A Return Mapping Algorithm for Isotropic and Anisotropic Plasticity Models Using a Line Search Method

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Abstract

The numerical integration of constitutive models in computational solid mechanics codes allows for the solution of boundary value problems involving complex material behavior. Metal plasticity models, in particular, have been instrumental in the development of these codes. Most plasticity models implemented in computational codes use an isotropic von Mises yield surface. The von Mises, or J_2 , yield surface has a simple predictor-corrector algorithm - the radial return algorithm - to integrate the model. For non-quadratic yield surfaces, including anisotropic yield surfaces, no simple algorithm exists. One isotropic non-quadratic yield surface that is useful for analysis is the Hosford yield surface [12]. The Hosford yield surface can model a von Mises yield surface, a Tresca yield surface, or a yield surface that lies between the two. For anisotropic yield surfaces many options exist, including Hill's 1948 model [11], and models by Karafillis and Boyce [14], Barlat and co-workers [2], and Cazacu et. al. [8] among others. None of these plasticity models are easily integrated using the Newton-Raphson algorithm. Two yield surface models are studied in this paper: Hosford [12] and Barlat [2]. Reliable and robust algorithms for integrating the models are presented. The algorithms include an efficient eigenvalue/eigenvector solver, scaling of the stresses in the yield function, and employing a line search algorithm for integration. The theory and implementation of the models, the details of the return mapping algorithms, and results that show the effectiveness of the algorithms are presented.

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1. Introduction

Material models for the plastic deformation of metals are necessary in general purpose finite element codes that are used for solid mechanics analysis. In fact, their usefulness can be seen by the vast number of plasticity models that are proposed in the literature and used in finite element calculations.

A fundamental feature of most plasticity models is a yield surface, which defines a region of stress space where the material behavior is elastic and, if associated flow is assumed, provides the plastic flow direction. The two most common yield surfaces are the Tresca yield surface and the von Mises yield surface - both of which are isotropic. The Tresca yield surface is arguably the simplest; it is based on yield occurring when the maximum shear stress reaches some critical value. Tresca is a very simple criterion that lends itself well to analytical problems, but it is less suitable to a numerical implementation due to sharp corners. With the von Mises yield surface yield occurs when the second invariant of the deviatoric stress reaches some critical value. Unlike Tresca this yield surface is well suited for computational codes and it underlies most plasticity models implemented in them. An alternative isotropic yield surface was developed by Hosford [12] to deal with experimental results in biaxial stretching that lay between the Tresca and von Mises surfaces. The Hosford yield surface has the feature that it can fit Tresca or von Mises, or fit a yield surface that lies between the two. It has seen little use in computational codes, probably due to the expensive eigenvalue/eigenvector calculations that are required and the difficulty in integrating it. A numerical implementation for the Hosford model is presented and its behavior is examined extensively in this work. The Tresca and von Mises models are treated as specializations of the Hosford model.

The number and variety of isotropic yield surface models is relatively small compared to anisotropic models. In this work attention will be restricted to a subset of anisotropic models, namely orthotropic models. Perhaps the most widely used orthotropic model is the Hill yield surface [11] which is used extensively to model sheet metal forming. The Hill model can be thought of as an orthotropic extension of the von Mises model. In an effort to better match results from sheet metal forming problems a number of alternative orthotropic models have been proposed including the models of Karafillis and Boyce [14], and Barlat and co-workers [1] [5] [7] [4] [6] [3] [2]. Cazacu et. al. [8] also address anisotropy along with tension/compression asymmetry in yield. In this paper the Yld2004-18p model of Barlat et. al. [2] is used given that it is perhaps one of the most general anisotropic yield surface descriptions in the literature.

Many of the plasticity models presented in the literature show little of their numerical implementation. The implementation of a model in a computational code - generally a finite element code - is extremely important if it will be used in the solution of solid mechanics problems. Specifically we are interested in the numerical integration of the plasticity model using a predictor-corrector algorithm to solve the closest point projection problem [13] [27] [25]. The key to these algorithms is the behavior of the return mapping that returns an elastic predictor, or trial stress state, to the yield surface. Of particular interest is the performance of the algorithms for very large trial stress states, a situation that can easily arise during the iterative solution in large deformation finite element problems.

When researchers address the numerical implementation of plasticity models, most look at the accuracy of the numerical integration. Krieg and Krieg [15] and Schreyer et. al. [22] look at the accuracy of the return mapping algorithm for a von Mises model in plane stress. Ortiz and Simo [19] analyze a fairly general return mapping algorithm for elastoplastic material models that use a predictor-corrector algorithm; they also look at the accuracy of the cutting plane algorithm

that they proposed [26]. Barlat and Yoon and co-workers [3] [29] [30] implement the 2D plane stress version of the anisotropic Barlat model (Yld2000-2d) using a “multi-stage return mapping” algorithm. The algorithm was analyzed for accuracy for moderate strain increments and the model was successfully used in a number of boundary value problems. Kroon and Faleskog [16] propose an isotropic yield surface that has J_2 and J_3 dependence which is similar to the Hosford yield surface. They use a bracketing scheme for the return mapping algorithm but do not provide an analysis of the algorithm. Line search algorithms have been used by a number of authors, for example Dutko et. al. [10] and Seifert and Schmidt [23], but little is presented on the line search algorithm itself. Perhaps the most general analysis of the integration of plasticity models using a line search algorithm is provided by Pérez-Foguet and Armero [20]. They look at a number of line search techniques and apply them to different yield surfaces. Their work is, for the most part, followed in this paper and applied to the Hosford and Barlat models.

This paper is organized as follows. First, the Hosford model and Barlat model are presented and expressions for the yield surfaces and their associated derivatives are given. Additional considerations necessary to numerically stabilize the problem are also discussed. Next, details of the model implementation into a computational code are shown for both models. This section includes the predictor-corrector algorithm which can be solved using Newton-Raphson or a line search method. The performance of the algorithms is presented comparing their convergence for trial stress states plotted in the π -plane. To highlight the return mapping algorithm, attention is restricted to perfect plasticity. However one case of a non-linear hardening law is used to show that the hardening has little effect on the convergence of the algorithms, slightly improving the convergence behavior. The results show the poor behavior for the Newton-Raphson algorithm and excellent behavior for the line search algorithm. Finally, the boundary value problem of internal pressurization of a cylinder in plane strain is presented to show the wide range of behavior that can be seen using alternative yield surface formulations.

2. Models

Two yield surface models are presented in this paper: the Hosford model [12] and the Barlat model [2]. They are similar models in that the yield surfaces are formulated using principal stresses. In the case of the isotropic Hosford model these are the principal values of the Cauchy stress while for the orthotropic Barlat model the principal values of transformed stresses are used. The Hosford model can, with the appropriate choice of parameters, be reduced to a Tresca or von Mises model. The Barlat model can, with the appropriate choice of parameters, be reduced to Hill or Hosford model, and by extension a Tresca or von Mises model.

2.1. Hosford Model

The Hosford yield surface [12] was proposed as a generalization of the von Mises and Tresca yield surfaces that is capable of fitting the experimental data for biaxial stress states which often lie between the other two yield surfaces. Having a model that is capable of fitting this data is very important in modeling and simulation since most numerical simulations will involve stress paths that are different from the experimental stress paths that are used to parameterize the model.

The Hosford model is the description of the yield surface for the material and an associated flow rule that supplies the plastic flow direction. The model does not address the form of the hardening law, which can be defined separately. The form of the yield surface used in this paper is given by

$$f(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \phi(\boldsymbol{\sigma}) - \sigma_y(\bar{\varepsilon}^p) = 0 \quad (1)$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\bar{\varepsilon}^p$ is the equivalent plastic strain, ϕ is the effective stress, and σ_y is the hardening function. The effective stress in the Hosford model is

$$\phi(\boldsymbol{\sigma}) = \left\{ \frac{1}{2} \left[|\sigma_1 - \sigma_2|^a + |\sigma_2 - \sigma_3|^a + |\sigma_3 - \sigma_1|^a \right] \right\}^{1/a} \quad (2)$$

where σ_i are the principal stresses and the exponent a is a material model parameter.

The form of the effective stress in (2) allows us to equate the consistency parameter with the rate of the equivalent plastic strain, $\dot{\bar{\varepsilon}}^p$. We start with the statement of equivalent plastic work

$$\boldsymbol{\sigma} : \dot{\bar{\varepsilon}}^p = \sigma_y \dot{\bar{\varepsilon}}^p \quad , \quad (3)$$

and assume that the plastic strain rate is normal to the yield surface

$$\dot{\bar{\varepsilon}}^p = \dot{\gamma} \frac{\partial \phi}{\partial \boldsymbol{\sigma}} \quad (4)$$

where $\dot{\gamma}$ is the consistency parameter. Using (4) in (3), and making use of the fact that ϕ is a homogeneous function of degree one in (1), we have

$$\dot{\gamma} \boldsymbol{\sigma} : \frac{\partial \phi}{\partial \boldsymbol{\sigma}} = \dot{\gamma} \phi = \sigma_y \dot{\bar{\varepsilon}}^p \quad \rightarrow \quad \dot{\gamma} = \dot{\bar{\varepsilon}}^p \quad . \quad (5)$$

In our implementation of the return mapping algorithm the increment in the consistency parameter will be equivalent to the increment in the equivalent plastic strain.

The choice of a affects the shape of the yield surface. If $a = 2$, or if $a = 4$, the yield surface is a von Mises yield surface. If $a = 1$, or as $a \rightarrow \infty$, the yield surface is a Tresca yield surface. If $a \in (1, 2)$, or if $a \in (4, \infty)$, the yield surface lies between a von Mises and Tresca yield surface.¹ In practice values of a are usually chosen that are greater than 4, and that will be assumed in this paper. Examples of Hosford yield surfaces with $a = 6$ and $a = 8$, which are useful modeling BCC and FCC metals (see [17]) are shown in the π -plane in Figure 1 along with a Tresca and von Mises yield surface. Finally, it should be noted that in the literature most authors restrict a to be an integer - and some an even integer - but no such restriction is required in this work.

Looking at the yield surfaces in Figure 1 we see that the yield point can change significantly depending on the exponent, a , and the load path. In uniaxial stress, tension or compression, the different yield surfaces give the same results, but for other load paths, including pure shear, the behavior of the models will be different. However, a bigger impact of the Hosford model is the modeling of associated plastic flow. The change in the normal to the yield surface for some load

¹For $a \in (2, 4)$ the yield surface lies outside the von Mises yield surface. The yield surface is convex, but it is not seen to match experimental data and will not be considered here.

paths can be significant when compared to a von Mises yield surface. This will give a different accumulation of plastic strain, elastic strain, and stress in a boundary value problem.

For the Hosford model the normal to the yield surface is most easily written in the basis of the principal stress directions

$$\boldsymbol{\sigma} = \sum_{i=1}^3 \sigma_i \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i \quad ; \quad \frac{\partial \phi}{\partial \boldsymbol{\sigma}} = \sum_{i=1}^3 \frac{\partial \phi}{\partial \sigma_i} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i \quad (6)$$

where $\hat{\mathbf{e}}_i$ are the principal stress directions, or the eigenvectors associated with the eigenvalues, σ_i , of the Cauchy stress.² The expression for the normal is found using results first presented by Chadwick and Ogden [9] on the derivatives of functions of principal values of tensors.

For the incremental solution presented below, not only are the first derivatives of ϕ needed but also the second derivatives, or Hessian. Expressions for the Hessian were also found by Chadwick and Ogden [9] and will be applied here. The components of the second derivative of ϕ with respect to $\boldsymbol{\sigma}$ are written in the basis of eigenvectors

$$\frac{\partial^2 \phi}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} = \mathcal{H}_{ijkl} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l \quad . \quad (7)$$

The only non-zero components in this basis are of the form

$$\begin{aligned} \mathcal{H}_{1111} &= \frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_1} \quad ; \quad \mathcal{H}_{1122} = \frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_2} \\ \mathcal{H}_{1212} &= \frac{\partial \phi / \partial \sigma_1 - \partial \phi / \partial \sigma_2}{2(\sigma_1 - \sigma_2)} \end{aligned} \quad (8)$$

with the other non-zero components found by permuting the indicies. The only difficulty with this form arises with the $\mathcal{H}_{1212}$ terms when we have two equal principal stresses. When this occurs we can take the limit

$$\mathcal{H}_{1212} = \lim_{\sigma_1 \rightarrow \sigma_2} \left[\frac{\partial \phi / \partial \sigma_1 - \partial \phi / \partial \sigma_2}{2(\sigma_1 - \sigma_2)} \right] = \frac{1}{2} \left(\frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_1} - \frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_2} \right) \quad . \quad (9)$$

This gives us a very simple and effective way of calculating the Hessian for ϕ when we know the eigenvalues and eigenvectors of $\boldsymbol{\sigma}$.

The derivatives of the effective stress are necessary to evaluate the normal and curvature for the yield surface which are used in the return mapping algorithm. In the case of the Hosford model,

²In indicial notation the standard summation convention is assumed where repeated indicies are summed from 1 to 3. Indices that are not repeated take values of 1, 2, or 3. In some cases the summation is *explicitly* stated when an index is repeated more than twice as in equation (6).

care must be taken when calculating the effective stress and the derivatives to ensure that the results are numerically stable when the exponent, a , is large.

The eigenvalues and eigenvectors of $\boldsymbol{\sigma}$ are found using the algorithm presented by Scherzinger and Dohrmann [21] and the effective stress is given by (2). For numerical stability, rather than use the principal stresses directly in (2) we scale them with the von Mises stress³

$$\sigma_{\text{vm}} = \sqrt{\frac{3}{2} \mathbf{s} : \mathbf{s}} \quad (10)$$

where $\mathbf{s}$ is the deviatoric stress

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3} (\text{tr} \boldsymbol{\sigma}) \mathbf{I} \quad (11)$$

There is nothing special about using the von Mises stress, and any other measure of the magnitude of stress can be used.

Scaling the principal stresses, σ_i , we have

$$\bar{\sigma}_i = \frac{\sigma_i}{\sigma_{\text{vm}}} \quad ; \quad \phi(\boldsymbol{\sigma}) = \sigma_{\text{vm}} \left\{ \frac{1}{2} \left[|\bar{\sigma}_1 - \bar{\sigma}_2|^a + |\bar{\sigma}_2 - \bar{\sigma}_3|^a + |\bar{\sigma}_3 - \bar{\sigma}_1|^a \right] \right\}^{1/a} \quad (12)$$

The scaling ensures that the terms in the brackets do not cause numerical problems when a is large. Once we have ϕ we seek expressions for (6) and (8).

The first derivative of the effective stress with respect to the first principal stress is

$$\frac{\partial \phi}{\partial \sigma_1} = \frac{\phi}{2\phi^a} \left[\frac{|\sigma_1 - \sigma_2|^a}{\sigma_1 - \sigma_2} - \frac{|\sigma_3 - \sigma_1|^a}{\sigma_3 - \sigma_1} \right] \quad (13)$$

Here we make use of a similar modification where we scale the stresses to improve the behavior of the numerical calculations. However, instead of using the von Mises stress we use the effective stress, ϕ , which we just calculated in (12)

$$\hat{\sigma}_i = \frac{\sigma_i}{\phi} \quad ; \quad \frac{\partial \phi}{\partial \sigma_1} = \frac{1}{2} \left[\frac{|\hat{\sigma}_1 - \hat{\sigma}_2|^a}{\hat{\sigma}_1 - \hat{\sigma}_2} - \frac{|\hat{\sigma}_3 - \hat{\sigma}_1|^a}{\hat{\sigma}_3 - \hat{\sigma}_1} \right] \quad (14)$$

with the derivatives with respect to σ_2 and σ_3 found in a similar manner. The use of ϕ is the natural choice for the scaling given the definition of the derivative in (13). Using similar techniques for numerical stability, the second derivatives of the effective stress are

³The “:” operation for tensors in indicial notation is given by $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$ when $\mathbf{A}$ and $\mathbf{B}$ are rank-two tensors and $\mathbf{A} : \mathbf{B} = A_{ijkl} B_{kl} \mathbf{e}_i \otimes \mathbf{e}_j$ when $\mathbf{A}$ is a rank-four tensor.

$$\begin{aligned}\frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_1} &= \frac{a-1}{\phi} \left\{ \frac{1}{2} \left[\frac{|\hat{\sigma}_1 - \hat{\sigma}_2|^a}{(\hat{\sigma}_1 - \hat{\sigma}_2)^2} + \frac{|\hat{\sigma}_3 - \hat{\sigma}_1|^a}{(\hat{\sigma}_3 - \hat{\sigma}_1)^2} \right] - \frac{\partial \phi}{\partial \sigma_1} \frac{\partial \phi}{\partial \sigma_1} \right\} \\ \frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_2} &= \frac{a-1}{\phi} \left\{ -\frac{1}{2} \frac{|\hat{\sigma}_1 - \hat{\sigma}_2|^a}{(\hat{\sigma}_1 - \hat{\sigma}_2)^2} - \frac{\partial \phi}{\partial \sigma_1} \frac{\partial \phi}{\partial \sigma_2} \right\}\end{aligned}\tag{15}$$

with the other derivatives found by permuting indicies.

Addressing one final numerical issue, we would like to eliminate the possible singularity in the denominators of (14) and (15). To do this we factor out the denominator and obtain

$$\begin{aligned}\frac{\partial \phi}{\partial \sigma_1} &= \frac{1}{2} \left[(\hat{\sigma}_1 - \hat{\sigma}_2) |\hat{\sigma}_1 - \hat{\sigma}_2|^{a-2} - (\hat{\sigma}_3 - \hat{\sigma}_1) |\hat{\sigma}_3 - \hat{\sigma}_1|^{a-2} \right] \\ \frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_1} &= \frac{a-1}{\phi} \left\{ \frac{1}{2} \left[|\hat{\sigma}_1 - \hat{\sigma}_2|^{a-2} + |\hat{\sigma}_3 - \hat{\sigma}_1|^{a-2} \right] - \frac{\partial \phi}{\partial \sigma_1} \frac{\partial \phi}{\partial \sigma_1} \right\} \\ \frac{\partial^2 \phi}{\partial \sigma_1 \partial \sigma_2} &= \frac{a-1}{\phi} \left\{ -\frac{1}{2} |\hat{\sigma}_1 - \hat{\sigma}_2|^{a-2} - \frac{\partial \phi}{\partial \sigma_1} \frac{\partial \phi}{\partial \sigma_2} \right\}\end{aligned}\tag{16}$$

with the other non-zero derivatives having the same form. Since we have restricted $a \in [4, \infty)$ this form does not have a singularity. All of the derivatives are evaluated using the three normalized stress differences: $\hat{\sigma}_1 - \hat{\sigma}_2$, $\hat{\sigma}_2 - \hat{\sigma}_3$, and $\hat{\sigma}_3 - \hat{\sigma}_1$.

2.2. Barlat Model

The orthotropic model due to Barlat [2] is also used in this paper. Its appeal is partly due to its ability to model the orthotropic response of Aluminum alloys. In addition to this, it is also an excellent model for studying the plastic deformation of metals since it can be reduced to the more common von Mises and Hill models with the appropriate parameterization. It can also be reduced to the Hosford model.

The Barlat model (Yld2004-18p) describes the yield surface for the material and an associated flow rule that supplies the plastic flow direction. The model does not address the form of the hardening law, which can be defined separately. The yield surface in this paper is again defined by (1) where ϕ is a homogeneous function of degree one. This form of the yield surface has the same property that the Hosford model has, i.e. $\dot{\gamma} = \dot{\epsilon}^p$.

The effective stress is similar to the Hosford model presented above, but with two transformations of the stress state that account for the anisotropy of the yield surface. The effective stress is

$$\begin{aligned}
\phi(\boldsymbol{\sigma}) = & \left\{ \frac{1}{4} \left[|s'_1 - s''_1|^a + |s'_1 - s''_2|^a + |s'_1 - s''_3|^a \right. \right. \\
& + |s'_2 - s''_1|^a + |s'_2 - s''_2|^a + |s'_2 - s''_3|^a \\
& \left. \left. + |s'_3 - s''_1|^a + |s'_3 - s''_2|^a + |s'_3 - s''_3|^a \right] \right\}^{1/a}
\end{aligned} \tag{17}$$

where a is a material model parameter, and s'_i and s''_i ($i = 1, 2, 3$) are the principal values (eigenvalues) of the two transformed stresses $\mathbf{s}'$ and $\mathbf{s}''$ with principal directions (eigenvectors) $\hat{\mathbf{e}}'_i$ and $\hat{\mathbf{e}}''_i$ respectively

$$\begin{aligned}
\mathbf{s}' &= \sum_{i=1}^3 s'_i \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_i = \mathbf{L}' : \boldsymbol{\sigma} \\
\mathbf{s}'' &= \sum_{i=1}^3 s''_i \hat{\mathbf{e}}''_i \otimes \hat{\mathbf{e}}''_i = \mathbf{L}'' : \boldsymbol{\sigma}
\end{aligned} \tag{18}$$

The transformations, $\mathbf{L}'$ and $\mathbf{L}''$, are fourth-order tensors that account for the anisotropy of the material, mapping the Cauchy stress to $\mathbf{s}'$ and $\mathbf{s}''$ respectively. To preserve a pressure independent yield surface, the transformations are written as

$$\mathbf{L}' = \mathbf{C}' : \boldsymbol{\Pi}' \quad ; \quad \mathbf{L}'' = \mathbf{C}'' : \boldsymbol{\Pi}' \tag{19}$$

where $\boldsymbol{\Pi}'$ is the symmetric deviatoric projection with components in a Cartesian coordinate system given by

$$\Pi'_{ijkl} = \frac{1}{2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3} \delta_{ij}\delta_{kl} \quad . \tag{20}$$

Since the deviatoric stress, $\mathbf{s}$, is the deviatoric projection of the Cauchy stress, i.e. $\mathbf{s} = \boldsymbol{\Pi}' : \boldsymbol{\sigma}$, we can write

$$\mathbf{s}' = \mathbf{C}' : \mathbf{s} \quad ; \quad \mathbf{s}'' = \mathbf{C}'' : \mathbf{s} \quad . \tag{21}$$

The fourth-order tensors $\mathbf{C}'$ and $\mathbf{C}''$ can be written in matrix notation with respect to the material coordinate system

$$\begin{aligned}
[\mathbf{C}'] &= \begin{bmatrix} 0 & -c'_{12} & -c'_{13} & 0 & 0 & 0 \\ -c'_{21} & 0 & -c'_{23} & 0 & 0 & 0 \\ -c'_{31} & -c'_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c'_{66} \end{bmatrix} \\
[\mathbf{C}''] &= \begin{bmatrix} 0 & -c''_{12} & -c''_{13} & 0 & 0 & 0 \\ -c''_{21} & 0 & -c''_{23} & 0 & 0 & 0 \\ -c''_{31} & -c''_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c''_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c''_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c''_{66} \end{bmatrix}
\end{aligned} \tag{22}$$

The 18 terms in $\mathbf{C}'$ and $\mathbf{C}''$ are material parameters giving the yield surface its common name, Yld2004-18p. The choice of the parameters in $\mathbf{C}'$ and $\mathbf{C}''$, along with the exponent a , affect the shape of the yield surface. If all of the 18 parameters in $\mathbf{C}'$ and $\mathbf{C}''$ are set to one, then the model reduces to an isotropic Hosford model; if $\mathbf{C}' = \mathbf{C}''$, $c'_{ij} = c'_{ji}$, and $a = 2$ or 4, then the model reduces to the Hill model.

The parameterization of the model affects the shape of the yield surface. Figure 2 shows the von Mises, Hill, and Barlat parameterizations for the same material. The material parameterization of 2090-T4 aluminum given by Barlat et. al. [2] is used (see Table 1). The model restricted to uniaxial stress in the three material directions and pure shear relative to those directions gives six yield stresses that are used to parameterize a Hill model (see Table 2); uniaxial stress in one material direction gives the parameterization for the von Mises model. Figure 2 shows that the yield stress for the three material directions is the same for the Barlat and Hill models and they are significantly different than the von Mises model. This is what we expect. For stress paths other than uniaxial stress the yield behavior for the three models is significantly different, even for the two orthotropic models. Furthermore, when looking at the normal to the yield surface, the plastic flow directions will be different virtually everywhere, including loading in uniaxial stress. This will give a different accumulation of plastic strain, elastic strain, and stress in a boundary value problem.

The normal to the yield surface, needed for the plastic flow direction, is found from the chain rule

$$\frac{\partial \phi}{\partial \boldsymbol{\sigma}} = \sum_{i=1}^3 \left(\frac{\partial \phi}{\partial s'_i} \frac{\partial s'_i}{\partial \mathbf{s}'} : \mathbf{L}' + \frac{\partial \phi}{\partial s''_i} \frac{\partial s''_i}{\partial \mathbf{s}''} : \mathbf{L}'' \right) \quad . \tag{23}$$

The derivative of the eigenvalues with respect to the transformed stresses are

$$\frac{\partial s'_i}{\partial \mathbf{s}'} = \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_i \quad ; \quad \frac{\partial s''_i}{\partial \mathbf{s}''} = \hat{\mathbf{e}}''_i \otimes \hat{\mathbf{e}}''_i \tag{24}$$

where there is no summation over the repeated indicies on the right hand side of each equation. This allows us to write (23) as

$$\frac{\partial \phi}{\partial \boldsymbol{\sigma}} = \sum_{i=1}^3 \left(\frac{\partial \phi}{\partial s'_i} (\hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_i) : \mathbf{L}' + \frac{\partial \phi}{\partial s''_i} (\hat{\mathbf{e}}''_i \otimes \hat{\mathbf{e}}''_i) : \mathbf{L}'' \right) . \quad (25)$$

As with the results of the Hosford model presented above, use is made of the results of Chadwick and Ogden [9] to evaluate the derivatives.

For the incremental solution presented below the Hessian is also needed. The Hessian for the Barlat yield surface is found by simply continuing the chain rule

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} = & \sum_{i=1}^3 \left\{ \sum_{j=1}^3 \left[\frac{\partial^2 \phi}{\partial s'_i \partial s'_j} \left(\frac{\partial s'_i}{\partial \mathbf{s}'} : \mathbf{L}' \right) \otimes \left(\frac{\partial s'_j}{\partial \mathbf{s}'} : \mathbf{L}' \right) + \frac{\partial^2 \phi}{\partial s'_i \partial s''_j} \left(\frac{\partial s''_i}{\partial \mathbf{s}''} : \mathbf{L}'' \right) \otimes \left(\frac{\partial s''_j}{\partial \mathbf{s}''} : \mathbf{L}'' \right) \right. \right. \\ & + \frac{\partial^2 \phi}{\partial s'_i \partial s''_j} \left(\frac{\partial s'_i}{\partial \mathbf{s}'} : \mathbf{L}' \right) \otimes \left(\frac{\partial s''_j}{\partial \mathbf{s}''} : \mathbf{L}'' \right) + \frac{\partial^2 \phi}{\partial s''_i \partial s'_j} \left(\frac{\partial s''_i}{\partial \mathbf{s}''} : \mathbf{L}'' \right) \otimes \left(\frac{\partial s'_j}{\partial \mathbf{s}'} : \mathbf{L}' \right) \left. \right] \\ & + \frac{\partial \phi}{\partial s'_i} \left(\mathbf{L}'^T : \frac{\partial^2 s'_i}{\partial \mathbf{s}' \partial \mathbf{s}'} : \mathbf{L}' \right) + \frac{\partial \phi}{\partial s''_i} \left(\mathbf{L}''^T : \frac{\partial^2 s''_i}{\partial \mathbf{s}'' \partial \mathbf{s}''} : \mathbf{L}'' \right) \left. \right\} . \quad (26) \end{aligned}$$

The normal and the Hessian require the calculation of $\partial \phi / \partial s'_i$, $\partial \phi / \partial s''_i$, $\partial^2 \phi / \partial s'_i \partial s'_j$, $\partial^2 \phi / \partial s'_i \partial s''_j$, and $\partial^2 \phi / \partial s''_i \partial s''_j$. The last two terms in the Hessian involve $\partial^2 s'_i / \partial \mathbf{s}' \partial \mathbf{s}'$, and $\partial^2 s''_i / \partial \mathbf{s}'' \partial \mathbf{s}''$. These terms are handled separately. This expression for the Hessian is *much* more complicated than the corresponding expression for the Hosford model.

The derivatives of the effective stress (17) with respect to the principal transformed stresses are similar to the results presented for the Hosford model. Again, care must be taken in computing the effective stress and its derivatives to ensure that the results are numerically stable when the exponent, a , is large. The eigenvalues and eigenvectors of $\mathbf{s}'$ and $\mathbf{s}''$ are found using the algorithm presented by Scherzinger and Dohrmann [21].

The effective stress is given by (17). For numerical stability the principal transformed stresses, s'_i and s''_i , are scaled by the von Mises stress given in (10). A different measure of the stress could also be used, or even some magnitude of the transformed stress. The only requirement is that the principal values of *both* transformed stresses be scaled with the same measure of the stress. Scaling the principal transformed stresses we have

$$\bar{s}'_i = \frac{s'_i}{\bar{\sigma}_{\text{vm}}} \quad ; \quad \bar{s}''_i = \frac{s''_i}{\bar{\sigma}_{\text{vm}}}$$

$$\begin{aligned}
\phi(\boldsymbol{\sigma}) = & \bar{\sigma}_{\text{vm}} \left\{ \frac{1}{4} \left[|\bar{s}'_1 - \bar{s}''_1|^a + |\bar{s}'_1 - \bar{s}''_2|^a + |\bar{s}'_1 - \bar{s}''_3|^a \right. \right. \\
& + |\bar{s}'_2 - \bar{s}''_1|^a + |\bar{s}'_2 - \bar{s}''_2|^a + |\bar{s}'_2 - \bar{s}''_3|^a \\
& \left. \left. + |\bar{s}'_3 - \bar{s}''_1|^a + |\bar{s}'_3 - \bar{s}''_2|^a + |\bar{s}'_3 - \bar{s}''_3|^a \right] \right\}^{1/a}
\end{aligned} \tag{27}$$

The scaling ensures that the stress values raised to a large power stay reasonable.

The first derivative of the effective stress for the Barlat model is similar to the Hosford model, but slightly more complicated. Taking the first derivative of (17) with respect to s'_1 we have

$$\frac{\partial \phi}{\partial s'_1} = \frac{\phi}{4\phi^a} \left[\frac{|s'_1 - s''_1|^a}{s'_1 - s''_1} + \frac{|s'_1 - s''_2|^a}{s'_1 - s''_2} + \frac{|s'_1 - s''_3|^a}{s'_1 - s''_3} \right] \tag{28}$$

Here we make use of a similar modification where we scale the stresses to improve the numerics for the calculations. Instead of using the von Mises stress we use the effective stress, ϕ , which we just calculated in (27))

$$\begin{aligned}
\hat{s}'_i &= \frac{s'_i}{\phi} \quad ; \quad \hat{s}''_i = \frac{s''_i}{\phi} \\
\frac{\partial \phi}{\partial s'_1} &= \frac{1}{4} \left[\frac{|\hat{s}'_1 - \hat{s}''_1|^a}{\hat{s}'_1 - \hat{s}''_1} + \frac{|\hat{s}'_1 - \hat{s}''_2|^a}{\hat{s}'_1 - \hat{s}''_2} + \frac{|\hat{s}'_1 - \hat{s}''_3|^a}{\hat{s}'_1 - \hat{s}''_3} \right]
\end{aligned} \tag{29}$$

The use of ϕ is the natural choice given the definition of the derivative in (28). Similar results are found for the derivatives with respect to s'_2 , s'_3 , s''_1 , s''_2 , and s''_3 .

Using similar techniques the second derivatives of the effective stress with respect to the principal transformed stresses are

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial s'_1 \partial s'_1} &= \frac{a-1}{\phi} \left\{ \frac{1}{4} \left[\frac{|\hat{s}'_1 - \hat{s}''_1|^a}{(\hat{s}'_1 - \hat{s}''_1)^2} + \frac{|\hat{s}'_1 - \hat{s}''_2|^a}{(\hat{s}'_1 - \hat{s}''_2)^2} + \frac{|\hat{s}'_1 - \hat{s}''_3|^a}{(\hat{s}'_1 - \hat{s}''_3)^2} \right] - \frac{\partial \phi}{\partial s'_1} \frac{\partial \phi}{\partial s'_1} \right\} \\
\frac{\partial^2 \phi}{\partial s'_1 \partial s'_2} &= -\frac{a-1}{\phi} \frac{\partial \phi}{\partial s'_1} \frac{\partial \phi}{\partial s'_2} \\
\frac{\partial^2 \phi}{\partial s'_1 \partial s''_1} &= \frac{a-1}{\phi} \left\{ -\frac{1}{4} \frac{|\hat{s}'_1 - \hat{s}''_1|^a}{(\hat{s}'_1 - \hat{s}''_1)^2} - \frac{\partial \phi}{\partial s'_1} \frac{\partial \phi}{\partial s''_1} \right\}
\end{aligned} \tag{30}$$

with the other non-zero derivatives having the same form.

As with the Hosford model, we would like to eliminate the possible singularity in the denominator of (29) and (30). Factoring out the denominator we find

$$\begin{aligned}
\frac{\partial \phi}{\partial s'_1} &= \frac{1}{4} \left[(\hat{s}'_1 - \hat{s}''_1) |\hat{s}'_1 - \hat{s}''_1|^{a-2} + (\hat{s}'_1 - \hat{s}''_2) |\hat{s}'_1 - \hat{s}''_2|^{a-2} + (\hat{s}'_1 - \hat{s}''_3) |\hat{s}'_1 - \hat{s}''_3|^{a-2} \right] \\
\frac{\partial^2 \phi}{\partial s'_1 \partial s'_1} &= \frac{a-1}{\phi} \left\{ \frac{1}{4} [|\hat{s}'_1 - \hat{s}''_1|^{a-2} + |\hat{s}'_1 - \hat{s}''_2|^{a-2} + |\hat{s}'_1 - \hat{s}''_3|^{a-2}] - \frac{\partial \phi}{\partial s'_1} \frac{\partial \phi}{\partial s'_1} \right\} \\
\frac{\partial^2 \phi}{\partial s'_1 \partial s'_2} &= -\frac{a-1}{\phi} \frac{\partial \phi}{\partial s'_1} \frac{\partial \phi}{\partial s'_2} \\
\frac{\partial^2 \phi}{\partial s'_1 \partial s''_1} &= \frac{a-1}{\phi} \left\{ -\frac{1}{4} |\hat{s}'_1 - \hat{s}''_1|^{a-2} - \frac{\partial \phi}{\partial s'_1} \frac{\partial \phi}{\partial s''_1} \right\}
\end{aligned} \tag{31}$$

with the other derivatives found by permuting the indicies. As long as $a \geq 2$ we have removed any singularities in these expressions. In this work we require $a \in [4, \infty)$.

For the last two terms in (26), those that involve $\partial^2 s'_i / \partial \mathbf{s}' \partial \mathbf{s}'$, and $\partial^2 s''_i / \partial \mathbf{s}'' \partial \mathbf{s}''$, we find

$$\begin{aligned}
\sum_{i=1}^3 \frac{\partial \phi}{\partial s'_i} \frac{\partial^2 s'_i}{\partial \mathbf{s}' \partial \mathbf{s}'} &= \frac{1}{2} \left[\frac{\partial \phi / \partial s'_1 - \partial \phi / \partial s'_2}{s'_1 - s'_2} \hat{\mathbf{E}}'_{1212} + \frac{\partial \phi / \partial s'_2 - \partial \phi / \partial s'_3}{s'_2 - s'_3} \hat{\mathbf{E}}'_{2323} \right. \\
&\quad \left. + \frac{\partial \phi / \partial s'_3 - \partial \phi / \partial s'_1}{s'_3 - s'_1} \hat{\mathbf{E}}'_{3131} \right]
\end{aligned} \tag{32}$$

where

$$\hat{\mathbf{E}}'_{1212} = \hat{\mathbf{e}}'_1 \otimes \hat{\mathbf{e}}'_2 \otimes \hat{\mathbf{e}}'_1 \otimes \hat{\mathbf{e}}'_2 + \hat{\mathbf{e}}'_1 \otimes \hat{\mathbf{e}}'_2 \otimes \hat{\mathbf{e}}'_2 \otimes \hat{\mathbf{e}}'_1 + \hat{\mathbf{e}}'_2 \otimes \hat{\mathbf{e}}'_1 \otimes \hat{\mathbf{e}}'_1 \otimes \hat{\mathbf{e}}'_2 + \hat{\mathbf{e}}'_2 \otimes \hat{\mathbf{e}}'_1 \otimes \hat{\mathbf{e}}'_2 \otimes \hat{\mathbf{e}}'_1 \tag{33}$$

with $\hat{\mathbf{E}}'_{2323}$ and $\hat{\mathbf{E}}'_{3131}$ similarly defined. A similar expression is found for the terms involving $\partial^2 s''_i / \partial \mathbf{s}'' \partial \mathbf{s}''$. The only difficulty in using this form of the derivatives in the Hessian is the singularity when, without loss of generality, $s'_1 = s'_2$. When this occurs we can take the limit to obtain

$$\lim_{s'_1 \rightarrow s'_2} \left[\frac{\partial \phi / \partial s'_1 - \partial \phi / \partial s'_2}{s'_1 - s'_2} \right] = \frac{\partial^2 \phi}{\partial s'_1 \partial s'_1} - \frac{\partial^2 \phi}{\partial s'_1 \partial s'_2} \quad . \tag{34}$$

3. Return Mapping Algorithm

The numerical integration of the models is based on an incremental backward Euler algorithm. The algorithm increments the stress state and the equivalent plastic strain from time t_n to time t_{n+1} with the variables at time t_n denoted by $\boldsymbol{\sigma}^n$ and $\bar{\varepsilon}_n^p$. The integration is done by formulating a return mapping algorithm using a Newton-Raphson method modified with a line search.

We start with a strain increment, $\Delta\boldsymbol{\varepsilon}$. A trial stress state, $\boldsymbol{\sigma}^{\text{tr}}$, is calculated assuming that the strain increment is elastic

$$\boldsymbol{\sigma}^{\text{tr}} = \boldsymbol{\sigma}^n + \mathbb{C} : \Delta\boldsymbol{\varepsilon} \quad (35)$$

where $\mathbb{C}$ is the fourth order tensor of elastic moduli. In this work the moduli are assumed to be constant and isotropic. If the trial stress state is such that $f(\boldsymbol{\sigma}^{\text{tr}}, \bar{\varepsilon}_n^p) \leq 0$, where f is the yield function given by (1), then the strain increment is elastic and $\boldsymbol{\sigma}^{n+1} = \boldsymbol{\sigma}^{\text{tr}}$ and $\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p$. However, if $f(\boldsymbol{\sigma}^{\text{tr}}, \bar{\varepsilon}_n^p) > 0$ then plastic deformation occurs and a return mapping algorithm is used to integrate the model and determine the amount of plastic deformation over the time step.

The return mapping is solved using a Newton-Raphson algorithm (hereafter referred to as the Newton algorithm) to solve the closest point projection problem (see Simo and Hughes [24]). To this end we consider two residuals which can be found by maximizing plastic dissipation. The first residual is the yield function

$$f(\boldsymbol{\sigma}, \Delta\gamma) = \phi(\boldsymbol{\sigma}) - \sigma_y(\Delta\gamma) \quad (36)$$

and the second residual is the plastic strain residual, $\mathbf{R}$

$$\mathbf{R}(\boldsymbol{\sigma}, \Delta\gamma) = \Delta\boldsymbol{\varepsilon}^p(\boldsymbol{\sigma}) - \Delta\gamma \frac{\partial f}{\partial \boldsymbol{\sigma}} \quad (37)$$

where $\Delta\gamma = \Delta t \dot{\gamma}$ denotes the incremental consistency parameter. We seek a solution, $\boldsymbol{\sigma}$ and $\Delta\gamma$, that makes both residuals zero. We do this with an iterative solution for $\boldsymbol{\sigma}$ and $\Delta\gamma$

$$\begin{aligned} \boldsymbol{\sigma}^{(k+1)} &= \boldsymbol{\sigma}^{(k)} + \Delta\boldsymbol{\sigma} \\ \Delta\gamma^{(k+1)} &= \Delta\gamma^{(k)} + \Delta(\Delta\gamma) \end{aligned} \quad (38)$$

where (k) denotes the iteration, $\Delta(\Delta\gamma)$ is the *change* in the incremental consistency parameter, and $\Delta\boldsymbol{\sigma}$ is the change in the stress. In addition to the residuals we can define the plastic strain increment, which appears in (37), as a function of the stress

$$\Delta\boldsymbol{\varepsilon}^p = -\mathbb{C}^{-1} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{tr}}) \quad (39)$$

which reduces the problem to that of finding $\boldsymbol{\sigma}$ and $\Delta\gamma$ for the initial strain increment or trial stress state.

Defining the Hessian of the return mapping algorithm⁴ for the k^{th} iteration, $\mathcal{L}^{(k)}$, through its inverse

$$\mathcal{L}_{(k)}^{-1} = \mathbb{C}^{-1} + \Delta\gamma^{(k)} \frac{\partial^2 \phi^{(k)}}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} \quad (40)$$

we get for the increment in $\Delta\gamma$,

$$\Delta(\Delta\gamma) = \frac{f^{(k)} - \mathbf{R}^{(k)} : \mathcal{L}^{(k)} : \frac{\partial \phi^{(k)}}{\partial \boldsymbol{\sigma}}}{\frac{\partial \phi^{(k)}}{\partial \boldsymbol{\sigma}} : \mathcal{L}^{(k)} : \frac{\partial \phi^{(k)}}{\partial \boldsymbol{\sigma}} + H'_{(k)}} \quad (41)$$

where $H'_{(k)}$ is the slope of the hardening curve at $\bar{\varepsilon}_{(k)}^p = \bar{\varepsilon}_n^p + \Delta\gamma^{(k)}$

$$H'_{(k)} = \frac{d\sigma_y}{d\bar{\varepsilon}^p} \left(\bar{\varepsilon}_{(k)}^p \right) \quad . \quad (42)$$

The stress increment is

$$\Delta\boldsymbol{\sigma} = -\mathcal{L}^{(k)} : \left(\mathbf{R}^{(k)} + \Delta(\Delta\gamma) \frac{\partial \phi^{(k)}}{\partial \boldsymbol{\sigma}} \right) \quad . \quad (43)$$

Finally, the plastic strain increment, which is used in the residual (37), is simply

$$\Delta\boldsymbol{\varepsilon}_{(k+1)}^p = \Delta\boldsymbol{\varepsilon}_{(k)}^p - \mathbb{C} : \Delta\boldsymbol{\sigma} \quad (44)$$

This iterative algorithm is followed until convergence.

For convergence we define a measure of the two residuals for iteration (k) , $\psi^{(k)}$, as

$$\psi^{(k)} = \frac{1}{2} \left[\mathbf{R}^{(k)} : \mathbf{R}^{(k)} + \left(\frac{f^{(k)}}{2\mu} \right)^2 \right] \quad . \quad (45)$$

$\psi^{(k)}$ is non-dimensional due to the scaling of $f^{(k)}$ with twice the shear modulus, 2μ , and results in a measure that is on the order of strain squared. If during the k^{th} iteration, $\psi^{(k)}$ is less than some tolerance the algorithm is converged.

Some algorithms present two tolerances, one for the plastic strain residual and one for the yield function. However, as will be seen with the line search algorithm, the expression in (45) also serves as the merit function for the line search. Given this we have chosen to also use it as our measure of convergence.

⁴Note that this is *not* the Hessian of ϕ that gives the curvature of the yield surface. It is the Hessian of the optimization problem for the return mapping algorithm.

It has been observed, and will be shown, that the Newton algorithm does not always converge for non-quadratic yield surfaces. A number of researchers have shown this and proposed line search algorithms to help convergence - see for example Dutko et. al. [10], Pérez-Foguet and Armero [20], and Seifert and Schmidt [23]. Implementing a line search algorithm using the Newton search directions ((41) and (43)) gives us the following iterative updates to the variables

$$\begin{aligned}\boldsymbol{\sigma}^{(k+1)} &= \boldsymbol{\sigma}^{(k)} + \alpha_{(k)} \Delta \boldsymbol{\sigma} \\ \Delta \gamma^{(k+1)} &= \Delta \gamma^{(k)} + \alpha_{(k)} \Delta (\Delta \gamma)\end{aligned}\tag{46}$$

where $\Delta \boldsymbol{\sigma}$ and $\Delta (\Delta \gamma)$ are given by (41) and (43), and $\alpha_{(k)}$ is the step size for the k^{th} iteration. If $\alpha_{(k)} = 1$ the algorithm is identical to the Newton algorithm. The line search algorithm adds one simple step to the return mapping algorithm: determine a reasonable step size, $\alpha_{(k)}$.

To this end we seek some measure of improvement in our solution for a given step size, and using $\psi^{(k)}$ we can determine if the solution has improved. Pérez-Foguet and Armero [20] refer to this as a *merit function* as it measures the merit of a given approximation to the solution. Since our search direction is known, the function $\psi^{(k)}$ for the k^{th} iteration is just a function of $\alpha_{(k)}$

$$\psi^{(k)}(\alpha_{(k)}) = \frac{1}{2} \left[\mathbf{R}^{(k)}(\alpha_{(k)}) : \mathbf{R}^{(k)}(\alpha_{(k)}) + \left(\frac{f^{(k)}(\alpha_{(k)})}{2\mu} \right)^2 \right] \quad . \tag{47}$$

We want a value for $\alpha_{(k)}$ such that $\alpha_{(k)} \in (0, 1]$ and $\alpha_{(k)}$ minimizes $\psi^{(k)}$. Unfortunately, determining the value of $\alpha_{(k)}$ that actually minimizes $\psi^{(k)}$ is a difficult problem; fortunately all we need is an *approximation* that provides a *sufficient improvement* in the solution. So rather than strictly minimize $\psi^{(k)}$ we find a suitable value of $\alpha_{(k)}$ that sufficiently reduces $\psi^{(k)}$. This is the key to an effective line search algorithm.

We can easily evaluate $\psi^{(k)}(0)$ and $\psi^{(k)}(1)$ - the first is evaluated using the starting residuals and the second using the solution from the Newton-Raphson algorithm. Taking advantage of the fact that the search direction is found using the Newton algorithm, the derivative of $\psi^{(k)}$ at $\alpha = 0$ is simply

$$\psi'_{(k)}(0) = -2\psi^{(k)}(0) \tag{48}$$

which allows us to generate a quadratic approximation to $\psi^{(k)}(\alpha_{(k)})$, which we call $\hat{\psi}^{(k)}(\alpha_{(k)})$

$$\hat{\psi}^{(k)}(\alpha_{(k)}) = \left(1 - 2\alpha_{(k)} + \alpha_{(k)}^2 \right) \psi^{(k)}(0) + \alpha_{(k)}^2 \psi^{(k)}(1) \quad . \tag{49}$$

The value of $\alpha_{(k)}$ that minimizes $\hat{\psi}^{(k)}$ is

$$\alpha_{(k)} = \frac{\psi^{(k)}(0)}{\psi^{(k)}(0) + \psi^{(k)}(1)} \quad . \tag{50}$$

If $\alpha_{(k)}$ does not result in a sufficient decrease in $\psi^{(k)}$, then we construct another approximation to the merit function and minimize that

$$\alpha_{(k)}^{(j+1)} = \frac{\psi^{(k)}(0)}{\psi^{(k)}(0) + \psi^{(k)}\left(\alpha_{(k)}^{(j)}\right)} \quad (51)$$

where j denotes the iteration for finding $\alpha_{(k)}$ and $\alpha_{(k)}^{(0)}$ is given by (50). This results in a second iterative procedure inside the first iterative procedure for the solution variables, $\sigma^{(k)}$ and $\Delta\gamma^{(k)}$. Given that $\psi^{(k)} > 0$, (48) tells us that there should be some neighborhood of $\alpha_{(k)} = 0$ where the merit function is decreasing. Therefore we expect this algorithm to find a suitable value for $\alpha_{(k)}$. A sufficient decrease in the merit function is determined by requiring the solution to satisfy the Goldstein's condition [18] [20]

$$\psi_{(j+1)}^{(k)} < \left(1 - 2\beta\alpha_{(k)}^{(j)}\right) \psi_{(j)}^{(k)} \quad (52)$$

where β is a parameter for the algorithm. We also do not want $\alpha_{(k)}^{(j)}$ to be too small, so we also set a minimum value for it using a parameter η

$$\alpha_{(k)}^{(j+1)} = \max \left\{ \eta\alpha_{(k)}^{(j)}, \alpha_{(k)}^{(j+1)} \right\} \quad (53)$$

For the results presented in this paper $\beta = 1 \times 10^{-4}$ and $\eta = 0.1$ as in Pérez-Foguet and Armero [20].

The line search algorithm can be thought of as only slightly modifying the Newton algorithm. First, a step based on the Newton algorithm is assumed. If $\psi^{(k)}(1) < (1 - 2\beta)\psi^{(k)}(0)$, then we continue with the Newton algorithm; if $\psi^{(k)}(1) > (1 - 2\beta)\psi^{(k)}(0)$, then we look for a solution with $\alpha_{(k)} < 1$ that minimizes the approximate quadratic function using (50).

4. Results

The return mapping algorithms presented above are tested by looking at trial stress states and recording how many iterations it takes the algorithm to converge. The trial stress states are chosen so that they cover the π -plane outside of the yield surface out to trial stresses that have an effective stress that is 30 times the yield stress. This is a large set of potential trial stress states. However, for a general purpose finite element code you can see deformation increments that give extremely large trial stress states, especially when trying to iterate to a global equilibrium solution. While we could reduce the global time step for these problems, we do not want to do this when the trial stress state is the result of an iteration on the way to an equilibrium solution; it is better to have an algorithm that can *always* solve the return mapping problem.

First, results for the isotropic Hosford model are presented. The model is complex enough to provide interesting results, but simple enough to easily interpret the behavior. Next, results are presented for the Barlat model. The same general behavior can be seen with the Barlat model with the addition of some interesting behavior for certain trial stress states. In all cases the Newton algorithm modified with a line search works unconditionally.

4.1. Hosford Model

The Hosford model provides an excellent yield surface to test return mapping algorithms. In the case of the Newton algorithm it has been found that the algorithm *unconditionally* converges only for the case of $a = 4$, i.e. for J_2 plasticity or a von Mises yield surface, where the Newton algorithm reduces to the radial return algorithm. However, if the exponent is such that the yield surface is no longer a von Mises yield surface, then the algorithm must determine not only the increment in the consistency parameter, but also the plastic flow direction. In this case the Newton algorithm is no longer a radial return algorithm and solving this problem is more difficult.

In this section the return mapping algorithm has been tested by comparing the Newton algorithm and the line search algorithm for the Hosford model with $a = 6, 8$, and 100 . Logan and Hosford [17] have shown that the exponents 6 and 8 are good at modeling BCC and FCC metals respectively, so they provide practical examples. The exponent $a = 100$ is very close to a Tresca yield surface and emphasizes the stable behavior of the the model implementation and line search algorithm for an extremely large exponent. For the results in this paper the elastic properties are $E = 70$ GPa and $\nu = 0.3$, and the yield stress is taken as 200 MPa. For simplicity, the return mapping algorithm is tested using perfect plasticity, so H' in (41) is zero. For the return mapping algorithm the hardening law is not the difficulty, rather the difficulty lies in solving for both the consistency parameter *and* the flow direction. However, for completeness, one case with hardening will also be examined.

4.1.1. Analysis of Trial Stress States

The behavior of the return mapping algorithm depends on the shape of the yield surface, which is dependent on the exponent, a . We examine three cases: $a = 6$, $a = 8$, and $a = 100$.

The exponent $a = 6$ is common for BCC metals [17]. The results for the Newton algorithm are shown in Figure 3 which shows the convergence behavior for trial stress states in the π -plane. The trial stress states are color coded depending on the number of iterations it takes to converge: 1-5, 6-10, 11-15, 16-20, 21-25, 26-30, 31-35, and 36-40 iterations. The more iterations needed to converge the darker the shade of blue. If it takes more than 40 iterations we assume the algorithm does not converge and the point is colored red. This convention for presenting results will be used throughout this paper. It can be seen that the results have six-fold symmetry, which we expect with an isotropic model. Approximately 96,000 trial stress states are examined for each plot.

For $a = 6$ most trial stress states converge for the Newton algorithm, especially those that are near to the yield surface. The algorithm has difficulty where the trial stress states return to a point near to the principal stress axes. This area of the yield surface has higher curvature than the rest of the yield surface, which shows that the Newton algorithm has difficulty where the flow direction can change significantly with a slight change in the stress state. The effect of the high curvature has been noted by other authors (see [20]) and will be more pronounced as a increases.

When the algorithm is modified to include a line search, the convergence improves dramatically as shown in Figure 3. Every trial stress state examined converges in less than 10 iterations - even points that return to areas of high curvature.

For the case where $a = 8$, for FCC metals [17], the comparison of the Newton algorithm and the line search algorithm is more dramatic as seen in Figure 4. The behavior of the Newton algorithm for $a = 8$ shows a significant increase in the number of trial stress states that do not converge. The trial stress states that do converge do so in less than 30 iterations. The cutoff for iterations is still 40, so this means no trial stress states converge between 30 and 40 iterations for the Newton

algorithm. Again, as with $a = 6$, we see that the most difficulty lies in regions where the trial stress will return to a region of high curvature on the yield surface.

The results for the line search algorithm in Figure 4 again show much better behavior. Every trial stress state converges. Most trial stress states converge in less than 10 iterations, but some, in areas of higher curvature, take more iterations. The dramatic improvement, and the fact that every trial stress state returns to the yield surface, means that the line search algorithm provides us with a reliable way to integrate the model.

For larger values of the exponent a the yield surface approaches a Tresca yield surface. As an extreme example the algorithms were run with $a = 100$. The results are shown in Figure 5. In these problems the yield surface is very nearly a Tresca yield surface with corners along the principal stress axes. These corners always present numerical difficulties. The difficulties are seen in the results for the Newton algorithm. Unless the trial stress state returns to one of the sides of the yield surface, or it lies exactly on a principal stress axis, the algorithm does not converge. The algorithm is not capable of returning the stress state to the yield surface if it needs to return near a corner.

The line search algorithm, on the other hand, performs very well as shown in 5. The algorithm does have some difficulty near the corners, but this region is quite small and again, most importantly, it always converges. The algorithm is very reliable which is the first requirement for implementation in a general purpose computational code.

Given the results in Figures 3 to 5 the line search algorithm with a search direction from Newton is a very reliable algorithm for integrating the Hosford model.

4.1.2. Hardening Law

To show the effect of hardening and compare the results to the previous results, the return mapping algorithm was tested for the Hosford models using a Voce hardening law. The Voce hardening law has the form

$$\sigma_y(\bar{\varepsilon}^p) = \sigma_y^0 + A [1 - \exp(-b\bar{\varepsilon}^p)] \quad (54)$$

For the analysis presented here $\sigma_y^0 = 200$ MPa, $A = 200$ MPa, and $b = 20$.

When the Voce hardening law is used with the return mapping algorithm, we expect some improvement in the results for the Newton algorithm. This is due to the fact that the yield surface will grow during the return mapping algorithm reducing the distance between the trial stress state and the yield surface. However, this reduction in distance is for practical problems very small compared to the original distance between the trial stress and the yield surface when the magnitude of the trial stress is 10, 20, or 30 times that of the yield surface. This is seen in Figure 6 which shows the results for the Hosford model with hardening and an exponent of $a = 8$ using the Newton algorithm. There is a *slight* improvement in the Newton algorithm when hardening is added, but the improvement does not come close to fixing the problem of non-convergence. Therefore it is our belief that hardening does not help the Newton algorithm, and for the purposes of analyzing the return mapping algorithm, it does not need to be included in the analysis.

4.1.3. Analysis of Return Mapping

Finally, it is useful when comparing the two algorithms to look in detail at the iterations in the return mapping algorithm for some trial stress state. For this purpose we choose $a = 8$ and one trial

stress state - one that does not converge for the Newton algorithm. What we want is to visualize *how* the return mapping algorithm for the line search is different from the Newton algorithm.

We pick a deviatoric trial stress state of $\sigma_1 = 788$ MPa, $\sigma_2 = -514$ MPa, and $\sigma_3 = -274$ MPa. Any hydrostatic pressure can be added to this stress state with no change in the results. The Newton algorithm applied to this trial stress state does not converge, but the line search algorithm does. The behavior of the two algorithms is shown in the π -plane in Figure 7a. The two algorithms share the same first two iterations. For the third iteration the two algorithms diverge. The Newton algorithm travels back and forth between two sextants of the π -plane as if it is trying to converge on the correct flow direction, or normal to the yield surface, but overshoots the correct solution each iteration. The line search algorithm, on the other hand, takes a smaller step for the third iteration and rapidly converges on the correct flow direction.

We can see why this happens when we look at the merit function, ψ , plotted versus the iteration count, as shown in Figure 7b, for this trial stress state. For the first two iterations both algorithms behave exactly the same. The step sizes in the line search algorithm for the first two iterations are $\alpha_{(1)} = \alpha_{(2)} = 1$. In other words the line search algorithm determines that the Newton algorithm gives a sufficient decrease in the merit function.

For the third iteration, however, the merit function evaluated with $\alpha = 1$ is greater than the merit function with $\alpha = 0$. In other words, the Newton iteration does not improve the residual and does not get us closer to the solution. The next iteration reduces the residual, then the following one increases it. Every two iterations form a pair of iterations where the merit function does not improve. This behavior matches the sextant switching shown in Figure 7a where the algorithm seems to be trying to find the correct flow direction, oscillating between two roughly equivalent solutions.

The line search, on the other hand, detects the decrease in the quality of the solution during the third iteration and cuts back the step. The details of the third iteration are shown in Figure 7c which shows the merit function for this iteration along with the quadratic approximation. For this iteration $\alpha = 0.322$ is found to minimize the quadratic approximation and give a sufficient decrease in ψ . The fourth iteration we see in Figure 7b that the merit function starts with a value less than the start of the third iteration. Similar behavior is seen for iterations 4, 5, and 6. After that the line search returns to the Newton algorithm, i.e. $\alpha_{(k)} = 1$, and it rapidly taking a total of ten iterations.

4.2. Barlat Model

The 18 parameter Barlat model (Yld2004-18p) provides a more difficult test for a return mapping algorithm than the Hosford model. In the case of the Newton algorithm the results are similar to what was found for the Hosford model; the algorithm *unconditionally* converges only for the case when $c'_{ij} = c''_{ij} = 1$ and $a = 2$ or $a = 4$, i.e. for von Mises plasticity where the Newton algorithm reduces to the radial return algorithm. However, when the model is isotropic, $c'_{ij} = c''_{ij} = 1$, but the exponent is not 2 or 4, then the model reduces to the Hosford model and, as shown previously, the Newton algorithm does not converge unconditionally. For an anisotropic model with any c'_{ij} or c''_{ij} and any exponent the Newton algorithm also does not converge. For the general form of Yld2004-18p the algorithm must determine the increment in the consistency parameter and the plastic flow direction, but unlike the isotropic model, the plastic flow direction lies in the full 5-dimensional deviatoric space. Finding the flow direction in this space is considerably more difficult.

In what follows the return mapping algorithm has been tested by comparing the Newton algorithm and the line search algorithm using the Newton search direction. Two data sets for the

Table 1: Material parameters for 2090-T3 Al and 6111-T4 Al for the 18 parameter Barlat model (Yld2004-18p). The parameters are from Barlat et. al. [2]. In both cases $a = 8$.

	2090-T3	6111-T4		2090-T3	6111-T4
c'_{12}	-0.069888	1.241024	c''_{12}	0.981171	0.775366
c'_{13}	0.936408	1.078271	c''_{13}	0.476741	0.922743
c'_{21}	0.079143	1.216463	c''_{21}	0.575316	0.765487
c'_{23}	1.003060	1.223867	c''_{23}	0.866827	0.793356
c'_{31}	0.524741	1.093105	c''_{31}	1.145010	0.918689
c'_{32}	1.363180	0.889161	c''_{32}	-0.079294	1.027625
c'_{44}	1.023770	0.501909	c''_{44}	1.051660	1.115833
c'_{55}	1.069060	0.557173	c''_{55}	1.147100	1.112273
c'_{66}	0.954322	1.349094	c''_{66}	1.404620	0.589787

Yld2004-18p were provided in Barlat et. al. [2] - one for 2090-T3 Al and one for 6111-T4 Al. The parameterizations from Barlat et. al. are shown in Table 1. The exponent for both parameterizations is $a = 8$. For the results in this paper the elastic properties are $E = 70$ GPa and $\nu = 0.3$, and the yield stress is taken as 200 MPa with perfect plasticity. These are the same properties we used for analyzing the Hosford model.

4.2.1. Analysis of Trial Stress States

The behavior of the return mapping algorithm depends on the shape of the yield surface, which is dependent on anisotropy parameters, c'_{ij} and c''_{ij} , and the exponent, a . We examine two cases for each material parameterization given in Table 1: a trial stress state where the principal stresses are aligned with the material axes, and a trial stress state where the principal stresses are not aligned with the material axes.

The results for the return mapping with a Newton algorithm and a line search algorithm are shown in Figures 8 and 9 for the 2090-T3 and 6111-T4 parameterizations, respectively, when the trial stress has principal stresses aligned with the material directions. The plots show the convergence behavior for different trial stress states in the π -plane. Approximately 96,000 trial stress states are examined for each plot. In every plot the yield surface is the small region at the center of the π -plane.

As before, the trial stress states are color coded depending on the number of iterations it takes to converge: 1-5, 6-10, 11-15, 16-20, 21-25, 26-30, 31-35, and 36-40 iterations. The more iterations an algorithm takes the darker the blue. If it takes more than 40 iterations we assume it does not converge and the point is colored red. It can be seen that, unlike the isotropic results shown previously, the results do not have six-fold symmetry. Since we are using an anisotropic model the results only show two-fold symmetry, the only symmetry being tension-compression symmetry. For the results in Figure 8 and Figure 9 the principal stress axes are aligned with the material axes.

There are many trial stress states that do not converge using the Newton algorithm, especially for the 2090-T3 Al parameterization shown in Figure 8. Of the trial stress states that converge using the Newton algorithm, the maximum number of iterations to converge is 27; a trial stress state converged with the Newton algorithm in 27 or fewer iterations or it never converged. This trend was seen even when the maximum number of iterations allowed was raised from 40 to 80 - no

more trial stress states converged with the Newton algorithm. This leads us to believe that in these cases the Newton algorithm becomes stuck, just like what was observed with the Hosford model. For the line search all trial stress states converge and the the maximum number of iterations to converge is 19. This shows a tremendous improvement in the convergence of the return mapping algorithm.

Similar results are seen for the 6111-T4 Al parameterization in Figure 9. Using the Newton algorithm more trial stress states converge for this parameterization than for the 2090-T3 parameterization, but not enough to think that the algorithm is robust. The maximum number of iterations to converge is 28; a trial stress state converged with the Newton algorithm in 28 or fewer iterations or it never converged. For the line search again we see that all trial stress states converge and the maximum number of iterations to converge is 19. This is very similar to what was seen for the 2090-T4 Al parameterization and again shows a tremendous improvement in the convergence of the return mapping algorithm.

The results in Figures 8 and 9 have the principal stress axes aligned with the material axes, which reduces the dimension of the return mapping algorithm since the principal stress axes will remain aligned with the material axes. To test the behavior for more general stress states the principal trial stress axes are rotated with respect to the material axes. Figures 10 and 11 show results for 2090-T4 and 6111-T3 Al for principal trial stress axes that are rotated 45° about the x_1 material axis.

First, it should be noted that the yield surfaces look very different in Figures 10 and 11 than in Figures 8 and 9. This occurs because the shape of the yield surface in the π -plane depends on the orientation of the stress state, or its eigenvectors, with respect to the material axes. When we visualize a stress state in the π -plane we are visualizing a five-dimensional space on a two-dimensional plane. The yield surface itself is a four-dimensional surface that appears as a one-dimensional surface in the π -plane. So the change in shape of the yield surface for an anisotropic material is expected; this is different than the yield surface for an isotropic material where there is no change in the shape.

Figure 10 shows the results for the 2090-T3 Al parameterization with the principal trial stress axes not aligned with the material axes. Both the Newton and the line search algorithms have more difficulty in this case, but the line search algorithm again performs significantly better than the Newton algorithm. The maximum number of iterations for the Newton algorithm before there is no convergence is 27. With the line search all trial stress states converge and the maximum number of iterations to converge is 22. There are regions where the two algorithms show the same behavior, but these regions are not large. The region where the Newton algorithm fails to converge is quite large.

Similarly, Figure 11 shows results for 6111-T4 Al parameterization. The maximum number of iterations for the Newton algorithm before there was no convergence is 27 again. All trial stress states converge with the line search algorithm with the maximum number of iterations being 28. Again we see regions where the two algorithms show the same behavior. For this material parameterization it appears that the Newton algorithm performs better, but it still has regions where the algorithm does not converge.

In summary, the cases where the trial stress state is not aligned with the material axes show more difficulty converging with the line search algorithm, which can be attributed to the larger space of possible solutions. But the algorithm always converges. The benefit of using a line search algorithm is apparent; the Newton algorithm is simply not reliable for non-quadratic yield surfaces in a general purpose numerical code.

4.2.2. Analysis of Return Mapping

It is useful when comparing the two algorithms to look in detail at the iterations in the return mapping algorithm. For this purpose we choose one parameterization of the model: the 2090-T3 Al. The first thing we want to do is visualize *how* the return mapping algorithm for the line search is different from the Newton algorithm, similar to what we did with the Hosford model.

To look at a single return mapping we choose a deviatoric trial stress state of $\sigma_1 = 788$ MPa, $\sigma_2 = -514$ MPa, and $\sigma_3 = -274$ MPa that is aligned with the material axes. Any hydrostatic pressure can be added to this stress state with no change in the results. The Newton algorithm applied to this trial stress state does not converge, but the line search algorithm does. The behavior of the two algorithms is shown in Figure 12a. The two algorithms share the same first two iterations. For the third iteration the two algorithms diverge. The Newton algorithm travels back and forth between two sextants of the π -plane trying to converge on the correct flow direction, or normal to the yield surface. Every iteration appears to overshoot the correct solution. The line search algorithm, on the other hand, takes a smaller step for the third iteration and rapidly converges to the correct flow direction, or normal to the yield surface.

We can see why this happens when we look at the merit function, ψ , plotted versus the iteration count, as shown in Figure 12b, for this trial stress state. For the first two iterations both algorithms behave exactly the same. However, at this point continuing with the Newton algorithm does not improve the residual and does not get closer to the solution. Every two Newton iterations form a pair of iterations where the merit function does not improve. This behavior matches the sextant switching shown in Figure 12a where it appears that the algorithm is trying to find the correct flow direction, oscillating between two roughly equivalent solutions. This behavior was also seen in the analysis of the Hosford model.

The line search, on the other hand, detects the lack of improvement in ψ in the third iteration and cuts back the step. The merit function for this iteration is shown in detail in Figure 12c along with the quadratic approximation to the merit function. Using the quadratic approximation a value of $\alpha = 0.407$ is found to give a sufficient decrease in ψ . The actual minimizer of the merit function is $\alpha = 0.424$, so in this case the approximate minimizer is fairly close. The result of this is seen in Figure 12b where we see that the merit function starts the fourth iteration with a value less than the start of the third iteration. For the fourth and subsequent iterations the Newton algorithm provides sufficient decrease in the merit function and, for this particular trial stress state, converges rapidly.

It is also interesting to look at the return map when the principal stress axes are *not* aligned with the material axes. To do this we used the same principal trial stresses, but rotated them by 20° about the (1,1,1) axis in the material coordinate system. By doing this we have the same starting point in the π -plane, because the principal stresses are the same, but a more complicated closest point projection problem to solve since the closest point on the yield surface is not a stress state that is aligned with the trial stress state.

The behavior of the Newton and line search algorithms for this trial stress state is shown in Figure 13a. Again the Newton algorithm does not converge, appearing to overcorrect every iteration. The line search algorithm, on the other hand, cuts back on the third iteration and quickly converges.

These results also show two new features unique to an anisotropic yield surface. First, the paths in the π -plane for the return mapping algorithms in Figure 13a are curved. This is due to the fact that the principal stresses *and* the principal stress orientation, change during the return mapping. Realizing that the π -plane represents a two-dimensional slice of five-dimensional deviatoric stress space, the curved stress paths are not surprising.

This same fact also causes the final converged yield surface to appear different than the original yield surface related to the trial stress state. Since there is no hardening in the model, the two yield surfaces are really the same yield surface, just viewed in the π -plane relative to different principal stress axes. Both projections of the yield surface are shown in Figure 13a. We can compare this to Figure 12a where the principal stress axes of the trial stress state are aligned with the material axes, and they stay aligned during the return mapping algorithm. This is not the case when the principal stresses of trial stress state are not aligned with the material axes.

The merit function for the non-aligned case is shown in Figure 13b. Here we notice two things. First, the line search algorithm cuts back on the first iteration. The merit function for this iteration and the quadratic approximation to it are shown in Figure 13c. Second, the convergence is much slower. Given that the solution gets close to the yield surface relatively quickly (see Figure 13a) we attributed the slow convergence to the difficulty of finding the plastic flow direction, which in a higher dimensional space with anisotropy seems reasonable.

5. Example Problem

An example problem is run to show the effect of using different yield surface models on the results of a simple boundary value problem. The models presented in this paper were implemented in the finite element code Sierra/SM [28]. The boundary value problem is the internal pressurization of a thin walled cylinder in plane strain. The geometry for the problem is shown in Figure 14. The cylinder in question has a radius, R , of 10 cm, and a thickness, h , of 1 mm. The cylinder is modeled using only 5 hexahedral elements through the thickness. Since we want the element size to be nearly uniform, the model is 0.2 mm in the axial, z , direction.

For the plane strain boundary condition we impose no z -displacement on the $\pm z$ faces. To impose axisymmetry we impose no normal displacement, $\mathbf{u} \cdot \mathbf{n}^+ = \mathbf{u} \cdot \mathbf{n}^- = 0$, on the lateral surfaces. Finally, the radial displacement, $\mathbf{u} \cdot \mathbf{e}_r$, is prescribed on the inside surface. We impose the displacement rather than the pressure since this boundary value problem will see a maximum load; if the pressure is prescribed then the finite element solution will stop at the maximum load and fail to converge using a standard solution strategy.⁵ With the prescribed displacement, the pressure is backed out from the current geometry and the resultant forces on the $\mathbf{n}^{+/-}$ faces. If $\mathbf{F}$ is the resultant force on one of the faces, $F = \mathbf{F} \cdot \mathbf{n}$, and $\bar{u}_r$ is the average radial displacement in the model, the pressure is

$$p = \frac{F}{(R + u_r)h_z} \quad (55)$$

The boundary value problem is run using Sierra/SM with implicit quasi-statics.

5.1. Hosford Model

Three isotropic models are used which show the effect of the exponent in the Hosford model: a J_2 plasticity model, the Hosford model with an exponent of $a = 8$, and the Tresca model, which for our purposes is approximated using the Hosford model with $a = 100$. The elastic response for

⁵If available, an arc-length method can be used for pressure control.

each has an elastic modulus of 70 GPa and a Poisson's ratio of 0.3. The yield stress, σ_y , for each model is 200 MPa. The hardening function is a Voce hardening law given by

$$\bar{\sigma} = \sigma_y + A(1 - \exp(-b\bar{\epsilon}^p)) \quad (56)$$

where A is 200 MPa, and b is 20. At large plastic strains the hardening curve saturates at 400 MPa, or twice the initial yield stress.

The pressure versus radial displacement curves are shown in Figure 15a. The radial displacement increases quickly after initial yield, reaching a maximum load which is related to the thinning of the cylinder wall and the circumferential stress. After the maximum load the pressure decreases. The results from the von Mises model show the highest maximum load, the Tresca model gives the lowest maximum load, and the Hosford model with $a = 8$ lies between the two. The maximum pressure using the von Mises model is 2.86 MPa, the maximum pressure using the Tresca model is 2.45 MPa, and the maximum pressure using the Hosford model is 2.68 MPa. The Hosford model is 6.4% lower than the von Mises model results, and the Tresca model is 14.5% lower. Depending on the quantity of interest for the boundary value problem, this could be significant.

In analyzing the boundary value problem we can assume plane stress, although there is some small through thickness stress. With this simplification the problem loads the cylinder in a biaxial stress state which we can examine in plane stress space. The stress paths in plane stress space are shown in Figure 15c. It is interesting to note that the stress path for the Hosford model lies between the Tresca and von Mises paths. It is also interesting to see that the stress path for the Tresca model is radial, i.e. the ratio of the axial stress to the circumferential stress is constant. A simple analytical analysis of the Tresca model applied to biaxial stretching shows this is indeed the case.

The results in Figure 15c also point to a significant difference in the axial stress, and as a result the axial force required for plane strain. This is seen in Figure 15b where the axial resultant force is plotted for the three models. The von Mises and Hosford models give much higher axial forces than the Tresca model, which is what we would expect given the stress paths in Figure 15b.

5.2. Barlat Model

Finally, the boundary value problem is run with the Barlat yield surface. The parameterization for the yield surface is taken from the parameterization for 2090-T3 Al in Barlat et. al. [2] and shown in Table 1. In addition to the parameterization for the 18 parameter Barlat model, the anisotropic Hill plasticity model was also parameterized. The effective stress for the Hill model is given by

Table 2: Material parameters for 2090-T3 Al and 6111-T4 Al for the Hill plasticity model when fit to the parameterization for the 18 parameter Barlat model. The parameters are from Barlat et. al. [2].

	2090-T3	6111-T4		2090-T3	6111-T4
F	0.69094	0.606623	L	2.85422	0.933164
G	0.206643	0.39339	M	3.67927	1.71113
H	0.790646	0.646767	N	2.19515	0.827801

$$\phi(\boldsymbol{\sigma}) = \left\{ F(\hat{\sigma}_{22} - \hat{\sigma}_{33})^2 + G(\hat{\sigma}_{33} - \hat{\sigma}_{11})^2 + H(\hat{\sigma}_{11} - \hat{\sigma}_{22})^2 \right. \\ \left. + 2L\hat{\sigma}_{23}^2 + 2M\hat{\sigma}_{31}^2 + 2N\hat{\sigma}_{12}^2 \right\}^{1/2} \quad (57)$$

where $\hat{\sigma}_{ij}$ are the stress components with respect to the principal material axes. The parameters for the Hill model were found by equating the yield stress in uniaxial stress along the three material directions, and equating the yield stress in shear relative to the material axes. The parameters for the Hill model are shown in Table 2. The orientation for the Hill and Barlat models is such that the x_1 material direction is aligned in the circumferential direction, the x_2 material direction is aligned in the axial direction, and the x_3 material direction is aligned in the radial direction. For completeness the isotropic J_2 is also compared. All three models have a yield stress of 200 MPa. The hardening function for all of the models is the Voce hardening law given by (56) with the parameters: $A = 200$ MPa, and $b = 20$.

The pressure versus radial displacement curves are shown in Figure 16a. The radial displacement increases quickly for all of the models after initial yield, reaching a maximum load. After the maximum load the pressure decreases. The results from the Hill model show a highest maximum load and the results from the Barlat model the lowest maximum load. The maximum pressure using the Hill model is 4.05 MPa, while the maximum pressure using the Barlat model is 3.47 MPa, or 14.3% lower than the Hill model results. The maximum pressure for the J_2 model lies between the other two.

These results are very significant. All three models are reasonable parameterizations based on a set of experiments done to parameterize the model. If uniaxial tests are performed only in the circumferential direction, the the J_2 model is often the model chosen for this type of analysis. If we allow for the model to be anisotropic, then we might assume a Hill parameterization and we would fit the model appropriately. But even this may not be accurate enough. The Barlat model, which is arguably the highest fidelity representation of the yield surface, gives the same results for uniaxial stress and shear as the Hill model, but *very* different results in this boundary value problem.

The axial forces required to enforce plane strain are shown in Figure 16b. They are also very different between the three models, with the axial forces calculated using the Barlat model being the highest, and much higher than the von Mises and Hill models which give similar results.

In analyzing the boundary value problem we assume plane stress, although there is some small through thickness stress. With this simplification we consider a biaxial stress state which we examine in plane stress space. The stress paths in plane stress space are shown in Figure 16c where the effect of the initial yield and the flow direction are seen. The maximum load can be correlated to the initial yield. The Barlat model yields first and has the lowest failure pressure, while the Hill model yields last and has the highest failure pressure. The flow direction significantly affects the axial stress, which in turn affects the axial force required for plane strain. The Barlat model has significantly higher axial stress and force.

6. Conclusions

A robust and reliable implementation of a return mapping algorithm for a non-quadratic, isotropic and anisotropic yield surfaces that uses a line search algorithm has been presented. The line search algorithm is based on the Newton algorithm, which is a standard approach for return mapping algorithms, with some minor modifications. The line search algorithm shows excellent convergence behavior, even for very large trial stress states, making the algorithm suitable for general purpose finite element codes.

Two important features of the implementation of the Hosford and Barlat models are the eigenvalue/eigenvector solution presented in [21], and the scaling of the stress when the exponent, a , is large to avoid numerical difficulties. The scaling of the stresses includes the scaling with the von Mises stress to find the effective stress, and the evaluation of the derivatives which are scaled with the effective stress. These careful considerations of the numerical implementation allow us to use the Hosford yield surface to model a Tresca yield surface when a is very large, in our case 100. While the presentation in this paper has been for small strain plasticity, or its extension to finite deformations using a hypoelastic formulation with an appropriate objective stress rate, there is no reason to believe that the algorithm cannot be applied to hyperelastic formulations using a multiplicative decomposition of the deformation gradient with similar results.

The plotting of iterations used in the return mapping shows the benefit of the line search algorithm in a dramatic fashion. By plotting the iterations to converge in the π -plane we can understand where the classic Newton algorithm has difficulty converging and see that the line search algorithm avoids those difficulties, e.g. near areas of high curvature. When we look at the return mapping for a single trial stress state we see how the line search improves the return mapping algorithm and have confidence that it is reliable.

The Hosford model and the Barlat model, with the return mapping algorithm using a line search, have been implemented in the Sierra/SM code [28] and successfully used in modeling the internal pressurization of a cylinder. The model, along with the return mapping algorithm, performs very well, and emphasizes the need to carefully consider yield surface shape and anisotropy in continuum modeling of plastic deformation.

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- [1] F. Barlat. Crystallographic texture, anisotropic yield surfaces and forming limits of sheet metals. *Materials Science and Engineering*, 91:55–72, 1987.
- [2] F. Barlat, H. Aretz, J. W. Yoon, J. C. Brem, and R. E. Dick. Linear transformation-based anisotropic yield functions. *International Journal of Plasticity*, 21:1009–1039, 2005.
- [3] F. Barlat, J. C. Brem, J. W. Yoon, K. Chung, R. E. Dick, D. J. Lege, F. Pourboghra, S. H. Choi, and E. Chu. Plane stress yield function for aluminum alloy sheets - part I: theory. *International Journal of Plasticity*, 19:1297–1319, 2003.
- [4] F. Barlat, D. J. Lege, and J. C. Brem. A six-component yield function for anisotropic materials. *International Journal of Plasticity*, 7:693–712, 1991.
- [5] F. Barlat and J. Lian. Plastic behavior and stretchability of sheet metals part I: a yield function for orthotropic sheets under plane stress conditions. *International Journal of Plasticity*, 5:51–66, 1989.
- [6] F. Barlat, Y. Maeda, K. Chung, M. Yanagawa, J. C. Brem, Y. Hayashida, D. J. Lege, K. Matsui, S. J. Murtha, S. Hattori, R. C. Becker, and S. Makosey. Yield function development for aluminum alloy sheets. *Journal of the Mechanics and Physics of Solids*, 45(11/12):1727–1763, 1997.
- [7] F. Barlat and O. Richmond. Prediction of tricomponent plane stress yield surfaces and associated flow and failure behavior of strongly textured F.C.C. polycrystalline sheets. *Materials Science and Engineering*, 95:15–29, 1989.
- [8] O. Cazacu, B. Plunkett, and F. Barlat. Orthotropic yield criterion for hexagonal closed packed metals. *International Journal of Plasticity*, 22:1171–1194, 2006.
- [9] P. Chadwick and R. W. Ogden. A theorem of tensor calculus and its application to isotropic elasticity. *Archive for Rational Mechanics and Analysis*, 44(1):54–68, 1971.
- [10] M. Dutko, D. Perić, and D. R. J. Owen. Universal anisotropic yield criterion based on superquadratic functional representation: Part I: Algorithmic issues and accuracy analysis. *Computer Methods in Applied Mechanics and Engineering*, 109:73–93, 1993.
- [11] R. Hill. A theory of the yielding and plastic flow of anisotropic metals. *Proceedings of the Royal Society of London: Series A.*, 193(1033):281–297, 1948.
- [12] W. F. Hosford. A generalized isotropic yield criterion. *Journal of Applied Mechanics*, 39:607–609, 1972.
- [13] T. J. R. Hughes. Numerical implementation of constitutive models: Rate independent deviatoric plasticity. In S. Nemat-Nasser, R. J. Asaro, and G. A. Hegemier, editors, *Theoretical Foundation for Large-Scale Computations for Nonlinear Material Behavior*. Martinus Nijhoff, Boston, MA, 1984.
- [14] A. P. Karafillis and M. C. Boyce. A general anisotropic yield criterion using bounds and a transformation weighting tensor. *Journal of the Mechanics and Physics of Solids*, 41(12):1859–1886, 1993.

- [15] R. D. Krieg and D. B. Krieg. Accuracies of numerical solution methods for the elastic-perfectly plastic model. *Journal of Pressure Vessel Technology*, 99:510–515, 1977.
- [16] M. Kroon and J. Faleskog. Numerical implementation of a J_2 - and J_3 -dependent plasticity model based on a spectral decomposition of the stress deviator. *Computational Mechanics*, 52(5):1059–1070, 2013.
- [17] R. W. Logan and W. F. Hosford. Upper-bound anisotropic yield locus calculations assuming $\langle 111 \rangle$ pencil glide. *International Journal of Mechanical Sciences*, 22:419–430, 1980.
- [18] J. Nocedal and S. J. Wright. *Numerical Optimization, Second Edition*. Springer, New York, 2006.
- [19] M. Ortiz and J. C. Simo. An analysis of a new class of integration algorithms for elastoplastic constitutive equations. *International Journal for Numerical Methods in Engineering*, 23:353–366, 1986.
- [20] A. Pérez-Foguet and F. Armero. On the formulation of closest-point projection algorithms in elastoplasticity - part II: Globally convergent schemes. *International Journal for Numerical Methods in Engineering*, 53:331–374, 2002.
- [21] W. M. Scherzinger and C. R. Dohrmann. A robust algorithm for finding the eigenvalues and eigenvectors of 3×3 symmetric matrices. *Computer Methods in Applied Mechanics and Engineering*, 197:4007–4015, 2008.
- [22] H. L. Schreyer, R. F. Kulak, and J. M. Kramer. Accurate numerical solutions for elastic-plastic models. *Journal of Pressure Vessel Technology*, 101:226–234, 1979.
- [23] T. Seifert and I. Schmidt. Line-search methods in general return mapping algorithms with application to porous plasticity. *International Journal for Numerical Methods in Engineering*, 73:1468–1495, 2008.
- [24] J. C. Simo and T. J. R. Hughes. *Computational Inelasticity*. Springer, New York, 1998.
- [25] J. C. Simo, J. G. Kennedy, and S. Govindjee. Non-smooth multisurface plasticity and viscoplasticity. loading/unloading conditions and numerical algorithms. *International Journal for Numerical Methods in Engineering*, 26:2161–2185, 1988.
- [26] J. C. Simo and M. Ortiz. A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive equations. *Computer Methods in Applied Mechanics and Engineering*, 49:221–245, 1985.
- [27] J. C. Simo and R. L. Taylor. Consistent tangent operators for rate-independent elastoplasticity. *Computer Methods in Applied Mechanics and Engineering*, 48:101–118, 1985.
- [28] Sierra/SM Development Team. Sierra/SM 4.36 User’s Guide. SAND Report 2015-2199, Sandia National Laboratories, Albuquerque, NM and Livermore, CA, 2015.
- [29] J. Yoon, F. Barlat, R. E. Dick, K. Chung, and T. J. Kang. Plane stress yield function for aluminum alloy sheets - part II: FE formulation and its implementation. *International Journal of Plasticity*, 20:495–522, 2004.

- [30] J. Yoon, D. Y. Yang, and K. Chung. Elasto-plastic finite element method based on incremental deformation theory and continuum based shell elements for planar anisotropic sheet materials. *Computer Methods in Applied Mechanics and Engineering*, 174:23–56, 1999.

Figure 1: The von Mises, Tresca, and Hosford yield surfaces in the π -plane. The Hosford yield surfaces are shown for $a = 6$ and $a = 8$.

Figure 2: The von Mises, Hill, and Barlat yield surfaces in the π -plane. The parameterization is for 2090-T4 Al given in Barlat et. al. [2]. The models are parameterized to give the same yield stress for uniaxial stress in the x_1 direction (σ_1).

Figure 3: Comparison of the Newton algorithm and the line search algorithm for the Hosford model with $a = 6$ and perfect plasticity. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 4: Comparison of the Newton algorithm and the line search algorithm for the Hosford model with $a = 8$ and perfect plasticity. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 5: Comparison of the Newton algorithm and the line search algorithm for the Hosford model with $a = 100$ and perfect plasticity. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 6: Comparison of the Newton algorithm for the Hosford model with $a = 8$ using perfect plasticity and a Voce hardening law. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 7: The return map for the trial stress state is shown in (a) for both the Newton algorithm (red) and the line search (blue). The merit function versus the iteration count is shown in (b) for the Hosford yield surface with $a = 8$ for the Newton algorithm (red) and the line search algorithm (blue). Figure (c) shows the quadratic approximation (green) and the actual merit function (red) along with the minimizers for both functions. The line search obviously performs better.

Figure 8: Return mapping for the Barlat model using the parameterization 2090-T3 Al given in Barlat et. al. [2]. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 9: Return mapping for the Barlat model using the parameterization 6111-T4 Al given in Barlat et. al. [2]. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 10: Return mapping for the Barlat model using the parameterization 2090-T3 Al given in Barlat et. al. [2]. The principal axes for the trial stress state has an orientation that is rotated 45° about the x_1 axis. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 11: Return mapping for the Barlat model using the parameterization 6111-T4 Al given in Barlat et. al. [2]. The principal axes for the trial stress state has an orientation that is rotated 45° about the x_1 axis. Each trial stress state is colored based on the number of iterations the algorithm takes to converge. If the algorithm takes more than 40 iterations, then the trial stress state is colored red denoting a trial stress state that does not converge.

Figure 12: The return map is plotted in the π -plane for the trial stress state in (a) for both the Newton algorithm (red) and the line search (blue) for the Barlat model using the parameterization for 2090-T3 Al with a trial stress state aligned with the material axes. The merit function versus the iteration count is shown in (b). The merit function for the Newton algorithm (red) and the line search algorithm (blue) are shown. Figure (c) shows the quadratic approximation (green) and the actual merit function (red) along with the minimizers for both functions for the third iteration. The line search obviously performs better.

Figure 13: The return map for the trial stress state is shown in (a) for both the Newton algorithm (red) and the line search (blue) for the Barlat model using the parameterization for 2090-T3 Al with a trial stress state that is not aligned with the material axes. The merit function versus the iteration count is shown in (b). The merit function for the Newton algorithm (red) and the line search algorithm (blue) are shown. Figure (c) shows the quadratic approximation (green) and the actual merit function (red) along with the minimizers for both functions for the first iteration. The line search obviously performs better insofar that it does find a solution, however it does take many more iterations than the same case where the axes are aligned (see Figure 12a).

Figure 14: Geometry for the internally pressurized cylinder. The model for the cylinder is a 2θ wedge, thickness h_z in the axial direction, with appropriate boundary conditions on the $\mathbf{n}^+$ and $\mathbf{n}^-$ faces.

(a)

(b)

(c)

Figure 15: The pressure (a) and axial force (b) for an internally pressurized cylinder in plane strain using the von Mises model (J_2), the Tresca model, and the Hosford model with $a = 8$. The stress paths in plane stress space are shown in (c). The yield surface shape has a large effect on the stress path and, ultimately, the pressure and axial force.

(a)

(b)

(c)

Figure 16: The pressure (a) and axial force (b) for an internally pressurized cylinder in plane strain using the von Mises model (J_2), the Hill model, and the Barlat model. The stress paths in plane stress space are shown in (c). The yield surface shape has a large effect on the stress path and, ultimately, the pressure and axial force.