illustrate the numerical results of the passive rotational case in this paper since the inaccurate configuration of $(d-2)$ -dimensional axis, where d is the number of the space dimension, does not make full mathematical sense and might introduce an extra stress concentration effect on such a $(d-2)$ -dimensional axis under a large external load. Recently, we developed a novel and more accurate numerical method for the passive rotational case, which will be the subject of a forthcoming paper.

8. Conclusion and Future work

We build an Eulerian-Lagrangian model for the fluid-structure interaction problem involving an elastic rotor based on the arbitrary Lagrangian Eulerian (ALE) approach. Using velocity as the principle unknown in the structure equation, we are able to deal with the no-slip condition on the interface of the fluid and

Figure 17: A realistic elastic rotor: the evolution of the magnitude of velocity with time in 1/4 of the fluid domain.

Figure 18: A realistic elastic rotor: the magnitude of the velocity field on the central cross-section along the flow direction (left), streamline (middle), and velocity (right) fields at 14.25s.

structure more flexibly with the technique of the Master-Slave Relation. In addition, with the variational formulation and the mixed finite element method, the interface conditions are automatically enforced. Our proposed novel ALE method can flexibly generate a rotational and deformable fluid mesh according to the boundary conditions arising from the ambient elastic rotor and the fixed fluid subdomain. The linear system resulting from linearization and discretization is proved to be well-posed. By means of the developed monolithic algorithm involving the relaxed fixed-point iteration, a series of satisfactory numerical results are illustrated and validated for the elastic hydro-turbine that is spinning around its axis of rotation and interacting with the fluid. This demonstrates that our model and numerical techniques are effective in exploring the interactional mechanism between the fluid and an elastic rotor.

There are several interesting issues to be explored in future work. In this paper, we only discussed the stability of spatial discretization. The stability of time integration, especially when rotating structures are involved, is an interesting research topic. Moreover, we will continue to develop the numerical method for passive rotation case that is more suited to the hydro-turbine. Further, we will study the FSI problem in which turbulence flow is involved. In addition, a fully nonlinear hyperelastic model will be also studied for the rotating structure in the future in order to handle the case of large deformation.

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