Efficient Computation of Dispersion Curves for Multilayered Waveguides and Half-Spaces

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Abstract

Motivated by the need to compute dispersion curves for layered media in the contexts of geophysical inversion and nondestructive testing, a novel discretization approach, termed complex-length finite element method (CFEM), is developed and shown to be more efficient than the existing finite element approaches. The new approach is exponentially convergent based on two key features: unconventional **stretching** of the mesh into complex space and midpoint integration for evaluating the contribution matrices. For modeling the layered half-spaces of infinite depth, we couple CFEM with the method of perfectly matched discrete layers (PMDL) to minimize the errors due to mesh truncation. A number of numerical examples are used to investigate the efficiency of the proposed methods. It is shown that the suggested combination of CFEM and PMDL drastically reduces the number of elements, while requiring minor modifications to the existing finite element codes. It is concluded that the methods' exponential convergence and sparse computation associated with linear finite elements, result in significant reduction in the overall computational cost.

Keywords Guided waves, Surface waves, Layered media, Dispersion curve, Thin layer method, Perfectly matched layers

1. Introduction

Propagation of guided waves in stratified media can be exploited for obtaining the structure information in a wide range of applications. One major group of waveguides includes layered half-spaces where seismic surface waves including Rayleigh and Love waves can be generated and propagated near the surface. Rayleigh waves can be easily excited and recorded along a free surface since they carry the near-surface wavefield energy [1] with large amplitudes and thus high signal to noise ratio [2]. Being sensitive to the surface topography, Rayleigh waves are widely employed in the areas of near surface geophysics [3-6], pavement structures characterization [7, 8] as well as geotechnical site characterization [9]. Love waves can also be exploited in near-surface inversion due to their simpler dispersion curves and higher signal-to-noise ratio and less dependency on initial models [10-12]. Furthermore the joint analysis of Rayleigh and Love waves through defining bi-objective problems can improve the predictions and reduce the uncertainties [13-15]. Lamb waves form another significant category of guided waves which exist in structures such as plates and beams, and are utilized for nondestructive evaluations such as identification of cracks in beams and slabs [16-19], damage detection in composite laminates [20] and thickness prediction of oil and gas pipelines [21].

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The aforementioned inverse problems are often solved through an optimization procedure that involves multiple forward solutions, specifically the computation of dispersion curves for multilayered waveguides or half-spaces. Existing methods to calculate the dispersion curves for layered media consist of the transfer (propagator) matrix method, the stiffness matrix method and the thin layer method. Transfer matrix method was proposed by Thomson and Haskell [22, 23]. This method involves a challenging root finding procedure for the highly nonlinear dispersion relation (the roots are in general complex-valued in the presence of material or radiation damping). There are also some issues associated with high frequencies and various improvements have been made so far [24-29], but many inconveniences still remain, which are mentioned in [30]. Stiffness matrix method is also based on closed form trigonometric expressions with some advantages over the transfer matrix method as discussed in [31].

Thin layer method (TLM) [32, 33] is based on the finite element discretization in the transverse (heterogeneous) direction while using analytical solutions in the homogeneous directions. This method approximates the transcendental functions in the stiffness matrix with algebraic functions. However in order to achieve the required accuracy, sufficient number of elements per wave length should be adopted, leading to increase in computational cost. While the TLM was originally proposed using linear elements, an example of its high-order variant can be found in [34].

This paper presents a fast and accurate forward solution procedure that can be used for both local and global inversion algorithms. Our method is similar to TLM in that it uses finite element semi-discretization, but with two important differences. Firstly, the bounded layers are discretized with the help of recently proposed Complex-length Finite Element Method (CFEM), which has exponential convergence and requires much fewer finite elements [35]. Secondly, the half-space is also modeled with an efficient discretization technique based on Perfectly Matched Discrete Layers (PMDL), which is proven to be an effective method for modeling unbounded domains (see e.g. [36]). We show that the combination of CFEM and PMDL results in orders of magnitude reduction in the cost of computing the dispersion curves.

The rest of the paper is organized as follows. Section 2 contains a summary of the proposed algorithms. In Section 3, we discuss the model problem and governing equations for multilayered waveguides, followed by the description of standard finite-element based technique to compute the dispersion curves. Sections 4 and 5 formulate the CFEM and PMDL methods respectively. In Section 6, various numerical examples are presented to illustrate the efficiency of the proposed techniques.

2. Overview of the proposed algorithms

Consider the multilayered waveguide and half-space shown in Figs. 1 (a) and 1 (b) respectively; the objective is either the solution at layer interfaces, or the wave dispersion relations in the horizontal direction. To this end, we propose an unconventional discretization in the vertical direction, which includes two main parts: (a) discretization of finite layers using the Complex-length Finite Element Method (CFEM), and (b) discretization of the half-space using Perfectly Matched Discrete Layers (PMDL). In what follows, we briefly summarize the two approaches and elaborate and justify them in the rest of the paper.

The basic idea of CFEM is to discretize each (homogeneous) layer with regular piecewise linear finite element mesh, but with two important modifications: (a) midpoint integration is used to compute the element contribution matrices, and (b) the element depths are chosen to be specially computed complex (conjugate) values in Table 1, resulting in unconventional stretching of the finite element mesh into the complex plane. These two key modifications of the mesh result in exponential convergence of the solution with respect to number of finite elements. The implication is that high accuracy can be obtained by much fewer CFEM elements in contrast with large number of conventional finite elements (see the schematic in Fig. 1(a)).

The idea of PMDL is similar to CFEM in that the half-space is discretized using midpoint integrated linear finite elements. However, in contrast to CFEM, a (truncated) mesh with rapidly growing element depth is used in the

vertically unbounded direction. The PMDL element depths are chosen to be complex-valued in the presence of vertically propagating modes, but can be purely real when the goal is to simulate vertically evanescent (surface) waves. Like CFEM, PMDL results in exponential convergence of the solution at the top of the half-space. A schematic of the PMDL mesh is shown in Fig. 1 (b). Again, owing to exponential convergence, PMDL requires much fewer elements, typically 2 to 4, compared to much larger number of elements required by conventional FEM.

Fig. 1 Overview of the proposed methods: (a) CFEM discretization of finite layers with complex conjugate depths. The resulting CFEM mesh is bent into complex plane (on the right), but requires **considerably less** elements in comparison with regular finite element mesh (on the left); (b) PMDL discretization of half-spaces. Typically, 2-4 PMDL elements with drastically graded mesh (on the right) is sufficient to capture the effect of the half-space, compared to much larger number of regular finite elements (on the left). Thus, CFEM and PMDL facilitate accurate simulation of layered waveguides and half-spaces.

3. Preliminaries: numerical solution of multilayered elastic media

3.1. Model problem

We consider a two-dimensional layered elastic waveguide in Fig. 2. Each layer is homogeneous and horizontally infinite, but the material properties and thicknesses may vary between different layers. The equation representing in-plane wave propagation for the harmonic waves of the form $\bar{\mathbf{u}}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, \omega) e^{-i\omega t} = \tilde{\mathbf{u}}(\mathbf{k}, \omega) e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}$ with no external body forces and damping, can be written as,

$$-\nabla_s^T \boldsymbol{\sigma} - \rho \omega^2 \mathbf{u} = \mathbf{0}, \quad (1)$$

where $\nabla_s^T = [\partial/\partial x \ 0 \ \partial/\partial z; 0 \ \partial/\partial z \ \partial/\partial x]$ is the generalized divergence operator, $\omega \in \mathbf{R}$ is the temporal frequency and $\mathbf{u} = \{u_x \ u_y\}^T$ is the infinitesimal in-plane displacement vector. Appropriate boundary conditions including Dirichlet (specified displacement), Neumann (specified traction) and Robin (mixed e.g. absorbing boundary condition) can be considered. More details on applying absorbing boundary conditions will be given in Section 5. The stress vector $\boldsymbol{\sigma} = \{\sigma_{xx} \ \sigma_{zz} \ \sigma_{xz}\}^T$ is related to the strain vector $\boldsymbol{\varepsilon} = \{\varepsilon_{xx} \ \varepsilon_{zz} \ \gamma_{xz}\}^T = \nabla_s \mathbf{u}$, through the linear relationship:

$$\boldsymbol{\sigma} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \boldsymbol{\varepsilon}. \quad (2)$$

For the special case of isotropic elasticity the entries can be expressed in terms of the shear wave velocity c_{shear} , pressure wave velocity c_{pressure} and density ρ :

$$D_{11} = D_{22} = \rho c_{\text{pressure}}^2, \quad D_{33} = \rho c_{\text{shear}}^2, \quad D_{12} = \rho(c_{\text{pressure}}^2 - 2c_{\text{shear}}^2) \quad \text{and} \quad D_{13} = D_{23} = 0. \quad (3)$$

Note that viscoelasticity can also be modeled through complex-valued wave velocities.

Fig. 2. Layered waveguide geometry of a composite plate.

Expanding the governing equation in (1) gives,

$$-\frac{\partial}{\partial x} \left(\mathbf{D}_{xx} \frac{\partial \mathbf{u}}{\partial x} \right) - \frac{\partial}{\partial x} \left(\mathbf{D}_{xz} \frac{\partial \mathbf{u}}{\partial z} \right) - \frac{\partial}{\partial z} \left(\mathbf{D}_{xz}^T \frac{\partial \mathbf{u}}{\partial x} \right) - \frac{\partial}{\partial z} \left(\mathbf{D}_{zz} \frac{\partial \mathbf{u}}{\partial z} \right) - (\rho \omega^2 \mathbf{I}) \mathbf{u} = \mathbf{0}, \quad (4)$$

where $\mathbf{I}$ denotes 2×2 identity matrix and $\mathbf{D}_{zz}$, $\mathbf{D}_{xz}$ and $\mathbf{D}_{xx}$ are the in-plane matrix coefficients that can be expressed in terms of the material property coefficients as follows,

$$\mathbf{D}_{xx} = \begin{bmatrix} D_{11} & D_{13} \\ D_{13} & D_{33} \end{bmatrix}, \quad \mathbf{D}_{xz} = \begin{bmatrix} D_{13} & D_{12} \\ D_{33} & D_{23} \end{bmatrix}, \quad \mathbf{D}_{zz} = \begin{bmatrix} D_{33} & D_{23} \\ D_{23} & D_{22} \end{bmatrix}. \quad (5)$$

3.2. Finite element approach

In this section we briefly review the finite element semi-discretization for obtaining dispersion curves. To this end, we apply the following weak form to (4) without external traction,

$$-\int_z \left(\delta \mathbf{u}^T \mathbf{D}_{xx} \frac{\partial^2 \mathbf{u}}{\partial x^2} \right) dz + \int_z \left(\frac{d(\delta \mathbf{u}^T)}{dz} \mathbf{D}_{xz}^T \frac{\partial \mathbf{u}}{\partial x} - \delta \mathbf{u}^T \mathbf{D}_{xz} \frac{\partial^2 \mathbf{u}}{\partial x \partial z} \right) dz + \int_z \left(\frac{d(\delta \mathbf{u}^T)}{dz} \mathbf{D}_{zz} \frac{\partial \mathbf{u}}{\partial z} - \delta \mathbf{u}^T \rho \omega^2 \mathbf{u} \right) dz = 0. \quad (6)$$

Using linear finite elements along the z direction, the semi-discretized equations can be obtained as,

$$\left(-\mathbf{A} \frac{\partial^2}{\partial x^2} + \mathbf{B} \frac{\partial}{\partial x} + \mathbf{C} \right) \mathbf{u} = \mathbf{0}, \quad (7)$$

where the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are obtained by assembling the element counterparts, which are given by:

$$\begin{aligned} \mathbf{A}_{el} &= \frac{L}{4} \begin{bmatrix} (1+\gamma^2) \mathbf{D}_{xx} & (1-\gamma^2) \mathbf{D}_{xx} \\ (1-\gamma^2) \mathbf{D}_{xx} & (1+\gamma^2) \mathbf{D}_{xx} \end{bmatrix}, \\ \mathbf{B}_{el} &= \frac{1}{2} \begin{bmatrix} -\mathbf{D}_{xz}^T & -\mathbf{D}_{xz}^T \\ \mathbf{D}_{xz}^T & \mathbf{D}_{xz}^T \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -\mathbf{D}_{xz} & \mathbf{D}_{xz} \\ -\mathbf{D}_{xz} & \mathbf{D}_{xz} \end{bmatrix}, \\ \mathbf{C}_{el} &= \frac{1}{L} \begin{bmatrix} \mathbf{D}_{zz} & -\mathbf{D}_{zz} \\ -\mathbf{D}_{zz} & \mathbf{D}_{zz} \end{bmatrix} - \frac{\omega^2 L}{4} \begin{bmatrix} (1+\gamma^2) \mathbf{I} & (1-\gamma^2) \mathbf{I} \\ (1-\gamma^2) \mathbf{I} & (1+\gamma^2) \mathbf{I} \end{bmatrix}. \end{aligned} \quad (8)$$

In the above matrices γ depends on the choice of the quadrature for evaluation of the integral in (6). In natural coordinates $\xi = \pm\gamma$ are the integration points, with $\gamma = \sqrt{1/3}$ representing Gauss quadrature, $\gamma = 1$ nodal integration, and $\gamma = 0$ mid-point integration.

Fourier transforming (7) in the x direction, we obtain the system of equations in the frequency-wavenumber domain as,

$$\mathbf{K}(\omega, k_x) = k_x^2 \mathbf{A} + ik_x \mathbf{B} + \mathbf{C}. \quad (9)$$

The dispersion relation is defined as nontrivial solutions of $\det(\mathbf{K}) = 0$, i.e. the (ω, k_x) pairs that make the stiffness matrix singular. For a given frequency ω , solving the quadratic eigenvalue problem $\mathbf{K}(\omega, k_x)\boldsymbol{\phi} = \mathbf{0}$ results in the horizontal wavenumbers k_x as the eigenvalues, and $\boldsymbol{\phi}$ are the eigenvectors.

By rearranging the horizontal and vertical degrees of freedom and multiplying the vertical components by $i = \sqrt{-1}$, the eigenvalue problem can be written as,

$$\left(k_x^2 \begin{bmatrix} \mathbf{A}_{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{zz} \end{bmatrix} + k_x \begin{bmatrix} \mathbf{0} & \mathbf{B}_{xz} \\ -\mathbf{B}_{zx} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{zz} \end{bmatrix} \right) \begin{Bmatrix} \boldsymbol{\phi}_x \\ i\boldsymbol{\phi}_z \end{Bmatrix} = \mathbf{0}, \quad (10)$$

where $\mathbf{B}_{zx} = -\mathbf{B}_{xz}^T$ which leads to real symmetric matrices using regular FEM. Due to the special structure of stiffness matrix, we can convert the quadratic eigenvalue problem in (10) to a generalized eigenvalue problem of the same size (see [31]) which yields:

$$\left(k_x^2 \begin{bmatrix} \mathbf{A}_{xx} & \mathbf{0} \\ \mathbf{B}_{xz}^T & \mathbf{A}_{zz} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{B}_{xz} \\ \mathbf{0} & \mathbf{C}_{zz} \end{bmatrix} \right) \begin{Bmatrix} \boldsymbol{\phi}_x \\ ik_x \boldsymbol{\phi}_z \end{Bmatrix} = \mathbf{0}. \quad (11)$$

To simplify the notation, we consider the following matrix form for the above eigenvalue problem as,

$$\bar{\mathbf{A}} \mathbf{N}_R \boldsymbol{\Lambda} + \bar{\mathbf{C}} \mathbf{N}_R = \mathbf{0} \quad \text{and} \quad \boldsymbol{\Lambda} \mathbf{N}_L^T \bar{\mathbf{A}} + \mathbf{N}_L^T \bar{\mathbf{C}} = \mathbf{0}, \quad (12)$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{xx} & \mathbf{0} \\ \mathbf{B}_{xz}^T & \mathbf{A}_{zz} \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{B}_{xz} \\ \mathbf{0} & \mathbf{C}_{zz} \end{bmatrix}, \quad (13)$$

where $\boldsymbol{\Lambda}_{n \times n} = \text{diag}\{\lambda_1 = (k_x^1)^2, \dots, \lambda_n = (k_x^n)^2\}$ is the diagonal matrix of the generalized eigenvalues, and $\mathbf{N}_R^{n \times n}$ and $\mathbf{N}_L^{n \times n}$ are the associated right and left eigenvector matrices respectively. The eigenvectors should satisfy the orthogonality conditions as described in [32]:

$$\mathbf{N}_L^T \bar{\mathbf{A}} \mathbf{N}_R = \boldsymbol{\Lambda}^{1/2} \quad \text{and} \quad \mathbf{N}_L^T \bar{\mathbf{C}} \mathbf{N}_R = -\boldsymbol{\Lambda}^{3/2}. \quad (14)$$

To this end $\mathbf{N}_R$ should be first obtained from the eigenvalue problem (11), then $\mathbf{N}_L$ can be computed as below which will satisfy the second eigenvalue problem in (12):

$$\mathbf{N}_R = \begin{bmatrix} \mathbf{N}_R^x \\ \mathbf{N}_R^z \end{bmatrix}, \quad \mathbf{N}_L = \begin{bmatrix} \mathbf{N}_R^x \boldsymbol{\Lambda}^{1/2} \\ \mathbf{N}_R^z \boldsymbol{\Lambda}^{-1/2} \end{bmatrix}. \quad (15)$$

The normalized eigenvectors satisfying the orthogonality conditions in (14) are obtained by post-multiplying the eigenvector matrices with the inverse square root of the diagonal matrix, $\boldsymbol{\Sigma} = \mathbf{N}_L^T \bar{\mathbf{A}} \mathbf{N}_R \boldsymbol{\Lambda}^{-1/2}$:

$$\tilde{\mathbf{N}}_R = \mathbf{N}_R \mathbf{\Sigma}^{-1/2} \quad \text{and} \quad \tilde{\mathbf{N}}_L = \mathbf{N}_L \mathbf{\Sigma}^{-1/2}. \quad (16)$$

For solving the eigenvalue problem (12), the entries of matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$ can be easily reordered to form block tri-diagonal matrices with blocks of size $n_e \times n_e$ where n_e is the number of finite element nodes. Discussion of solving block tri-diagonal eigenvalue problems can be found in [37] and the references therein.

To obtain the dispersion curves over a frequency range of interest, the phase velocity $c_p = \omega / k$ should be calculated using different values of real wavenumbers k_x^j ($j=1, \dots, m$) which correspond to different modes at each frequency. Note that we do not consider material damping and focus on propagating modes, hence for the sake of obtaining the dispersion curves only real-valued wavenumbers are considered. However to calculate the accurate displacement response especially near the source, we consider both forward propagating modes with positive wavenumbers, and forward decaying modes with complex-valued wavenumbers (with positive imaginary part). As an example, for a unit vertical line load, by superposition of different modes, we can get the displacement response in the frequency-space domain as [38]:

$$u_z(x, \omega) = \sum_{j=1}^{N_R} \frac{1}{2ik_x^j} \phi_{z_j}^s \phi_{z_j}^r e^{ik_x^j x}, \quad x \geq 0, \quad (17)$$

where N_R is the number of included modes, and $\phi_{z_j}^s$, $\phi_{z_j}^r$ denote the vertical components of the normalized right eigenvectors in (16) at the source and receiver elevations, respectively.

4. Complex-length finite element method for discretizing finite layers

So far we discussed the procedure of obtaining the dispersion curves by solving the eigenvalue problem of semi-discretized waveguide or half-space. The main drawback of this approach, which is also called as thin layer method, is the fact that discretization should be fine enough to achieve the required accuracy. The consequence of fine discretization is a significant increase in computational cost. The computational effort for solving the Eigenvalue problem in (12) grows as $O(n^3)$ where n is the system size. In addition, solving the inverse problem for finding the medium parameters, multiplies the cost by the number of frequencies in the entire range of the interest followed by the number of forward solutions required for the optimization approach. This can result in an intensive computational effort especially when global optimization methods are employed where a large number of forward solutions are required.

Motivated by the aforementioned reasons, we introduce a new finite element discretization that we term complex-length finite element (CFEM) which has exponential convergence, as opposed to algebraic convergence of regular FEM. Therefore, for a given accuracy and compared to regular FEM, using CFEM reduces the size of the eigenvalue problem drastically. Note that the exponential convergence can also be obtained by employing spectral finite elements (SFEM) however it imposes global computation [39] which is in contrast to CFEM which preserves the sparse computation of linear FEM. For a complete discussion on attributes of CFEM, we refer the reader to [35]; here we primarily focus on solving the eigenvalue problems for the layered media.

CFEM is essentially finite element semi-discretization, similar to the approach in Section 3.2, but with two important differences: (a) the finite elements are discretized with linear finite elements with midpoint integration, and (b) the element depths are chosen to be complex-valued and in such a way that the layer stiffness is accurately captured. In the remainder of this section, we summarize the derivation of CFEM, starting with the exact layer stiffness, followed by the special properties of midpoint integrated linear finite elements, ending with the choice of

element depths leading to exponential convergence. As shown in Fig. 3, we focus on the CFEM discretization of a single homogeneous layer; the same procedure would be followed for all the layers in the system.

Fig. 3. (a) Geometry of an elastic layer in a stratified medium and (b) discretized layer with mid-point integrated finite elements.

4.1. Vertical Dispersion Relation

While the proposed approach is applicable to 3D problems, we focus on the 2D model problem introduced in the previous section. Starting with the governing differential equation (4), and Fourier transforming in the x direction with the same convention used for Eq. (9), we get the 1D vector equation in z direction,

$$\frac{\partial}{\partial z} \left(\mathbf{A} \frac{\partial \mathbf{u}}{\partial z} \right) + \frac{\partial}{\partial z} (\mathbf{B}_1(k_x) \mathbf{u}) + \mathbf{B}_2(k_x) \frac{\partial \mathbf{u}}{\partial z} + \mathbf{D}(k_x, \omega) \mathbf{u} = \mathbf{0}, \quad (18)$$

where $\mathbf{A} = \mathbf{D}_{zz}$, $\mathbf{B}_1 = ik_x \mathbf{D}_{xz}^T$, $\mathbf{B}_2 = ik_x \mathbf{D}_{xz}$, and $\mathbf{D} = -k_x^2 \mathbf{D}_{xx} + \rho \omega^2 \mathbf{I}$ are 2×2 matrices. Eq. (18) admits individual modes of the form $\mathbf{u} = \boldsymbol{\phi} e^{ik_z z}$, where k_z is the vertical wavenumber. By substituting $\mathbf{u}$ in (18), we get the governing dispersion relation for k_z :

$$\left(-k_z^2 \mathbf{A} + ik_z (\mathbf{B}_1 + \mathbf{B}_2) + \mathbf{D} \right) \boldsymbol{\phi} = \mathbf{0}. \quad (19)$$

Note that this dispersion relation is for wavenumbers in the vertical direction, as opposed to the dispersion relation for horizontal wavenumbers in Eq. (9). The remainder of the paper revolves more around the k_z as opposed to k_x since the focus is the discretization in the vertical direction.

For an elastic layer from $z=0$ to $z=L$, the traction takes the form,

$$\mathbf{F}|_{z=z_0} = - \left(\mathbf{A} \frac{\partial \mathbf{u}}{\partial z} + \mathbf{B}_1 \mathbf{u} \right) \Big|_{z=z_0} \equiv - (ik_z \mathbf{A} + \mathbf{B}_1) \mathbf{u}|_{z=z_0}, \quad (20)$$

Using the expression in (20), the displacement and stress at the top and bottom of the layer are given by:

$$\begin{aligned} \mathbf{u}_{z=0}^{\text{exact}} &= \mathbf{u}, & \mathbf{F}_{z=0}^{\text{exact}} &= - (ik_z \mathbf{A} + \mathbf{B}_1) \mathbf{u}, \\ \mathbf{u}_{z=L}^{\text{exact}} &= e^{ik_z L} \mathbf{u}, & \mathbf{F}_{z=L}^{\text{exact}} &= -e^{ik_z L} (ik_z \mathbf{A} + \mathbf{B}_1) \mathbf{u}. \end{aligned} \quad (21)$$

4.2. Properties of midpoint integrated linear finite elements

The first key property of midpoint integrated finite element layer is the *impedance-preserving property*, which is defined as follows: when the exact stiffness of the element $[0, L]$ is augmented with the exact stiffness of half-space with same material properties, and the interior node at $z=L$ is eliminated, the exact stiffness of half-space

$[0, \infty)$ is recovered at $z=0$. Since the dynamic stiffness of the half-space is preserved by adding a layer, the half-space impedance is also preserved, and hence the name impedance-preserving property. Clearly, exact layer stiffness satisfies this property; this follows from the simple physical argument that when a layer is added to a half-space with same material properties, we obtain the same half-space as depicted in Fig. 4.

This impedance-preserving property is mathematically written as,

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} + \mathbf{K}_{\text{HS}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_L \end{Bmatrix} = \begin{Bmatrix} \mathbf{K}_{\text{HS}} \mathbf{u}_0 \\ \mathbf{0} \end{Bmatrix}, \quad (22)$$

where $\mathbf{u}_0$ and $\mathbf{u}_L$ are the displacements at $x=0$ and $x=L$, respectively (see Fig. 4). $\mathbf{K}_{kl}$ ($k, l \in \{1, 2\}$) correspond to the stiffness of the layer and,

$$\mathbf{K}_{\text{HS}} = -(ik_z \mathbf{A} + \mathbf{B}_1), \quad (23)$$

is the stiffness of the bottom half-space (see (20)).

Fig. 4. (a) Mid-point integrated layer $[0, L]$ plus the half-space $[L, \infty)$ and (b) the entire half-space $[0, \infty)$.

Impedance preserving property ensures that the impedance of both half-spaces are identical at the top surface.

The impedance-preserving property will not be satisfied if the layer stiffness is approximated by a regular finite element due to the discretization error. However, it turns out that if the layer is discretized with linear finite elements with midpoint integration, the impedance-preserving property is satisfied. This simple but powerful observation is a key ingredient **in the** CFEM development. This can be proved by verifying (22) for the discretized layer stiffness using midpoint integration with the following form,

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} = \frac{1}{L} \begin{bmatrix} \mathbf{A} & -\mathbf{A} \\ -\mathbf{A} & \mathbf{A} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\mathbf{B}_1 & -\mathbf{B}_1 \\ \mathbf{B}_1 & \mathbf{B}_1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -\mathbf{B}_2 & \mathbf{B}_2 \\ -\mathbf{B}_2 & \mathbf{B}_2 \end{bmatrix} - \frac{L}{4} \begin{bmatrix} \mathbf{D} & \mathbf{D} \\ \mathbf{D} & \mathbf{D} \end{bmatrix}. \quad (24)$$

Note that Eq. (22) can be alternatively written as,

$$\begin{bmatrix} \mathbf{K}_{11} - \mathbf{K}_{\text{HS}} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} + \mathbf{K}_{\text{HS}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_L \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \quad (25)$$

Substituting (23) and (24) in (25) yields,

$$\left(\frac{1}{L} \begin{bmatrix} (1 + ik_z L) \mathbf{A} & -\mathbf{A} \\ -\mathbf{A} & (1 - ik_z L) \mathbf{A} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (\mathbf{B}_1 + \mathbf{B}_2) & -(\mathbf{B}_1 + \mathbf{B}_2) \\ (\mathbf{B}_1 + \mathbf{B}_2) & -(\mathbf{B}_1 + \mathbf{B}_2) \end{bmatrix} - \frac{L}{4} \begin{bmatrix} \mathbf{D} & \mathbf{D} \\ \mathbf{D} & \mathbf{D} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_L \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \quad (26)$$

As $\mathbf{u}_0$ and $\mathbf{u}_L$ satisfy the dispersion relation (19) we have,

$$(\mathbf{B}_1 + \mathbf{B}_2) \mathbf{u}_0 = -(ik_z \mathbf{A} + \mathbf{D} / ik_z) \mathbf{u}_0 \quad \text{and} \quad (\mathbf{B}_1 + \mathbf{B}_2) \mathbf{u}_L = -(ik_z \mathbf{A} + \mathbf{D} / ik_z) \mathbf{u}_L. \quad (27)$$

Substituting these into Eq. (26) and pre-multiplying by the nonsingular matrix $[L/2 \quad -L/2; -ik_z \quad -ik_z]$ gives,

$$\begin{bmatrix} (1+ik_z L/2)\mathbf{A} & (-1+ik_z L/2)\mathbf{A} \\ (1+ik_z L/2)\mathbf{D} & (-1+ik_z L/2)\mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_L \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \quad (28)$$

which is satisfied with,

$$\mathbf{u}_L = P \mathbf{u}_0 = \left(\frac{1+ik_z L/2}{1-ik_z L/2} \right) \mathbf{u}_0, \quad (29)$$

where P is the propagation factor of the displacement using one element. So far we showed the recovery of the exact half-space stiffness after adding a mid-point integrated linear finite element to a half-space, *irrespective* of the finite element depth L . Besides, we obtained the propagation factor of the displacement field for the mid-point integrated element as given in (29). While this derivation for displacement propagation factor is also presented in an earlier paper [40], as seen in the next subsection, CFEM validity requires the propagation factor for displacement-traction pairs. We now proceed to derive the propagation factor of the traction. To this end we look into the relation between the traction and displacement for the mid-point integrated element,

$$\begin{Bmatrix} \mathbf{F}_0 \\ -\mathbf{F}_L \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_L \end{Bmatrix}, \quad (30)$$

where the stiffness matrix of the element is given in (24). Substituting (29) in (25), we get,

$$\begin{Bmatrix} \mathbf{F}_0 \\ -\mathbf{F}_L \end{Bmatrix} = \left(\begin{bmatrix} \mathbf{K}_{11} - \mathbf{K}_{\text{HS}} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} + \mathbf{K}_{\text{HS}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{\text{HS}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_{\text{HS}} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_L \end{Bmatrix} \stackrel{\text{Imp.-Pres. Property}}{=} \begin{bmatrix} \mathbf{K}_{\text{HS}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_{\text{HS}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ P\mathbf{u}_0 \end{Bmatrix}, \quad (31)$$

which gives the traction $\mathbf{F}_0$ and $\mathbf{F}_L$ at the top and bottom of the elements as,

$$\mathbf{F}_0 = -(ik_z \mathbf{A} + \mathbf{B}_1) \mathbf{u}_0, \quad \mathbf{F}_L = -P(ik_z \mathbf{A} + \mathbf{B}_1) \mathbf{u}_0 = P \mathbf{F}_0. \quad (32)$$

Comparing the above with (29), we note that the propagation factors are the same for displacements and tractions. Additionally, comparing (29) and (32) to (21) shows this special property that a midpoint integrated linear finite element essentially approximates the exponential propagator in Eq. (21) with the rational function in Eq. (29):

$$e^{ik_z L} \approx P = \left(\frac{1+ik_z L/2}{1-ik_z L/2} \right). \quad (33)$$

This approximation is fundamental to the construction of CFEM as discussed in the following section.

4.3. CFEM construction

Clearly, Eq. (33) is not a good approximation, especially for large $k_z L$, indicating that a single element cannot accurately represent the propagator for the layer, and hence its stiffness. When the layer is discretized with multiple elements $[0, L_1]$, $[L_1, L_1 + L_2]$, $\dots$ $[L_{n-1}, L_{n-1} + L_n = L]$, iterative application of Eqs. (29) and (32) gives the total propagator P as:

$$P = \prod_{j=1}^n P_j = \prod_{j=1}^n \left(\frac{1+ik_z L_j/2}{1-ik_z L_j/2} \right). \quad (34)$$

Comparing $\mathbf{u}_L$ and $\mathbf{F}_L$ for CFEM in (29) and (32) with the exact solution in (21), shows that CFEM propagator P is in fact the relative approximant of $e^{ik_z L}$, i.e.,

$$e^{ik_z L} \approx P = \prod_{j=1}^n \left(\frac{1 + ik_z L_j / 2}{1 - ik_z L_j / 2} \right). \quad (35)$$

Essentially P is a rational function of the form $Q(k_z)/Q(-k_z)$ where Q is a polynomial of degree n with roots $2i/L_j$. Relative approximants are well studied subject of rational approximation theory [41], and there is a variety of approximants convergent on the real axis at least exponentially with respect to n . Obviously, $Q(k_z)/Q(-k_z)$ must not have poles and residues on the real axis in order to be a good approximant of the exponent there; indicating that L_j must be complex, hence the method is named complex-length finite element method (CFEM).

In this paper, we consider the relative (diagonal) Padé approximant matching the first $2n$ terms of Taylor expansion at $k_z = 0$. In other words, the 0 to $2n^{th}$ derivatives of the approximant must match with that of the exact solution at $k_z = 0$, i.e.,

$$\left. \frac{d^j P_{\text{Padé}}}{d(k_z)^j} \right|_{k_z=0} = \left. \frac{d^j (e^{ik_z L})}{d(k_z)^j} \right|_{k_z=0}, \quad j=0, \dots, 2n. \quad (36)$$

For the specific Padé approximant considered above the roots $x_j = 2/L_j$ are in fact the roots of the n^{th} -degree polynomial given below (see e.g. [42] for details):

$$\sum_{j=0}^n \frac{(2n-j)!}{j!(n-j)!} (-x)^j = 0. \quad (37)$$

Table 1 gives the element depths for varying mesh size, for a layer with unit thickness. To obtain the discretization for a general layer, the values in Table 1 needs to be simply scaled with the total depth of the layer.

Table 1 Element depths for varying number of elements (n). The layer thickness is assumed to be unity. For other cases, the element depths simply scale with layer depth.

$n=1$	1.000000000000000	$n=2$	0.500000000000000 $\pm$ 0.28867513459481 i
	0.28468557688388 $\pm$ 0.27159985141630 i		0.18313248053143 $\pm$ 0.23132522602625 i
$n=3$	0.43062884623222	$n=4$	0.31686751946856 $\pm$ 0.09488202514221 i
	0.12803667831541 $\pm$ 0.19668213834621 i		0.09489061789607 $\pm$ 0.16944514819433 i
$n=5$	0.23485450871940 $\pm$ 0.12209940763707 i	$n=6$	0.17914640739749 $\pm$ 0.12594324946340 i
	0.27421762593037		0.22596297470643 $\pm$ 0.04614135671779 i
	0.07338559568636 $\pm$ 0.14811940741461 i		0.05861791492234 $\pm$ 0.13119236974564 i
$n=7$	0.14065739395847 $\pm$ 0.12154781833235 i	$n=8$	0.11325833004971 $\pm$ 0.11445496413908 i
	0.18538954553266 $\pm$ 0.06776497788782 i		0.15337794771885 $\pm$ 0.07709430353886 i
	0.20113492964499		0.17474580730908 $\pm$ 0.02713226173787 i

$n=9$	$0.04802049907890 \pm 0.11752570488080 i$	$n=10$	$0.04014472910062 \pm 0.10630697796687 i$
	$0.09316287173966 \pm 0.10679717990370 i$		$0.07802273616547 \pm 0.09940003581225 i$
	$0.12840441052931 \pm 0.08014721079992 i$		$0.10881874816902 \pm 0.07996015060428 i$
	$0.15100957026047 \pm 0.04281546509628 i$		$0.13078256453612 \pm 0.05163001172938 i$
	0.158805296783284		$0.14223122202876 \pm 0.01782841382104 i$
$n=11$	$0.03412261657800 \pm 0.09695789626293 i$	$n=12$	$0.02940803944815 \pm 0.08906181395662 i$
	$0.06634486381072 \pm 0.09256005182226 i$		$0.05715192456673 \pm 0.08635352530097 i$
	$0.09329088025857 \pm 0.07811366645707 i$		$0.08082582076929 \pm 0.07545289071966 i$
	$0.11386072467713 \pm 0.05628616844255 i$		$0.09976290741316 \pm 0.05839885662872 i$
	$0.12678425930221 \pm 0.02942684799389 i$		$0.11301395814524 \pm 0.03687019624343 i$
	0.131193310746699		$0.11983734965741 \pm 0.01259866487095 i$
$n=13$	$0.02564318775369 \pm 0.08231355321087 i$	$n=14$	$0.02258550311646 \pm 0.07648569373967 i$
	$0.04978573946440 \pm 0.08076582096948 i$		$0.04379127631258 \pm 0.07574740948845 i$
	$0.07069390925651 \pm 0.07243833050796 i$		$0.06236027779351 \pm 0.06932300610809 i$
	$0.08799247146698 \pm 0.05894563008011 i$		$0.07811546938314 \pm 0.05852853926583 i$
	$0.10097469339377 \pm 0.04152986688324 i$		$0.09053178744825 \pm 0.04431195125450 i$
	$0.10902977699627 \pm 0.02144314210934 i$		$0.09911464460927 \pm 0.02760255210502 i$
	0.11176044333671		$0.10350104133675 \pm 0.00937153174338 i$
$n=15$	$0.02006570730347 \pm 0.07140591920396 i$	$n=16$	$0.01796266496341 \pm 0.06694162366630 i$
	$0.03884638966705 \pm 0.07123849065223 i$		$0.03471809507502 \pm 0.06717964470869 i$
	$0.05542991160544 \pm 0.06624511011050 i$		$0.04960794930403 \pm 0.06327803486714 i$
	$0.06977487505357 \pm 0.05752432367232 i$		$0.06268398209122 \pm 0.05617188543383 i$
	$0.08149231310591 \pm 0.04581962272804 i$		$0.07365957371625 \pm 0.04645858465651 i$
	$0.09018490369015 \pm 0.03183750267162 i$		$0.08221667350938 \pm 0.03468717833984 i$
	$0.09553514496841 \pm 0.01630985840134 i$		$0.08808374597041 \pm 0.02141958057287 i$
	0.09734150921194		$0.09106731537025 \pm 0.00724175169196 i$

It is worthwhile mentioning that the CFEM mesh has the following properties:

1. $\sum L_j = L$ (follows directly from evaluating (36) for $j=1$). This indicates that the end points match with the physical edges of the domain.
2. The CFEM mesh (L_j) is independent of k_z , indicating that the same mesh can be used for the range of complex k_z , and thus for any combination of ω and k_x , making CFEM useful for obtaining accurate dispersion relation from Eq. (9), which is the primary goal of the paper.
3. A peculiar property of CFEM mesh is that the stiffness is invariant with respect to the ordering of the elements. This implies that the mesh is not unique; we order the mesh so that it is symmetric and smooth (gradually varying lengths) in the complex plane. Such a mesh is obtained by ordering the elements with increasing or decreasing phase angle of their depths. The obtained mesh is shown in Fig. 1 (a) for different layers using different number of elements.

4. Our method can be considered as a standard finite element method with complex coordinate stretching. Thus, it shares similarities with perfectly matched layers (PML) used in unbounded domain modeling, which is often viewed in terms of complex coordinate stretching. The difference is that PML stretching leaves invariant one boundary point, whereas both boundary points are invariant in our method. PML finite element depths do not come in conjugate pairs and the stiffness matrix is not Hermitian, consistent with PML's need to absorb energy. On the other hand, our method propagates all the information from one end of the domain to the other end without any loss of energy [35]. This is partly a consequence of complex conjugate pairs of element depths.
5. An implication of CFEM's high accuracy is that it has superior dispersion properties, but it must be noted that the solution from CFEM is accurate only on layer edges. The solution within the CFEM mesh of each layer has no direct physical significance. While it is known that midpoint integration has worse numerical dispersion compared to regular integration, the key here is the combination of midpoint integration and particular choice of (complex-valued) element depths, which results in highly accurate solution at the edges of the mesh.

More details and proofs of the CFEM properties can be found in [35]. In summary, the implementation guideline for modelling finite layers using CFEM is as follows:

1. Divide the finite part of the layered media into multiple layers bounded by points where (a) the material properties are discontinuous, (b) loads are applied, or (c) the solution is desired.
2. Discretize each layer of depth L with n linear finite elements with the depth $L \times L_j$, ($j=1, \dots, n$), where L_j can be obtained from Table 1, for $n=1, \dots, 16$ (see Equation (38) below and the related discussion for choosing n).
3. Combine all the meshes for all layers and perform finite element analysis on the entire domain. Utilize midpoint integration to evaluate the contribution matrices.
4. Extract the results from the layer interfaces (the field variables within the CFEM mesh for each layer have no immediate physical meaning). If dispersion curves are needed, simply solve the eigenvalue problem using the assembled matrices.

For vertical discretization of waveguides, the number of CFEM layers in each layer, n_i , can be estimated by calculating the bounds of the real and imaginary parts of the vertical wavenumber inside each layer:

$$n_i = \max(k_{zR_i}^{\max}, k_{zI_i}^{\max}) H_i G / 2\pi, \quad \text{for } (i=1, \dots, N_{\text{Layer}}), \quad (38)$$

$$k_{zR_i}^{\max} = \omega_{\max} \sqrt{(c_{\text{shear}}^i)^{-2} - (c_p^{\max})^{-2}}, \quad k_{zI_i}^{\max} = \omega_{\max} \sqrt{(c_p^{\min})^{-2} - (c_{\text{pressure}}^i)^{-2}},$$

where $k_{zR_i}^{\max}$ and $k_{zI_i}^{\max}$ are the bounds of the real and imaginary parts, $\omega_{\max}$ is the maximum temporal frequency, H_i is the thickness of the layer, G is the number of elements per wavelength, and $c_p^{\max}$ and $c_p^{\min}$ are the maximum and minimum phase velocities. $c_p^{\max}$ is simply the upper limit of the range of interest for the phase velocities. On the other hand, $c_p^{\min}$ can be chosen as the minimum Rayleigh wave velocity among all the layers (note that, even when phase velocity is less than Rayleigh wave speed, e.g. flexural modes in plates at low frequencies, Equation (38) is

valid because of the $\omega_{\max}$ term in the expression for $k_{zI_i}^{\max}$).

Based on numerical experiments in Section 6, number of elements per wavelength $G = 2 \sim 3$ appears to be appropriate for a target relative error of $0.1\% \sim 1\%$ (the low value of G is attributed to the exponential convergence of CFEM).

5. Treatment of half-spaces using Perfectly Matched Discrete Layers

Layered half-spaces as shown in Fig. 1 are ubiquitous, especially in the context of geophysics. The radiation of energy through unbounded layer has a significant effect on the dispersion curves and should be properly captured, which is often done with the help of absorbing boundary conditions (ABCs). In this paper we adopt the method of Perfectly Matched Discrete Layer (PMDL) which is a recently developed ABC compatible with finite element or finite difference discretization of the bounded domains [43].

Like CFEM, PMDL is a mesh of midpoint integrated finite elements with properly chosen element depths to mimic the effect of the bottom (homogeneous) half space; a schematic is shown in Fig. 5. While PMDL development precedes CFEM development, instead of following the original derivation, we construct PMDL by building on some of the discussion around CFEM in Section 4.

The accuracy of any ABC, including PMDL, can be evaluated with the help of reflection coefficient R , the ratio of the amplitude of reflected wave to that of the incident wave (both evaluated at the interior-exterior interface shown in Fig. 5). The ideal value of R is of course zero, since any wave going into the homogeneous half-space never gets reflected. On the other hand, any approximate ABC would have artificial reflections and $R \neq 0$. We first derive the expression of R for the PMDL, and then choose the element depths to minimize. Note that since we are dealing with vector systems, the reflection coefficient **must be** replaced by a reflection matrix $\mathbf{R}$.

The reflections from a finite element mesh similar to that shown in Fig. 5 have two components: the reflections from the element interfaces, and the reflections from the truncation boundary at the bottom. Both are discussed below.

We first note that *there are no reflections at element interfaces when midpoint integration is used*. This follows from a simple physical argument around the impedance-preserving property discussed in Section 4.2. From the impedance-preserving property, a half-space discretized with multiple mid-point integrated linear finite elements followed by an exact half-space would behave like the exact half-space as far as the top boundary is concerned. This means that any incident wave at the top of the composite half-space will have no reflections, immediately implying that there are no reflections at the element interfaces inside the discretized half-space. In other words, there is perfect impedance matching of the finite element mesh and the exact half-space. This is the main reason why the method is named perfectly matched *discrete* layers. The name is adopted from the well-known perfectly matched layers – PML, but the key difference here is that perfect matching is preserved through the finite element discretization which leads to increased efficiency (note that the PML in its original form has been explored in the context of the TLM [44]).

Fig. 5. (a) Bottom homogeneous half-space, (b) complex coordinate stretching, (c) discretization with midpoint integrated layers and (d) truncation with homogeneous Dirichlet boundary condition.

The lack of reflections at finite element interfaces makes the analysis of the reflection matrix for PMDL rather elegant. When an incident wave enters the PMDL mesh with n elements, the wave propagates through each of the PMDL elements, reaches the truncation boundary, gets reflected, and travels back through each of the PMDL elements. The reflection matrix can be mathematically written as (with modal degrees of freedom),

$$\mathbf{R} = \mathbf{P}_{\text{up}} \mathbf{R}_T \mathbf{P}_{\text{down}}, \quad (39)$$

where $\mathbf{P}_{\text{down}}$ is the 2×2 downward propagator matrix, $\mathbf{R}_T$ is the 2×2 reflection matrix at the truncation boundary and $\mathbf{P}_{\text{up}}$ is the 2×2 upward propagator matrix. Following Eq. (35), we have the expression for the propagators,

$$\mathbf{P}_{\text{down}} = \begin{bmatrix} \prod_{j=1}^n \left(\frac{1 + ik_z^1 L_j / 2}{1 - ik_z^1 L_j / 2} \right) & 0 \\ 0 & \prod_{j=1}^n \left(\frac{1 + ik_z^2 L_j / 2}{1 - ik_z^2 L_j / 2} \right) \end{bmatrix}, \quad \mathbf{P}_{\text{up}} = \begin{bmatrix} \prod_{j=1}^n \left(\frac{1 + ik_z^3 L_j / 2}{1 - ik_z^3 L_j / 2} \right) & 0 \\ 0 & \prod_{j=1}^n \left(\frac{1 + ik_z^4 L_j / 2}{1 - ik_z^4 L_j / 2} \right) \end{bmatrix}, \quad (40)$$

where k_z^i are the vertical wavenumbers for down-going ($i = 1, 2$) and up-coming ($i = 3, 4$) modes respectively, thus $k_z^{i+2} = -k_z^i$ ($i = 1, 2$). Substituting (40) into (39), we have the coefficients of the final reflection matrix:

$$R_{lm} = \left(\prod_{j=1}^n \left(\frac{1 + ik_z^l L_j / 2}{1 - ik_z^l L_j / 2} \right) \left(\frac{1 + ik_z^m L_j / 2}{1 - ik_z^m L_j / 2} \right) \right) R_{T,lm}, \quad l, m \in \{1, 2\}. \quad (41)$$

Note that there may be mode coupling at the truncation boundary and $\mathbf{R}_T$ is not necessarily diagonal. Notwithstanding this, since $\mathbf{R}_T$ is bounded (typically of the order 1), the reflection matrix $\mathbf{R}$ can be made arbitrarily small by focusing on the product in Eq. (41). For propagating waves (with real k_z^i), the product can be made small by choosing imaginary or complex L_j . However, we are interested in waves propagating along the layers, which tend to be evanescent in the vertical direction. Thus, k_z^i are either purely imaginary or complex, and real L_j would suffice. The choice of PMDL mesh depths for evanescent waves (in the context of static analysis) is elaborately analyzed in [45], and a simplified procedure based on geometric gradation of L_j is provided:

$$L_j = L_1 \alpha^{j-1} \quad (j = 1, \dots, n), \quad L_1 = \frac{2}{\sqrt{\alpha^{n-1} k_z^{\max} k_z^{\min}}}. \quad (42)$$

With this simplification, the PMDL parameters reduce to the geometric ratio α and the number of layers n , which are then chosen to minimize the overall error; specifically Figs. 9 and 11 in [45] can be used to obtain α and n for a given target error. Also $k_z^{\max}$ and $k_z^{\min}$ can be estimated for each problem.

In summary, the implementation guideline for modelling the infinite layer using PMDL method is as follows:

1. For modeling the surface waves which are evanescent in the vertical direction, discretize the infinite layer using a real-valued graded mesh with the optimized lengths obtained from Eq. (42). The number of elements depends on the targeted error in obtaining the half-space stiffness. Use midpoint integration to evaluate the contribution matrices. The graded mesh using PMDL layers is schematically shown in Fig. 1 (b) and compared to regular FEM.
2. For modeling both vertically propagative and evanescent waves which is important for obtaining the displacement response especially at the short offset, use PMDL layers with imaginary/complex lengths. Appropriate criteria for choosing the lengths are not discussed in this paper but can be found in [40, 46] for elasticity problems and in [43] for scalar field problems.

Remark: Simulation of Love waves in layered half-spaces is similar to Rayleigh waves and in fact simpler; the (vector) in-plane elastic wave equation is replaced by (scalar) anti-plane shear wave equation. Both CFEM and PMDL constructions in Sections 4 and 5 are immediately applicable for Love waves, since the derivations for the scalar equations can be considered as a special case of the general vector equations.

6. Numerical examples

To evaluate the effectiveness of the proposed methods, we consider the elastic medium (see Fig. 1) with three layers on an infinite half-space. This represents the earth model frequently used for near surface imaging. Properties of the layers are reported in table 2. In this section we separately investigate the performance of CFEM for discretization of the interior layers, and the performance of PMDL for modeling the half-space.

Table 2 Parameters of a three-layer model on a half-space.

Layer number	c_{shear} (m/s)	ν	ρ (kg/m ³)	h (m)
1	200	0.35	1800	6
2	300	0.35	1800	4
3	400	0.35	1800	8
Half-Space	500	0.35	1800	∞

6.1. Performance of CFEM for obtaining dispersion curves

Dispersion curves of the first 10 modes are computed for the frequency band of $f = 1-100$ Hz using regular and complex-length FEM. The results are shown and compared with the reference solution in Fig. 6 using 27 regular and complex-length elements (9, 6 and 12 elements for the top three finite layers). The reference solution is obtained by discretizing each layer with 31-noded (order 30th polynomial) spectral finite element method (SFEM) using Lobatto polynomials as discussed in [39]. In both cases, we used 10 PMDL elements to model the infinite half-space with $L_1 = 1$ and $\alpha = 2$ to generate the optimum graded mesh using (42). Note that 10 PMDL elements are employed here to practically eliminate the error due to modeling of the half-space, since the focus of this section is to compare the performance of the regular and complex-length FEM for discretizing the interior layers (the next section is devoted to performance of PMDL).

Fig. 6. Phase velocity curves using (a) complex-length FEM and (b) regular FEM with 27 elements.

Fig. 6 indicates that regular FEM has significant errors for higher frequencies and modes, whereas CFEM shows good agreement with the reference solution for the entire frequency range for all the considered modes. To quantify

the accuracy of the dispersion curves, the relative error in the phase velocity is plotted against the frequency in Fig. 7. It can be seen that using 27 regular elements will lead to noticeable errors even for the first mode. On the other hand, Fig. 7 (a) shows that using the same number of complex elements keeps the error for all the 10 modes below 0.5%.

Fig. 7. Relative error in the phase velocity using (a) complex-length FEM and (b) regular FEM with 27 elements (note that the vertical scales are different).

In Fig. 8 we show the dispersion curves and relative errors using 90 regular finite elements. It can be seen that the error for the higher modes is still above 1% in the high frequency range and using CFEM with 27 elements is still superior. The main reason for considering the higher modes in this example is the fact that they are more sensitive to the wave velocity distribution across the layers, which is often the objective of inversion problems. Besides, for some frequency ranges, higher modes can sometimes become more dominant than the fundamental mode (see e.g. [47]).

Fig. 8. (a) Phase velocity curves and (b) relative error in phase velocity using regular FEM with 90 elements.

We now proceed to study the convergence of the relative error, defined as $e_r = \|\mathbf{c}_p - \mathbf{c}_p^{ref}\|_2 / \|\mathbf{c}_p^{ref}\|_2$, where $\mathbf{c}_p, \mathbf{c}_p^{ref}$ vectors contain the calculated and the reference phase velocities of the first 10 modes for all the considered frequencies. Fig. 9 contains the convergence of e_r , for both regular and complex-length FEM; one can notice the exponential convergence of the CFEM as compared to algebraic convergence of the regular FEM. Fig. 9 also shows that for reaching 0.1% relative error, 27 complex elements (54 degrees of freedom) are sufficient as opposed to 200

elements (400 degrees of freedom) are required for regular FEM. Note that exponential convergence of CFEM allows us to achieve even smaller error by adding just a few element per layer, whereas regular FEM requires a large number of additional elements. As mentioned earlier in Section 4, solving the eigenvalue problem for obtaining the dispersion curves has $O(n^3)$ complexity. Thus, CFEM, which requires much fewer elements, can significantly improve the efficiency, especially in the context of global search optimization algorithms.

Fig. 9. Convergence curves of phase velocity using regular and complex-length FEM.

To investigate the efficiency of CFEM further, we compared the convergence of CFEM with that of spectral finite element method (SFEM) using Lobatto polynomials. We discretize each layer by one spectral element and increase the polynomial order of the shape functions which increases the number of degrees of freedom per layer. For CFEM we equally increase the number of degrees of freedom by increasing the number of complex-length finite elements. The results are reported in Fig. 10, which indicate that SFEM and CFEM have identical convergence behavior with respect to the number of degrees of freedom. However since CFEM uses linear finite elements, the resulting matrices are tridiagonal (block tri-diagonal for vector equations), whereas SFEM leads to full matrices for each layer, which indicates that CFEM would be more efficient than SFEM.

Fig. 10. Convergence curves of phase velocity using the complex-length and spectral FEM.

As one might expect there is an additional cost when using CFEM due to the complex arithmetic. However this computational overhead is negligible compared to the savings due to significantly more sparse matrices adopting efficient block tri-diagonal eigensolvers (see e.g. [37] and the references therein). Moreover, complex arithmetic is often unavoidable due to the physics of the problem, e.g. material damping, or radiation of waves into the far field. Another advantage of CFEM over SEFM is that CFEM is simpler to implement and demands minimal modifications to the existing codes.

6.2. Performance of CFEM for obtaining surface displacements

Surface displacement response has important implications for solving inverse problems since many times the measurements are at the surface. Consequently we compared the accuracy of surface displacements from CFEM with those from regular FEM, at the frequency $f = 100$ Hz for a unit vertical line load applied at the surface (Eq. (17) is used for computing the surface displacements). To compare CFEM and FEM performance for discretization of finite layers, 10 PMDL layers, deemed sufficient for good accuracy, are used for both cases. PMDL layers are chosen to be complex with the uniform depth of $L_j = 2i / k^{ref}$ ($j = 1, \dots, 10$) where $k^{ref} = 10^{-3}(1+i)\omega$ (to treat both propagating and decaying waves).

The real and imaginary parts of the surface displacement for the offset range $r = 10 - 20$ m are shown in Figs. 11 and 12 and compared with the reference solution. As seen from these figures, CFEM with 27 elements gives accurate solutions for both real and imaginary parts of the surface displacement. However, using even 135 regular finite elements, the response has not converged.

Fig. 11. Real part of the surface displacement response using (a) complex-length FEM with 27 elements ($N = 27$) and (b) regular FEM with 135 elements ($N = 135$) at $f = 100$ Hz .

Fig. 12. Imaginary part of the surface displacement response using (a) complex-length FEM with 27 elements ($N = 27$) and (b) regular FEM with 135 elements ($N = 135$) at $f = 100$ Hz .

As a quantitative comparison, we studied the convergence of the surface displacement response using the relative error $e_r = \|\mathbf{u} - \mathbf{u}^{ref}\|_2 / \|\mathbf{u}^{ref}\|_2$, where $\mathbf{u}$ and $\mathbf{u}^{ref}$ are calculated and the reference surface displacements in the offset range $r = 10 - 20$ m. We observe from Fig. 13 that CFEM gives exponential convergence for the displacement response, while regular FEM delivers only algebraic convergence.

Fig. 13. Convergence curves of the surface displacement using the complex-length and regular FEM at $f = 100\text{Hz}$.

6.3. Performance of PMDL for modeling infinite half-spaces

We now turn to layered half-spaces and examine the effectiveness of CFEM+PMDL combination, with emphasis on PMDL. We consider a single layer on a half-space, with properties given in Table 3. In order to solely focus on the error due to discretization of the half-space, we discretize the top layer with a 31-noded (order 30th polynomial) spectral finite element.

Table 3 Parameters of a one-layer model on a half-space.

Layer number	c_{shear} (m/s)	ν	ρ (kg/m ³)	h (m)
1	200	0.35	1800	2
Half-Space	500	0.35	1800	∞

Aiming for 1% relative error and following the procedure discussed in Section 5, one can obtain the optimized PMDL parameters as $n = 4$, $L_1 = 1.1$ and $\alpha = 3$ which defines the graded mesh as discussed earlier in connection with Eq. (42).

Fig. 14. Phase velocity curves using (a) PMDL method and (b) regular FEM with the graded mesh parameters $n = 4$, $L_1 = 1.1$ and $\alpha = 3$.

Taking into account a broader range of frequencies, $f = 1 - 150$ Hz, we compared the reference dispersion curves with those obtained using regular FEM and PMDL layers for discretizing the half-space. Fig. 14 reveals the superior

performance of PMDL method compared to regular FEM. Note that the same optimized parameters of the PMDL graded mesh are also exploited for regular FEM which is better performing than a uniform mesh. Therefore the major difference here originates from the mid-point integration scheme which is shown to eliminate the discretization error in half-space impedance.

Fig. 15. Relative error in phase velocity using (a) PMDL method and (b) regular FEM with the graded mesh parameters $n = 4$, $L_1 = 1.1$ and $\alpha = 3$ (note that the vertical scales are different).

The relative errors in the phase velocities are plotted in Fig. 15. It is first observed that using PMDL, the error is always less than target error of 1%, for the entire frequency range. On the other hand, as expected, few regular finite elements even with grading, are not accurate enough (Fig. 15 (b)). It can also be seen that the fundamental mode enters the half-space in the low frequency region but remains largely in the top layer at higher frequencies where the error vanishes due to the use of SFEM. On the other hand it is interesting to observe that the higher modes penetrate deeper into the half-space at high frequencies, resulting in higher error when the half-space is not modeled accurately (see Fig. 15 (b)).

Fig. 16. (a) Relative error in phase velocity using (a) PMDL method with the graded mesh parameters $n = 6$, $L_1 = 1$, $\alpha = 2$ and (b) regular FEM with the graded mesh parameters $n = 20$, $L_1 = 0.8$, $\alpha = 1.1$ (note that the vertical scales are different).

Note that, similar to CFEM, PMDL has exponential convergence, indicating that addition of each PMDL element will reduce the error significantly. This is verified in Fig. 16 (a) which shows that adding only 2 more PMDL elements decreases the relative error to below 0.05% for the entire range. In contrast, Fig. 16 (b) demonstrates that increasing the number of regular (graded) finite elements to 20, with the optimized parameters $L_1 = 0.8$ and $\alpha = 1.1$, is still not successful in getting the error below that in Fig. 15 (a), which uses only 4 PMDL elements.

6.4. Performance of CFEM for obtaining Lamb wave dispersion curves

Dispersion curves for Lamb waves in composite plates can be computed using CFEM, but without PMDL as there is no half-space. As an example we consider the composite plate consisting of four layers with material parameters in Table 4. The relative error of the dispersion curves obtained from CFEM and regular FEM are shown in Fig. 17, for the frequency range $f = 0 - 500$ kHz ; clearly, consistent with previous observations, CFEM, even with 40 elements (12, 8, 12 and 8 elements for four layers), has superior performance when compared to regular FEM with 140 elements.

Table 4 Parameters of a four-layer composite plate.

Layer number	c_{shear} (m/s)	ν	ρ (kg/m ³)	h (mm)
1	1000	0.3	5000	7.5
2	2000	0.3	5000	5
3	3000	0.3	5000	7.5
4	4000	0.3	5000	5

Fig. 17. Phase velocity of Lamb waves using (a) complex-length FEM with 40 elements and (b) regular FEM with 140 elements (note that the vertical scales are different).

Conclusions

We presented the complex-length finite element method (CFEM) for multilayered waveguides to improve the efficiency of forward modeling. By employing midpoint integration rules and unconventional **stretching** of the finite element mesh into complex space, we achieved exponential convergence of the dispersion curves. Additionally, by coupling CFEM with perfectly matched discrete layers (PMDL), we were able to extend this method to layered half-spaces that are unbounded in the vertical direction. Exponential convergence of CFEM (and CFEM+PMDL) is verified with a number of numerical experiments involving the computation of dispersion curves and surface displacements. The examples also indicated that, for practical error tolerance requirements, the proposed method reduces the number of elements by an order of magnitude, leading to substantial efficiency gains in the computing the dispersion curves, which requires the solution of $O(n^3)$ eigenvalue problem. The method is also attractive in that its implementation requires minor modifications of existing finite element codes. The theory is presented here for the general vector (elastic) equations for modeling Rayleigh and Lamb waves, but Love (scalar) wave dispersion can be considered as a special case of the presented vector wave analysis. In closing, given the efficiency gains and ease of implementation, we argue that the proposed approach can be used in place of existing forward solution procedures typically used in inversion of layered systems.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1016514. The authors wish to thank Professor Eduardo Kausel for openly reviewing the manuscript and providing several excellent suggestions that led to the improvement of the paper.

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