The non-symmetric Nitsche method for the parameter-free imposition of weak boundary and coupling conditions in immersed finite elements

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Abstract

We explore the use of the non-symmetric Nitsche method for the weak imposition of boundary and coupling conditions along interfaces that intersect through a finite element mesh. In contrast to symmetric Nitsche methods, it does not require stabilization and therefore does not depend on the appropriate estimation of stabilization parameters. We first review the available mathematical background, recollecting relevant aspects of the method from a numerical analysis viewpoint. We then compare accuracy and convergence of symmetric and non-symmetric Nitsche methods for a Laplace problem, a Kirchhoff plate, and in 3D elasticity. Our numerical experiments confirm that the non-symmetric method leads to reduced accuracy in the L^2 error, but exhibits superior accuracy and robustness for derivative quantities such as diffusive flux, bending moments or stress. Based on our numerical evidence, the non-symmetric Nitsche method is a viable alternative for problems with diffusion-type operators, in particular when the accuracy of derivative quantities is of primary interest.

Keywords: Immersed finite element methods, Weak boundary and coupling conditions, Non-symmetric Nitsche method

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1. Introduction

Immersed domain finite element methods approximate the solution of boundary value problems using non-boundary-fitted meshes that do not need to conform to the boundary of the domain on which the problem is defined. Their primary goal is to increase the geometric flexibility of the finite element method and to alleviate mesh related obstacles that often appear for geometrically complex domains. For example, immersed finite element concepts have been employed to resolve multi-phase flow interfaces [1, 2], to deal with trimmed CAD surfaces [3–6] and image based geometries [7–9], to prevent mesh updating and mesh distortion [10–13], or to handle fluid–structure interaction problems involving large displacements and contact [14–17]. Immersed schemes require two fundamental components beyond standard finite element technology. First, they need to be able to evaluate integrals in cut elements. In this context, the importance of geometrically faithful quadrature has been recently highlighted [15, 18–24]. The second component is the enforcement of Dirichlet boundary and interface conditions at immersed boundaries.

For the latter, an important family of techniques that has attracted large attention in recent years revolves around the original idea of Nitsche who developed a variationally consistent method for weakly enforcing Dirichlet boundary conditions [25–27]. In contrast to Lagrange multiplier methods [28–31], the symmetric Nitsche formulation is free of auxiliary fields, which simplifies the theory and reduces computational cost. The variational consistency of the symmetric Nitsche method allows the reinterpretation of the penalty parameter as a mesh dependent stabilization parameter that needs be chosen sufficiently large as to maintain stability of the bilinear form. Suitable stabilization parameters can be estimated by an eigenvalue approach on a global level for the complete mesh [32] or on a local level for each intersected element [33, 34]. For interface problems, an additional weighting of consistency terms at the interface can improve the accuracy with respect to Nitsche’s classical formulation [35–37].

Symmetric Nitsche-type methods are accurate and robust, but their performance crucially depends on appropriate estimates of the stabilization parameters involved. If estimates are too large, the method degrades to a penalty method, which adversely influences consistency, accuracy and robustness. If they are too small, stability is lost. Moreover, accurate estimation techniques are often delicate from an algorithmic viewpoint [27, 36, 37]. Therefore, there has been an increasing interest in methods that can enforce boundary and interface conditions without mesh dependent stabilization parameters [38–41].

In this paper, we explore the use of the non-symmetric Nitsche method for the parameter-free weak enforcement of boundary and interface conditions in immersed finite element schemes. The non-symmetric Nitsche method was introduced as a Discontinuous Galerkin (DG) method by BAUMANN, ODEN and coworkers [42–45] and is therefore also often referred to as the Baumann-Babuška-Oden method. It is based on variationally consistent numerical flux conditions that are introduced in such a way that the criterion for stability is (weakly) satisfied. Therefore, it does not require the introduction of additional stabilization terms with associated parameters and, in contrast to symmetric Nitsche methods, its performance does not depend on the accuracy of the variational estimate or the reliability

and robustness of associated numerical algorithms. On the other hand, the non-symmetric Nitsche method leads to unsymmetric system matrices, convergence is currently only proved for basis functions of degree $p \geq 2$, and convergence rates of the L^2 error are suboptimal [46–50]. A stabilization term can mitigate the reduced L^2 accuracy [51–54], but again leads to the question of estimating appropriate parameters.

BURMAN and coworkers presented an improved analysis for the non-symmetric Nitsche method for pure diffusion and convection-diffusion problems [55, 56], focusing on weak boundary conditions on boundary-fitted continuous finite element meshes. In particular, it was shown that the non-symmetric Nitsche method is stable and optimally convergent in the H^1 -norm for all $p \geq 1$ and that the rate of the L^2 error is suboptimal with only half a power of h . These more favorable results are supported by more recent work in [57–59]. In this paper, we provide numerical evidence that the non-symmetric Nitsche method can also be reliably applied in the context of non-matching and non-boundary-fitted discretizations of problems with diffusion-type operators.

Our paper is structured as follows: Section 2 first reviews the formulation of the symmetric and non-symmetric Nitsche methods. Section 3 provides a concise review of the mathematical background established in a DG context [47–49] and beyond [55]. Our summary illustrates that for non-boundary-fitted discretizations that are at least quadratic and focus on elliptic problems, the available analysis framework establishes variational consistency, conservativity of the diffusive flux, and bounds in the L^2 and H^1 error norms. We also discuss convergence of both H^1 and L^2 errors. In Section 4, we present a series of numerical experiments that corroborate the competitive performance of the non-symmetric Nitsche method in comparison with recent symmetric Nitsche variants for problems based on second- and fourth-order diffusion-type operators. The examples include a Laplace problem, a Kirchhoff plate, and a double-layered elastic thick shell under internal pressure, discretized by Cartesian meshes based on quadratic and cubic B-splines. Section 5 puts analysis and numerical results into perspective and motivates future work.

2. The non-symmetric Nitsche method for unfitted discretizations

The non-symmetric Nitsche method was originally introduced as a Discontinuous Galerkin method. In the following, we review its derivation for the Poisson problem in the context of unfitted finite element meshes. We also compare the parameter-free non-symmetric formulation with the classical symmetric form that requires stabilization parameters.

2.1. A simple model problem

We consider the following Poisson problem

$$-\Delta u = f \quad \text{on } \Omega \quad (1)$$

$$u = g \quad \text{on } \Gamma_D \quad (2)$$

In addition to the Dirichlet boundary Γ_D , we assume a number of interfaces $\Gamma^\star$ that divide the domain Ω into several well-behaved subdomains $\mathcal{K}$. We assume that the boundary $\partial\mathcal{K}$

Figure 1: Example domain Ω divided into two subdomains and discretized by independent unfitted meshes. The plus/minus signs on the normal vectors refer to the left subdomain in green.

of each subdomain can be partitioned into sections with sufficient regularity. We define u^+ and $\mathbf{n}^+$ as the value of the primary variable and the outward unit normal on $\partial\mathcal{K}$, and u^- and $\mathbf{n}^-$ as the value of the primary variable and the outward unit normal of the neighboring subdomain, if the boundary point belongs to Γ^* . We can then formulate for each subdomain $\mathcal{K}$ the following boundary and interface conditions

$$u^+ - g = 0 \quad \text{on } \partial\mathcal{K} \subset \Gamma_D \quad (3)$$

$$u^+ - u^- = 0 \quad \text{on } \partial\mathcal{K} \subset \Gamma^* \quad (4)$$

$$\nabla u^+ \cdot \mathbf{n}^+ + \nabla u^- \cdot \mathbf{n}^- = 0 \quad \text{on } \partial\mathcal{K} \subset \Gamma^* \quad (5)$$

We note that conditions (4) and (5) could be nonzero functions [37], which we exclude in the present work. Figure 1 illustrates a simple two-domain example.

2.2. Variational formulation of the non-symmetric Nitsche method

Following the unified framework in [47], we start the derivation of the variational form of Nitsche-type methods by rewriting the problem (1) as a first-order system

$$\sigma = \nabla u, \quad -\nabla \cdot \sigma = f \quad (6)$$

Multiplying the first and second equations by suitable test functions τ and v , respectively, and performing integration by parts on each subdomain $\mathcal{K}$, we find

$$\int_{\mathcal{K}} \sigma \cdot \tau \, d\Omega = - \int_{\mathcal{K}} u \nabla \cdot \tau \, d\Omega + \int_{\partial\mathcal{K}} u \mathbf{n}^+ \cdot \tau \, d\Gamma \quad (7)$$

$$\int_{\mathcal{K}} \sigma \cdot \nabla v \, d\Omega = \int_{\mathcal{K}} f v \, d\Omega + \int_{\partial\mathcal{K}} \sigma \cdot \mathbf{n}^+ v \, d\Gamma \quad (8)$$

The solution space for u and σ associated with each subdomain $\mathcal{K}$ is $\mathcal{S} = L^2(\mathcal{K})$, where L^2 is the space of square integrable functions. **We note that in line with [47], 3.1, we assume that the traces on $\partial\mathcal{K}$ are well-defined.** The test function space for v and τ associated with each

subdomain $\mathcal{K}$ is $\mathcal{V} = H^1(\mathcal{K})$, where H^1 is the Sobolev space of square integrable functions with square integrable first derivatives.

We then discretize (7) and (8) such that $u_h \in \mathcal{S}_h \subset \mathcal{S}$ and $\sigma_h \in \mathcal{V}_h \subset \mathcal{V}$, arriving at the *flux formulation* [47, 60]: Find u_h and σ_h such that we have

$$\int_{\mathcal{K}} \sigma_h \cdot \tau \, d\Omega = - \int_{\mathcal{K}} u_h \nabla \cdot \tau \, d\Omega + \int_{\partial\mathcal{K}} \hat{u} \mathbf{n}^+ \cdot \tau \, d\Gamma \quad (9)$$

$$\int_{\mathcal{K}} \sigma_h \cdot \nabla v \, d\Omega = \int_{\mathcal{K}} f v \, d\Omega + \int_{\partial\mathcal{K}} \hat{\sigma} \cdot \mathbf{n}^+ v \, d\Gamma \quad (10)$$

where the numerical fluxes $\hat{\sigma}$ and $\hat{u}$ are approximations to $\sigma = \nabla u$ and to u , respectively, on the boundary $\partial\mathcal{K}$. In a DG context, each subdomain $\mathcal{K}$ is typically associated with one element and its boundary $\partial\mathcal{K}$ is identical to the element faces. Here, we employ definitions (9) and (10) to general discretizations of $\mathcal{K}$, including unfitted meshes with finite elements whose boundaries do not conform to $\partial\mathcal{K}$ [53, 54, 61]. Elements can be arbitrarily intersected by the Dirichlet boundary Γ_D and the immersed interface Γ^* as shown in Fig. 1.

In the next step, we design expressions in terms of σ_h and u_h for the numerical fluxes. To arrive at the non-symmetric Nitsche method, we choose

$$\hat{u} = \frac{3}{2}u_h^+ - \frac{1}{2}u_h^- \quad (11)$$

$$\hat{\sigma} = \frac{1}{2}(\nabla u_h^+ + \nabla u_h^-) \quad (12)$$

for all boundaries $\partial\mathcal{K} \subset \Gamma^*$ that are part of interior interfaces, and

$$\hat{u} = 2u_h^+ - g \quad (13)$$

$$\hat{\sigma} = \nabla u_h^+ \quad (14)$$

for all boundaries $\partial\mathcal{K} \subset \Gamma_D$ on the outer Dirichlet boundary.

The final form of the non-symmetric Nitsche method is the primal formulation of (9) and (10), which can be obtained by relating σ_h and τ to u_h and v . To this end, we first consider the integration by parts formula

$$- \int_{\mathcal{K}} u_h \nabla \cdot \tau \, d\Omega = \int_{\mathcal{K}} \nabla u_h \cdot \tau \, d\Omega - \int_{\partial\mathcal{K}} u_h \mathbf{n}^+ \cdot \tau \, d\Gamma \quad (15)$$

where we restrict $u_h \in \mathcal{V}_h \subset H^1(\mathcal{K})$. Inserting (15) and the flux approximations (11) and (13) into (9), and identifying $\tau = \nabla v$ yields the following expression

$$\begin{aligned} \int_{\mathcal{K}} \sigma_h \cdot \nabla v \, d\Omega &= \int_{\mathcal{K}} \nabla u_h \cdot \nabla v \, d\Omega + \int_{\partial\mathcal{K} \subset \Gamma^*} \frac{1}{2} (u_h^+ - u_h^-) \mathbf{n}^+ \cdot \nabla v \, d\Gamma + \\ &\quad \int_{\partial\mathcal{K} \subset \Gamma_D} (u_h - g) \nabla v \cdot \mathbf{n}^+ - \nabla u_h \cdot \mathbf{n}^+ v \, d\Gamma \end{aligned} \quad (16)$$

Inserting the flux approximations (12) and (14) into (10), relating the result to the left-hand side of (16) and summing over all subdomains $\mathcal{K}$ yields the following primal formulation: Find u_h such that we have $B(u_h, v) = l(v)$, where

$$B(u_h, v) = \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla u_h \cdot \nabla v \, d\Omega + \int_{\Gamma^*} \llbracket u_h \rrbracket \{ \nabla v \} d\Gamma - \int_{\Gamma^*} \{ \nabla u_h \} \llbracket v \rrbracket d\Gamma + \int_{\Gamma_D} u_h \nabla v \cdot \mathbf{n}^+ \, d\Gamma - \int_{\Gamma_D} \nabla u_h \cdot \mathbf{n}^+ v \, d\Gamma \quad (17)$$

$$l(v) = \int_{\Omega} f v \, d\Omega + \int_{\Gamma_D} g \nabla v \cdot \mathbf{n}^+ \, d\Gamma \quad (18)$$

For a compact notation, we use the jump operator defined for a scalar quantity as

$$\llbracket u_h \rrbracket = u_h^+ \mathbf{n}^+ + u_h^- \mathbf{n}^- \quad (19)$$

and the average operator defined for a vector quantity as

$$\{ \nabla u_h \} = \frac{1}{2} (\nabla u_h^+ + \nabla u_h^-) \quad (20)$$

2.3. Comparison with the symmetric Nitsche method

To arrive at the classical form of Nitsche's method, we choose another set of numerical fluxes in (9) and (10). They read

$$\widehat{u} = \frac{1}{2} u_h^+ + \frac{1}{2} u_h^- \quad (21)$$

$$\widehat{\sigma} = \{ \nabla u_h \} - \frac{1}{2} \alpha \llbracket u_h \rrbracket \quad (22)$$

for all boundaries $\partial\mathcal{K} \subset \Gamma^*$ that are part of interior interfaces, and

$$\widehat{u} = g \quad (23)$$

$$\widehat{\sigma} = \nabla u_h^+ - \beta (u_h^+ - g) \quad (24)$$

for all boundaries $\partial\mathcal{K} \subset \Gamma_D$ on the outer Dirichlet boundary.

Using the new flux approximations (22) to (24) in the procedure above yields: Find u_h such that we have $B(u_h, v) = l(v)$, where

$$B(u_h, v) = \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla u_h \cdot \nabla v \, d\Omega - \int_{\Gamma^*} \llbracket u_h \rrbracket \{ \nabla v \} d\Gamma - \int_{\Gamma^*} \{ \nabla u_h \} \llbracket v \rrbracket d\Gamma - \int_{\Gamma_D} u_h \nabla v \cdot \mathbf{n}^+ \, d\Gamma - \int_{\Gamma_D} \nabla u_h \cdot \mathbf{n}^+ v \, d\Gamma + \alpha \int_{\Gamma^*} \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, d\Gamma + \beta \int_{\Gamma_D} u v \, d\Gamma \quad (25)$$

$$l(v) = \int_{\Omega} f v \, d\Omega - \int_{\Gamma_D} g \nabla v \cdot \mathbf{n}^+ \, d\Gamma + \beta \int_{\Gamma_D} g v \, d\Gamma \quad (26)$$

We observe that in contrast to the non-symmetric form (17) and (18), the symmetric Nitsche method includes additional parameters α and β . They ensure that (25) satisfies the stability criterion (34) (see Section 3.2). For optimal performance of the method, α and β need to be chosen as small as possible. Element-wise configuration dependent stabilization parameters can be estimated based on a local eigenvalue problem [4, 27, 33, 37]. For unfitted discretizations, the values required for local stability per element depend strongly on how immersed surfaces intersect that particular element.

3. Overview of analysis results for the non-symmetric Nitsche method

In the next step, we summarize analysis results for the non-symmetric Nitsche method available in the literature and discuss their extensibility to non-boundary-fitted and non-matching discretizations. We illustrate the main ideas for the Poisson model (1), as most results are only available for second-order elliptic problems. For a detailed and comprehensive analysis, we refer to the work of BAUMANN, ODEN et al. [42–45], to the seminal contributions on DG methods by ARNOLD, BREZZI, COCKBURN and MARINI [46, 47] and RIVIERE, WHEELER and GIRAULT [48, 49], and to the work of BURMAN [55].

3.1. Consistency and adjoint consistency

Consistency of the non-symmetric Nitsche method with respect to the strong form of the original problem (1) and its constraints (3) to (5) can be easily demonstrated by considering the primal formulation (17) and (18). **Replacing u_h by u in (17), where u is the solution of (1)-(2),** integrating by parts and bringing all terms on the left-hand side, we find

$$\begin{aligned} - \sum_{\mathcal{K}} \int_{\mathcal{K}} (\Delta u + f) v \, d\Omega + \int_{\Gamma^*} \llbracket u \rrbracket \{ \nabla v \} \, d\Gamma + \int_{\Gamma^*} \llbracket \nabla u \rrbracket \{ v \} \, d\Gamma \\ + \int_{\Gamma_D} (u - g) \nabla v \cdot \mathbf{n}^+ \, d\Gamma = 0 \end{aligned} \quad (27)$$

Since (27) must hold for arbitrary test functions v on each subdomain $\mathcal{K}$, and **hence also for arbitrary averages $\{v\}$ and $\{\nabla v\}$** on the inner boundaries $\partial\mathcal{K} \subset \Gamma^*$, each term in (27) must individually be zero. With this argument, we recover the original equation (1) from the first term, the Dirichlet boundary condition (3) from the last term, and the interface constraints (4) and (5) from the two terms in the center.

Considering the adjoint problem to (1),

$$-\Delta w = f \quad \text{on } \Omega, \quad w = 0 \quad \text{on } \Gamma_D \quad (28)$$

with its solution $w \in H^2(\Omega)$, we can establish adjoint consistency by showing that

$$B(v, w) = \int_{\Omega} f v \, d\Omega \quad (29)$$

A straightforward computation of the left-hand side shows that the non-symmetric Nitsche method is *not* adjoint consistent. With $w|_{\Gamma_D} = 0$, $\{\nabla w\} = \nabla w$ and $\llbracket w \rrbracket = 0$ and integration by parts, we find

$$\begin{aligned} B(v, w) &= \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla v \cdot \nabla w \, d\Omega + \int_{\Gamma^*} \llbracket v \rrbracket \cdot \nabla w \, d\Gamma + \int_{\Gamma_D} v \nabla w \cdot \mathbf{n}^+ \, d\Gamma \\ &= \sum_{\mathcal{K}} \left(- \int_{\mathcal{K}} v \Delta w \, d\Omega + \int_{\partial \mathcal{K}} v \nabla w \cdot \mathbf{n}^+ \, d\Gamma \right) + \int_{\Gamma^*} \llbracket v \rrbracket \cdot \nabla w \, d\Gamma + \int_{\Gamma_D} v \nabla w \cdot \mathbf{n}^+ \, d\Gamma \\ &= \int_{\Omega} f v \, d\Omega + 2 \int_{\Gamma^*} \llbracket v \rrbracket \cdot \nabla w \, d\Gamma + 2 \int_{\Gamma_D} v \nabla w \cdot \mathbf{n}^+ \, d\Gamma \end{aligned} \quad (30)$$

which obviously does not satisfy condition (29).

Remark 1: It can be shown that conservativity of the numerical fluxes, i.e. $\llbracket \hat{\sigma} \rrbracket = 0$ and $\llbracket \hat{u} \rrbracket = 0$, implies adjoint consistency (see e.g. 3.3, [47]). Considering the numerical fluxes (11) and (12) of the non-symmetric Nitsche method, we observe that

$$\llbracket \hat{\sigma} \rrbracket = \frac{1}{2}(\nabla u_h^+ + \nabla u_h^-) \cdot \mathbf{n}^+ + \frac{1}{2}(\nabla u_h^+ + \nabla u_h^-) \cdot \mathbf{n}^- = 0 \quad (31)$$

$$\llbracket \hat{u} \rrbracket = \left(\frac{3}{2}u_h^+ - \frac{1}{2}u_h^- \right) \mathbf{n}^+ + \left(\frac{3}{2}u_h^- - \frac{1}{2}u_h^+ \right) \mathbf{n}^- = 2u_h^+ \mathbf{n}^+ + 2u_h^- \mathbf{n}^- \quad (32)$$

The vector flux $\hat{\sigma}$ is conservative, while the scalar flux $\hat{u}$ of the primal variable is consistent, but not conservative, since the jump in the primal variable does not vanish.

3.2. Boundedness and weak stability

Boundedness and stability are important ingredients for obtaining error estimates. However, both properties pose difficulties in the case of the non-symmetric Nitsche method. The bilinear forms of many Discontinuous Galerkin methods can be bounded in the form

$$B(w, v) \leq C_b \llbracket w \rrbracket \cdot \llbracket v \rrbracket \quad (33)$$

and satisfy the stability condition

$$B(v, v) \geq C_s \llbracket v \rrbracket^2 \quad (34)$$

with respect to the norm

$$\|v\|^2 = \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla v \cdot \nabla v \, d\Omega + \int_{\Gamma^*} \llbracket v \rrbracket \cdot \llbracket v \rrbracket \, d\Gamma \quad (35)$$

where $v \in \mathcal{V}_h \subset H^1(\mathcal{K})$ (see e.g. 4.2, [47]).

For the non-symmetric Nitsche method, the bilinear form (17) can in general not be bounded with respect to the norm (35). For $B(v, v)$ as defined in (17), the surface terms cancel each other and hence control on piecewise constant functions in $\mathcal{V}_h$ is lost. This results in the following identity

$$B(v, v) = |v|_1^2 \quad (36)$$

with respect to the seminorm

$$|v|_1^2 = \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla v \cdot \nabla v \, d\Omega \quad (37)$$

Based on (36), the non-symmetric Nitsche method can be classified as *weakly* stable [47].

Remark 2: The stability condition (34) can be satisfied if stabilization terms of the form

$$B(u, v)_{stab} = \alpha \int_{\Gamma^*} \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, d\Gamma + \beta \int_{\Gamma_D} u v \, d\Gamma \quad (38)$$

$$l(v)_{stab} = \beta \int_{\Gamma_D} g v \, d\Gamma \quad (39)$$

are added to the variational formulation (17) and (18), **with α and β being sufficiently large**. The corresponding DG method is known as the non-symmetric interior penalty Galerkin (NIPG) method [47, 49].

3.3. Error estimates in the H^1 norm

Due to the missing bounds on $B(v, v)$, the standard machinery for determining error estimates cannot be applied to the non-symmetric Nitsche method. Instead, special analysis ideas are necessary. In the context of fitted DG discretizations, [48, 49], RIVIERE, WHEELER and GIRAULT showed how to obtain optimal order error estimates in the H^1 -norm under the assumption that the polynomial degree is $p \geq 2$. Their main idea is based on the construction of an interpolant $u_I \in \mathcal{V}_h \subset H^1(\mathcal{K})$ of the exact solution u that can be discontinuous across Γ^* , but satisfies the following properties

$$\int_{\Gamma^*} \{\nabla(u - u_I)\} \cdot \mathbf{n}^+ \, d\Gamma = 0, \quad \text{on } \Gamma^* \quad (40)$$

$$\int_{\Gamma_D} \nabla(u - u_I) \cdot \mathbf{n}^+ \, d\Gamma = 0, \quad \text{on } \Gamma_D \quad (41)$$

Such an interpolant u_I only exists for bases of polynomial degree $p \geq 2$ [48, 49], so that their analysis does not hold for linear basis functions.

A fundamental prerequisite for applying the available DG analysis to our case is that relations (40) and (41) can be extended to non-matching and non-boundary-fitted discretizations, where elements between patches are discontinuous, but in each patch continuous (see Fig. 1). To the best of our knowledge, this seems unclear at this point. In each patch $\mathcal{K}$, polynomials on different elements along the interface may not be chosen independently because of the continuity constraints of neighboring elements [55]. However, in [55], an interpolant similar to (41) with both approximation power and zero average normal gradient has been constructed along the boundary of a fitted mesh with continuous finite elements (the construction relies on specific assumptions about the configuration of boundary elements).

In the scope of this work, we hypothesize that (40) and (41) exist for non-matching and non-boundary-fitted continuous discretizations. For the non-symmetric Nitsche method, (40) and (41) do not have to hold for a single element, but for the complete interface in a weak sense, since convergence is based on the mesh refinement of each discontinuous patch $\mathcal{K}$. Therefore, it seems reasonable to assume that (40) and (41) exist on each patch boundary $\partial\mathcal{K}$ after (minimal) mesh refinement. In other words, we just refine until we have enough flexibility in the function space on each discontinuous patch to find an interpolant that satisfies (40) and (41). This could also explain why the non-symmetric Nitsche method works well with linear basis functions (see Section 4.6 for a numerical example). This is in contrast to the DG case, where (40) and (41) must hold for each element, since with mesh refinement the number of discontinuous patches $\mathcal{K}$ (i.e., the elements) increases, so that a refinement of each patch is not possible.

Proceeding with the analysis, we consider the piecewise constant components v_0 of any test function v , obtained by an orthogonal L^2 projection of v onto the space of piecewise constant functions. Since we know that $\llbracket v_0 \rrbracket$ and v_0 are piecewise constant and $\nabla v_0 = 0$, it is easy to see that each term of the following expression

$$\begin{aligned} B(u - u_I, v_0) &= \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla(u - u_I) \cdot \nabla v_0 \, d\Omega + \int_{\Gamma^*} \llbracket (u - u_I) \rrbracket \{ \nabla v_0 \} \, d\Gamma \\ &+ \int_{\Gamma_D} (u - u_I) \nabla v_0 \cdot \mathbf{n}^+ \, d\Gamma - \llbracket v_0 \rrbracket \int_{\Gamma^*} \{ \nabla(u - u_I) \} \, d\Gamma - v_0 \int_{\Gamma_D} \nabla(u - u_I) \cdot \mathbf{n}^+ \, d\Gamma = 0 \end{aligned} \quad (42)$$

must be zero. We emphasize that for the last two terms, we require properties (40) and (41), which we assume to exist for sufficiently refined patches $\mathcal{K}$. We can then write

$$B(u - u_I, v) = B(u - u_I, v - v_0) \leq C \llbracket u - u_I \rrbracket \llbracket v - v_0 \rrbracket \leq C \llbracket u - u_I \rrbracket |v|_1 \quad (43)$$

where we have employed the obvious estimate $\llbracket v - v_0 \rrbracket \leq C |v|_1$ [47]. We note that for obtaining estimate (43), the definition of the norm (35) must be augmented by an additional term (see [47], 4.1 for details). Using weak stability (36), Galerkin orthogonality and (43),

we find

$$|u_h - u_I|_1^2 = B(u_h - u_I, u_h - u_I) = B(u - u_I, u_h - u_I) \leq C \|u - u_I\| |u_h - u_I|_1 \quad (44)$$

With a suitable bound on the approximation error $\|u - u_I\| \leq C_a h^p |u|_{p+1}$ (see 4.3, [47]), we find an optimal error estimate in the H^1 norm as follows

$$|u - u_h|_1 \leq C h^p |u|_{p+1} \quad (45)$$

3.4. Error estimates in the L^2 norm

In [47], ARNOLD, BREZZI, COCKBURN and MARINI showed how to obtain an error estimate for the non-symmetric Nitsche method in the L^2 -norm, which can be outlined as follows. We define ψ as the solution of the adjoint problem to (1),

$$-\Delta \psi = u - u_h \quad \text{on } \Omega, \quad \psi = 0 \quad \text{on } \Gamma_D \quad (46)$$

We assume that this problem satisfies elliptic regularity, i.e., the solution $\psi \in H^2(\Omega)$ satisfies $\|\psi\|_2 \leq C \|u - u_h\|_0$ [47], where the constant depends solely on the regularity of the domain Ω . We can then write

$$\begin{aligned} \|u - u_h\|_0^2 &= \sum_{\mathcal{K}} \int_{\mathcal{K}} -\Delta \psi (u - u_h) \, d\Omega = B(\psi, u - u_h) \\ &= B(\psi, u - u_h) + B(u - u_h, \psi) - B(u - u_h, \psi) \\ &= 2 \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla \psi \cdot \nabla (u - u_h) \, d\Omega - B(u - u_h, \psi - \psi_I) \end{aligned} \quad (47)$$

where we have used (17) and Galerkin orthogonality to find the terms in the last row. An estimate for this term can be obtained by using (45) and elliptic regularity as follows

$$\begin{aligned} \sum_{\mathcal{K}} \int_{\mathcal{K}} \nabla \psi \cdot \nabla (u - u_h) \, d\Omega &\leq |\psi|_1 |u - u_h|_1 \leq C \|\psi\|_2 |u - u_h|_1 \\ &\leq C \|u - u_h\|_0 |u - u_h|_1 \leq C \|u - u_h\|_0 h^p |u|_{p+1} \end{aligned} \quad (48)$$

It is shown in [47] that the second term in (47) converges with order h^{p+1} , so that (48) dominates the L^2 estimate. Combining (47) and (48), we therefore obtain the following suboptimal error estimate in the L^2 norm

$$\|u - u_h\|_0 \leq C h^p |u|_{p+1} \quad (49)$$

The suboptimality of (49) can be directly linked to the missing adjoint consistency of the non-symmetric Nitsche method [47, 48].

We note that in [55], BURMAN showed that there are better analytical results for the non-symmetric Nitsche method when used as a means of applying boundary conditions to

(a) Cartesian mesh with unfitted boundary (red line) and corresponding solution field for $p=2$.

(b) Recursive Gaussian quadrature in cut elements (finite cell method).

Figure 2: Laplace problem with unfitted Dirichlet boundary.

boundary-fitted continuous Galerkin discretizations. In this case, the convergence rate of the L^2 error is suboptimal with only half a power of h .

4. Numerical experiments

To corroborate the numerical analysis results of the previous section, we present a series of numerical experiments that illustrate the performance of the non-symmetric Nitsche method. The examples include a Laplace problem, a Kirchhoff plate, and a 3D elastic thick spherical shell. We consider different discretization scenarios that cover boundary and coupling conditions along straight and curved immersed surfaces. We employ 2D and 3D Cartesian meshes of quadratic and cubic B-splines and standard linear basis functions. We also compare results obtained with the non-symmetric method to results obtained with two different variants of the symmetric Nitsche method. On the one hand, we use a sophisticated variant with element-wise estimates and weighting across interfaces for optimal fine-tuning of stabilization parameters, which represents the current gold-standard. On the other hand, we also use the simplest variant possible, based on an empirically chosen global stabilization parameter. The results of the simple variant are of interest, because this method is still widely used within the computational mechanics community.

4.1. Laplace problem: Boundary conditions

We consider the model problem (1) with $f = 0$ on a square domain $\Omega \in [0, 1]^2$. Imposing nonzero boundary conditions $u(x, 0) = \sin(\pi x)$ and $u = 0$ on all other boundaries, we obtain the analytical solution [33]

$$u_{ex}(x, y) = [\cosh(\pi y) - \coth(\pi) \sinh(\pi y)] \sin(\pi x) \quad (50)$$

We discretize the domain Ω with a Cartesian mesh that defines B-spline basis functions [62]. We use a straight line rotated by $\pi/8$ about the center point to trim away a portion of

Figure 3: Element-wise stabilization parameters β , estimated from the local eigenvalue problem (51).

the mesh, creating an immersed boundary. The corresponding nonzero boundary condition can be easily derived from (50). Figure 2a illustrates the trimmed mesh, the immersed boundary, and the corresponding solution field obtained with quadratic B-splines. We use the recursive quadrature approach applied in the finite cell method [9, 63] to integrate intersected elements. To ensure accuracy, we employ 8 levels of quadrature sub-cells in all 2D examples. Figure 2b illustrates the resulting quadrature points for the example at hand. To integrate over the immersed boundary, we divide the straight line into 1D sub-elements irrespective of the underlying Cartesian mesh.

For weakly imposing boundary conditions at both boundary-fitted and immersed parts of the surface, we employ the boundary condition part of the non-symmetric Nitsche method (17) and (18). We compare its performance with two variants of the symmetric form of Nitsche's method (25) and (26) that estimate the boundary stabilization parameter β locally per element [33] or globally for the whole mesh [32]. The estimation procedure is based on a generalized eigenvalue problem

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{B}\mathbf{x} \quad (51)$$

defined in each element with support at the boundary. For the Laplace problem, the matrices in (51) are defined as follows

$$[\mathbf{A}]_{ij} = \int_{\Gamma^e} (\nabla N_i \cdot \mathbf{n}^+) (\nabla N_j \cdot \mathbf{n}^+) \, d\Gamma \quad (52)$$

$$[\mathbf{B}]_{ij} = \int_{\Omega^e} \nabla N_i \cdot \nabla N_j \, d\Omega \quad (53)$$

where Ω^e is the portion of the cut element with support in the domain, Γ^e is the portion of the boundary with support in the cut element, and N_i are the element basis functions. Reliable solution procedures for (51) require special techniques [27, 37], because $\mathbf{B}$ is rank deficient by construction and the presence of small cut elements deteriorates the conditioning

(a) Non-symmetric Nitsche method (parameter-free)

(b) Symmetric Nitsche method (local stab.)

(c) Symmetric Nitsche method (global stab.)

Figure 4: Distribution of the absolute error of the solution field. The error is amplified by the same factor in all plots for better visibility.

of the problem. The largest eigenvalue represents an estimate for the minimum value of the stabilization parameter β that ensures stability. Figure 3 shows element-wise estimates of β for the example mesh along the immersed boundary. We observe that the difference in the absolute value for β varies significantly as smaller cut elements require a larger stabilization parameter. We multiply the element-wise eigenvalues β by an additional safety factor of 2 to ensure that the local estimates are conservative. For the second variant of the symmetric Nitsche method, we empirically determine a global stabilization parameter that yields the best possible result.

Figure 4 plots the absolute error distribution $|u_{ex} - u_{FEM}|$ over the domain obtained with a 12×12 mesh and quadratic B-splines. We observe that the error of the non-symmetric Nitsche method is larger than the error of the two symmetric variants of Nitsche's method. This is confirmed in Figs. 5a and 5b that show the convergence of the L^2 error for quadratic and cubic B-spline basis functions as the Cartesian mesh is uniformly refined. The level of L^2 accuracy for the non-symmetric Nitsche method is reduced by a constant factor, but optimal rates of convergence are still achieved (contrary to the suboptimal rates predicted in Section 3.3). Figures 6a and 6b show the corresponding convergence curves for the error in the H^1 semi-norm, which are optimal between the three methods.

An important aspect that we would like to mention at this point is the significant dependence of the solution accuracy on the accuracy of volume and surface quadrature in intersected elements. Figure 7a plots the solution error in the H^1 semi-norm for different levels of quadrature sub-cells. With increasing number of sub-cells, the integration accuracy in each intersected element is increased [9]. We observe that all methods depend on accurate volume integration in the same way, requiring at least 5 levels of recursive quadrature sub-cells to achieve the best possible H^1 error limit. Figure 7b plots the solution error in the H^1 semi-norm for different numbers of quadrature elements along the immersed boundary, which shows the same dependence on accurate surface quadrature for all three methods.

(a) Quadratic B-splines.

(b) Cubic B-splines.

Figure 5: Convergence of the relative error in the L^2 norm for the Laplace problem with weak boundary conditions when the mesh uniformly refined.

(a) Quadratic B-splines.

(b) Cubic B-splines.

Figure 6: Convergence of the relative error in the H^1 semi-norm for the Laplace problem with weak boundary conditions when the mesh uniformly refined.

We note that the symmetric Nitsche method with local stabilization parameters seems to be more sensitive to quadrature error, which additionally affects the evaluation of integrals (52) and (53). For a more comprehensive account on the importance of geometrically faithful surface and volume quadrature in immersed finite element methods, we refer the interested reader to a series of recent papers [15, 18–24]. It is also worthwhile to mention in this context that with the quadrature scheme described, the $\mathbf{A}$ matrix of the eigenvalue problem (51) may be nonzero while the $\mathbf{B}$ matrix is zero (or vice versa). This happens if there is a boundary quadrature point in an element that is cut so unfortunately that it cannot be

(a) Volume quadrature of intersected elements. (b) Surface quadrature at immersed boundary.

Figure 7: Dependence of the error in the H^1 semi-norm on quadrature accuracy in intersected elements and at immersed boundaries for the Laplace problem with weak boundary conditions.

resolved by the finest level of recursive volume quadrature (or vice versa). In our implementation, we make sure that any such element is removed from the list of cut elements and treated as if the element was completely outside the domain.

4.2. Accuracy and conservativity of the diffusive flux

For many applications, the local accuracy of the diffusive flux at the boundary is of primary importance. In the context of Nitsche-type methods, the definition of the discrete diffusive flux on the Dirichlet boundary uses either the field derivative only,

$$q_{diff} = \nabla u_h \cdot \mathbf{n}^+ \quad (54)$$

or it can include the influence of the stabilization parameter,

$$q_{diff} = \nabla u_h \cdot \mathbf{n}^+ - \beta (u_h - g) \quad (55)$$

The second definition (55) can be motivated by conservation considerations [64–66]. Choosing the test function to be $v = 1$ and neglecting all coupling terms in the variational form (25) and (26) of the symmetric Nitsche method, we obtain the following discrete conservation law for the case of weak boundary conditions:

$$-\int_{\Omega} f \, d\Omega - \int_{\Gamma_D} \nabla u_h \cdot \mathbf{n}^+ \, d\Gamma + \beta \int_{\Gamma_D} u_h \, d\Gamma - \beta \int_{\Gamma_D} g \, d\Gamma = 0 \quad (56)$$

It shows that definition (55) of the diffusive flux is conservative in the sense that its boundary integral matches the volume integral of the source term f . Moreover, definition (55) is consistent with the definition of the numerical flux (24). Going through a similar procedure

(a) *Symmetric Nitsche / local stab.*

(b) *Symmetric Nitsche / global stab.*

Figure 8: Accuracy of field and conservative definitions of the diffusive flux in Nitsche's method, evaluated by computing the L^2 error on the immersed boundary for different quadratic B-spline meshes.

(a) *Quadratic B-splines.*

(b) *Cubic B-splines.*

Figure 9: Convergence of the diffusive flux at the immersed boundary in terms of the relative error in the L^2 norm for the Laplace problem with weak boundary conditions.

with the variational form (17) and (18), we verify that for the non-symmetric Nitsche method the definition of the diffusive flux (54) is conservative (see also Remark 1 in Section 3.1).

Figures 8a and 8b assess the accuracy of the flux definitions (54) and (55) for the symmetric form of Nitsche's method with local and global stabilization parameters, respectively. The L^2 norm of the error of the diffusive flux is computed along the immersed boundary for different Cartesian meshes of quadratic B-splines. We observe that the conservative flux definition (54) exhibits an accuracy advantage, when local stabilization parameters are employed. However, in the case of global stabilization parameters its accuracy is signifi-

Figure 10: Absolute value of the error in the diffusive flux along the immersed boundary.

cantly reduced as compared to the diffusive flux definition (55) based on the derivative of the solution field only. In the remainder of this study, we will compute both flux definitions for symmetric Nitsche methods and use the more accurate result for comparison to the non-symmetric Nitsche method.

Figure 9 illustrates the accuracy of the diffusive flux in terms of the L^2 error evaluated over the immersed boundary for all three methods. We observe that the non-symmetric Nitsche method is more accurate than both symmetric variants of Nitsche's method. It benefits from its independence of additional parameters and the conservativity of its flux definition (54). This is further illustrated in Figs. 10a and 10b that plot the absolute value of the error in the diffusive flux along the immersed boundary for two different mesh sizes.

4.3. Laplace problem: Coupling

In the next step, we study the performance of the non-symmetric Nitsche method for coupling two unfitted discretizations along a trimming interface. Figure 11a illustrates the two trimmed Cartesian meshes of different size, the immersed interface, and the corresponding solution field obtained with quadratic B-splines. We again use recursive Gaussian quadrature to integrate intersected elements (see Fig. 11b).

We compare the performance of the non-symmetric Nitsche method (17) and (18) with a symmetric variant of Nitsche's method recently introduced by ANNAVARAPU et al. [35–37]. The method is based on a weighting of the consistency terms at immersed interfaces, which has been shown to improve the accuracy and robustness of the Nitsche approach in the presence of intersected elements. The weighting concept requires the adjustment of the interface term in the bilinear form of the classical formulation (25) as follows

$$B_{\Gamma^*}(u_h, v) = - \int_{\Gamma^*} \llbracket u_h \rrbracket \cdot \langle \nabla v \rangle_\gamma \, d\Gamma - \int_{\Gamma^*} \langle \nabla u_h \rangle_\gamma \cdot \llbracket v \rrbracket \, d\Gamma + \alpha \int_{\Gamma^*} \llbracket u_h \rrbracket \cdot \llbracket v \rrbracket \, d\Gamma \quad (57)$$

(a) Unfitted meshes with coupling interface (red line) and solution field for $p=2$.

(b) Recursive quadrature points in cut elements for the two trimmed Cartesian B-spline meshes.

Figure 11: Laplace problem with immersed coupling interface.

where the weighting operator across the interface is defined as

$$\langle \nabla u \rangle_\gamma = \gamma \nabla u^+ + (1 - \gamma) \nabla u^- \quad (58)$$

The method makes use of one-sided inequalities to establish estimates of local stabilization parameters that can be computed from separate eigenvalue problems (51) on each side of the interface. This constitutes a significant advantage from an implementation point of view, as the contribution of each immersed mesh to the discrete system can be computed and assembled separately. The only quantity that needs to be communicated at each interface quadrature point is the eigenvalue C^- of the intersected element on the opposite side.

Following [37], we compute the stabilization parameter α and the weighting factor γ^+ at

Figure 12: Element-wise maximum eigenvalues, computed separately on each side of the immersed interface from the local eigenvalue problem (51).

Figure 13: Distribution of the absolute error of the solution field for the weakly coupled Laplace problem based on two trimmed Cartesian meshes. The error is amplified in all plots for better visibility.

each location of the interface as

$$\alpha = \frac{1}{1/C^+ + 1/C^-} \quad (59)$$

$$\gamma = \frac{1/C^+}{1/C^+ + 1/C^-} \quad (60)$$

where C^+ and C^- are the element-wise maximum eigenvalues computed on the current and opposite side of the interface, respectively. Figure 12 shows the results of the eigenvalue computations on each side of the interface, illustrating that the size of the eigenvalues depends strongly on the size of the cut element. The weighted definition (59) of α prevents that a large eigenvalue on one side takes control of the interface stabilization.

Figure 14: Convergence of the error in the L^2 norm for the weakly coupled Laplace problem.

(a) Quadratic B-splines.

(b) Cubic B-splines.

Figure 15: Convergence of the error in the H^1 semi-norm for the weakly coupled Laplace problem.

(a) Quadratic B-splines.

(b) Cubic B-splines.

Figure 16: Convergence of the diffusive flux at the immersed interface in terms of the relative error in the L^2 norm for the weakly coupled Laplace problem.

We compare the non-symmetric Nitsche method with the weighted variant of Nitsche's method (57) based on local estimates and the classical form of Nitsche's method (25) that uses $\gamma = 0.5$ and a global stabilization parameter. Figure 13 plots the absolute error distribution on two trimmed Cartesian meshes of quadratic B-splines with 12×12 and 23×23 elements. The error of the solution field itself is larger for the non-symmetric Nitsche method than for the two symmetric variants of Nitsche's method. Figures 14a and 14b show the convergence of the L^2 error for quadratic and cubic B-spline basis functions as the Cartesian mesh is uniformly refined. They confirm the reduced level of L^2 accuracy of the non-symmetric Nitsche method, but again show optimal rates. Figures 15a and 15b show the

corresponding convergence curves for the relative error in the H^1 semi-norm, which are optimal for all three methods.

Figure 16 illustrates the accuracy of the diffusive flux in terms of the error in the L^2 norm for all three methods. We evaluate the flux at the immersed interface, using the basis functions of the coarser trimmed Cartesian mesh. We observe again that the non-symmetric Nitsche method is more accurate than both symmetric variants of Nitsche's method for quadratic and cubic meshes, benefiting from the absence of additional stabilization parameters that influence the performance of its symmetric counterparts.

Remark 3 (Convergence of the flux in the L^∞ norm): In Fig. 17a, we plot the error for the interfacial diffusive flux in the L^∞ norm. We observe that for the simple Nitsche method with global stabilization parameter, the error of the flux in the L^∞ norm does not converge for finer meshes. This indicates that Nitsche's method with global stabilization is not suitable for embedded domain simulations that require high-fidelity accuracy at the embedded interface. The weighted approach successfully removes this deficiency, but the accuracy of the non-symmetric method is still slightly better.

Figure 17: Convergence of the diffusive flux at the immersed interface in the L^∞ norm for the weakly coupled Laplace problem, discretized with quadratic B-splines.

4.4. Kirchhoff plate: Boundary conditions

In the next step, we examine the performance of the non-symmetric Nitsche method for the solution of a fourth order problem. We consider a Kirchhoff plate [67], for which the strong form of the Dirichlet problem is

$$\nabla \cdot \nabla \cdot \mathbf{m} = f \quad \text{on } \Omega \quad (61)$$

$$\left. \begin{array}{l} w = g \\ \mathbf{n}^+ \cdot \nabla w = \theta_n \end{array} \right\} \quad \text{on } \Gamma_D \quad (62)$$

where g and θ_n are the boundary deflection and rotations normal to the Dirichlet boundary Γ_D . We assume a square plate defined by the domain $\Omega \in [0, 1]^2$, transversely loaded by

$$f = \sin(\pi x) \sin(\pi y) \quad (63)$$

The corresponding analytical deflection of the plate is given by

$$w(x, y) = \frac{1}{4\pi^4 D} \sin(\pi x) \sin(\pi y) \quad (64)$$

where $D = (Ez^3)/(12(1 - \nu^2))$ is the isotropic bending stiffness. We choose plate thickness $z = 0.01$, Young's modulus $E = 100$ and Poisson's ratio $\nu = 0.3$. The values for g and θ_n in (62) as well as $g_{,t}$ can be computed from (64) at each point of the boundary. We recall that the second-order moment and curvature tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\kappa}$ are defined as follows

$$m_{ij} = D (\nu \delta_{ij} w_{,kk} + (1 - \nu) w_{,ij}) \quad (65)$$

$$\kappa_{ij} = \frac{1}{2} (w_{,ij} + w_{,ji}) \quad (66)$$

with indices $\{i, j, k\} = \{1, 2\}$.

We first consider weak boundary conditions. Following the procedures of Section 2, the discretized variational formulation of the non-symmetric Nitsche method for the Kirchhoff plate reads: Find w_h such that we have $B(w_h, v) = l(v)$, where

$$\begin{aligned} B(w_h, v) = & \sum_{\mathcal{K}} \int_{\mathcal{K}} \mathbf{m} : \hat{\boldsymbol{\kappa}} \, d\Omega \\ & - \int_{\Gamma_D} (\mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{n}^+) \mathbf{n}^+ \cdot \nabla v \, d\Gamma + \int_{\Gamma_D} \nabla w_h \cdot \mathbf{n}^+ (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{n}^+) \, d\Gamma \\ & - \int_{\Gamma_D} (\mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{t}^+) \mathbf{t}^+ \cdot \nabla v \, d\Gamma + \int_{\Gamma_D} \nabla w_h \cdot \mathbf{t}^+ (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{t}^+) \, d\Gamma \\ & + \int_{\Gamma_D} \mathbf{n}^+ \cdot \nabla \cdot \mathbf{m} \, v \, d\Gamma - \int_{\Gamma_D} w_h \mathbf{n}^+ \cdot \nabla \cdot \hat{\mathbf{m}} \, d\Gamma \end{aligned} \quad (67)$$

$$\begin{aligned} l(v) = & \int_{\Omega} f v \, d\Omega + \int_{\Gamma_D} \theta_n (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{n}^+) \, d\Gamma \\ & + \int_{\Gamma_D} g_{,t} (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{t}^+) \, d\Gamma - \int_{\Gamma_D} g \mathbf{n}^+ \cdot \nabla \cdot \hat{\mathbf{m}} \, d\Gamma \end{aligned} \quad (68)$$

where $\mathbf{t}$ is the tangent vector. Using Cartesian meshes with cubic B-spline basis functions, we have trial and test functions w_h and v that belong to $\mathcal{W}_h \subset H^3(\mathcal{K})$, where H^3 is the Sobolev space of square integrable functions with square integrable third derivatives. Note that we mark quantities derived with test functions v with a hat. We use the same meshes and quadrature approach as in Section 4.1. Figure 18 illustrates an example mesh, the im-

(a) Deflection w .

(b) Twist moment m_{12} .

Figure 18: Kirchhoff plate problem with weak boundary conditions: Immersed Cartesian mesh and solutions fields for cubic B-splines.

mersed boundary and corresponding solution fields obtained with the non-symmetric Nitsche method.

Using the same discretizations and following [27, 68], the discretized variational formulation of the symmetric Nitsche method for the Kirchhoff plate reads

$$\begin{aligned}
 B(w_h, v) = & \sum_{\mathcal{K}} \int_{\mathcal{K}} \mathbf{m} : \hat{\mathbf{k}} \, d\Omega \\
 & - \int_{\Gamma_D} (\mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{n}^+) \mathbf{n}^+ \cdot \nabla v \, d\Gamma - \int_{\Gamma_D} \nabla w_h \cdot \mathbf{n}^+ (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{n}^+) \, d\Gamma \\
 & - \int_{\Gamma_D} (\mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{t}^+) \mathbf{t}^+ \cdot \nabla v \, d\Gamma - \int_{\Gamma_D} \nabla w_h \cdot \mathbf{t}^+ (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{t}^+) \, d\Gamma \\
 & + \int_{\Gamma_D} \mathbf{n}^+ \cdot \nabla \cdot \mathbf{m} \, v \, d\Gamma + \int_{\Gamma_D} w_h \mathbf{n}^+ \cdot \nabla \cdot \hat{\mathbf{m}} \, d\Gamma \\
 & + \frac{\beta h^2}{12} \int_{\Gamma_D} \nabla w_h \cdot \mathbf{n}^+ \nabla v \cdot \mathbf{n}^+ \, d\Gamma + \frac{\beta h^2}{12} \int_{\Gamma_D} \nabla w_h \cdot \mathbf{t}^+ \nabla v \cdot \mathbf{t}^+ \, d\Gamma + \beta \int_{\Gamma_D} w_h v \, d\Gamma \quad (69)
 \end{aligned}$$

$$\begin{aligned}
 l(v) = & \int_{\Omega} f v \, d\Omega - \int_{\Gamma_D} \theta_n (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{n}^+) \, d\Gamma \\
 & - \int_{\Gamma_D} g_{,t} (\mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{t}^+) \, d\Gamma + \int_{\Gamma_D} g \mathbf{n}^+ \cdot \nabla \cdot \hat{\mathbf{m}} \, d\Gamma \\
 & + \frac{\beta h^2}{12} \int_{\Gamma_D} \theta_n \nabla w \cdot \mathbf{n}^+ \, d\Gamma + \frac{\beta h^2}{12} \int_{\Gamma_D} g_{,t} \nabla w \cdot \mathbf{t}^+ \, d\Gamma + \beta \int_{\Gamma_D} g v \, d\Gamma \quad (70)
 \end{aligned}$$

In contrast to (67) and (68), the symmetric form of Nitsche's method requires stabiliza-

Figure 19: Distribution of the absolute error of the solution field. The error is amplified by the same factor in all plots for better visibility.

tion terms with one stabilization parameter β . We can estimate β from an element-wise generalized eigenvalue problem (51), where the matrices $\mathbf{A}$ and $\mathbf{B}$ are computed as

$$\mathbf{A} = \int_{\Gamma^e} \frac{12}{h^2} \left[(\mathbf{n} \mathbf{D} \mathbf{b})^T (\mathbf{n} \mathbf{D} \mathbf{b}) + (\mathbf{t} \mathbf{D} \mathbf{b})^T (\mathbf{t} \mathbf{D} \mathbf{b}) \right] + (\mathbf{n} \mathbf{D} \mathbf{c})^T (\mathbf{n} \mathbf{D} \mathbf{c}) \, d\Gamma \quad (71)$$

$$\mathbf{B} = \int_{\Omega^e} \mathbf{b}^T \mathbf{D} \mathbf{b} \, d\Omega \quad (72)$$

$\mathbf{D}$ and $\mathbf{b}$ are the moment-curvature and deflection-curvature operators [69], and $\mathbf{n}$ and $\mathbf{t}$ are operators that extract the normal and tangential components of a moment. The maximum eigenvalue is an element-wise estimate for β . We also use a second variant, for which we empirically determine a suitable global stabilization parameter.

Figure 19 plots the absolute error distribution $|w_{ex} - w_{FEM}|$ obtained with a 12×12 mesh for each method. Figures 20a and 20b show the convergence of the L^2 and H^2 errors for uniform mesh refinement. Figure 21 illustrates the accuracy of the diffusive flux, i.e. the moment $q_{diff} = (\mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{n}^+)$ normal to the immersed boundary, in terms of the L^2 error and point-wise. All plots indicate a similar behavior of the non-symmetric Nitsche method as shown in previous sections, indicating that the method also works well for fourth-order problems. While the accuracy of the solution itself is again slightly reduced in comparison to the symmetric variant of Nitsche's method with local stabilization, it yields the same H^2 accuracy in terms of error constants and rate of convergence. We also observe again that the non-symmetric Nitsche method exhibits superior accuracy in the diffusive flux at the boundary, benefiting from the absence of additional stabilization parameters that have a significant impact on the flux accuracy in both symmetric variants. We note that Fig. 21 should be interpreted in the sense that the non-symmetric Nitsche method is over-performing in this particular case, for some reason we do not know yet, and not in the sense that the accuracy of the symmetric Nitsche method with local eigenvalue stabilization is suboptimal (as it might look like).

(a) Relative error in the L^2 norm.

(b) Relative error in the H^2 semi-norm.

Figure 20: Convergence with uniform mesh refinement for the Kirchhof plate problem with weakly enforced boundary conditions.

(a) Convergence of the normal moment in the relative L^2 error.

(b) Absolute error distribution along the immersed boundary in the 12×12 mesh.

Figure 21: Accuracy of the diffusive flux in terms of the moment $q_{diff} = (\mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{n}^+)$ normal to the immersed boundary.

We emphasize that for the plate problem, the symmetric Nitsche method with global stabilization often achieves a significantly lower level of accuracy than the other two methods. As stability is governed by the smallest cut element, the global stabilization parameter is too large for the bulk of the cut elements. Therefore, the method tends to degenerate to a penalty-type method that emphasizes accuracy directly at the immersed Dirichlet boundary at the expense of the accuracy within the domain. This mechanism is very well reflected in the error distribution shown in Fig. 19c.

Figure 22: Examples of solution fields, computed with the non-symmetric Nitsche method on two Cartesian meshes with 6×6 and 11×11 elements coupled along an immersed interface (see Fig. 11).

4.5. Kirchhoff plate: Coupling

We then consider the case of weak coupling along an immersed interface. To this end, we directly extend the formulation of the non-symmetric Nitsche method (67) and (68) to the coupling case. The resulting discretized bilinear form reads

$$\begin{aligned}
 B(w_h, v) = & \sum_{\mathcal{K}} \int_{\mathcal{K}} \mathbf{m} : \hat{\mathbf{k}} \, d\Omega \\
 & - \int_{\Gamma^*} \{ \mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{n}^+ \} \llbracket \nabla v \rrbracket_n \, d\Gamma + \int_{\Gamma^*} \llbracket \nabla w_h \rrbracket_n \{ \mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{n}^+ \} \, d\Gamma \\
 & - \int_{\Gamma^*} \{ \mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{t}^+ \} \llbracket \nabla v \rrbracket_t \, d\Gamma + \int_{\Gamma^*} \llbracket \nabla w_h \rrbracket_t \{ \mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{t}^+ \} \, d\Gamma \\
 & + \int_{\Gamma^*} \{ \mathbf{n}^+ \cdot \nabla \cdot \mathbf{m} \} \llbracket v \rrbracket \, d\Gamma - \int_{\Gamma^*} \llbracket w_h \rrbracket \{ \mathbf{n}^+ \cdot \nabla \cdot \hat{\mathbf{m}} \} \, d\Gamma \quad (73)
 \end{aligned}$$

where we use the jump and average operators (19) and (20) for scalar quantities and the following jump operators for vector quantities

$$\llbracket \nabla w \rrbracket_n = (\nabla w^+ \cdot \mathbf{n}^+ + \nabla w^- \cdot \mathbf{n}^-) \quad (74)$$

$$\llbracket \nabla w \rrbracket_t = (\nabla w^+ \cdot \mathbf{t}^+ + \nabla w^- \cdot \mathbf{t}^-) \quad (75)$$

As we assume a perfectly bonded interface with the same material properties at both sides, the right-hand side $l(v)$ consists of the transverse forcing term only. The consistency of (73) can be examined by applying the procedure shown in Section 3.1. **We note that for notational conciseness, we omitted the terms to enforce Dirichlet boundary conditions on Γ_D in (73). They should be taken over from (67).**

Following [27, 37], we also extend the formulation of the weighted symmetric variant of

Nitsche's method to the coupling case. The resulting discretized bilinear form reads

$$\begin{aligned}
B(w_h, v) = & \sum_{\mathcal{K}} \int_{\mathcal{K}} \mathbf{m} : \hat{\mathbf{k}} \, d\Omega \\
& - \int_{\Gamma^*} \langle \mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{n}^+ \rangle_{\gamma} \llbracket \nabla v \rrbracket_n \, d\Gamma - \int_{\Gamma^*} \llbracket \nabla w_h \rrbracket_n \langle \mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{n}^+ \rangle_{\gamma} \, d\Gamma \\
& - \int_{\Gamma^*} \langle \mathbf{n}^+ \cdot \mathbf{m} \cdot \mathbf{t}^+ \rangle_{\gamma} \llbracket \nabla v \rrbracket_t \, d\Gamma - \int_{\Gamma^*} \llbracket \nabla w_h \rrbracket_t \langle \mathbf{n}^+ \cdot \hat{\mathbf{m}} \cdot \mathbf{t}^+ \rangle_{\gamma} \, d\Gamma \\
& + \int_{\Gamma^*} \langle \mathbf{n}^+ \cdot \nabla \cdot \mathbf{m} \rangle_{\gamma} \llbracket v \rrbracket \, d\Gamma + \int_{\Gamma^*} \llbracket w_h \rrbracket \langle \mathbf{n}^+ \cdot \nabla \cdot \hat{\mathbf{m}} \rangle_{\gamma} \, d\Gamma \\
& + \frac{\alpha h^2}{12} \int_{\Gamma^*} \llbracket \nabla w_h \rrbracket_n \llbracket \nabla v \rrbracket_n \, d\Gamma + \frac{\alpha h^2}{12} \int_{\Gamma^*} \llbracket \nabla w_h \rrbracket_t \llbracket \nabla v \rrbracket_t \, d\Gamma + \alpha \int_{\Gamma^*} \llbracket w_h \rrbracket \llbracket v \rrbracket \, d\Gamma \quad (76)
\end{aligned}$$

which uses the weighting operator (58) and jump operators (19), (74), and (75). We obtain estimates of the parameters α and γ from (59) and (60). The required maximum eigenvalues C^+ and C^- are computed from separate generalized eigenvalue problems that are defined in each cut element on each side of the immersed interface based on relations (71) and (72). For Nitsche's method with a global stabilization parameter, we use the classical weighting $\gamma = 0.5$ and empirically find an optimal value for α . **We note again that the terms to enforce Dirichlet boundary conditions on Γ_D in (76) should be taken over from (69).**

Figure 22 plots the deflections and the twist moment based on mixed first derivatives obtained with the non-symmetric Nitsche method. The fields are computed on two Cartesian meshes with 6×6 and 11×11 elements coupled along the immersed interface shown in Fig. 11. We observe that even for this rather coarse discretization the solution fields are smooth and continuous at the immersed interface without exhibiting any visible jumps or oscillations. Figure 23 plots the absolute error distribution $|w_{ex} - w_{FEM}|$ on two trimmed and coupled Cartesian meshes with 12×12 and 23×23 elements for each method. Figures 24a and 24b show the convergence of the L^2 and H^2 errors as the Cartesian mesh is uniformly

Figure 23: Distribution of the error in the deflection for the weakly coupled Kirchhoff plate problem based on two trimmed Cartesian meshes. The error is amplified in all plots for better visibility.

(a) Relative error in the L^2 norm.

(b) Relative error in the H^1 semi-norm.

Figure 24: Convergence of error norms for the weakly coupled Kirchhoff plate problem.

refined. Contrary to all previous results, the non-symmetric Nitsche method achieves the same accuracy for the deflection field as the symmetric Nitsche method with local stabilization. This includes optimal rates of convergence in both error norms and a uniform distribution of the error over the domain with comparable maximum values. In contrast, the symmetric Nitsche method based on global stabilization concentrates error around the immersed interface due to its penalty-like behavior. This leads to larger errors in comparison to the other two methods.

4.6. Double layered spherical thick shell

In the last example, we extend our study to three-dimensional problems with curved immersed interfaces. To this end, we examine a double layered spherical thick shell under internal pressure. The geometry and the location of the material interface are shown in Fig. 25. The radii that define the inner material layer are $r_i = 50$ and $r_m = 75$, and the corresponding radii for the outer material layer are $r_m = 75$ and $r_o = 100$. Young's moduli for the inner and outer layer are $E_i = 10,000$ and $E_o = 20,000$, respectively, and Poisson's ratio is $\nu = 0.3$ for the complete structure. We apply a pressure of $p_i = 50$ at the inner surface of the shell. Due to symmetry, only one eighth of the original geometry is considered. From an analytical calculation, the total strain energy of the octant of the shell can be computed as $1.636\,013\,453\,835\,590 \times 10^4$ [70].

We discretize the embedding cube with a Cartesian mesh that defines standard linear hexahedral basis functions and quadratic and cubic B-splines. The variational form of the non-symmetric Nitsche method for linear elasticity reads: Find $\mathbf{u}_h$ such that we have $B(\mathbf{u}_h, \delta \mathbf{u}) = l(\delta \mathbf{u})$, where

$$B(\mathbf{u}_h, \delta \mathbf{u}) = \sum_{\mathcal{K}} \int_{\mathcal{K}} \boldsymbol{\sigma}_h : \delta \boldsymbol{\varepsilon} \, d\Omega + \int_{\Gamma^*} \llbracket \mathbf{u}_h \rrbracket : \{\delta \boldsymbol{\sigma}\} \, d\Gamma - \int_{\Gamma^*} \{\boldsymbol{\sigma}_h\} : \llbracket \delta \mathbf{u} \rrbracket \, d\Gamma$$

Figure 25: Geometry of the double layered spherical shell [71] (interface highlighted in magenta).

$$+ \int_{\Gamma_D} \mathbf{u}_h \cdot (\delta \boldsymbol{\sigma} \cdot \mathbf{n}^+) \, d\Gamma - \int_{\Gamma_D} (\boldsymbol{\sigma}_h \cdot \mathbf{n}^+) \cdot \delta \mathbf{u} \, d\Gamma \quad (77)$$

$$l(v) = \int_{\Omega} \mathbf{f} \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma_D} \hat{\mathbf{u}} \cdot (\delta \boldsymbol{\sigma} \cdot \mathbf{n}^+) \, d\Gamma + \int_{\Gamma_N} \hat{\mathbf{t}} \cdot \delta \mathbf{u} \, d\Gamma \quad (78)$$

where $\mathbf{u}$ and $\delta \mathbf{u}$ are the displacement vector and the vector of virtual displacements; $\boldsymbol{\sigma}$, $\delta \boldsymbol{\sigma}$ and $\delta \boldsymbol{\varepsilon}$ are the stress, virtual stress and virtual strain tensors; and $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$ are the prescribed displacements and tractions at the Dirichlet and Neumann boundaries Γ_D and Γ_N , respectively. The jump and average operators in (77) are defined as

$$[[\mathbf{u}]] = \mathbf{u}^+ \otimes \mathbf{n}^+ + \mathbf{u}^- \otimes \mathbf{n}^- \quad (79)$$

$$\{\boldsymbol{\sigma}\} = \frac{1}{2} (\boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^-) \quad (80)$$

Analogous to the numerical analysis framework outlined in Section 3, the formulation (77) and (78) can be shown to satisfy consistency and coercivity (i.e., stability), and to converge with optimal rates in the strain energy norm [72].

We use the non-symmetric Nitsche method to impose perfect-bond interface conditions at the immersed interface that couple the two material layers. At the inner and outer surface of the shell, we impose pressure and zero-traction boundary conditions, respectively. We again use the recursive quadrature scheme of the finite cell method to perform integration in cut elements and over immersed interfaces. Figure 26 illustrates the sub-cell structure and the corresponding quadrature points for volume and surface integration. We emphasize again that the recursive quadrature scheme does not affect the basis functions, which are still defined on the regular Cartesian mesh.

(a) Adaptive sub-cells for volume integration in the $8 \times 8 \times 8$ Cartesian mesh.

(b) Corresponding volume and surface quadrature points (blue and magenta).

Figure 26: Numerical integration of cut elements and immersed surfaces in three dimensions with the finite cell method [9].

Figure 27: Von Mises stress obtained with the non-symmetric Nitsche method on a $8 \times 8 \times 8$ mesh and quadratic B-splines.

Figure 27 plots the distribution of the von Mises stresses obtained with the non-symmetric Nitsche method with a $8 \times 8 \times 8$ mesh and quadratic B-splines. We observe that the discontinuity due to the jump in stresses in circumferential direction can be represented accurately and without oscillations on coarse meshes. Figures 28a and 28b compare the convergence in strain energy for linear basis functions and quadratic and cubic B-splines obtained with the non-symmetric Nitsche method and the symmetric variant of Nitsche's method with global stabilization (see e.g. [4, 72] for details in the Nitsche formulation). For the latter, we choose suitable stabilization parameters empirically. We observe that the non-symmetric Nitsche method achieves the same accuracy as the symmetric variant for quadratic and cubic basis functions. For linear basis functions not covered by our numerical analysis framework, we obtain still optimal rates of convergence, but observe a slightly reduced error constant. This is in line with our observations for further test cases (e.g., on steady Navier-Stokes flow with

Figure 28: Convergence of the relative error in energy norm for the double-layer hollow sphere.

stabilized equal-order linear triangles) that are not reported here.

5. Summary, conclusions and outlook

In this paper, we explored the use of the non-symmetric Nitsche method for weakly imposing boundary and interface conditions in problems based on diffusion-type operators, with particular emphasis on immersed finite element methods. The non-symmetric Nitsche method is attractive, because it does not depend on stabilization parameters. We started by reviewing the numerical analysis framework of the non-symmetric Nitsche method in the context of Discontinuous Galerkin methods and beyond. The available analysis results for the non-symmetric Nitsche method can be summarized as follows:

Strong points:

- The method is variationally consistent.
- The method is conservative with respect to the flux based on the first derivatives.
- The error can be bounded in both L^2 and H^1 norms.
- Optimal convergence is achieved in the H^1 norm.
- The method is free of any stabilization parameter.

Weak points:

- The missing adjoint consistency leads to suboptimal convergence in the L^2 norm.
- The method is not conservative with respect to flux based on the primary variable.
- Symmetry of the discrete system matrix is lost.

We then presented a series of numerical experiments for a 2D second-order Laplace problem, a fourth-order Kirchhoff plate, and a 3D elastic spherical thick shell. Our results confirm that the non-symmetric Nitsche method is stable in all problems examined, does

not show any spurious oscillations, but exhibits a reduced level of accuracy in the L^2 error. Contrary to the theoretical results, we still observed optimal rates of convergence for the L^2 error. This is in agreement with the study of BURMAN who notes in [55] that he has “not managed to construct an example exhibiting the suboptimal convergence order” when enforcing boundary conditions on fitted meshes with the non-symmetric Nitsche method. On the other hand, our results demonstrate that the non-symmetric Nitsche method leads to the same optimal accuracy as its symmetric counterpart in global H^1 and H^2 error norms and provides superior accuracy in the diffusive flux at immersed boundaries. They also indicate that the non-symmetric method works well for linear basis functions, which is not covered by the DG analysis framework, but supported by more recent analysis results [55].

Our results demonstrate the potential of the non-symmetric Nitsche method in the context of immersed finite elements when put in the following perspective. In many applications, the derivative norms are decisive. A salient example is displacement based structural analysis, where from an engineering viewpoint the accuracy of the derivatives of the primal variable, i.e. the stress, is much more important than the accuracy of the primal variable itself, i.e. the displacement vector. Therefore, the optimal convergence in H^1 (and in H^2 for the Kirchhoff plate) delivered by the non-symmetric Nitsche method is of primary importance, while its reduced L^2 accuracy is acceptable. We note that isogeometric collocation [73] is another recent analysis technology with optimal accuracy in the derivatives, but reduced convergence rates in the L^2 error that has been successfully applied for structural analysis [74–76]. The conservativity and superior accuracy for the diffusive flux at immersed boundaries, e.g., in terms of boundary stresses or moments, is a significant added benefit of the non-symmetric Nitsche method. With increasing computing power and its proliferation, aspects such as automation and robustness of simulations are likely to gain more importance over the pure computational efficiency of a numerical method. From this point of view, the absence of stabilization parameters in the non-symmetric Nitsche method can be a considerable advantage that is more important than the additional memory and solution time due to the missing symmetry of the system matrix. This is particularly true for immersed methods, where stabilization parameters are very sensitive to cut element scenarios with significant impact on local accuracy and stability.

In summary, we think that the parameter-free non-symmetric Nitsche method can be a viable alternative to symmetric variants of Nitsche’s method for problems with diffusion-type operators, in particular in situations, where the accuracy of derivative quantities is of primary importance. **An issue that still needs to be investigated in this context is the behavior of the eigenspectrum that in the non-symmetric case also potentially spans the complex plane.** From a more fundamental point of view, Future work will explore its use in unfitted discretizations of flow problems, for which encouraging initial work exists [55, 57, 59]. In this context, the non-symmetric Nitsche method seems particularly appealing, because advection-diffusion-type problems naturally lead to unsymmetric system matrices.

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Appendix A. Global stabilization parameters

To support full traceability of the current study, we provide a list of maximum eigenvalues for the two basic Laplace problems with an embedded boundary and an embedded interface. Each entry represents the maximum eigenvalue that was computed for each given discretization and polynomial degree with the element-wise strategy and has been used to derive the global stabilization parameter for the simple Nitsche approach.

Size of (coarser) unfitted mesh	Example Section 4.1: Laplace w embedded boundary		Example Section 4.3: Laplace w embedded interface	
	Quadratics ($p=2$)	Cubics ($p=3$)	Quadratics ($p=2$)	Cubics ($p=3$)
3×3	261.66	524.26	435.51	873.50
6×6	368.55	744.74	2313.3	4268.2
12×12	1778.1	3509.5	1.1466e+04	2.1269e+04
24×24	3568.8	7008.0	9.1398e+04	6.4249e+04
48×48	1.6677e+04	2.9478e+04	1.4411e+05	3.0440e+05
96×96	8.5783e+04	1.0121e+05	5.1516e+05	6.7931e+05

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