A Nitsche method for wave propagation problems in time domain

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Abstract

We propose a new Nitsche-type approach for the weak enforcement of Dirichlet and Neumann boundary conditions in the context of time-domain wave propagation problems in mixed form. A peculiar feature of the proposed method is that, due to the hyperbolic structure of the problem considered, two penalty parameters are introduced, corresponding to Dirichlet and Neumann conditions, respectively. A stability and convergence estimate is also provided, in the case of a discontinuous-in-time Galerkin space-time integrator. The spatial discretization used is based on a stabilized method with equal order interpolation for all solution components. In principle, however, the proposed methodology is not confined to stabilized methods. We conclude with an extensive set of tests to validate the robustness and accuracy of the proposed approach.

Key words: Weak boundary conditions, wave equation, stabilized methods, variational multiscale analysis.

1. Introduction

The *weak* enforcement of boundary conditions consists in using the variational formulation associated with a partial differential equation (PDE) problem, rather than directly incorporating the boundary conditions in the function spaces used to represent the solution.

In finite element methods, boundary conditions are more often enforced in the strong sense, meaning that the space of test and trial functions are appropriately modified to accommodate the boundary conditions. This is particularly true in the case of Dirichlet-type boundary conditions. An alternative approach is to impose the boundary conditions weakly, through the variational formulation itself. The computational data structure of a finite element method is often times simplified by the use of weak boundary conditions, as it is not necessary to directly prescribe the values of the numerical solution's degrees of freedom on the boundary. The reader can already realize this fact considering *no slip* boundary conditions for the Navier-Stokes equations. As will be discussed momentarily, the advantages are even greater in the context of wave propagation problems, which require inviscid, zero normal flux boundary conditions, and in nonlinear systems, in which strong enforcement of boundary conditions may lead to lack of discrete conservation principles.

Weak boundary conditions can be traced back to the late 1960s. Lions [56] considered the problem of solving elliptic PDEs with very rough Dirichlet boundary data, and proposed a formulation in which the Dirichlet boundary condition is replaced by a Robin condition depending on an artificial penalty. Aubin [2] extended this approach in the framework of finite difference approximations of nonlinear problems. A consistent and optimal penalty formulation for Dirichlet boundary conditions for elliptic problems had been proposed by Nitsche [59]. More recently, the authors in [5, 6] utilized weak enforcement of boundary conditions for advection-diffusion problems, the Navier-Stokes equations, and turbulence, and showed improved results in the presence of boundary layers.

In [67], a new formulation for the weak enforcement of boundary conditions for the mixed form of a linear longitudinal wave propagation problem was presented. This formulation lacked a formal proof of stability and convergence,

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but showed improved performance with respect to formulations enforcing boundary conditions strongly, especially near corners and edges. That formulation was based on a mid-point time integration scheme, and stabilized linear finite elements in space.

In this work, we propose a new class of algorithms, for which a formal proof of stability and convergence can be established. The proposed methods are based on a space-time approach, and generalize to high-order polynomial interpolations in space and time. Specifically we introduce a Nitsche-type approach for both Dirichlet and Neumann boundary conditions. A peculiar aspect of the proposed method is the application of a penalty parameter not only to the Dirichlet, but also to the Neumann boundary conditions, with the purpose of enforcing a tight control on all boundary conditions in the stability and convergence analysis. This aspect makes the Nitsche approach pursued here different from almost any other Nitsche method in the literature. Although developed in complete independence, the proposed approach was found to have some connections to the discontinuous Galerkin (DG) framework for Friedrichs systems developed by Ern and Guermond [13] and Houston and collaborators [36]. We would also like to point the reader to the very recent and interesting work of Annavarapu *et al.* [1], in which a Nitsche method is used in the context of explicit time-dependent computations with embedded interfaces. As a side note, we will not consider non-reflecting boundary conditions, for which considerable literature exists (see [19–32, 35, 54, 60] and references therein.)

The simplest possible boundary conditions in the context of longitudinal waves involve enforcing either the pressure or the normal velocity component. Although in engineering mechanics the pressure is considered as the diagonal component of a stress tensor, in longitudinal wave problems pressure boundary conditions are of Dirichlet type, while the normal velocity boundary conditions are of Neumann type. This situation is somewhat reversed with respect to waves in solids, in which the full velocity component can be imposed as a Dirichlet boundary condition, while the stress boundary condition is of Neumann type. This is due to the fact that the spatial differential operators in the mixed equations for longitudinal waves have a structure analogous to the Darcy flow operator, while spatial differential operators in the mixed equations for waves in solids have a structure similar to the Stokes flow operator.

Codina analyzed in [8] a stabilized method in which *both* pressure and normal velocity boundary conditions were enforced *strongly*. The authors in [7, 9–11, 40, 41, 44], instead, using finite elements of Raviart-Thomas type, enforced the pressure (Dirichlet) boundary condition *weakly* while the normal velocity (Neumann) boundary condition was enforced *strongly*, using a spatial discretization often employed for Darcy flow problems. In the work of [4], three classes of stabilized methods using orthogonal sub-scales are presented, and appropriate choices of the stabilization parameters are designed to mimic variational formulations in which both pressure and normal velocity are imposed strongly, as well as the case in which either one or the other are imposed weakly.

However the mentioned literature does not cover the interesting case in which *both* boundary conditions are enforced *weakly*, which is the scope of the present work. Imposing both Dirichlet and Neumann boundary conditions weakly results in a considerably simpler formulation to implement. The proposed strategy has also the advantage that equal-order approximations can be employed for pressure and velocity (thanks to the use of a variational multiscale stabilized formulation), and that imposition of the normal velocity boundary condition is incorporated in the variational form. Equal-order interpolations can be quite useful, since Raviart-Thomas finite elements generate non-diagonal mass matrices which involve linear matrix solves in the time-integration procedure. Recently, mass lumping techniques have been proposed to improve efficiency in computations with Raviart-Thomas approximation spaces [7, 9]. Needless to say that equal-order continuous finite elements allow for very efficient and robust mass lumping strategies. Weak boundary conditions are particularly useful when considering complex geometry domains, in which the normal velocity component has to be enforced on curved surfaces. In this case it can be quite tedious to strongly constrain the components of the velocity at the nodes so that the normal velocity boundary condition is satisfied strongly. All these aspects are exacerbated in the nonlinear case, where, in addition, strong enforcement of boundary conditions may hinder the global conservation properties of equal-order interpolation finite element methods.

We would also like to stress that the proposed work on weak boundary conditions for the mixed form of the linear wave equation, has many important applications in time-domain forward and inverse problems. The mixed form of the wave equation under consideration is also often the preferred approach in fluid/structure interaction problems for time-domain acoustics [15–17]. In addition, the mixed form of the wave equation is often used as a simplified model for more complex nonlinear equations, such as, among many others, the compressible flow equations of Lagrangian shock hydrodynamics [57, 63–66, 68–71], nonlinear acoustics, shallow water flows, and meteorological flows, [12, 33, 34, 45–53, 55, 58, 61, 73, 74].

The rest of the exposition is organized as follows: Section 2 describes the general setting of the problem considered; Section 3 introduces the Nitsche approach in the general abstract sense; Section 4 details the implementation of the proposed Nitsche method using space-time integrators, and relative analysis of stability and convergence; Section 5 presents two explicit space-time integration strategies, through which the Nitsche approach is implemented. Numerical testing is performed in Section 6 and conclusions are summarized in Section 7.

2. Equations for longitudinal wave propagation

We consider the equations for the propagation of longitudinal waves in a non-dissipative medium. These equations describe, for example, the propagation of waves in elastic fluids (acoustics), and related problems. We specifically consider the equation in mixed form:

$$\rho \partial_t \mathbf{v} + \nabla p = \rho \mathbf{b} , \quad (1a)$$

$$\partial_t p + \rho c_s^2 \nabla \cdot \mathbf{v} = 0 . \quad (1b)$$

Here, $\mathbf{v}$ is the velocity, p is the pressure, ∇ and $\nabla \cdot$ are the gradient and divergence operators, ∂_t indicates derivation with respect to time, and $\mathbf{b}$ is a body force. These equations describe the propagation of disturbances in a medium of density ρ and wave speed c_s , both assumed positive and finite. Complete specification of the problem requires initial conditions on Ω

$$\mathbf{v}(t = 0) = \mathbf{v}_0 , \quad (2a)$$

$$p(t = 0) = p_0 , \quad (2b)$$

and appropriate boundary conditions. Assuming that the boundary $\Gamma = \partial\Omega$ is partitioned as $\overline{\Gamma^g \cup \Gamma^h}$, $\Gamma^g \cap \Gamma^h = \emptyset$, pressure boundary conditions are enforced on the Dirichlet boundary Γ^g and normal velocity boundary conditions are enforced on the Neumann boundary Γ^h :

$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma^h} = h(\mathbf{x}, t) , \quad (3a)$$

$$p|_{\Gamma^g} = g(\mathbf{x}, t) , \quad (3b)$$

where $\mathbf{n}$ is the outward-pointing normal to the spatial boundary $\partial\Omega$. The structure of equations (1a)–(1b) is common to many other mechanical systems, such as non-dissipative acoustics, gravity wave propagation in shallow water flows, simplified models for global circulation of meteorological flows. In the case of acoustics, these equations can be derived by linearization of the compressible Euler equations characterizing inviscid fluids, as shown in [75, p. 158] or also [71]. Denoting by $\zeta = \rho c_s$ the *characteristic impedance* of the medium, the previous equations can be rearranged by introducing the parameter $\chi = \rho \zeta^{-2}$:

$$\rho \partial_t \mathbf{v} + \nabla p = \rho \mathbf{b} , \quad (4)$$

$$\chi \partial_t p + \nabla \cdot \mathbf{v} = 0 , \quad (5)$$

or, in vector operator form,

$$\left(\begin{bmatrix} \rho \mathbf{I}_{n_d \times n_d} & \mathbf{0}_{n_d \times 1} \\ \mathbf{0}_{1 \times n_d} & \chi \end{bmatrix} \frac{\partial}{\partial t} + \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \nabla \\ \nabla \cdot & 0 \end{bmatrix} \right) \begin{Bmatrix} \mathbf{v} \\ p \end{Bmatrix} = \begin{Bmatrix} \rho \mathbf{b} \\ 0 \end{Bmatrix} . \quad (6)$$

where n_d is the number of space dimensions. Denoting by $\mathbf{Y} = \{\mathbf{v}^T, p\}^T$ the solution state vector, equation (6) can be rewritten as

$$\mathcal{L}_t \mathbf{Y} = \mathbf{F} , \quad (7)$$

where

$$\mathcal{L}_t = \mathbf{A}_0 \partial_t + \mathcal{L}, \quad (8)$$

$$\mathcal{L} = \mathbf{A}_i \partial_{x_i}, \quad (9)$$

$$\mathbf{A}_0 = \begin{bmatrix} \rho \mathbf{I}_{n_d \times n_d} & \mathbf{0}_{n_d \times 1} \\ \mathbf{0}_{1 \times n_d} & \chi \end{bmatrix}, \quad (10)$$

$$\mathbf{A}_i = \begin{bmatrix} & & \delta_{1i} \\ & \mathbf{0}_{n_d \times n_d} & \delta_{2i} \\ & & \delta_{3i} \\ \delta_{1i} & \delta_{2i} & \delta_{3i} & 0 \end{bmatrix}, \quad (11)$$

$$\mathbf{F} = \begin{bmatrix} \rho \mathbf{b} \\ 0 \end{bmatrix}. \quad (12)$$

Note that $\mathbf{A}_i \in \mathbb{R}^{(n_d+1) \times (n_d+1)}$ are symmetric for $i = 0, \dots, n_d$.

2.1. General notation: Inner products and boundary functionals

We denote by

$$(v, w)_\omega = \int_\omega v w \, d\omega \quad \text{and} \quad (\mathbf{v}, \mathbf{w})_\omega = \int_\omega \mathbf{v} \cdot \mathbf{w} \, d\omega \quad (13)$$

the $L^2(\omega)$ - and $(L^2(\omega))^{n_d}$ -inner products on the interior of the domain $\omega \subset \Omega$, and by

$$\langle v, w \rangle_\gamma = \int_\gamma v w \, d\gamma, \quad \gamma \subset \Gamma, \quad (14)$$

a boundary functional on $\gamma \subset \Gamma$. Consider now the space-time domain $\mathbf{Q} = \Omega \times]0, T[\subset \mathbb{R}^{n_d+1}$, as depicted in Figure 1. Let $\partial\mathbf{Q}$ be the boundary of $\mathbf{Q}$, which is also decomposed in a space-like boundary $\mathbf{P} = \partial\Omega \times]0, T[$ and a time-like boundary $\Omega \times \{0\}$ and $\Omega \times \{T\}$. It is possible to extend the previous definitions of inner products and boundary functions in the space-time domain, as follows:

$$(v, w)_\mathbf{Q} = \int_0^T \left(\int_\Omega v w \, d\Omega \right) dt \quad \text{and} \quad (\mathbf{v}, \mathbf{w})_\mathbf{Q} = \int_0^T \left(\int_\Omega \mathbf{v} \cdot \mathbf{w} \, d\Omega \right) dt. \quad (15)$$

Analogously,

$$\langle v, w \rangle_\mathbf{P} = \int_0^T \left(\int_\Gamma v w \, d\Gamma \right) dt, \quad (16)$$

$$\langle v, w \rangle_{\mathbf{P}^g} = \int_0^T \left(\int_{\Gamma^g} v w \, d\Gamma^g \right) dt, \quad (17)$$

$$\langle v, w \rangle_{\mathbf{P}^h} = \int_0^T \left(\int_{\Gamma^h} v w \, d\Gamma^h \right) dt. \quad (18)$$

2.2. Well posedness of the abstract problem

An analysis of the well-posedness of the wave propagation problem in mixed form can be pursued as suggested in the book by Ern and Guermond [14, p. 312], in which the Hille-Yosida theorem is leveraged in the case $\mathcal{L}$ is a maximal and monotone operator. This approach is followed in [8], and we report below the main results. We consider the case of homogenous Dirichlet and Neumann conditions, as the non-homogenous case can be reconnected to the

Figure 1: Space-time domain Q .

homogenous case through standard arguments. Analogously, we assume $\mathbf{A}_0 = \mathbf{I}$, since, with an appropriate rescaling of the solution, the following results can be generalized to a more general form of $\mathbf{A}_0$.

Let us define first a Hilbert space $L = \{\mathbf{W} = \{\psi, \phi\}^T \mid \psi \in (L^2(\Omega))^{n_d}, \phi \in L^2(\Omega)\}$, and the domain of the operator $\mathcal{L}$ as $V = D(\mathcal{L}) = \{\mathbf{W} = \{\psi, \phi\}^T \mid \psi \in H^{div}(\Omega) \text{ and } \psi \cdot \mathbf{n} = 0 \text{ on } \Gamma_h, \phi \in H^1(\Omega) \text{ and } \phi = 0 \text{ on } \Gamma_g\}$. By classical arguments, $V = D(\mathcal{L})$ is dense in L , and it is not very difficult to show that $\mathcal{L}$ is monotone, that is, $(\mathcal{L}\mathbf{W}, \mathbf{W})_L \geq 0$, where $(\cdot, \cdot)_L$ is the inner product associated with L . In fact, $(\mathcal{L}\mathbf{W}, \mathbf{W})_L = (\psi, \nabla\phi)_\Omega + (\nabla \cdot \psi, \phi)_\Omega = \langle \psi \cdot \mathbf{n}, \phi \rangle_\Gamma = 0$, due to the homogeneous boundary conditions. Note that the space V is also a Hilbert space, with inner product $(\tilde{\mathbf{W}}, \mathbf{W})_V = (\tilde{\psi}, \psi)_{H^{div}(\Omega)} + (\tilde{\phi}, \phi)_{H^1(\Omega)}$, and, as a consequence, a special argument can be applied to verify that $\mathcal{L}$ is maximal, as indicated again in [14, p. 314]. Namely, it is sufficient to show that there exist two constants c_1 and c_2 such that

$$\forall \mathbf{W} \in V = D(\mathcal{L}), \|\mathcal{L}\mathbf{W}\|_L = \sup_{\tilde{\mathbf{W}} \in L} \frac{(\mathcal{L}\mathbf{W}, \tilde{\mathbf{W}})_L}{\|\tilde{\mathbf{W}}\|_L} \geq c_1 \|\mathbf{W}\|_V - c_2 \|\tilde{\mathbf{W}}\|_L. \quad (19)$$

Now, setting again $\mathbf{W} = \{\psi, \phi\}^T$, we have $\|\mathcal{L}\mathbf{W}\|_L = \|\nabla \cdot \psi\|_\Omega + \|\nabla\phi\|_\Omega = \|\mathbf{W}\|_V - \|\mathbf{W}\|_L$, so $c_1 = c_2 = 1$. Since $\mathcal{L}$ is maximal and monotone, we can apply the Hille-Yosida theorem, to prove existence and uniqueness of the solution in time. Namely:

Theorem (L^2 -well posedness, Hille-Yosida). *For all $\mathbf{F} \in C^1([0, T]; L)$, and $Y_0 \in D(\mathcal{L})$, the problem (7) with homogeneous boundary conditions has a unique solution. Moreover, there is a constant c , independent of T , such that:*

$$\|\mathbf{Y}\|_{C^0([0, T]; L)} \leq c (\|\mathbf{Y}_0\|_L + T \|\mathbf{F}\|_{C^0([0, T]; L)}), \quad (20)$$

$$\|\mathbf{Y}\|_{C^1([0, T]; L)} + \|\mathbf{Y}\|_{C^0([0, T]; V)} \leq c (\|\mathbf{Y}_0\|_L + T \|\mathbf{F}\|_{C^0([0, T]; L)}). \quad (21)$$

The previous theorem ensures the well posedness of the abstract problem. Note that also in this case the estimate depends on the time horizon T , and so it is not meaningful for long times.

3. Weak versus strong boundary conditions: A comparison

In what follows, we present a weak approach to the enforcement of boundary conditions. To begin the discussion, we summarize the case of strong imposition of boundary conditions, to be used as a reference for further comparisons.

3.1. Strong enforcement of boundary conditions

For the sake of clarity and without loss of generality, here we consider only the case of a semi-discrete formulation, in which the equations are discretized in space but not in time. We begin our discussion by introducing a variational formulation in which the boundary conditions for both velocity and pressure are enforced *strongly*, as suggested for example in [8]. In this case, the function spaces for the velocity and pressure directly embed the appropriate Neumann and Dirichlet boundary conditions, namely:

$$\mathcal{S}_v = \{v \in H^{div}(\Omega) : v \cdot n = h \text{ on } \Gamma_h\}, \quad (22)$$

$$\mathcal{S}_p = \{p \in H^1(\Omega) : p = g \text{ on } \Gamma_g\}, \quad (23)$$

$$\mathcal{V}_v = \{\psi \in H^{div}(\Omega) : \psi \cdot n = 0 \text{ on } \Gamma_h\}, \quad (24)$$

$$\mathcal{V}_p = \{\phi \in H^1(\Omega) \text{ and } \phi = 0 \text{ on } \Gamma_g\}, \quad (25)$$

and the following variational formulation can be derived:

Find $v \in \mathcal{S}_v$ and $p \in \mathcal{S}_p$, such that, for all $\psi \in \mathcal{V}_v$ and $\phi \in \mathcal{V}_p$,

$$0 = (\psi, \rho \partial_t v - \rho b)_\Omega - (\nabla \cdot \psi, p)_\Omega + \langle \psi \cdot n, p \rangle_{\Gamma_h}, \quad (26)$$

$$0 = (\phi, \chi \partial_t p)_\Omega - (\nabla \phi, v)_\Omega + \langle \phi, v \cdot n \rangle_{\Gamma_g}. \quad (27)$$

Integrating by parts the gradient terms in (26)–(27) is not necessary, since the solution has sufficient regularity, but this will make more evident the differences with respect to the case of weak enforcement of the boundary conditions. There are of course intermediate approaches, in which only the velocity or pressure are enforced strongly, and less regularity is required for the solution, as discussed for example in [4, 7, 9–11, 40, 41, 44].

3.2. Weak enforcement of the boundary conditions: An abstract introduction

An alternative strategy is to avoid embedding the boundary conditions in the function spaces, and simply require

$$\mathcal{S}_v = \mathcal{V}_v = H^{div}(\Omega), \quad (28)$$

$$\mathcal{S}_p = \mathcal{V}_p = H^1(\Omega). \quad (29)$$

Integrating by parts the strong form of the equation tested on the appropriate test functions, we obtain:

$$0 = (\psi, \rho \partial_t v - \rho b)_\Omega - (\nabla \cdot \psi, p)_\Omega + \langle \psi \cdot n, \hat{p} \rangle_\Gamma, \quad (30)$$

$$0 = (\phi, \chi \partial_t p)_\Omega - (\nabla \phi, v)_\Omega + \langle \phi, \widehat{v \cdot n} \rangle_\Gamma. \quad (31)$$

The terms $\hat{p}$ and $\widehat{v \cdot n}$ are boundary traces of the solution. By appropriately defining these trace terms it is possible to derive an abstract formulation, in which the boundary conditions are enforced weakly. The formulation we propose involves the following choices:

$$\hat{p} = \begin{cases} g & \text{on } \Gamma_g, \\ p + \alpha \rho (v \cdot n - h) & \text{on } \Gamma_h, \end{cases} \quad (32)$$

$$\widehat{v \cdot n} = \begin{cases} v \cdot n + \alpha \chi (p - g) & \text{on } \Gamma_g, \\ h & \text{on } \Gamma_h, \end{cases} \quad (33)$$

in which $\alpha = \tilde{\alpha} c_s$, and the terms containing $\tilde{\alpha}$ are penalties of the corresponding boundary conditions, and act as additional stabilizing mechanisms.

Remark. The rationale for using the same scaling parameter α for both penalty terms is to respect the wave structure of the problem, in which strong coupling between the pressure and velocity update equations occurs. Adopting two different parameters (e.g., α_v and α_p) would imply an unequal weight of one boundary condition with respect to the other. Taking this route to an extreme, it is clear that for large values of one of the two penalties one could recover the limit of strong velocity or strong pressure boundary conditions.

Remark. The Nitsche stabilization term α is dimensional and defined to be proportional to the speed of sound c_s , but not explicitly dependent on a length scale. This scaling, induced by the hyperbolic structure of the wave propagation equations, is somewhat atypical if compared to the state of the practice for Laplace and Poisson operators. However, there are examples of similar scalings in the literature of compressible Lagrangian hydrodynamics [66], where shock capturing viscosities are introduced to induce artificial acoustic dissipation. An alternative choice for α could have been the ratio of length and time scales, such as $h_e/\Delta t$, which, in the present context of explicit time integration, leads to very similar results to c_s .

The previous choice yields the final abstract formulation

$$0 = (\boldsymbol{\psi}, \rho \partial_t \mathbf{v} - \rho \mathbf{b})_\Gamma - (\nabla \cdot \boldsymbol{\psi}, p)_\Omega + \langle \boldsymbol{\psi} \cdot \mathbf{n}, g \rangle_{\Gamma^g} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, p \rangle_{\Gamma^h} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \alpha \rho (\mathbf{v} \cdot \mathbf{n} - h) \rangle_{\Gamma^h}, \quad (34)$$

$$0 = (\phi, \chi \partial_t p)_\Gamma - (\nabla \phi, \mathbf{v})_\Omega + \langle \phi, h \rangle_{\Gamma^h} + \langle \phi, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma^g} + \langle \phi, \alpha \chi (p - g) \rangle_{\Gamma^g}, \quad (35)$$

which can be compared with (26)–(27) to highlight differences. Integrating by parts once more in space leads to the Euler-Lagrange equations

$$0 = (\boldsymbol{\psi}, \rho \partial_t \mathbf{v} + \nabla p - \rho \mathbf{b})_\Gamma - \langle \boldsymbol{\psi} \cdot \mathbf{n}, p - g \rangle_{\Gamma^g} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \alpha \rho (\mathbf{v} \cdot \mathbf{n} - h) \rangle_{\Gamma^h}, \quad (36)$$

$$0 = (\phi, \chi \partial_t p + \nabla \cdot \mathbf{v})_\Gamma - \langle \phi, \mathbf{v} \cdot \mathbf{n} - h \rangle_{\Gamma^h} + \langle \phi, \alpha \chi (p - g) \rangle_{\Gamma^g}, \quad (37)$$

which exactly match the specification of problem (1a)–(1b) with boundary conditions (3b)–(3a), and confirm the variational consistency of the proposed approach.

3.3. A detour: A Nitsche displacement-based formulation and energy principles

For the sake of completeness, we present in this section the Nitsche approach in the context of a displacement-based formulation, in which the pressure/displacement pair $\{p, \mathbf{u}\}$ is used in place of the pressure/velocity pair $\{p, \mathbf{v}\}$. This section can be skipped in a first reading of this manuscript, since the implementation of this variational form will not be pursued further. We think it is however quite interesting from a theoretical perspective, since it casts the Nitsche method within the framework of energy minimization principles. Consider the mixed formulation in which we substitute $\dot{\mathbf{u}} = \mathbf{v}$:

$$0 = (\boldsymbol{\psi}, \rho \partial_t \mathbf{u} + \nabla p - \rho \mathbf{b})_\Omega, \quad (38)$$

$$0 = (\phi, \chi p + \nabla \cdot \mathbf{u})_\Omega, \quad (39)$$

and associated boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma^h} = \tilde{h}_n(\mathbf{x}, t), \quad (40a)$$

$$p|_{\Gamma^g} = g(\mathbf{x}, t). \quad (40b)$$

Here $\tilde{h}_n(\mathbf{x}, t)$ is chosen so that $\mathbf{v} \cdot \mathbf{n}|_{\Gamma^h} = h(\mathbf{x}, t)$. This system has been studied for example in [41], where Raviart-Thomas finite elements were used in conjunction with strong boundary conditions. The interesting feature of this variational formulation, is that it can be derived in the context of Hellinger-Reissner energy functionals for elasticity, and that Nitsche's weak enforcement of boundary conditions can be interpreted as an approximation to an augmented Lagrangian approach. The Lagrangian of the system $\mathcal{L}$ may be built by adding the work of the Lagrange multipliers λ to the Hellinger-Reissner energy functional:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\rho \dot{\mathbf{u}}, \dot{\mathbf{u}})_\Omega + \frac{1}{2}(\chi p, p)_\Omega + (p, \nabla \cdot \mathbf{u})_\Omega + (\mathbf{u}, \mathbf{b})_\Omega - \langle g, \mathbf{u} \cdot \mathbf{n} \rangle_{\Gamma^g} + \langle \lambda, \mathbf{u} \cdot \mathbf{n} - \tilde{h}_n \rangle_{\Gamma^h} \\ & + \frac{1}{2} \langle \alpha \rho (\mathbf{u} \cdot \mathbf{n} - \tilde{h}_n)^2, 1 \rangle_{\Gamma^h} + \frac{1}{2} \langle \alpha \chi (p - g)^2, 1 \rangle_{\Gamma^g}. \end{aligned} \quad (41)$$

Imposing the stationarity conditions leads to the variational equations

$$\begin{aligned} 0 = \delta \mathcal{L} = & -(\delta \mathbf{u}, \rho \ddot{\mathbf{u}})_{\mathbf{Q}} + (\delta p, \chi p)_{\mathbf{Q}} + (\delta p, \nabla \cdot \mathbf{u})_{\mathbf{Q}} + (p, \nabla \cdot \delta \mathbf{u})_{\mathbf{Q}} + (\delta \mathbf{u}, \mathbf{b})_{\mathbf{Q}} \\ & - \langle g, \delta \mathbf{u} \cdot \mathbf{n} \rangle_{\mathbf{P}_g} + \langle \delta \mathbf{u} \cdot \mathbf{n}, \lambda \rangle_{\mathbf{P}_h} + \langle \delta \lambda, \mathbf{u} \cdot \mathbf{n} - \tilde{h}_n \rangle_{\mathbf{P}_h} \\ & + \langle \alpha \rho \delta \mathbf{u} \cdot \mathbf{n}, \mathbf{u} \cdot \mathbf{n} - \tilde{h}_n \rangle_{\mathbf{P}_h} + \langle \alpha \chi \delta p, p - g \rangle_{\mathbf{P}_g} . \end{aligned} \quad (42)$$

Boundary conditions on the normal displacements are typically used in solid mechanics to characterize frictionless contact. In this case, the Lagrange multiplier λ corresponds to the pressure necessary to enforce this constraint, namely,

$$\lambda = -p , \quad (43)$$

which can be substituted back into equation (42), to obtain:

$$\begin{aligned} 0 = & -(\delta \mathbf{u}, \rho \ddot{\mathbf{u}})_{\mathbf{Q}} + (\delta p, \chi p)_{\mathbf{Q}} + (\delta p, \nabla \cdot \mathbf{u})_{\mathbf{Q}} + (p, \nabla \cdot \delta \mathbf{u})_{\mathbf{Q}} + (\delta \mathbf{u}, \mathbf{b})_{\mathbf{Q}} - \langle g, \delta \mathbf{u} \cdot \mathbf{n} \rangle_{\mathbf{P}_g} \\ & - \langle \delta \mathbf{u} \cdot \mathbf{n}, p \rangle_{\mathbf{P}_h} - \langle \delta p, \mathbf{u} \cdot \mathbf{n} - \tilde{h}_n \rangle_{\mathbf{P}_h} + \langle \alpha \rho \delta \mathbf{u} \cdot \mathbf{n}, \mathbf{u} \cdot \mathbf{n} - \tilde{h}_n \rangle_{\mathbf{P}_h} + \langle \alpha \chi \delta p, p - g \rangle_{\mathbf{P}_g} . \end{aligned} \quad (44)$$

The variations of pressure and displacements can be understood as the corresponding weighting functions, namely, $\delta \mathbf{u} \triangleq \boldsymbol{\psi}$ and $\delta p \triangleq \phi$. Substituting these into equation (44), and integrating by parts the term containing the divergence of the displacements yields

$$0 = (\boldsymbol{\psi}, \rho \ddot{\mathbf{u}} - \rho \mathbf{b})_{\mathbf{Q}} - (\nabla \cdot \boldsymbol{\psi}, p)_{\mathbf{Q}} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, g \rangle_{\mathbf{P}_g} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, p \rangle_{\mathbf{P}_h} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \alpha \rho (\mathbf{u} \cdot \mathbf{n} - \tilde{h}_n) \rangle_{\mathbf{P}_h} , \quad (45)$$

$$0 = (\phi, \chi p)_{\mathbf{Q}} - (\nabla \phi, \mathbf{u})_{\mathbf{Q}} + \langle \phi, \tilde{h}_n \rangle_{\mathbf{P}_h} + \langle \phi, \mathbf{u} \cdot \mathbf{n} \rangle_{\mathbf{P}_g} + \langle \phi, \alpha \chi (p - g) \rangle_{\mathbf{P}_g} , \quad (46)$$

which represent a Nitsche-type variational formulation for the pressure/displacement pair of unknowns. It is easy to observe the similarity between (45)–(46) and (34)–(35). In what follows, we prefer to use the system of equations for the pressure/velocity pair (34)–(35), because these are amenable to simpler and more general time integration techniques, as shown in the next sections.

4. A space-time Nitsche approach to boundary conditions

4.1. Space/time form of the equations and non-dissipative Friedrichs systems

In the space-time continuum, assigning $x_0 = t$, one can rewrite

$$\mathcal{L}_t \mathbf{Y} = \sum_{i=0}^{n_d} \mathbf{A}_i \partial_{x_i} \mathbf{Y} , \quad (47)$$

with $\partial_{x_0} = \partial_t$. Define $\mathbf{D} = \sum_{i=0}^{n_d} n_i \mathbf{A}_i$, where $n_1, \dots, n_{n_d}$ are components of the unit normal $\mathbf{n}$ in space, and n_0 is the temporal normal, equal to +1 on the top boundary of the space-time domain $\mathbf{Q}$ (i.e. on $\Omega \times \{T\}$) and -1 on the bottom boundary of the space-time domain $\mathbf{Q}$ of Figure 1 (i.e. on $\Omega \times \{0\}$). Hence,

$$\mathbf{D} = \begin{cases} \sum_{i=1}^{n_d} n_i \mathbf{A}_i = \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \mathbf{n} \\ \mathbf{n}^T & 0 \end{bmatrix} , & \text{on } \mathbf{P} = \partial \Omega \times [0, T] , \\ \mathbf{A}_0 , & \text{on } \Omega(T) , \\ -\mathbf{A}_0 , & \text{on } \Omega(0) . \end{cases} \quad (48)$$

$$\quad (49)$$

$$\quad (50)$$

Using the classic formalism of Friedrichs systems, it is possible to recast the entire initial/boundary value problem as a space/time boundary value problem. We opt for this notation as it is very compact, and well suited to the subsequent

development of space/time variational methods. Accordingly, equations (1) can be rewritten as follows, with initial conditions (2) and boundary conditions (3) combined as space/time boundary conditions:

$$\sum_{i=0}^{n_d} A_i \partial_{x_i} Y = F, \quad \text{in } Q, \quad (51a)$$

$$\frac{1}{2} (M - D) (Y - H) = 0, \quad \text{on } \partial Q, \quad (51b)$$

with

$$H = \left\{ \begin{array}{l} \chi_{\Gamma_h} h n \\ \chi_{\Gamma_g} g \end{array} \right\} \text{ on } \Gamma \text{ for } t \geq 0, \quad \text{and} \quad H = Y_0 = \left\{ \begin{array}{l} v_0 \\ p_0 \end{array} \right\} \text{ on } \Omega \text{ for } t = 0, \quad (51c)$$

where χ_γ is the characteristic function of the set γ , equal to 1 if $x \in \gamma$ and zero otherwise. In Appendix A, it is shown that the set of equations for acoustic wave propagation (1a)–(1b) is a *non-dissipative* Friedrichs system, that is, a degenerate Friedrichs system. For appropriate choices of the matrix $M \in \mathbb{R}^{(n_d+1) \times (n_d+1)}$ the initial/boundary conditions (2)–(3) can be recovered. Namely,

$$\text{Initial condition on } \partial Q = \Omega \times \{0\} : M = -D = A_0 \Rightarrow \frac{1}{2}(M - D) = A_0 \text{ (invertible)}, \quad (52a)$$

$$\text{No condition on } \partial Q = \Omega \times \{T\} : M = D = A_0 \Rightarrow M - D = 0, \quad (52b)$$

$$\text{Dirichlet condition on } \partial Q = P^g : M = \begin{bmatrix} 0_{n_d \times n_d} & -n \\ n^T & 0 \end{bmatrix} \Rightarrow \frac{1}{2}(M - D) = \begin{bmatrix} 0_{n_d \times n_d} & -n \\ 0_{1 \times n_d} & 0 \end{bmatrix} \Rightarrow p n = g n, \quad (52c)$$

$$\text{Neumann condition on } \partial Q = P^h : M = \begin{bmatrix} 0_{n_d \times n_d} & n \\ -n^T & 0 \end{bmatrix} \Rightarrow \frac{1}{2}(M - D) = \begin{bmatrix} 0_{n_d \times n_d} & 0_{n_d \times 1} \\ -n^T & 0 \end{bmatrix} \Rightarrow v \cdot n = h. \quad (52d)$$

Defining

$$I_g = \begin{bmatrix} 0_{n_d \times n_d} & 0_{n_d \times 1} \\ 0_{1 \times n_d} & 1 \end{bmatrix} \geq 0 \quad \text{and} \quad I_h = \begin{bmatrix} n \otimes n & 0_{n_d \times 1} \\ 0_{1 \times n_d} & 0 \end{bmatrix} \geq 0, \quad (53)$$

and using the equalities

$$\left\{ \begin{array}{l} p n \\ 0 \end{array} \right\} = -\frac{1}{2} (M - D)^* Y \quad \text{on } P^h, \quad (54)$$

$$\left\{ \begin{array}{l} 0_{1 \times n_d} \\ v \cdot n \end{array} \right\} = -\frac{1}{2} (M - D)^* Y \quad \text{on } P^g, \quad (55)$$

$$\left\{ \begin{array}{l} g n \\ 0 \end{array} \right\} = -\frac{1}{2} (M - D)^* H \quad \text{on } P^g, \quad (56)$$

$$\left\{ \begin{array}{l} 0_{1 \times n_d} \\ h \end{array} \right\} = -\frac{1}{2} (M - D)^* H \quad \text{on } P^h, \quad (57)$$

and observing that $M_{P^g}^* = M_{P^h}$ (here E^* denotes the transpose of a matrix E), it is possible to recast (34)–(35) in system form:

$$\begin{aligned} 0 &= (W, A_0 \dot{Y})_Q - (\partial_{x_i} W, A_i Y)_Q - (W, F)_Q \\ &\quad - \frac{1}{2} \langle W, (M - D)^* Y + (M - D) H \rangle_{P_h^g} - \frac{1}{2} \langle W, (M - D)^* Y + (M - D) H \rangle_{P_h^h} \\ &\quad + \langle W, \alpha I_g A_0 (Y - H) \rangle_{P_h^g} + \langle W, \alpha I_h A_0 (Y - H) \rangle_{P_h^h}. \end{aligned} \quad (58)$$

This form of the problem will be the basis for two families of discrete space-time methods to be presented next.

Figure 2: Slicing the space-time domain Q into space-time slabs Q_n

4.2. Two families of stabilized space-time Nitsche methods

The previous variational equations are cast in the semi-discrete approach. In order to pursue a fully discrete formulation, we consider a space-time finite element approximation. Consider then the partition of the time-interval $I =]0, T[$ (see Fig.(2)),

$$0 = t_0 < t_1 < \dots < t_{N+1} = T.$$

The n^{th} space-time slab Q_n and its boundary P_n are defined as

$$Q_n = \Omega \times I_n, \quad (59)$$

$$P_n = \Gamma \times I_n, \quad (60)$$

and we denote by

$$P_n^g = \Gamma^g \times I_n, \quad (61)$$

$$P_n^h = \Gamma^h \times I_n, \quad (62)$$

the corresponding spatial boundary portions on the space-like boundary of the space-time slab. Correspondingly, we denote by $(v, w)_{Q_n}$ and $(\mathbf{v}, \mathbf{w})_{Q_n}$ the $L^2(Q_n)$ - and $(L^2(Q_n))^{n_d}$ -inner products on the interior of the space-time slab Q_n and by $\langle v, w \rangle_{P_n}$, $\langle v, w \rangle_{P_n^g}$, and $\langle v, w \rangle_{P_n^h}$ the space-time boundary functionals on the space-time boundary P_n and its components P_n^g and P_n^h , respectively. These definitions are analogous to (15)–(18).

We proceed next to describe two classes of space-time numerical methods, based on equal-order, globally continuous interpolation in space for both the trial and test spaces, but different interpolations in time. In the first class, the same interpolation spaces are also used in time, for both the test and trial functions. These spaces are discontinuous in time, leading to an equal-order discontinuous Galerkin formulation in time. In the second class, the space of trial functions is continuous, and the space of test functions is of one polynomial order lower in time, and discontinuous, leading to a Petrov-Galerkin formulation in time.

4.2.1. Discontinuous-in-time space-time formulation

Within each space-time element $\mathbf{Q}_n^e = \Omega^e \times I_n$, the following choice of trial/test function spaces is made for the velocity and pressure, respectively:

$$\mathcal{S}_v^h = \mathcal{V}_v^h = \left\{ \psi^h \in (C^0(\mathbf{Q}_n))^n : \psi^h|_{\mathbf{Q}_n^e} \in (\mathcal{P}_k(\Omega_n^e))^n \times \mathcal{P}_p(I_n) \right\}, \quad (63)$$

$$\mathcal{S}_p^h = \mathcal{V}_p^h = \left\{ \phi^h \in C^0(\mathbf{Q}_n) : \phi^h|_{\mathbf{Q}_n^e} \in \mathcal{P}_k(\Omega_n^e) \times \mathcal{P}_p(I_n) \right\}. \quad (64)$$

Specifically, the trial and weighting functions are approximated by k^{th} -order polynomials in space and p^{th} -order in time. These functions are further assumed C^0 -continuous over each space-time slab, but are allowed to be discontinuous across the time-interfaces between slabs. The proposed space-time discretization leads to a Bubnov-Galerkin methods in time, with accuracy of order $2p + 1$ at the time sample points t_n^- (with $p \geq 0$) [3, 18, 42], while the order of accuracy in space will be presented in the next section. On the n^{th} space-time slab, the weak form of the governing equations using Nitsche approach to boundary conditions can be written as

$$\mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{Y})_n = \mathbf{L}_{\text{STAB}}(\mathbf{W})_n, \quad (65)$$

with $\mathbf{W} = \{\psi^T, \phi\}^T$ and

$$\mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{Y})_n = \mathbf{B}(\mathbf{W}, \mathbf{Y})_n + (\tau \mathcal{L}_t \mathbf{W}, \mathcal{L}_t \mathbf{Y})_{\mathbf{Q}_n}, \quad (66)$$

$$\mathbf{L}_{\text{STAB}}(\mathbf{W})_n = \mathbf{L}(\mathbf{W})_n + (\tau \mathcal{L}_t \mathbf{W}, \mathbf{F})_{\mathbf{Q}_n}. \quad (67)$$

Here $\tau = \tilde{\tau} \mathbf{A}_0^{-1}$ and

$$\begin{aligned} \mathbf{B}(\mathbf{W}, \mathbf{Y})_n &= -(\dot{\mathbf{W}}, \mathbf{A}_0 \mathbf{Y})_{\mathbf{Q}_n} - (\partial_{x_i} \mathbf{W}, \mathbf{A}_i \mathbf{Y})_{\mathbf{Q}_n} + (\mathbf{W}(t_{n+1}^-), \mathbf{A}_0 \mathbf{Y}(t_{n+1}^-))_{\Omega} \\ &\quad - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{Y} \rangle_{\mathbf{P}_n^s} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{Y} \rangle_{\mathbf{P}_n^h} \\ &\quad + \langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{Y} \rangle_{\mathbf{P}_n^s} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{Y} \rangle_{\mathbf{P}_n^h}, \end{aligned} \quad (68)$$

$$\begin{aligned} \mathbf{L}(\mathbf{W})_n &= (\mathbf{W}, \mathbf{F})_{\mathbf{Q}_n} + (\mathbf{W}(t_n^+), \mathbf{A}_0 \mathbf{Y}(t_n^-))_{\Omega} \\ &\quad + \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}_n^s} + \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}_n^h} \\ &\quad + \langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{H} \rangle_{\mathbf{P}_n^s} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{H} \rangle_{\mathbf{P}_n^h}, \end{aligned} \quad (69)$$

with $t^\pm = \lim_{\epsilon \rightarrow 0} t \pm \epsilon$. Summing over all space-time slab $\Omega \times [t_n, t_{n+1}]$, we obtain

$$\mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{Y}) = \mathbf{L}_{\text{STAB}}(\mathbf{W}), \quad (70)$$

where

$$\begin{aligned} \mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{Y}) &= -(\dot{\mathbf{W}}, \mathbf{A}_0 \mathbf{Y})_{\mathbf{Q}} - (\partial_{x_i} \mathbf{W}, \mathbf{A}_i \mathbf{Y})_{\mathbf{Q}} \\ &\quad + \sum_{n=0}^N \left\{ (\mathbf{W}(t_{n+1}^-), \mathbf{A}_0 \mathbf{Y}(t_{n+1}^-))_{\Omega} - (\mathbf{W}(t_n^+), \mathbf{A}_0 \mathbf{Y}(t_n^-))_{\Omega} \right\} \\ &\quad - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{Y} \rangle_{\mathbf{P}^s} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{Y} \rangle_{\mathbf{P}^h} \\ &\quad + \langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{Y} \rangle_{\mathbf{P}^s} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{Y} \rangle_{\mathbf{P}^h} \\ &\quad + (\tau \mathcal{L}_t \mathbf{W}, \mathcal{L}_t \mathbf{Y})_{\mathbf{Q}}, \end{aligned} \quad (71)$$

$$\begin{aligned} \mathbf{L}_{\text{STAB}}(\mathbf{W}) &= (\mathbf{W}, \mathbf{F})_{\mathbf{Q}} \\ &\quad + \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}^s} + \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}^h} \\ &\quad + \langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{H} \rangle_{\mathbf{P}^s} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{H} \rangle_{\mathbf{P}^h} \\ &\quad + (\tau \mathcal{L}_t \mathbf{W}, \mathbf{F})_{\mathbf{Q}}. \end{aligned} \quad (72)$$

4.2.2. Continuous-in-time space-time formulation

An alternative approach is to use a Petrov-Galerkin method in time, in which the trial space is represented by C^0 -continuous functions throughout the entire space-time domain, while leaving the test space discontinuous-in-time. Namely, the following choice is made for the velocity and pressure trial/test functions spaces:

$$\mathcal{S}_v^h = \left\{ \psi^h \in (C^0(\mathbf{Q}))^{n_d} : \psi^h|_{\mathbf{Q}_n^e} \in (\mathcal{P}_k(\Omega_n^e))^{n_d} \times \mathcal{P}_p(I_n) \right\}, \quad (73)$$

$$\mathcal{V}_v^h = \left\{ \psi^h \in (C^0(\mathbf{Q}_n))^{n_d} : \psi^h|_{\mathbf{Q}_n^e} \in (\mathcal{P}_k(\Omega_n^e))^{n_d} \times \mathcal{P}_{p-1}(I_n) \right\}, \quad (74)$$

$$\mathcal{S}_p^h = \left\{ \phi^h \in C^0(\mathbf{Q}) : \phi^h|_{\mathbf{Q}_n^e} \in \mathcal{P}_k(\Omega_n^e) \times \mathcal{P}_p(I_n) \right\}, \quad (75)$$

$$\mathcal{V}_p^h = \left\{ \phi^h \in C^0(\mathbf{Q}_n) : \phi^h|_{\mathbf{Q}_n^e} \in \mathcal{P}_k(\Omega_n^e) \times \mathcal{P}_{p-1}(I_n) \right\}. \quad (76)$$

On the n^{th} space-time slab, the final form of the continuous-in-time stabilized formulation reads then

$$\mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{Y})_n = \mathbf{L}_{\text{STAB}}(\mathbf{W})_n, \quad (77)$$

where

$$\begin{aligned} \mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{Y})_n = & -(\dot{\mathbf{W}}, \mathbf{A}_0 \mathbf{Y})_{\mathbf{Q}_n} - (\partial_{x_i} \mathbf{W}, \mathbf{A}_i \mathbf{Y})_{\mathbf{Q}_n} + (\mathbf{W}(t_{n+1}), \mathbf{A}_0 \mathbf{Y}(t_{n+1}))_{\Omega} \\ & - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{Y} \rangle_{\mathbf{P}_n^g} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{Y} \rangle_{\mathbf{P}_n^h} \\ & + \langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{Y} \rangle_{\mathbf{P}_n^g} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{Y} \rangle_{\mathbf{P}_n^h} \\ & + (\tau \mathcal{L}_t \mathbf{W}, \mathcal{L}_t \mathbf{Y})_{\mathbf{Q}_n}, \end{aligned} \quad (78)$$

$$\begin{aligned} \mathbf{L}_{\text{STAB}}(\mathbf{W})_n = & (\mathbf{W}, \mathbf{F})_{\mathbf{Q}_n} + (\mathbf{W}(t_n), \mathbf{A}_0 \mathbf{Y}(t_n))_{\Omega} \\ & + \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}_n^g} + \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}_n^h} \\ & + \langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{H} \rangle_{\mathbf{P}_n^g} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{H} \rangle_{\mathbf{P}_n^h} \\ & + (\tau \mathcal{L}_t \mathbf{W}, \mathcal{L}_t \mathbf{Y})_{\mathbf{Q}_n}, \end{aligned} \quad (79)$$

The time discretization outlined above is accurate of order $2p$ at the time sample point t_n , with $p \geq 1$. In the case $p = 1$, for which $\dot{\mathbf{W}} = \mathbf{0}$, the proposed method collapses to the well-known Crank-Nicolson and mid-point time integrators. We further discuss this case in Section (5.2), in the context of a very simple explicit time integrator.

4.3. Analysis of the discontinuous-in-time stabilized method

In this section, we derive a stability and convergence estimate for the discontinuous-in-time stabilized Galerkin approximation. To start, let us consider the bilinear form in the case $\mathbf{W} = \mathbf{Y}$, a choice that is possible in view of the boundary conditions and finite element spaces adopted:

$$\begin{aligned} \mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{W}) = & -(\dot{\mathbf{W}}, \mathbf{A}_0 \mathbf{W})_{\mathbf{Q}} + \sum_{n=0}^N \left\{ (\mathbf{W}(t_{n+1}^-), \mathbf{A}_0 \mathbf{W}(t_{n+1}^-))_{\Omega} - (\mathbf{W}(t_n^+), \mathbf{A}_0 \mathbf{W}(t_n^-))_{\Omega} \right\} \\ & - (\partial_{x_i} \mathbf{W}, \mathbf{A}_i \mathbf{W})_{\mathbf{Q}} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{\mathbf{P}^g} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{\mathbf{P}^h} \\ & + \langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{W} \rangle_{\mathbf{P}^g} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{W} \rangle_{\mathbf{P}^h} \\ & + (\tau \mathcal{L}_t \mathbf{W}, \mathcal{L}_t \mathbf{W})_{\mathbf{Q}}. \end{aligned} \quad (80)$$

We now consider separately all the terms that compose equation (80). The terms associated with the time derivatives and jumps yield:

$$\begin{aligned}
& \sum_{n=0}^N \left\{ -(\dot{\mathbf{W}}, \mathbf{A}_0 \mathbf{W})_{Q_n} + (\mathbf{W}(t_{n+1}^-), \mathbf{A}_0 \mathbf{W}(t_{n+1}^-))_{\Omega} - (\mathbf{W}(t_n^+), \mathbf{A}_0 \mathbf{W}(t_n^+))_{\Omega} \right\} \\
&= \sum_{n=0}^N \left(-\frac{1}{2} \|\mathbf{A}_0^{1/2} \mathbf{W}\|_{0,\Omega}^2 \Big|_{t_n^+}^{t_{n+1}^-} + \|\mathbf{A}_0^{1/2} \mathbf{W}(t_{n+1}^-)\|_{0,\Omega}^2 - (\mathbf{W}(t_n^+), \mathbf{A}_0 \mathbf{W}(t_n^+))_{0,\Omega} \right) \\
&= \sum_{n=0}^N \left(\frac{1}{2} \|\mathbf{A}_0^{1/2} \mathbf{W}(t_n^+)\|_{0,\Omega}^2 + \frac{1}{2} \|\mathbf{A}_0^{1/2} \mathbf{W}(t_{n+1}^-)\|_{0,\Omega}^2 - (\mathbf{W}(t_n^+), \mathbf{A}_0 \mathbf{W}(t_n^+))_{\Omega} \right) \\
&= \frac{1}{2} \left(\|\mathbf{A}_0^{1/2} \mathbf{W}(0^+)\|_{0,\Omega}^2 + \|\mathbf{A}_0^{1/2} \mathbf{W}(T^-)\|_{0,\Omega}^2 + \sum_{n=1}^N \|\mathbf{A}_0^{1/2} \llbracket \mathbf{W}(t_n) \rrbracket\|_{0,\Omega}^2 \right), \tag{81}
\end{aligned}$$

where $\llbracket \mathbf{W}(t_n) \rrbracket = \mathbf{W}(t_n^+) - \mathbf{W}(t_n^-)$. In view of the fact that M_{P_g} and M_{P_h} are skew symmetric, the space derivative and space-like boundary terms can be manipulated to yield:

$$\begin{aligned}
& -(\partial_{x_i} \mathbf{W}, \mathbf{A}_i \mathbf{W})_Q - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{P_g} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{P_h} \\
&= -\frac{1}{2} (1, \partial_{x_i} (\mathbf{W} \cdot \mathbf{A}_i \mathbf{W}))_Q - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{P_g} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{P_h} \\
&= -\frac{1}{2} \langle \mathbf{W}, \mathbf{A}_i \mathbf{n}_i \mathbf{W} \rangle_{P_g \cup P_h} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{P_g} - \frac{1}{2} \langle \mathbf{W}, (\mathbf{M} - \mathbf{D})^* \mathbf{W} \rangle_{P_h} \\
&= -\frac{1}{2} \langle \mathbf{W}, \mathbf{M}^* \mathbf{W} \rangle_{P_g} - \frac{1}{2} \langle \mathbf{W}, \mathbf{M}^* \mathbf{W} \rangle_{P_h} \\
&= 0. \tag{82}
\end{aligned}$$

The penalty terms simplify to

$$\langle \mathbf{W}, \alpha \mathbf{I}_g \mathbf{A}_0 \mathbf{W} \rangle_{P_g} + \langle \mathbf{W}, \alpha \mathbf{I}_h \mathbf{A}_0 \mathbf{W} \rangle_{P_h} = \alpha \|\mathbf{I}_g^{1/2} \mathbf{A}_0^{1/2} \mathbf{W}\|_{0,P_g}^2 + \alpha \|\mathbf{I}_h^{1/2} \mathbf{A}_0^{1/2} \mathbf{W}\|_{0,P_h}^2. \tag{83}$$

Since $\tau = \tilde{\tau} \mathbf{A}_0^{-1}$ and $\alpha = \tilde{\alpha} c_s$, with $\tilde{\tau} = c_{\tau} \frac{h}{2c_s}$, we have

$$\begin{aligned}
\mathbf{B}_{\text{STAB}}(\mathbf{W}, \mathbf{W}) &= \frac{1}{2} \left(\|\mathbf{A}_0^{1/2} \mathbf{W}(0^+)\|_{0,\Omega}^2 + \|\mathbf{A}_0^{1/2} \mathbf{W}(T^-)\|_{0,\Omega}^2 + \sum_{n=1}^N \|\mathbf{A}_0^{1/2} \llbracket \mathbf{W}(t_n) \rrbracket\|_{0,\Omega}^2 \right) \\
&\quad + \tilde{\alpha} \|c_s^{1/2} \mathbf{I}_g^{1/2} \mathbf{A}_0^{1/2} \mathbf{W}\|_{0,P_g}^2 + \tilde{\alpha} \|c_s^{1/2} \mathbf{I}_h^{1/2} \mathbf{A}_0^{1/2} \mathbf{W}\|_{0,P_h}^2 + \tilde{\tau} \|\mathbf{A}_0^{-1/2} \mathcal{L}_t \mathbf{W}\|_{0,Q}^2 \\
&= \|\mathbf{W}\|_{\text{STAB}}^2. \tag{84}
\end{aligned}$$

Remark. Note that (84) implies control over the L^2 -norm of the error at time T^- . By selecting $T^- = t_1, t_2, \dots, t_n, t_{n+1}, \dots$, we obtain that the L^2 -norm of the error is controlled at each time t_n^- .

We can now proceed with the error analysis, by decomposing the error, $\mathbf{e} = \mathbf{Y}^h - \mathbf{Y}$ as

$$\mathbf{e} = \mathbf{Y}^h - \tilde{\mathbf{Y}}^h + \tilde{\mathbf{Y}}^h - \mathbf{Y} = \mathbf{e}^h + \boldsymbol{\eta}, \tag{85}$$

where $\mathbf{e}^h = \{(\mathbf{e}_v^h)^T, \mathbf{e}_p^h\}^T \in \mathcal{V}^h$ is the part of the error in the finite element space and $\boldsymbol{\eta} = \{\boldsymbol{\eta}_v^T, \boldsymbol{\eta}_p\}^T$ is the interpolation

error. We can use the stability result and consistency statement, and write

$$\begin{aligned}
\|e^h\|_{\text{STAB}}^2 &= \mathbf{B}_{\text{STAB}}(e^h, e^h) \\
&= \mathbf{B}_{\text{STAB}}(e^h, e) - \mathbf{B}_{\text{STAB}}(e^h, \eta) \\
&\leq \left| \mathbf{B}_{\text{STAB}}(e^h, \eta) \right| \\
&= \left| -(\dot{e}^h, \mathbf{A}_0 \eta)_{\mathbf{Q}} - (\partial_{x_i} e^h, \mathbf{A}_i \eta)_{\mathbf{Q}} \right. \\
&\quad + \sum_{n=0}^N \left\{ (e^h(t_{n+1}^-), \mathbf{A}_0 \eta(t_{n+1}^-))_{\Omega} - (e^h(t_n^+), \mathbf{A}_0 \eta(t_n^-))_{\Omega} \right\} \\
&\quad - \frac{1}{2} \langle e^h, (\mathbf{M} - \mathbf{D})^* \eta \rangle_{\mathbf{P}^g} + \langle e^h, \alpha \mathbf{I}_g \mathbf{A}_0 \eta \rangle_{\mathbf{P}^g} \\
&\quad - \frac{1}{2} \langle e^h, (\mathbf{M} - \mathbf{D})^* \eta \rangle_{\mathbf{P}^h} + \langle e^h, \alpha \mathbf{I}_h \mathbf{A}_0 \eta \rangle_{\mathbf{P}^h} \\
&\quad \left. + (\tau \mathcal{L}_t e^h, \mathcal{L}_t \eta)_{\mathbf{Q}} \right|. \tag{86}
\end{aligned}$$

The terms in space-time domain interior can be bounded by ϵ -inequality,

$$\begin{aligned}
&\left| -(\dot{e}^h, \mathbf{A}_0 \eta)_{\mathbf{Q}} - (\partial_{x_i} e^h, \mathbf{A}_i \eta)_{\mathbf{Q}} + (\tau \mathcal{L}_t e^h, \mathcal{L}_t \eta)_{\mathbf{Q}} \right| \\
&\leq \left| (\mathcal{L}_t e^h, \eta) \right| + \left| (\tau \mathcal{L}_t e^h, \mathcal{L}_t \eta)_{\mathbf{Q}} \right| \\
&\leq \frac{\epsilon_1}{2} \|\mathbf{A}_0^{-1/2} \mathcal{L}_t e^h\|_{0,\mathbf{Q}}^2 + \frac{1}{2\epsilon_1} \|\mathbf{A}_0^{1/2} \eta\|_{0,\mathbf{Q}}^2 + \frac{\epsilon_2}{2} \tilde{\tau} \|\mathbf{A}_0^{-1/2} \mathcal{L}_t e^h\|_{0,\mathbf{Q}}^2 + \frac{\tilde{\tau}}{2\epsilon_2} \|\mathbf{A}_0^{-1/2} \mathcal{L}_t \eta\|^2. \tag{87}
\end{aligned}$$

The terms associated with time-like boundaries can be handled in a similar way:

$$\begin{aligned}
&\left| \sum_{n=0}^N \left\{ (e^h(t_{n+1}^-), \mathbf{A}_0 \eta(t_{n+1}^-))_{\Omega} - (e^h(t_n^+), \mathbf{A}_0 \eta(t_n^-))_{\Omega} \right\} \right| \\
&= \left| (e^h(T^-), \mathbf{A}_0 \eta(T^-))_{\Omega} - (e^h(0^+), \mathbf{A}_0 \eta(0^-))_{\Omega} - \sum_{n=1}^N (e^h(t_n), \mathbf{A}_0 \eta(t_n^-))_{\Omega} \right| \\
&\leq \frac{\epsilon_3}{2} \|\mathbf{A}_0^{1/2} e^h(T^-)\|_{0,\mathbf{Q}}^2 + \frac{1}{2\epsilon_3} \|\mathbf{A}_0^{1/2} \eta(T^-)\|_{0,\mathbf{Q}}^2 + \frac{\epsilon_4}{2} \|\mathbf{A}_0^{1/2} e^h(0^+)\|_{0,\mathbf{Q}}^2 + \frac{1}{2\epsilon_4} \|\mathbf{A}_0^{1/2} \eta(0^-)\|_{0,\mathbf{Q}}^2 \\
&\quad + \sum_{n=1}^N \left(\frac{\epsilon_5}{2} \|\mathbf{A}_0^{1/2} e^h(t_n)\|_{0,\mathbf{Q}}^2 + \frac{1}{2\epsilon_5} \|\mathbf{A}_0^{1/2} \eta(t_n^-)\|_{0,\mathbf{Q}}^2 \right). \tag{88}
\end{aligned}$$

Analogously, we can apply ϵ -inequality to the terms at space-like boundaries:

$$\begin{aligned}
& \left| -\frac{1}{2} \langle \mathbf{e}^h, (\mathbf{M} - \mathbf{D})^* \boldsymbol{\eta} \rangle_{\mathbf{P}^g} + \langle \mathbf{e}^h, \alpha \mathbf{I}_g \mathbf{A}_0 \boldsymbol{\eta} \rangle_{\mathbf{P}^g} \right| \\
&= \left| \langle \chi^{1/2} \mathbf{e}_p^h, \chi^{-1/2} \boldsymbol{\eta}_v \cdot \mathbf{n} \rangle_{\mathbf{P}^g} + \langle \mathbf{e}^h, \alpha \mathbf{I}_g \mathbf{A}_0 \boldsymbol{\eta} \rangle_{\mathbf{P}^g} \right| \\
&\leq \frac{\epsilon_6}{2} \langle \mathbf{e}_p^h, \chi \mathbf{e}_p^h \rangle_{\mathbf{P}^g} + \frac{1}{2\epsilon_6} \langle \boldsymbol{\eta}_v \cdot \mathbf{n}, \chi^{-1} \boldsymbol{\eta}_v \cdot \mathbf{n} \rangle_{\mathbf{P}^g} + \frac{\epsilon_7}{2} \tilde{\alpha} \|c_s^{1/2} \mathbf{I}_g^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^g}^2 + \frac{\tilde{\alpha}}{2\epsilon_7} \|c_s^{1/2} \mathbf{I}_g^{1/2} \mathbf{A}_0^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^g}^2 \\
&\leq \frac{\epsilon_6}{2} \|\mathbf{I}_{\mathbf{P}^g}^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^g}^2 + \frac{1}{2\epsilon_6} \|\chi^{-1/2} \mathbf{I}_h^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^g}^2 + \frac{\epsilon_7}{2} \tilde{\alpha} \|c_s^{1/2} \mathbf{I}_g^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^g}^2 + \frac{\tilde{\alpha}}{2\epsilon_7} \|c_s^{1/2} \mathbf{I}_g^{1/2} \mathbf{A}_0^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^g}^2, \quad (89)
\end{aligned}$$

$$\begin{aligned}
& \left| -\frac{1}{2} \langle \mathbf{e}^h, (\mathbf{M} - \mathbf{D})^* \boldsymbol{\eta} \rangle_{\mathbf{P}^h} + \langle \mathbf{e}^h, \alpha \mathbf{I}_h \mathbf{A}_0 \boldsymbol{\eta} \rangle_{\mathbf{P}^h} \right| \\
&= \left| \langle \rho^{1/2} \mathbf{e}_v^h \cdot \mathbf{n}, \rho^{-1/2} \boldsymbol{\eta}_p \rangle_{\mathbf{P}^h} + \langle \mathbf{e}^h, \alpha \mathbf{I}_h \mathbf{A}_0 \boldsymbol{\eta} \rangle_{\mathbf{P}^h} \right| \\
&\leq \frac{\epsilon_8}{2} \langle \mathbf{e}_v^h \cdot \mathbf{n}, \rho \mathbf{e}_v^h \cdot \mathbf{n} \rangle_{\mathbf{P}^h} + \frac{1}{2\epsilon_8} \langle \boldsymbol{\eta}_p, \rho^{-1} \boldsymbol{\eta}_p \rangle_{\mathbf{P}^h} + \frac{\epsilon_9}{2} \tilde{\alpha} \|c_s^{1/2} \mathbf{I}_h^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^h}^2 + \frac{\tilde{\alpha}}{2\epsilon_9} \|c_s^{1/2} \mathbf{I}_h^{1/2} \mathbf{A}_0^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^h}^2 \\
&\leq \frac{\epsilon_8}{2} \|\mathbf{I}_{\mathbf{P}^h}^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^h}^2 + \frac{1}{2\epsilon_8} \|\rho^{-1/2} \mathbf{I}_g^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^h}^2 + \frac{\epsilon_9}{2} \tilde{\alpha} \|c_s^{1/2} \mathbf{I}_h^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^h}^2 + \frac{\tilde{\alpha}}{2\epsilon_9} \|c_s^{1/2} \mathbf{I}_h^{1/2} \mathbf{A}_0^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^h}^2. \quad (90)
\end{aligned}$$

In summary

$$\begin{aligned}
\|\mathbf{e}^h\|_{\text{STAB}}^2 &\leq \left(\frac{\epsilon_1}{2} + \frac{\epsilon_2}{2} \tilde{\tau} \right) \|\mathbf{A}_0^{-1/2} \mathcal{L}^t \mathbf{e}^h\|_{0,\Omega}^2 + \frac{\epsilon_3}{2} \|\mathbf{A}_0^{1/2} \mathbf{e}^h(T^-)\|_{0,\Omega}^2 + \frac{\epsilon_4}{2} \|\mathbf{A}_0^{1/2} \mathbf{e}^h(0^+)\|_{0,\Omega}^2 \\
&+ \sum_{n=1}^N \frac{\epsilon_5}{2} \|\mathbf{A}_0^{1/2} [\![\mathbf{e}^h(t_n)]\!]\|_{0,\Omega}^2 + \left(\frac{\epsilon_6}{2} + \frac{\epsilon_7}{2} \tilde{\alpha} c_s \right) \|\mathbf{I}_g^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^g}^2 \\
&+ \left(\frac{\epsilon_8}{2} + \frac{\epsilon_9}{2} \tilde{\alpha} c_s \right) \|\mathbf{I}_h^{1/2} \mathbf{A}_0^{1/2} \mathbf{e}^h\|_{0,\mathbf{P}^h}^2 + \frac{1}{2\epsilon_1} \|\mathbf{A}_0^{1/2} \boldsymbol{\eta}\|_{0,\Omega}^2 + \frac{\tilde{\tau}}{2\epsilon_2} \|\mathbf{A}_0^{-1/2} \mathcal{L}^t \boldsymbol{\eta}\|_{0,\Omega}^2 \\
&+ \frac{1}{2\epsilon_3} \|\mathbf{A}_0^{1/2} \boldsymbol{\eta}(T^-)\|_{0,\Omega}^2 + \frac{1}{2\epsilon_4} \|\mathbf{A}_0^{1/2} \boldsymbol{\eta}(0^+)\|_{0,\Omega}^2 + \sum_{n=1}^N \frac{1}{2\epsilon_5} \|\mathbf{A}_0^{1/2} \boldsymbol{\eta}(t_n^-)\|_{0,\Omega}^2 \\
&+ \frac{1}{2\epsilon_6} \|\chi^{-1/2} \mathbf{I}_{\mathbf{P}^h}^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^g}^2 + \frac{\tilde{\alpha}}{2\epsilon_7} \|c_s^{1/2} \mathbf{I}_{\mathbf{P}^g}^{1/2} \mathbf{A}_0^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^g}^2 \\
&+ \frac{1}{2\epsilon_8} \|\rho^{-1/2} \mathbf{I}_g^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^h}^2 + \frac{\tilde{\alpha}}{2\epsilon_9} \|c_s^{1/2} \mathbf{I}_{\mathbf{P}^h}^{1/2} \mathbf{A}_0^{1/2} \boldsymbol{\eta}\|_{0,\mathbf{P}^h}^2. \quad (91)
\end{aligned}$$

Finally, we choose the ϵ_i 's in (91), so that the coefficients of the $\mathbf{e}^h$ -terms on the right hand side equal half of $\|\mathbf{e}^h\|_{\text{STAB}}^2$ on the left hand side. Hence:

$$\epsilon_3 = \epsilon_4 = \epsilon_5 = \frac{1}{2}, \quad (92)$$

$$\epsilon_1 = \frac{\tilde{\tau}}{2} \text{ and } \epsilon_2 = \frac{1}{2} \implies \frac{\epsilon_1}{2} + \frac{\epsilon_2 \tilde{\tau}}{2} = \frac{\tilde{\tau}}{2}, \quad (93)$$

$$\epsilon_6 = \frac{\tilde{\alpha} c_s}{2} \text{ and } \epsilon_7 = \frac{1}{2} \implies \frac{\epsilon_6}{2} + \frac{\epsilon_7 \tilde{\alpha} c_s}{2} = \frac{\tilde{\alpha} c_s}{2}, \quad (94)$$

$$\epsilon_8 = \frac{\tilde{\alpha} c_s}{2} \text{ and } \epsilon_9 = \frac{1}{2} \implies \frac{\epsilon_8}{2} + \frac{\epsilon_9 \tilde{\alpha} c_s}{2} = \frac{\tilde{\alpha} c_s}{2}. \quad (95)$$

We can now hide the e^h -terms into the left hand side:

$$\begin{aligned} \frac{1}{2} \|e^h\|_{\text{STAB}}^2 &\leq \|A_0^{1/2} \eta(T^-)\|_{0,\Omega}^2 + \|A_0^{1/2} \eta(0^-)\|_{0,\Omega}^2 + \sum_{n=1}^N \|A_0^{1/2} \eta(t_n^-)\|_{0,\Omega}^2 \\ &\quad + \frac{1}{\tilde{\tau}} \|A_0^{1/2} \eta\|_{0,\mathcal{Q}}^2 + \tilde{\tau} \|A_0^{-1/2} \mathcal{L}_t \eta\|_{0,\mathcal{Q}}^2 + \frac{1}{\tilde{\alpha}} \|(c_s \chi)^{-1/2} I_{\mathcal{P}^h}^{1/2} \eta\|_{0,\mathcal{P}^g}^2 \\ &\quad + \tilde{\alpha} \|c_s^{1/2} I_{\mathcal{P}^g}^{1/2} A_0^{1/2} \eta\|_{0,\mathcal{P}^g}^2 + \frac{1}{\tilde{\alpha}} \|\rho^{-1/2} c_s^{1/2} I_g^{1/2} \eta\|_{0,\mathcal{P}^h}^2 + \tilde{\alpha} \|c_s^{1/2} I_{\mathcal{P}^h}^{1/2} A_0^{1/2} \eta\|_{0,\mathcal{P}^h}^2. \end{aligned} \quad (96)$$

Using fairly standard interpolation estimates in space-time [37–39], it is not difficult to bound all the terms on the right hand side of the previous inequality. We will now make the assumption that the time steps $\Delta t = \max_n t_n = t_{n+1} - t_n$ and the minimum and maximum mesh spacing are related by a CFL-like condition, that is $h \sim c_s \Delta t_n$, where h is the representative mesh size. This is actually a fairly standard situation when explicit time integration is adopted, and more generally, when time-accurate computations are sought. For the interior of the space-time integrals, we then have

$$\|\eta\|_{\mathcal{Q},s} \leq c h^{k+1-s} \|Y\|_{\mathcal{Q},k+1}, \quad (97)$$

where $\|\cdot\|_{\mathcal{Q},s}$ is the Sobolev norm of order s on the space-time domain $\mathcal{Q}$. Analogous estimates would hold for the space-like, and time-like boundary terms. In summary, we can obtain

$$\|e^h\|_{\text{STAB}}^2 = O(h^{2k+2}) + O((N+1)h h^{2k+1}) + O(h^{2k+1}) + O(h^{2k+2}) = O(h^{2k+1}) \quad (98)$$

that is

$$\|e^h\|_{\text{STAB}} \leq C(\mathcal{Q}) h^{k+1/2} \|Y\|_{\mathcal{Q},k+1}, \quad (99)$$

which should be intended as an estimate in terms of a space-time mesh size, as h and Δt_n are proportional to each other through the speed of sound. Standard interpolation estimates imply that the error $\eta \leq C_I(\mathcal{Q}) h^{k+1} \|Y\|_{\mathcal{Q},k+1}$. Using the triangle inequality, and defining $c = \min(C, C_I)$, yields a final estimate for the overall solution error:

$$\|e\|_{\text{STAB}} \leq c h^{k+1/2} \|Y\|_{\mathcal{Q},k+1}. \quad (100)$$

Remark. Note that the estimate (100) and the definition of the norm $\|e\|_{\text{STAB}}$ implies control over the L^2 -norm of the error at time T^- . By repeatedly applying the estimate and selecting $T^- = t_1^-, t_2^-, \dots, t_n^-, t_{n+1}^-, \dots, t_{N+1}^- = T^-$, we obtain that the L^2 -norm of the error is controlled at each time instant t_n^- . This result follows from the fact that the proposed space-time algorithm can be decomposed in a sequence of steps over each of the separate space-time slabs. In practical applications, rather than using the norm $\|e\|_{\text{STAB}}$, it is then possible to use the more intuitive $l^1([0, T] : (L^2(\Omega)))$ -norm, given by

$$\|e\|_{l^1([0,T]:(L^2(\Omega)))} = \frac{1}{N+1} \sum_{i=0}^{N+1} \|e(t_i^-)\|_{L^2(\Omega)}, \quad (101)$$

where $t_{N+1} = T$.

5. Two explicit predictor/multi-corrector methods

We now introduce two families of explicit time integrators, corresponding to the discontinuous-in-time and continuous-in-time space-time methods introduced in the previous section. Both explicit integrators are based on a predictor/multi-corrector strategy. Specifically, we first consider an explicit time integrator for the discontinuous-in-time approach, in combination with linear interpolation in time and space for both the test and trial spaces. Then, we will also introduce a predictor/corrector method based on the continuous-in-time approach, in which the trial space is approximated using piecewise linear, globally continuous interpolation, while the test space is approximated by piecewise constant discontinuous functions. This second approach leads to space/time integrators analogous to the second-order, two-stage TVD Runge-Kutta and Crank-Nicholson schemes.

The two explicit time integrators introduced in this and the following sections are prototypical, and can be easily generalized to higher-order polynomial interpolation spaces.

5.1. Discontinuous Galerkin, linear-in-time interpolation

We consider first the case of equal order interpolation in space and time for the pressure and velocity fields, by means of piece-wise linear functions. Namely,

$$\mathbf{Y}^h(\mathbf{x}, t) = \sum_{A=1}^{n_{np}} N_A(\mathbf{x}) \left(\pi_n(t) \mathbf{Y}_{A;(n+1)} + \tilde{\pi}_n(t) \tilde{\mathbf{Y}}_{A;n} \right), \quad (102)$$

$$\mathbf{W}^h(\mathbf{x}, t) = \sum_{A=1}^{n_{np}} N_A(\mathbf{x}) \left(\pi_n(t) \mathbf{W}_{A;(n+1)} + \tilde{\pi}_n(t) \tilde{\mathbf{W}}_{A;n} \right), \quad (103)$$

where $N_A(\mathbf{x})$ is the piecewise linear shape function in space associated with node A in the global node numbering and $\pi_n(t)$ and $\tilde{\pi}_n(t)$ are the linear shape functions in time, defined as

$$\pi_n(t) = (t - t_n) / \Delta t, \quad (104)$$

$$\tilde{\pi}_n(t) = (t_{n+1} - t) / \Delta t. \quad (105)$$

In the previous expressions, n_{np} is the number of spatial nodes. Also, $\mathbf{Y}_{A;(n+1)}$ $\tilde{\mathbf{Y}}_{A;n}$ are vectors of solution degrees of freedom at t_{n+1}^- and t_n^+ , respectively, and an analogous notation holds for $\mathbf{W}_{A;(n+1)}$ and $\tilde{\mathbf{W}}_{A;n}$. Substituting (102)–(105) into (65), and taking $\mathbf{W}_{A;(n+1)}$ and $\tilde{\mathbf{W}}_{A;n}$ equal to $\mathbf{1}_{n_d+1}^T$ and $\mathbf{0}_{n_d+1}^T$, and subsequently $\mathbf{0}_{n_d+1}^T$ and $\mathbf{1}_{n_d+1}^T$, yields:

$$\begin{aligned} 0 = & - (N_A(\mathbf{x}) / \Delta t, \mathbf{A}_0 \mathbf{Y}^h)_{Q_n} + (N_A(\mathbf{x}), \mathbf{A}_0 \mathbf{Y}^h(t_{n+1}^-))_{\Omega} - (\pi_n \partial_{x_i} N_A(\mathbf{x}), \mathbf{A}_i \mathbf{Y}^h)_{Q_n} - (N_A(\mathbf{x}) \pi_n, \mathbf{F})_{Q_n} \\ & - \frac{1}{2} \langle N_A(\mathbf{x}) \pi_n, (\mathbf{M} - \mathbf{D})^* \mathbf{Y}^h \rangle_{P_n^s} - \frac{1}{2} \langle N_A(\mathbf{x}) \pi_n, (\mathbf{M} - \mathbf{D})^* \mathbf{Y}^h \rangle_{P_n^h} \\ & - \frac{1}{2} \langle N_A(\mathbf{x}) \pi_n, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{P_n^s} - \frac{1}{2} \langle N_A(\mathbf{x}) \pi_n, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{P_n^h} \\ & + \langle N_A(\mathbf{x}) \pi_n, \alpha \mathbf{I}_g \mathbf{A}_0 (\mathbf{Y}^h - \mathbf{H}) \rangle_{P_n^s} + \langle N_A(\mathbf{x}) \pi_n, \alpha \mathbf{I}_h \mathbf{A}_0 (\mathbf{Y}^h - \mathbf{H}) \rangle_{P_n^h} \\ & + (\tau \mathcal{L}_t N_A(\mathbf{x}) \pi_n, \mathcal{L}_t \mathbf{Y}^h - \mathbf{F})_{Q_n}, \end{aligned} \quad (106)$$

and, analogously,

$$\begin{aligned} 0 = & (N_A(\mathbf{x}) / \Delta t, \mathbf{A}_0 \mathbf{Y}^h)_{Q_n} - (N_A(\mathbf{x}), \mathbf{A}_0 \mathbf{Y}^h(t_n^+))_{\Omega} - (\tilde{\pi}_n \partial_{x_i} N_A(\mathbf{x}), \mathbf{A}_i \mathbf{Y}^h)_{Q_n} - (N_A(\mathbf{x}) \tilde{\pi}_n, \mathbf{F})_{Q_n} \\ & - \frac{1}{2} \langle N_A(\mathbf{x}) \tilde{\pi}_n, (\mathbf{M} - \mathbf{D})^* \mathbf{Y}^h \rangle_{P_n^s} - \frac{1}{2} \langle N_A(\mathbf{x}) \tilde{\pi}_n, (\mathbf{M} - \mathbf{D})^* \mathbf{Y}^h \rangle_{P_n^h} \\ & - \frac{1}{2} \langle N_A(\mathbf{x}) \tilde{\pi}_n, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{P_n^s} - \frac{1}{2} \langle N_A(\mathbf{x}) \tilde{\pi}_n, (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{P_n^h} \\ & + \langle N_A(\mathbf{x}) \tilde{\pi}_n, \alpha \mathbf{I}_g \mathbf{A}_0 (\mathbf{Y}^h - \mathbf{H}) \rangle_{P_n^s} + \langle N_A(\mathbf{x}) \tilde{\pi}_n, \alpha \mathbf{I}_h \mathbf{A}_0 (\mathbf{Y}^h - \mathbf{H}) \rangle_{P_n^h} \\ & + (\tau \mathcal{L}_t N_A(\mathbf{x}) \tilde{\pi}_n, \mathcal{L}_t \mathbf{Y}^h - \mathbf{F})_{Q_n}, \end{aligned} \quad (107)$$

with $N_A(\mathbf{x}) = N_A(\mathbf{x}) \mathbf{1}_{n_d+1}^T$. In the previous two equations, we also have denoted $\tau = \tilde{\tau} \mathbf{A}_0^{-1}$, where

$$\tilde{\tau} = c_{\tau} \frac{h_e}{2c_s}. \quad (108)$$

Here h_e is a characteristic mesh length, which could be taken as $(\text{meas}(\Omega_e))^{1/n_d}$, with $\text{meas}(\Omega_e)$ the area/volume measure of element e , in two/three dimensions, respectively.

The algorithm just presented requires the inversion of a matrix to compute the solution at time t_{n+1}^- and t_n^+ . However, a fully explicit procedure can be recovered by resorting to a predictor/multi-corrector approach. For this purpose, a number of preliminary definitions are needed.

The states of the system at time $t = t_{n+1}^-$ and $t = t_n^+$ are denoted by the vectors $\mathbf{Y}_{n+1} = [\mathbf{v}_{n+1}^T, \mathbf{p}_{n+1}^T]^T$ and $\tilde{\mathbf{Y}}_n = [\tilde{\mathbf{v}}_n^T, \tilde{\mathbf{p}}_n^T]^T$ respectively. It is easily seen that $2 \times n_{np}$ equations are to be solved, respectively, for $2 \times n_{np}$ unknowns

associated with $\mathbf{v}^h$ and p^h . It will also be useful to define separately the velocity and pressure residuals:

$$\text{Res}_v^h = \rho (\partial_t \mathbf{v}^h - \mathbf{b}) + \nabla p^h, \quad (109)$$

$$\text{Res}_p^h = \chi \partial_t p^h + \nabla \cdot \mathbf{v}^h. \quad (110)$$

Note that equation (106) contains no information from the previous space-time slab, namely the solution $\mathbf{Y}(t_n^-)$ is not present. This may cause severe problems if a straightforward explicit predictor/multi-corrector implementation of the method is sought [72]. Rather than integrating in time (106) and (107), we will integrate their sum and difference, as also suggested in [72]. We present below the detailed predictor/multi-corrector strategy for explicit time integration.

5.1.1. Velocity update at t_{n+1}^-

The velocity update at time t_{n+1}^- is performed by summing equations (106) and (107), and considering the entries of $\mathbf{Y}$ relative to the velocity degrees of freedom. In particular, in this first step of the update, we consider $\mathbf{v}_{n+1}^h$ as unknown, whereas $\tilde{\mathbf{v}}_n^h$ and $\mathbf{v}_n^h$ are assumed known. This yield:

$$\begin{aligned} \mathbf{0} = & (N_A, \rho N_B (\mathbf{v}_{B;(n+1)}^h - \mathbf{v}_{B;n}^h))_{\Omega} - (\nabla N_A, p^h)_{Q_n} - (N_A, \rho \mathbf{b})_{Q_n} + (\nabla N_A, \tilde{\tau} \chi^{-1} \text{Res}_p^h)_{Q_n} \\ & + \langle N_A, g \mathbf{n} \rangle_{P_n^s} + \langle N_A, p^h \mathbf{n} \rangle_{P_n^h} + \langle N_A, \alpha \rho (\mathbf{v}^h \cdot \mathbf{n} - h) \mathbf{n} \rangle_{P_n^h}. \end{aligned} \quad (111)$$

The variational form (111) reduces to the following discrete equations for the nodal vector $\mathbf{v}_{n+1}$ of velocity degrees-of-freedom:

$$[\mathbf{M}_v] (\mathbf{v}_{n+1} - \mathbf{v}_n) + \mathbf{F}_v = \mathbf{0}. \quad (112)$$

Here mass lumping has been applied to $[\mathbf{M}_v] = [\text{diag}(\mathbf{M}, \mathbf{M}, \mathbf{M})]$, a diagonal $[(n_d \times n_{np}) \times (n_d \times n_{np})]$ -matrix. The vector $\mathbf{M}$ is defined as

$$\mathbf{M} = \{M_A\}, \quad (113)$$

$$M_A = (N_A, \rho)_{\Omega}. \quad (114)$$

Analogously, $\mathbf{F}_v$ is a $(n_d \times n_{np})$ -vector:

$$\mathbf{F}_v = \{F_{v;A}\}, \quad (115)$$

$$\begin{aligned} F_{v;A} = & - (\nabla N_A, p^h)_{Q_n} - (N_A, \rho \mathbf{b})_{Q_n} + (\nabla N_A, \tilde{\tau} \chi^{-1} \text{Res}_p^h)_{Q_n} \\ & + \langle N_A, g \mathbf{n} \rangle_{P_n^s} + \langle N_A, p^h \mathbf{n} \rangle_{P_n^h} + \langle N_A, \alpha \rho (\mathbf{v}^h \cdot \mathbf{n} - h) \mathbf{n} \rangle_{P_n^h}. \end{aligned} \quad (116)$$

5.1.2. Velocity update at t_n^+

Consider now the difference between equations (107) and (106):

$$\begin{aligned} \mathbf{0} = & (N_A, \rho N_B (\tilde{\mathbf{v}}_{B;n}^h - \mathbf{v}_{B;n}^h))_{\Omega} + (\nabla N_A \Pi(t), p^h)_{Q_n} \\ & - (N_A \Pi(t), \rho \mathbf{b})_{Q_n} + \langle N_A \Pi(t), g \mathbf{n} \rangle_{P_n^s} + \langle N_A \Pi(t), p^h \mathbf{n} \rangle_{P_n^h} \\ & - (2/\Delta t N_A, \tilde{\tau} \text{Res}_v^h)_{Q_n} + (\nabla N_A \Pi(t), \tilde{\tau} \chi^{-1} \text{Res}_p^h)_{Q_n} \\ & + \langle N_A \Pi(t), \alpha \rho (\mathbf{v}^h \cdot \mathbf{n} - h) \mathbf{n} \rangle_{P_n^h}, \end{aligned} \quad (117)$$

where

$$\Pi(t) = \tilde{\pi}_n(t) - \pi_n(t). \quad (118)$$

In equation (117) only the variable $\tilde{\mathbf{v}}_n^h$ is treated as unknown, whereas $\mathbf{v}_{(n+1)}^h$ and $\mathbf{v}_n^h$ are assumed as known. We make one additional modification to the equations with respect to the standard implicit version of the time integration scheme, in that we remove the residual term $-(2/\Delta t N_A, \tilde{\tau} \text{Res}_v^h)_{Q_n}$, as it typically creates problems in the convergence

of the fixed-point iteration used in the predictor/multi-corrector strategy. Fundamentally, in the limit of small time steps, the scaling of $2/\Delta t$ may steer the iterates away from the fixed point. Thus, equation (117) yields the following discrete equations, for the nodal vector $\tilde{\mathbf{v}}_n$:

$$[\mathbf{M}_v] (\tilde{\mathbf{v}}_n - \mathbf{v}_n) + \tilde{\mathbf{F}}_v = \mathbf{0} , \quad (119)$$

where $\tilde{\mathbf{F}}_v$ is a $(n_d \times n_{np})$ -vector:

$$\tilde{\mathbf{F}}_v = \{\tilde{\mathbf{F}}_{v:A}\} , \quad (120)$$

$$\begin{aligned} \tilde{\mathbf{F}}_{v:A} = & - \left(\nabla N_A \Pi(t) , p^h \right)_{Q_n} - (N_A \Pi(t) , \rho \mathbf{b})_{Q_n} + \left(\nabla N_A \Pi(t) , \tilde{\tau} \chi^{-1} \text{Res}_p^h \right)_{Q_n} \\ & + \langle N_A \Pi(t) , g \mathbf{n} \rangle_{P_n^g} + \langle N_A \Pi(t) , p^h \mathbf{n} \rangle_{P_n^h} + \langle N_A \Pi(t) , \alpha \rho (\mathbf{v}^h \cdot \mathbf{n} - h) \mathbf{n} \rangle_{P_n^h} . \end{aligned} \quad (121)$$

5.1.3. Pressure update at t_{n+1}^-

Similarly to the case of the velocity update, the sum of (106) and (107) yields, in the case of the pressure equations:

$$\begin{aligned} 0 = & \left(N_A , \chi N_B (p_{B;(n+1)}^h - p_{B;n}^h) \right)_{\Omega} - \left(\nabla N_A , v^h \right)_{Q_n} \\ & + \langle N_A , \mathbf{v}^h \cdot \mathbf{n} \rangle_{P_n^g} + \langle N_A , h \rangle_{P_n^h} + \left(\nabla N_A , \tilde{\tau} \rho^{-1} \text{Res}_v^h \right)_{Q_n} \\ & + \langle N_A , \alpha \chi (p^h - g) \rangle_{P_n^g} , \end{aligned} \quad (122)$$

where $p_{(n+1)}^h$ is treated as unknown, whereas $\tilde{p}_n^h$ and p_n^h are assumed known. Therefore, the update equation for the nodal vector $\mathbf{p}_{n+1}$ of pressure degrees-of-freedom reads:

$$[\mathbf{M}_p] (\mathbf{p}_{n+1} - \mathbf{p}_n) + \mathbf{W}_p = \mathbf{0} . \quad (123)$$

Here mass lumping has been applied to $[\mathbf{M}_p] = [\text{diag}(\mathbf{M}_\chi)]$, a diagonal $[n_{np} \times n_{np}]$ -matrix. The vector $\mathbf{M}_\chi$ is defined as

$$\mathbf{M}_\chi = \{\mathbf{M}_{\chi:A}\} , \quad (124)$$

$$\mathbf{M}_{\chi:A} = (N_A, \chi)_{\Omega} . \quad (125)$$

And $\mathbf{W}_p$ is a n_{np} -dimensional vector defined, namely,

$$\mathbf{W}_p = \{\mathbf{W}_{p:A}\} , \quad (126)$$

$$\begin{aligned} \mathbf{W}_{p:A} = & - \left(\nabla N_A , v^h \right)_{Q_n} + \left(\nabla N_A , \tilde{\tau} \rho^{-1} \text{Res}_v^h \right)_{Q_n} \\ & + \langle N_A , \mathbf{v}^h \cdot \mathbf{n} \rangle_{P_n^g} + \langle N_A , h \rangle_{P_n^h} + \langle N_A , \alpha \chi (p^h - g) \rangle_{P_n^g} . \end{aligned} \quad (127)$$

5.1.4. Pressure update at t_n^+

To conclude, the difference of (107) and (106) yields

$$\begin{aligned} 0 = & \left(N_A , \chi N_B (\tilde{p}_{B;n}^h - p_{B;n}^h) \right)_{\Omega} - \left(\nabla N_A \Pi(t) , v^h \right)_{Q_n} \\ & + \langle N_A \Pi(t) , \mathbf{v}^h \cdot \mathbf{n} \rangle_{P_n^g} + \langle N_A \Pi(t) , h \rangle_{P_n^h} \\ & - \left(2/\Delta t N_A , \tilde{\tau} \text{Res}_p^h \right)_{Q_n} + \left(\nabla N_A \Pi(t) , \tilde{\tau} \rho^{-1} \text{Res}_v^h \right)_{Q_n} \\ & + \langle N_A \Pi(t) , \alpha \chi (p^h - g) \rangle_{P_n^g} , \end{aligned} \quad (128)$$

where only $\tilde{p}_n^h$ is treated as the unknown. Also in the case, as already done for the velocity update, we proceed to drop the term $-(2/\Delta t N_A , \tilde{\tau} \text{Res}_p^h)_{Q_n}$, which in certain cases may prevent convergence of the fixed-point iteration.

Table 1: Outline of the predictor/multi-corrector algorithm: Discontinuous-in-time. Note that, because of mass lumping, all matrices are diagonal, so that all inverse operations are just vector divisions. Three iterations were typically used in the computations.

```

Retrieve loop parameters:  $n_{\text{step}}, i_{\text{max}}$ 
Initialize all variables with initial conditions
Form  $\mathbf{M}, \mathbf{M}_\chi, [\mathbf{M}_v]$ , and  $[\mathbf{M}_p]$ 
For  $n = 0, \dots, n_{\text{step}}$  (Time-step loop begins)
  Set  $\Delta t$  (respecting the CFL condition)
  Predictor:  $\mathbf{Y}_{n+1}^{(0)} = \mathbf{Y}_n$   $\tilde{\mathbf{Y}}_n^{(0)} = \mathbf{Y}_n$ 
  For  $i = 0, \dots, i_{\text{max}} - 1$  (Multi-corr. loop begins)
    Assembly:  $\mathbf{F}_v^{(i)}$ 
    Primary velocity update:  $\mathbf{v}_{n+1}^{(i+1)} = \mathbf{v}_n - [\mathbf{M}_v]^{-1} \mathbf{F}_v^{(i)}$ 
    Assembly:  $\tilde{\mathbf{F}}_v^{(i+1)}$ 
    Secondary velocity update:  $\tilde{\mathbf{v}}_n^{(i+1)} = \mathbf{v}_n - [\mathbf{M}_v]^{-1} \tilde{\mathbf{F}}_v^{(i+1)}$ 
    Assembly:  $\mathbf{W}_p^{(i,i+1)}$ 
    Primary Pressure update:  $\mathbf{p}_{n+1}^{(i+1)} = \mathbf{p}_n - [\mathbf{M}_p]^{-1} \mathbf{W}_p^{(i,i+1)}$ 
    Assembly:  $\tilde{\mathbf{W}}_p^{(i,i+1)}$ 
    Secondary Pressure update:  $\tilde{\mathbf{p}}_n^{(i+1)} = \mathbf{p}_n - [\mathbf{M}_p]^{-1} \tilde{\mathbf{W}}_p^{(i,i+1)}$ 
  End (Multi-corrector loop ends)
  Time update:  $\mathbf{Y}_{n+1} = \mathbf{Y}_{n+1}^{(i_{\text{max}})}$   $\tilde{\mathbf{Y}}_n = \tilde{\mathbf{Y}}_n^{(i_{\text{max}})}$ 
End (Time-step loop ends)
Exit

```

Of course, the inclusion of this term in the case of fully implicit space-time integration strategies does not seem problematic [72]. The update equation for the nodal vector $\tilde{\mathbf{p}}_n$ of pressure degrees-of-freedom becomes:

$$[\mathbf{M}_p] (\tilde{\mathbf{p}}_n - \mathbf{p}_n) + \tilde{\mathbf{W}}_p = 0, \quad (129)$$

where $\tilde{\mathbf{W}}_p$ is a n_{np} -dimensional vector defined as

$$\tilde{\mathbf{W}}_p = \{\tilde{\mathbf{W}}_{p:A}\}, \quad (130)$$

$$\begin{aligned} \tilde{\mathbf{W}}_{p:A} = & - \left(\nabla N_A \Pi(t), \mathbf{v}^h \right)_{Q_n} + \left(\nabla N_A \Pi(t), \tilde{\tau} \rho^{-1} \text{Res}_v^h \right)_{Q_n} \\ & + \langle N_A \Pi(t), \mathbf{v}^h \cdot \mathbf{n} \rangle_{P_n^g} + \langle N_A \Pi(t), h \rangle_{P_n^h} + \langle N_A \Pi(t), \alpha \chi (p^h - g) \rangle_{P_n^g}. \end{aligned} \quad (131)$$

5.1.5. Summary of the time-integration procedure

Table 1 summaries the proposed approach. $\mathbf{F}_v^{(i)}$ indicates the evaluation of $\mathbf{F}_v$ using the state $\mathbf{Y}_{n+1}$ and the state $\tilde{\mathbf{Y}}_n$ at iterate (i) . The definition of the iterate of the force vector $\tilde{\mathbf{F}}_v^{(i,i+1)}$, work vector $\mathbf{W}_p^{(i,i+1)}$ and $\tilde{\mathbf{W}}_p^{(i,i+1)}$ are somewhat different, since updated variables are used in the i^{th} iteration.

5.2. Continuous Galerkin, linear-in-time interpolation

We now consider the case of the Petrov-Galerkin in time method, in which linear-in-time, globally continuous interpolation is used for the solution space, while constant-in-time, globally discontinuous interpolation is used for the test function space. In space, both test and trial functions are interpolated with piecewise linear, globally continuous

functions. Namely,

$$\mathbf{Y}^h(\mathbf{x}, t) = \sum_{A=1}^{n_{np}} N_A(\mathbf{x}) (\pi_n(t) \mathbf{Y}_{A;(n+1)} + \tilde{\pi}_n(t) \mathbf{Y}_{A;n}) , \quad (132)$$

$$\mathbf{W}^h(\mathbf{x}, t) = \sum_{A=1}^{n_{np}} N_A(\mathbf{x}) \mathbf{W}_{A;(n+1)} , \quad (133)$$

Substituting the above approximations into (77), we obtain

$$\begin{aligned} 0 = & (N_A(\mathbf{x}) , \mathbf{A}_0 \mathbf{Y}^h(t_{n+1}))_{\Omega} - (N_A(\mathbf{x}) , \mathbf{A}_0 \mathbf{Y}^h(t_n))_{\Omega} \\ & - (\partial_{x_i} N_A(\mathbf{x}) , \mathbf{A}_i \mathbf{Y}^h)_{\mathbf{Q}_n} - (N_A(\mathbf{x}) , \mathbf{F})_{\mathbf{Q}_n} \\ & - \frac{1}{2} \langle N_A(\mathbf{x}) , (\mathbf{M} - \mathbf{D})^* \mathbf{Y}^h \rangle_{\mathbf{P}_n^g} - \frac{1}{2} \langle N_A(\mathbf{x}) , (\mathbf{M} - \mathbf{D})^* \mathbf{Y}^h \rangle_{\mathbf{P}_n^h} \\ & - \frac{1}{2} \langle N_A(\mathbf{x}) , (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}_n^g} - \frac{1}{2} \langle N_A(\mathbf{x}) , (\mathbf{M} - \mathbf{D}) \mathbf{H} \rangle_{\mathbf{P}_n^h} \\ & + \langle N_A(\mathbf{x}) , \alpha \mathbf{I}_g \mathbf{A}_0 (\mathbf{Y}^h - \mathbf{H}) \rangle_{\mathbf{P}_n^g} + \langle N_A(\mathbf{x}) , \alpha \mathbf{I}_h \mathbf{A}_0 (\mathbf{Y}^h - \mathbf{H}) \rangle_{\mathbf{P}_n^h} \\ & + (\tau \mathcal{L}_t N_A(\mathbf{x}) , \mathcal{L}_t \mathbf{Y}^h - \mathbf{F})_{\mathbf{Q}_n} , \end{aligned} \quad (134)$$

Remark. Notice that (134) is identical to the sum of (106) and (107), which constitute the first stage of velocity and pressure update in the discontinuous-in-time predictor/multi-corrector. This fact outlines the nature of the latter, since, in practice, the first stage of the discontinuous-in-time approach resembles a mid-point like integration step, and the second stage a higher order correction.

Equation (134) can then be integrated in time, using single-point quadratures. In this case, a mid-point type integrator is obtained, which bears similarities to the Crack-Nicholson integrator, and more specifically, in its predictor/multi-corrector version, to the second-order, two-stage TVD Runge-Kutta integrator. Denoting the mid-point value of a quantity f by $f_{n+1/2} = (f_n + f_{n+1})$, where $f_n = f(t_n)$, we have:

$$\begin{aligned} \mathbf{0} = & (N_A , \rho ((\mathbf{v}_{n+1}^h - \mathbf{v}_n^h) / \Delta t - \mathbf{b}))_{\Omega} - (\nabla N_A , p_{n+1/2}^h)_{\Omega} \\ & + \langle N_A , g \mathbf{n} \rangle_{\Gamma^g} + \langle N_A , p_{n+1/2}^h \mathbf{n} \rangle_{\Gamma^h} \\ & + (\nabla N_A , \tilde{\tau} ((p_{n+1}^h - p_n^h) / \Delta t + \chi^{-1} \nabla \cdot \mathbf{v}_{n+1/2}^h))_{\Omega} \\ & + \langle N_A , \alpha \rho (\mathbf{v}_{n+1/2}^h \cdot \mathbf{n} - h) \mathbf{n} \rangle_{\Gamma^h} , \end{aligned} \quad (135)$$

$$\begin{aligned} 0 = & (N_A , \chi (p_{n+1}^h - p_n^h) / \Delta t)_{\Omega} - (\nabla N_A , \mathbf{v}_{n+1/2}^h)_{\Omega} \\ & + \langle N_A , \mathbf{v}_{n+1/2}^h \cdot \mathbf{n} \rangle_{\Gamma^g} + \langle N_A , h \rangle_{\Gamma^h} \\ & + (\nabla N_A , \tilde{\tau} ((\mathbf{v}_{n+1}^h - \mathbf{v}_n^h) / \Delta t + \rho^{-1} \nabla p_{n+1/2}^h - \mathbf{b}))_{\Omega} \\ & + \langle N_A , \alpha \chi (p_{n+1/2}^h - g) \rangle_{\Gamma^g} , \end{aligned} \quad (136)$$

where Δt is the time step.

An explicit procedure for time integration is obtained by adopting the predictor/multi-corrector algorithm detailed and analyzed in [71]. For the sake of completeness, we briefly summarized it below, and in Table 2.

5.2.1. Velocity update at t_{n+1}

The velocity update step reads

$$\begin{aligned} \mathbf{0} = & (N_A , \rho ((\mathbf{v}_{n+1}^{h;(i+1)}) - \mathbf{v}_n^h) / \Delta t - \mathbf{b}))_{\Omega} - (\nabla N_A , p_{n+1/2}^{h;(i)})_{\Omega} \\ & + \langle N_A , g \mathbf{n} \rangle_{\Gamma^g} + \langle N_A , p_{n+1/2}^{h;(i)} \mathbf{n} \rangle_{\Gamma^h} \\ & + (\nabla N_A , \tilde{\tau} ((p_{n+1}^{h;(i)} - p_n^h) / \Delta t + \chi^{-1} \nabla \cdot \mathbf{v}_{n+1/2}^{h;(i)}))_{\Omega} \\ & + \langle N_A , \alpha \rho (\mathbf{v}_{n+1/2}^{h;(i)} \cdot \mathbf{n} - h) \mathbf{n} \rangle_{\Gamma^h} . \end{aligned} \quad (137)$$

Table 2: Outline of the predictor/multi-corrector algorithm: Continuous-in-time. Note that, because of mass lumping, all matrices are diagonal, so that all inverse operations are just vector divisions. Three iterations were typically used in the computations.

```

Retrieve loop parameters:  $n_{\text{step}}, i_{\text{max}}$ 
Initialize all variables with initial conditions
Form  $\mathbf{M}_0, \mathbf{M}_\chi, [\mathbf{M}_v]$ , and  $[\mathbf{M}_p]$ 
For  $n = 0, \dots, n_{\text{step}}$  (Time-step loop begins)
  Set  $\Delta t$  (respecting the CFL condition)
  Predictor:  $\mathbf{Y}_{n+1}^{(0)} = \mathbf{Y}_n$ 
  For  $i = 0, \dots, i_{\text{max}} - 1$  (Multi-corr. loop begins)
    Assembly:  $\mathbf{F}_{n+1/2}^{(i)}$ 
    Velocity update:  $\mathbf{v}_{n+1}^{(i+1)} = \mathbf{v}_n - \Delta t[\mathbf{M}_v]^{-1}\mathbf{F}_{n+1/2}^{(i)}$ 
    Assembly:  $\mathbf{W}_{n+1/2}^{(i,i+1)}$ 
    Pressure update:  $\mathbf{p}_{n+1}^{(i+1)} = \mathbf{p}_n - \Delta t[\mathbf{M}_p]^{-1}\mathbf{W}_{n+1/2}^{(i,i+1)}$ 
  End (Multi-corrector loop ends)
  Time update:  $\mathbf{Y}_{n+1} = \mathbf{Y}_{n+1}^{(i_{\text{max}})}$ 
End (Time-step loop ends)
Exit

```

The variational form (137) which reduces to the following discrete equations for the nodal vector $\mathbf{v}_{n+1}$ of velocity degrees-of-freedom:

$$[\mathbf{M}_v](\mathbf{v}_{n+1}^{(i+1)} - \mathbf{v}_n) + \Delta t \mathbf{F}_{n+1/2}^{(i)} = \mathbf{0}, \quad (138)$$

where $\mathbf{F}_{n+1/2}^{(i)}$ is a $(n_d \times n_{np})$ -vector

$$\mathbf{F}_{n+1/2}^{(i)} = \{\mathbf{F}_{n+1/2;A}^{(i)}\}, \quad (139)$$

$$\begin{aligned} \mathbf{F}_{n+1/2;A}^{(i)} = & -(\nabla N_A, p_{n+1/2}^{h;(i)})_\Omega - (N_A, \rho \mathbf{b})_\Omega + \langle N_A, g \mathbf{n} \rangle_{\Gamma^g} + \langle N_A, p_{n+1/2}^{h;(i)} \mathbf{n} \rangle_{\Gamma^h} \\ & + (\nabla N_A, \tilde{\tau}((p_{n+1}^{h;(i)} - p_n^h)/\Delta t + \chi^{-1} \nabla \cdot \mathbf{v}_{n+1/2}^{h;(i)}))_\Omega \\ & + \langle N_A, \alpha \rho (\mathbf{v}_{n+1/2}^{h;(i)} \cdot \mathbf{n} - h) \mathbf{n} \rangle_{\Gamma^h}. \end{aligned} \quad (140)$$

5.2.2. Pressure update at t_{n+1}

The pressure update step is analogous,

$$\begin{aligned} 0 = & (N_A, \chi(p_{n+1}^{h;(i+1)} - p_n^h))_\Omega - (\nabla N_A, \mathbf{v}_{n+1/2}^{h;(i+1)})_\Omega \\ & + \langle N_A, \mathbf{v}_{n+1/2}^{h;(i+1)} \cdot \mathbf{n} \rangle_{\Gamma^g} + \langle N_A, h \rangle_{\Gamma^h} \\ & + (\nabla N_A, \tilde{\tau}((\mathbf{v}_{n+1}^{h;(i+1)} - \mathbf{v}_n^h)/\Delta t + \rho^{-1} \nabla p_{n+1/2}^{h;(i)} - \mathbf{b}))_{\Omega_n} \\ & + \langle N_A, \alpha \chi (p_{n+1/2}^{h;(i)} - g) \rangle_{\Gamma^g}, \end{aligned} \quad (141)$$

and leads to

$$[\mathbf{M}_p](\mathbf{p}_{n+1}^{(i+1)} - \mathbf{p}_n) + \Delta t \mathbf{W}_{n+1/2}^{(i,i+1)} = \mathbf{0}, \quad (142)$$

Figure 3: Explicit discontinuous-in-time space-time algorithm with *consistent* mass matrices for the acoustic pulse propagation problem. Convergence rates of the numerical error $|||e^h|||_{\text{STAB}}$. Results for various choices of the stabilization parameter c_τ and Nitsche penalty $\tilde{\alpha}$ are presented. On the left, $\tilde{\alpha} = 0.0$, and on the right, $\tilde{\alpha} = 0.2$: In both cases the expected order of convergence 1.5 is verified.

where $\mathbf{W}_{n+1/2}^{(i;i+1)}$ is a n_{np} -dimensional vector defined as

$$\mathbf{W}_{n+1/2}^{(i;i+1)} = \{\mathbf{W}_{n+1/2;A}^{(i;i+1)}\}, \quad (143)$$

$$\begin{aligned} \mathbf{W}_{n+1/2;A}^{(i;i+1)} = & -(\nabla N_A, \mathbf{v}_{n+1/2}^{h;(i+1)})_\Omega + \langle N_A, \mathbf{v}_{n+1/2}^{h;(i+1)} \cdot \mathbf{n} \rangle_{\Gamma^g} + \langle N_A, h \rangle_{\Gamma^h} \\ & + (\nabla N_A, \tilde{\tau}((\mathbf{v}_{n+1}^{h;(i+1)} - \mathbf{v}_n^h)/\Delta t + \rho^{-1} \nabla p_{n+1/2}^{h;(i)} - \mathbf{b}))_{\Omega_n} \\ & + \langle N_A, \alpha \chi(p_{n+1/2}^{h;(i)} - g) \rangle_{\Gamma^g}. \end{aligned} \quad (144)$$

Remark. Note that the explicit procedure just described for the continuous-in-time space-time algorithm is actually identical to the updates at t_{n+1}^- of velocity and pressure for the discontinuous-in-time algorithm. This becomes evident by comparing (140) and (121), and (144) and (127).

6. Numerical results

We present a number of numerical results, to confirm the proposed approach is a robust, stable, and accurate strategy for boundary condition enforcement on general domains. All computations were run with two corrector passes at the Courant-Lewy-Friedrichs (CFL) condition of 0.9. The density ρ and wave speed c_s are both equal to the unit constant. We start the presentation of numerical results with a verification test illustrating the order of convergence of the method.

6.1. Convergence rates for a wave propagation problem

We define a simple wave propagation problem in two dimensions. The domain is given by the rectangle $[0, 1] \times [0, 10]$. The exact solutions for velocity and pressure are one-dimensional in nature:

$$\mathbf{v} = \omega \mathbf{e}_x, \quad (145)$$

$$p = 1 + \omega, \quad (146)$$

with $\mathbf{e}_x$ the unit vector in the x -direction and

$$\omega(x) = \begin{cases} \frac{1}{100} \left(1 - \cos\left(\frac{2\pi(x-t-x_0)}{\lambda}\right) \right), & 0 \leq x - t - x_0 \leq \lambda, \\ 0, & \text{otherwise,} \end{cases} \quad (147)$$

where $x_0 = \lambda = 2.5$ and $t \in [0, 5]$. Zero normal velocity boundary conditions are weakly enforced on the entire boundary. The domain is meshed using a fully unstructured triangular mesh, which is progressively and hierarchically

Table 3: Convergence of the error norm $\|e^h\|_{\text{STAB}}$, for the discontinuous-in-time explicit integrator with *consistent* mass matrix.

Nodes	c_τ	$\tilde{\alpha} = 0.0$		$\tilde{\alpha} = 0.2$	
		norm	rate	norm	rate
255	0.05	6.18e-04		6.19e-04	
	0.1	6.89e-04		6.92e-04	
	0.2	9.80e-04		9.84e-04	
909	0.05	1.73e-04	2.00	1.74e-04	2.00
	0.1	1.99e-04	1.98	1.99e-04	1.98
	0.2	2.73e-04	2.01	2.74e-04	2.01
3417	0.05	5.37e-05	1.88	5.37e-05	1.88
	0.1	6.02e-05	1.90	6.03e-05	1.90
	0.2	7.46e-05	1.98	7.54e-05	1.97
13233	0.05	1.91e-05	1.75	1.91e-05	1.75
	0.1	2.11e-05	1.76	2.11e-05	1.76
	0.2	2.51e-05	1.79	2.57e-05	1.78
52065	0.05	7.0244e-06	1.71	7.02e-06	1.71
	0.1	7.74e-06	1.71	7.75e-06	1.71
	0.2	9.35e-06	1.70	9.62e-06	1.69

refined from 400 to 102,400 elements to check convergence rates. We consider the evolution of the problem until $T = 4$, that is, the instant of time at which the wave is about to leave the spatial domain.

In Figure 3 we show the results for the numerical error with norm $\|e^h\|_{\text{STAB}}$, for various choices of the Nitsche stabilization parameter. These results, which are also presented in Table 3, have been obtained for the discontinuous-in-time explicit algorithm with consistent mass matrices. The expected converge rate of 1.5 can be easily verified in both the picture and the table. However, the stability norm is not very practical in evaluating the performance of the method. For this reason we also consider the error in the discrete norm $\|e\|_{l^1([0,T];(L^2(\Omega)))}$ defined in (101), more relevant in applications (this norm is also the one used in Rauch's theorem on well-posedness of general Friedrichs' systems in Section A.) Note that the convergence result proved for the discontinuous-in-time space-time method implies that the $l^1([0,T] : (L^2(\Omega)))$ -norm of the error should converge at least with order 1.5 (see also the Remark at the end of Section 4.3).

Tables 4 and 5, and, respectively, Figures 4 and 5, present results in this norm with and without mass lumping applied to $\mathbf{M}$ and $\mathbf{M}_\chi$, and different values for the stabilization parameter c_τ and the penalty parameter $\tilde{\alpha}$. Examining in detail Tables 4 and 5, it becomes clear that, at least for this test, the influence of the penalty parameter $\tilde{\alpha}$ is very moderate on the convergence rates, which are substantially unchanged. We also note that the choice $\tilde{\alpha} = 0$ does not hinder the stability of the computations.

The rates of convergence are very close to second order for the pressure and velocity when consistent mass matrices are used. Mass lumping instead produced a degradation of the convergence rates to approximately order 1.8 and 1.7 for the pressure and velocity, respectively. This result may also be a byproduct of a more pronounced accumulation of dispersion error over long time durations, a well-known issue in the case of mass lumping.

We also observe that there is very little influence of the stabilization parameter c_τ on the convergence rates and overall error values, provided c_τ is chosen sufficiently large to ensure stability. More specifically the errors in velocity and pressure tend to decrease with an increase of c_τ when using lumped mass matrices, but the opposite trend is observed in the case of consistent mass matrices. We stress however that in any case these variations are extremely small, as they affect only the third digit of the error, and for this reason are not significant or relevant in practical applications.

Remark. We observed that in the case of lumped mass matrices we were able to run stable computations over a wider range of the parameter c_τ with respect to the case of consistent mass matrices. This can be attributed to the introduction, with the mass lumping procedure, of higher levels of numerical dissipation, which in turn allow a wider

(a) Velocity. On the left, $\tilde{\alpha} = 0.0$, and on the right, $\tilde{\alpha} = 0.2$.

(b) Pressure. On the left, $\tilde{\alpha} = 0.0$, and on the right, $\tilde{\alpha} = 0.2$.

Figure 4: Explicit discontinuous-in-time space-time algorithm with mass *lumping*: Convergence rates for the acoustic pulse propagation problem. Results for various choices of the stabilization parameter c_τ and Nitsche penalty $\tilde{\alpha}$ are presented. The rates are about 1.80 in the norm $l^1([0, T] : (L^2(\Omega)))$ of the solution. Compare with Figure 5, where the same problem is solved using consistent mass matrices.

range of stable choices for the parameter c_τ .

6.1.1. Comparison between continuous-/discontinuous-in-time explicit algorithms

In this section, a comparison between the discontinuous-in-time and the continuous-in-time explicit algorithms is presented. As already mentioned in Sections 4.2.1 and 4.2.2, the discontinuous-in-time formulation is more accurate as a time integrator than the continuous-in-time formulation if interpolation of the same order is used for the solution. In fact, in the case of linear interpolation, the continuous-in-time method is second order in time, at the time sampling point t_n , while the discontinuous-in-time method is third order in time at the time sampling point t_n^- . However, the spatial error is of lower order and in practice dominates the computations.

Trends similar to the case of the discontinuous-in-time integrator can be observed also for the continuous-in-time integrator, including the degradation of convergence rates in the case of mass lumping. Comparing the values of the errors on similar grids, it becomes apparent that the magnitude of the errors are on the order of 120-130% of the errors for the discontinuous in time algorithm, most likely because of the reduced accuracy in time. Most importantly, we observed that we had to reduce the values of the constant $c_\tau = 0.1$ to obtain similar solutions. This is expected though, since the stabilization parameter c_τ may in principle need to be adjusted to the specific time integrator adopted.

Note also that the continuous-in-time algorithm showed even less sensitivity to changes in the value of the stabilization parameter c_τ and penalty parameter $\tilde{\alpha}$, as all the error curves in Figure 6 and 7 are virtually overlapped. For

(a) Velocity. On the left, $\tilde{\alpha} = 0.0$, and on the right, $\tilde{\alpha} = 0.2$.

(b) Pressure. On the left, $\tilde{\alpha} = 0.0$, and on the right, $\tilde{\alpha} = 0.2$.

Figure 5: Explicit discontinuous-in-time space-time algorithm with *consistent* mass matrices: Convergence rates for the acoustic pulse propagation problem. Results for various choices of the stabilization parameter c_τ and Nitsche penalty $\tilde{\alpha}$ are presented. The norm $l^1([0, T] : (L^2(\Omega)))$ of the solution shows second order convergence. Compare with Figure 4, where the same problem is solved using lumped mass matrices.

this reason, Table 6 presents a comparison of the continuous-in-time algorithm run only for $c_\tau = 0.1$, with and without mass lumping, and with $\tilde{\alpha} = 0.05$. In the same table and figure, a comparison with the case of strong imposition of boundary conditions is also presented. We observed that the proposed weak enforcement of boundary conditions leads to results virtually indistinguishable from the strong boundary condition case.

6.2. Robustness tests

In this section, a number of tests with progressively more complex geometries is presented, to demonstrate the performance of the method in terms of robustness. All computations were conducted with mass lumping, $c_\tau = 0.4$ and $\tilde{\alpha} = 0.05$.

6.2.1. Waveguide with a curved obstruction

The same initial conditions and boundary conditions found in the previous verification example are used, but with a domain of more complex geometry, in which a curved boundary with a reentrant corner is present. The domain is obtained as a modification of the rectangle $[0, 1] \times [0, 10]$, in which a portion of elliptic shape is carved out.

The appearance of the corner implies that the enforcement of a zero velocity at the corner location, a constraint that will only be enforced weakly. This feature makes this test very valuable as a very first step in the assessment of algorithmic robustness.

Table 4: Error convergence error in the norm $l^1([0, T] : (L^2(\Omega)))$ error, for the discontinuous-in-time explicit integrator with *lumped* mass matrix.

Nodes	c_τ	$\tilde{\alpha} = 0.0$				$\tilde{\alpha} = 0.2$			
		pressure	rate	velocity	rate	pressure	rate	velocity	rate
255	0.05	9.85e-3		1.03e-2		9.74e-3		1.01e-2	
	0.1	9.60e-3		9.98e-3		9.52e-3		9.87e-3	
	0.2	9.22e-3		9.57e-3		9.16e-3		9.49e-3	
	0.4	8.63e-3		8.97e-3		8.57e-3		8.89e-3	
909	0.05	2.90e-3	1.92	3.14e-3	1.86	2.90e-3	1.91	3.13e-3	1.84
	0.1	2.84e-3	1.92	3.07e-3	1.85	2.84e-3	1.90	3.07e-3	1.84
	0.2	2.73e-3	1.92	2.96e-3	1.85	2.73e-3	1.91	2.96e-3	1.83
	0.4	2.54e-3	1.93	2.77e-3	1.85	2.53e-3	1.92	2.77e-3	1.84
3417	0.05	8.29e-4	1.89	9.38e-4	1.83	8.29e-4	1.89	9.37e-4	1.82
	0.1	8.12e-4	1.89	9.20e-4	1.82	8.12e-4	1.89	9.19e-4	1.82
	0.2	7.81e-4	1.89	8.89e-4	1.82	7.80e-4	1.89	8.88e-4	1.82
	0.4	7.23e-4	1.90	8.33e-4	1.81	7.22e-4	1.90	8.33e-4	1.81
13233	0.05	2.40e-4	1.83	2.90e-4	1.74	2.40e-4	1.83	2.89e-4	1.74
	0.1	2.35e-4	1.83	2.84e-4	1.74	2.35e-4	1.83	2.84e-4	1.74
	0.2	2.26e-4	1.83	2.75e-4	1.73	2.26e-4	1.83	2.75e-4	1.73
	0.4	2.08e-4	1.84	2.58e-4	1.73	2.08e-4	1.84	2.58e-4	1.73
52065	0.05	7.08e-5	1.78	9.32e-5	1.66	7.08e-5	1.78	9.31e-5	1.66
	0.1	6.94e-5	1.78	9.15e-5	1.65	6.93e-5	1.78	9.14e-5	1.65
	0.2	6.66e-5	1.78	8.87e-5	1.65	6.66e-5	1.78	8.86e-5	1.65
	0.4	6.13e-5	1.79	8.37e-5	1.64	6.13e-5	1.79	8.37e-5	1.64

In Figure 9, the contour and elevation plots of the pressure solution for the discontinuous-in-time explicit algorithm are presented at three specific instants in time. The smooth contour lines show that no spurious oscillations are present in the pressure.

For comparison, we also repeated the simulation using the continuous-in-time explicit algorithm, and the corresponding results are illustrated in the Figure 8. Also in this case the solution does not show any oscillations.

Similar considerations hold for the velocity, depicted in the left plot of Figure 10 for the discontinuous-in-time algorithm. Note that because the normal component of the velocity is enforced weakly at the boundary, there is no guarantee it will vanish exactly, as evident near the lower corner of the domain. However, the velocity at the corner is small, and will converge to the exact solution as the grid is refined. The right plot of Figure 10 is showing the solution with the strong enforcement of the normal velocity boundary conditions, for reference.

6.2.2. Two-dimensional problem in complex geometry

This second robustness test involves the interaction of an initially radially symmetric wave in a two-dimensional domain of complex geometry. The domain consists of the rectangle $[0, 10] \times [0, 6.5]$, with a number of embedded rectangular and elliptic cavities, at different angles with respect to the coordinate axes. Computations were performed on a grid of 9628 elements and 5126 nodes. This test is clearly of increased geometric complexity with respect to the previous, and, in addition, the sharp corners present may induce singularities in the velocity profile.

In fact, when corners have outer angles larger than 180° , the velocity becomes singular (it is easy to convince ourselves that this must be true by just considering a smoothed corner, and taking the limit of a vanishing radius of curvature).

Table 5: Error convergence error in the norm $l^1([0, T] : (L^2(\Omega)))$ error, for the discontinuous-in-time explicit integrator with *consistent* mass matrix.

Nodes	c_τ	$\tilde{\alpha} = 0.0$				$\tilde{\alpha} = 0.2$			
		pressure	rate	velocity	rate	pressure	rate	velocity	rate
255	0.05	1.23e-3		1.25e-3		1.22e-3		1.24e-3	
	0.1	1.31e-3		1.34e-3		1.31e-3		1.34e-3	
	0.2	1.55e-3		1.60e-3		1.54e-3		1.59e-3	
909	0.05	2.95e-4	2.24	2.98e-4	2.25	2.95e-4	2.24	2.98e-4	2.24
	0.1	3.15e-4	2.25	3.21e-4	2.25	3.15e-4	2.24	3.20e-4	2.25
	0.2	3.72e-4	2.24	3.85e-4	2.24	3.72e-4	2.24	3.84e-4	2.23
3417	0.05	7.22e-5	2.13	7.29e-5	2.13	7.21e-5	2.13	7.28e-5	2.13
	0.1	7.70e-5	2.13	7.87e-5	2.12	7.70e-5	2.13	7.86e-5	2.12
	0.2	9.21e-5	2.11	9.59e-5	2.10	9.20e-5	2.11	9.58e-5	2.10
13233	0.05	1.78e-5	2.07	1.81e-5	2.06	1.78e-5	2.07	1.81e-5	2.06
	0.1	1.92e-5	2.05	1.98e-5	2.04	1.92e-5	2.05	1.98e-5	2.04
	0.2	2.36e-5	2.01	2.49e-5	1.99	2.36e-5	2.01	2.50e-5	1.99
52065	0.05	4.42e-6	2.03	4.51e-6	2.03	4.42e-6	2.03	4.51e-6	2.03
	0.1	4.87e-6	2.00	5.08e-6	1.99	4.87e-6	2.00	5.08e-6	1.99
	0.2	6.28e-6	1.94	6.68e-6	1.92	6.28e-6	1.94	6.71e-6	1.92

Table 6: Convergence in norm $l^1([0, T] : (L^2(\Omega)))$ error, mid-point time integrator with lumped mass matrix, consistent mass matrix and weak and strong boundary conditions.

	Nodes	Weak boundary conditions								Strong boundary conditions			
		Lumped				Consistent				Consistent			
		pressure	rate	velocity	rate	pressure	rate	velocity	rate	pressure	rate	velocity	rate
T=2	255	3.0e-3		3.1e-3		6.5e-4		6.7e-4		6.6e-4		6.7e-4	
	909	8.8e-4	1.86	1.0e-3	1.75	1.7e-4	2.15	1.7e-4	2.16	1.7e-4	2.16	1.7e-4	2.16
	3417	2.6e-4	1.85	3.2e-4	1.74	4.3e-5	2.05	4.4e-5	2.05	4.3e-5	2.06	4.4e-5	2.05
	13233	7.6e-5	1.81	1.1e-4	1.66	1.1e-5	1.99	1.1e-5	1.98	1.1e-5	1.99	1.1e-5	1.98
	52065	2.3e-5	1.77	3.6e-5	1.58	2.9e-6	1.94	3.0e-6	1.93	2.9e-6	1.94	3.0e-6	1.94
T=4	255	1.1e-2		1.1e-2		1.6e-3		1.6e-3		1.6e-3		1.6e-3	
	909	3.3e-3	1.85	3.5e-3	1.80	4.1e-4	2.16	4.2e-4	2.17	4.1e-4	2.16	4.2e-4	2.16
	3417	9.5e-4	1.86	1.0e-3	1.81	1.1e-4	2.03	1.1e-4	2.02	1.1e-4	2.03	1.1e-4	2.03
	13233	2.8e-4	1.82	3.2e-4	1.75	2.9e-5	1.94	2.9e-5	1.94	2.9e-5	1.94	2.9e-5	1.94
	52065	8.2e-5	1.78	1.0e-4	1.68	7.9e-6	1.88	8.1e-6	1.88	7.9e-6	1.88	8.1e-6	1.88

(a) Velocity. $\tilde{\alpha} = 0.0$ (left) and $\tilde{\alpha} = 0.2$ (right).

(b) Pressure. $\tilde{\alpha} = 0.0$ (left) and $\tilde{\alpha} = 0.2$ (right).

Figure 6: Continuous-in-time space time algorithm: Convergence rate for the verification test problem using *lumped* mass matrix (W and S represent weakly and strongly enforced boundary conditions respectively). Results for various choices of the stabilization parameter c_τ , Nitsche penalty $\tilde{\alpha}$ and the enforcement of boundary conditions are presented, for the norm $l^1([0, T] : (L^2(\Omega)))$ of the solution. Compare with Figure 7, where the same problem is solved using consistent mass matrices. Compare also with Figure 4, where the discontinuous-in-time approach is used in combination with mass lumping.

(a) Velocity. $\tilde{\alpha} = 0.0$ (left) and $\tilde{\alpha} = 0.2$ (right).

(b) Pressure. $\tilde{\alpha} = 0.0$ (left) and $\tilde{\alpha} = 0.2$ (right).

Figure 7: Continuous-in-time space time algorithm: Convergence rate for the verification test problem using consistent mass matrix (W and S represent weakly and strongly enforced boundary conditions respectively). Results for various choices of the stabilization parameter c_τ , Nitsche penalty $\tilde{\alpha}$ and the enforcement of boundary conditions are presented, for the norm $l^1([0, T] : (L^2(\Omega)))$ of the solution. Compare with Figure 6, where the same problem is solved using lumped mass matrices. Compare also with Figure 5, where the discontinuous-in-time approach is used in combination with consistent mass matrices.

(a) $t = 0$, pressure elevation plot.

(b) $t = 0$: Pressure contour lines.

(c) $t = 2.7$, pressure elevation plot.

(d) $t = 2.7$, pressure elevation plot.

(e) $t = 7.4$, pressure elevation plot.

(f) $t = 7.4$, pressure elevation plot.

Figure 8: Discontinuous-in-time explicit integrator: Pressure elevation plots and pressure contours for the propagation of an acoustics wave in a waveguide with a curved obstruction

(a) $t = 0$, pressure elevation plot.

(b) $t = 0$: Pressure contour lines.

(c) $t = 2.7$, pressure elevation plot.

(d) $t = 2.7$, pressure elevation plot.

(e) $t = 7.4$, pressure elevation plot.

(f) $t = 7.4$, pressure elevation plot.

Figure 9: Continuous-in-time space time algorithm: Pressure elevation plots and pressure contours for the propagation of an acoustics wave in a waveguide with a curved obstruction

The initial condition is given by the fields:

$$\mathbf{v}_0 = \omega \frac{\mathbf{r}}{r}, \quad (148)$$

$$p_0 = 1 + \omega, \quad (149)$$

where

$$\omega(x) = \begin{cases} \frac{1}{10} \sin^2\left(\pi \frac{r}{r_0}\right), & r < r_0, \\ 0, & \text{otherwise}, \end{cases} \quad (150)$$

Figure 10: Velocity vector plot for the propagation of an acoustic wave in a waveguide with a curved obstruction. On the left, the discontinuous-in-time explicit integrator with weak boundary conditions, and on the right, for reference, the continuous-in-time explicit integrator with strong boundary conditions.

with $\mathbf{r}$ the radial vector of magnitude $r = \|\mathbf{r}\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, where $\{x_0, y_0\} = \{10, 0\}$ is the lower right corner. A zero normal velocity boundary condition is imposed on all boundaries.

It is easy to observe that also in this test case, the solution is free from spurious oscillations throughout the many reverberations of the wave against the various boundaries, as shown in the pressure plots of Figures 11 and 12. In particular, the results in Figure 11 were obtained using mass lumping, while the results in Figure 12 were obtained with consistent mass matrices. Comparing these results, slightly larger values of dissipation and dispersion errors for the mass lumping case are observable, in the form of lower values of the solution peaks and some spurious reverberations. These are however milder discrepancies, and the results with and without mass lumping show fairly good agreement.

6.2.3. Three-dimensional test in complex geometry

The last test we present involves the propagation of an acoustic pulse in a three dimensional geometry. The geometry of the problem with the grid superposed is presented in Figure 13, and consists of a cylindrical crown with inner diameter of 0.0088 and outer diameter of 0.0128. The length of the cylindrical crown is 0.0137, and two sets of six cylindrical holes each of diameter 0.0028 are carved out from it. In particular, the angular distance between the hole axes is 60 degrees, and the two sets of wholes are staggered by an angle of 30 degrees. Three grids have been considered, namely, the R_0 grid, with 1,608 nodes and 5,864 tetrahedra, the R_1 grid, with 10,678 nodes and 49,021 tetrahedra, and the R_2 grid, with 70,498 nodes and 365,228 tetrahedra.

The initial solution utilized for this series of tests is the same as the pulse given in section 6.1, but with different parameters. Specifically, we choose now the parameters $x_0 = 0.00162$ and $\lambda = -0.016119$.

In Figure 14, the pressure component of the solution of the discontinuous-in-time explicit algorithm is presented for various levels of grid resolution. It is noticeable the convergence of the solution and the absence of spurious oscillations.

For the R_2 grid, the pressure and x -component of the velocity are presented in Figure 15, while iso-surfaces of the same quantities are presented in Figure 16. These more detailed plots confirm the absence of oscillations in the numerical solution, in spite the complex geometry chosen presents many edges and corners.

(a) $t = 0$, pressure contours (left) and iso-surfaces (right).

(b) $t = 3.2$, pressure contours (left) and iso-surfaces (right).

(c) $t = 7.1$, pressure contours (left) and iso-surfaces (right).

Figure 11: Pressure contours for the propagation of an acoustics wave in a complex two-dimensional domain: Discontinuous-in-time explicit algorithm using lumped mass matrix.

(a) $t = 0$, pressure contours (left) and iso-surfaces (right).

(b) $t = 3.20312$, pressure contours (left) and iso-surfaces (right).

(c) $t = 7.08077$, pressure contours (left) and iso-surfaces (right).

Figure 12: Pressure contours for the propagation of an acoustics wave in a complex two-dimensional domain: Discontinuous-in-time explicit algorithm using consistent mass matrix.

(a) R_0 mesh: 1,608 nodes and 5,864 tetrahedra.

(b) R_1 mesh: 10,678 nodes and 49,021 tetrahedra.

(c) R_2 mesh: 70,498 nodes and 365,228 tetrahedra.

Figure 13: A series of progressively and hierarchically refined meshes

Figures 17 and 18 show instead a comparison of the discontinuous-in-time an continuous-in-time explicit time integrators. The solutions with both integrators match very closely, and also the continuous-in-time algorithm does not show any signs of spurious numerical oscillations.

7. Summary

We have presented a Nitsche-type approach to the enforcement of boundary conditions in the context of wave-propagation problems. We implemented the proposed method using a space-time formulation, for which a stability and convergence estimates can be been established. We also introduced two families of explicit time integrators derived from the space-time framework, and we considered a variety of tests, of progressively more complex geometry. Both families of time integrators confirmed the performance of the approach, in terms of robustness, accuracy, and overall quality.

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(a) R_0 mesh, $t = 0.0120739$ (left) and $t = 0.0268111$ (right).

(b) R_1 mesh, $t = 0.0120739$ (left) and $t = 0.0268111$ (right).

(c) R_2 mesh, $t = 0.0120739$ (left) and $t = 0.0268111$ (right).

Figure 14: Discontinuous-in-time explicit integrator: Pressure solution at two different instants of the simulation of a propagating acoustic pulse in a complex geometrical shape.

(a) $t = 0.00514915$, pressure (left) and x -component of the velocity (right).

(b) $t = 0.0210613$, pressure (left) and x -component of the velocity (right).

(c) $t = 0.0337435$, pressure (left) and x -component of the velocity (right).

Figure 15: Discontinuous-in-time explicit algorithm: Pressure and velocity contours at three instants in time for a propagating acoustic pulse in a complex geometrical shape. Solutions are computed on the R_2 mesh.

(a) $t = 0.00514915$: Pressure contours (left) and x -component of the velocity iso-surfaces (right).

(b) $t = 0.0210613$: Pressure contours (left) and x -component of the velocity iso-surfaces (right).

(c) $t = 0.0337435$: Pressure contours (left) and x -component of the velocity iso-surfaces (right).

Figure 16: Discontinuous-in-time explicit algorithm: Pressure and velocity iso-surfaces at three instants in time for a propagating acoustic pulse in a complex geometrical shape. Solutions are computed on the R_2 mesh.

(a) Discontinuous-in-time explicit integrator: Pressure contours (left) and pressure iso-surfaces (right).

(b) Continuous-in-time space time algorithm: Pressure contours (left) and pressure iso-surfaces (right).

Figure 17: Comparison of the discontinuous-in-time and the continuous-in-time explicit algorithms on a three-dimensional complex domain. Pressure contours and iso-surfaces at $t = 0.00514915$ for the R_2 grid.

(a) Discontinuous-in-time explicit integrator: x -component of the velocity contours (left) and iso-surfaces (right).

(b) Continuous-in-time space time algorithm: x -component of the velocity contours (left) and iso-surfaces (right).

Figure 18: Comparison of the discontinuous-in-time and the continuous-in-time explicit algorithms on a three-dimensional complex domain (R_2 grid.) x -component of the velocity contours and iso-surfaces at $t = 0.00514915$.

Appendix

A. Acoustic wave propagation equations as a non-dissipative Friedrichs' system

In what follows, we will consider the case of space/time homogenous boundary conditions, without loss of generality, as it is always possible to transform the non-homogenous problem, into a homogenous one with a modified source term F . Equations (51a) and (51b) can be formally written as a Friedrichs' system, provided a matrix $K \in \mathbb{R}^{(n_d+1) \times (n_d+1)}$:

$$\sum_{i=0}^{n_d} A_i \partial_{x_i} Y + KY = F, \quad \text{in } Q, \quad (151a)$$

$$(M - D)Y = 0, \quad \text{on } \partial Q, \quad (151b)$$

where A_i , K , M are real-valued $(n_d + 1) \times (n_d + 1)$ matrices and Y is an $(n_d + 1)$ -dimensional vector. (In our case $K = 0$.) The system of equations (151a) and (151b) is a Friedrichs' system if and only if the following properties hold:

- 1) $A_i(x)$ is symmetric, $i = 0, \dots, n_d$
- 2) $M + M^* \geq 0$, on ∂Q
- 3) $K + K^* - \sum_{i=0}^{n_d} \frac{\partial A_i}{\partial x_i} \geq \bar{\sigma} I$, in Q
- 4) $\text{Ker}(D - M) + \text{Ker}(D + M) = \mathbb{R}^{(n_d+1)}$, on ∂Q

where $\bar{\sigma}$ is a positive constant, I is the identity matrix, and for matrices E and F we write $E \geq F$ to mean that $E - F$ is positive semi-definite and $\text{Ker} E = \{W \in \mathbb{R}^{(n_d+1)} : EW = 0\}$. The system (151a)–(151b) satisfies the conditions of a Friedrichs system, with the exception of the third condition, which holds in a weaker sense, with $\bar{\sigma} = 0$.

- 1) A_i , $i = 0, \dots, n_d$, are symmetric and A_0 is symmetric positive definite.

$$3) \quad K + K^* - \sum_{i=0}^{n_d} \frac{\partial A_i}{\partial x_i} = 0 \geq \bar{\sigma} I \quad \text{for } \bar{\sigma} = 0.$$

Conditions 2) and 4) need to be verified for every portion of ∂Q :

A) Initial conditions $p = 0$ and $v = 0$ are imposed on $\Omega \times \{0\}$. Note that for $t = 0$,

$$D = -A_0 = - \begin{bmatrix} \rho I_{n_d \times n_d} & 0_{n_d \times 1} \\ 0_{1 \times n_d} & \chi \end{bmatrix}. \quad (152)$$

Choose $M|_{\Omega \times \{0\}} = -D = A_0$. Hence,

$$(M|_{\Omega \times \{0\}} - D)Y_0 = 0 \iff 2A_0Y_0 = 0 \iff \begin{cases} p = 0, \\ v = 0, \end{cases}$$

which yields the following results:

$$2) \quad M|_{\Omega \times \{0\}} + (M|_{\Omega \times \{0\}})^* = 2A_0 \geq 0.$$

- 4) Also, $\text{Ker}(D - M|_{\Omega \times \{0\}}) = \{0\}$ and $\text{Ker}(D + M|_{\Omega \times \{0\}}) = \text{Ker}\{0_{(n_d+1) \times (n_d+1)}\} = \mathbb{R}^{(n_d+1)}$, where $0_{(n_d+1) \times (n_d+1)}$ is a matrix of size $(n_d + 1) \times (n_d + 1)$ with zero entries. Hence, $\text{Ker}(D - M|_{\Omega \times \{0\}}) + \text{Ker}(D + M|_{\Omega \times \{0\}}) = \mathbb{R}^{(n_d+1)}$.

B) No boundary conditions are imposed on $\Omega \times \{T\}$. Since, for $t = T$,

$$\mathbf{D} = \mathbf{A}_0 = \begin{bmatrix} \rho \mathbf{I}_{n_d \times n_d} & \mathbf{0}_{n_d \times 1} \\ \mathbf{0}_{1 \times n_d} & \chi \end{bmatrix}, \quad (153)$$

then, choosing $\mathbf{M}|_{\Omega \times \{T\}} = \mathbf{D} = \mathbf{A}_0$ allows to avoid imposing any restriction on the boundary $\Omega(T)$. We can also conclude:

$$2) \mathbf{M}|_{\Omega \times \{T\}} + (\mathbf{M}|_{\Omega \times \{T\}})^* = 2\mathbf{A}_0 \geq 0.$$

$$4) \text{ Similarly to the previous case, } \text{Ker}(\mathbf{D} - \mathbf{M}|_{\Omega \times \{T\}}) = \text{Ker}\{\mathbf{0}_{(n_d+1) \times (n_d+1)}\} = \mathbb{R}^{(n_d+1)} \text{ and we also have } \text{Ker}(\mathbf{D} + \mathbf{M}|_{\Omega \times \{T\}}) = \{0\}, \text{ from which } \text{Ker}(\mathbf{D} - \mathbf{M}|_{\Omega \times \{T\}}) + \text{Ker}(\mathbf{D} + \mathbf{M}|_{\Omega \times \{T\}}) = \mathbb{R}^{(n_d+1)}.$$

C) Homogenous Dirichlet boundary conditions: $p = 0$ on $\mathbb{P}^g = \Gamma^g \times]0, T[$. On $\mathbb{P}^g$, we have that

$$\mathbf{D} = \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \mathbf{n} \\ \mathbf{n}^T & 0 \end{bmatrix}. \quad (154)$$

Let's choose

$$\mathbf{M}|_{\mathbb{P}^g} = \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & -\mathbf{n} \\ \mathbf{n}^T & 0 \end{bmatrix}, \quad (155)$$

which is skew symmetric. Hence,

$$(\mathbf{M}|_{\mathbb{P}^g} - \mathbf{D})\mathbf{Y} = \mathbf{0} \iff \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & -2\mathbf{n} \\ \mathbf{0}_{1 \times n_d} & 0 \end{bmatrix} \mathbf{Y} = \mathbf{0} \iff p = 0.$$

We can now verify the remaining conditions:

$$2) \mathbf{M}|_{\mathbb{P}^g} + (\mathbf{M}|_{\mathbb{P}^g})^* = \mathbf{0}_{(n_d+1) \times (n_d+1)} \geq 0.$$

$$4) \text{ The } \text{Ker}(\mathbf{D} - \mathbf{M}|_{\mathbb{P}^g}) \text{ can be obtained by solving the system}$$

$$\begin{aligned} (\mathbf{D} - \mathbf{M}|_{\mathbb{P}^g})\mathbf{W} = \mathbf{0} &\implies \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & -2\mathbf{n} \\ \mathbf{0}_{1 \times n_d} & 0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \vdots \\ W_N \end{Bmatrix} = \mathbf{0} \\ &\implies \mathbf{n}W_4 = 0 \implies W_4 = 0 \text{ since } \|\mathbf{n}\| = 1. \end{aligned}$$

This conditions is only on the last component of the vector $\mathbf{W}$, while the first three component of $\mathbf{W}$ are free. Then, $\text{Ker}(\mathbf{D} + \mathbf{M}|_{\mathbb{P}^g})$ is obtained by solving

$$\begin{aligned} (\mathbf{D} + \mathbf{M}|_{\mathbb{P}^g})\mathbf{W} = \mathbf{0} &\implies \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \mathbf{0}_{n_d \times 1} \\ 2\mathbf{n}^T & 0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \vdots \\ W_N \end{Bmatrix} = \mathbf{0} \\ &\implies \mathbf{n}^T \begin{Bmatrix} W_1 \\ \vdots \\ W_{n_d} \end{Bmatrix} = 0. \end{aligned}$$

Hence, $\text{Ker}(\mathbf{D} + \mathbf{M}|_{\mathbb{P}^g})$ is represented by vector functions $\{\boldsymbol{\psi}^T, \phi\}^T$ such that $\boldsymbol{\psi} \cdot \mathbf{n} = 0$, with no constraints on ϕ , the last component of $\mathbf{W}$. It is then straightforward to conclude

$$\text{Ker}(\mathbf{D} - \mathbf{M}|_{\mathbb{P}^g}) + \text{Ker}(\mathbf{D} + \mathbf{M}|_{\mathbb{P}^g}) = \mathbb{R}^N.$$

D) Homogenous Neumann boundary conditions, $\mathbf{v} \cdot \mathbf{n} = 0$, on $\mathbf{P}^h = \Gamma^h \times]0, T[$. On $\mathbf{P}^h$,

$$\mathbf{D} = \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \mathbf{n} \\ \mathbf{n}^T & 0 \end{bmatrix}, \quad (156)$$

and if we choose

$$\mathbf{M}|_{\mathbf{P}^h} = \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \mathbf{n} \\ -\mathbf{n}^T & 0 \end{bmatrix}, \quad (157)$$

we have

$$(\mathbf{M}|_{\mathbf{P}^h} - \mathbf{D})\mathbf{Y} = \mathbf{0} \iff \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \mathbf{0}_{n_d \times 1} \\ -2\mathbf{n}^T & 0 \end{bmatrix} \mathbf{Y} = \mathbf{0} \iff \mathbf{v} \cdot \mathbf{n} = 0.$$

We then proceed to verify:

$$2) \mathbf{M}|_{\mathbf{P}^h} + (\mathbf{M}|_{\mathbf{P}^h})^* = \mathbf{0}_{(n_d+1) \times (n_d+1)} \geq 0.$$

4) Since $\mathbf{M}|_{\mathbf{P}^h} = -\mathbf{M}|_{\mathbf{P}^g}$ and $\mathbf{M}|_{\mathbf{P}^h} = (\mathbf{M}|_{\mathbf{P}^g})^*$, the situation is somewhat specular to the case of the homogenous Dirichlet conditions. $\text{Ker}(\mathbf{D} - \mathbf{M}|_{\mathbf{P}^h})$ is the set of vectors such that:

$$\begin{aligned} (\mathbf{D} - \mathbf{M}|_{\mathbf{P}^h})\mathbf{W} = \mathbf{0} &\implies \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & \mathbf{0}_{n_d \times 1} \\ 2\mathbf{n}^T & 0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \vdots \\ W_N \end{Bmatrix} = \mathbf{0} \\ &\implies \mathbf{n}^T \begin{Bmatrix} W_1 \\ \vdots \\ W_{n_d} \end{Bmatrix} = 0. \end{aligned}$$

Hence, $\text{Ker}(\mathbf{D} + \mathbf{M}|_{\mathbf{P}^h})$ is represented by vector functions $\{\boldsymbol{\psi}^T, \phi\}^T$ such that $\boldsymbol{\psi} \cdot \mathbf{n} = 0$, with no constraints on ϕ , the last component of $\mathbf{W}$. Similarly, $\text{Ker}(\mathbf{D} - \mathbf{M}|_{\mathbf{P}^h})$, is obtained by solving the system

$$\begin{aligned} (\mathbf{D} + \mathbf{M}|_{\mathbf{P}^h})\mathbf{W} = \mathbf{0} &\implies \begin{bmatrix} \mathbf{0}_{n_d \times n_d} & -2\mathbf{n} \\ \mathbf{0}_{1 \times n_d} & 0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \vdots \\ W_N \end{Bmatrix} = \mathbf{0} \\ &\implies \mathbf{n}W_4 = 0 \implies W_4 = 0 \text{ since } \|\mathbf{n}\| = 1, \end{aligned}$$

which leaves the first three components of $\mathbf{W}$ unconstrained. These spaces are the same as in the case of homogenous Dirichlet boundary conditions, and consequently,

$$\text{Ker}(\mathbf{D} - \mathbf{M}|_{\mathbf{P}^h}) + \text{Ker}(\mathbf{D} + \mathbf{M}|_{\mathbf{P}^h}) = \mathbb{R}^N.$$

In conclusion, the mixed form of the wave propagation equations falls under the broader class of *non-dissipative* Friedrichs systems. A Friedrichs system (in the strict sense, i.e., with $\mathbf{K} \neq \mathbf{0}$) is guaranteed the existence and uniqueness of the solution, under basic regularity assumptions on the forcing term and initial and boundary conditions. as indicated in the following theorem, which also defines the functional setting used in what follows:

Theorem (L^2 -well posedness, Rauch [62]). *Let (51a)–(51b) satisfy conditions (1)–(4) above. Then for any $\mathbf{F} \in L^1([0, T] : (L^2(\Omega))^{n_d+1})$ and $\mathbf{Y}|_{t=0} \in (L^2(\Omega))^{n_d+1}$, there is a unique $\mathbf{Y} \in (L^2(\mathbf{Q}))^{n_d+1}$ solution of (51a)–(51b). Furthermore, $\mathbf{Y} \in C([0, T] : (L^2(\Omega))^{n_d+1})$, and the following estimate hold:*

$$\sup_{0 \leq t \leq T} \|\mathbf{Y}(t)\|_{(L^2(\Omega))^{n_d+1}} \leq c \|\mathbf{F}\|_{L^1([0, T]; (L^2(\Omega))^{n_d+1})} + \|\mathbf{Y}(0)\|_{(L^2(\Omega))^{n_d+1}} \quad (158)$$

Remark. With minor adjustments (see, e.g., [36, 43]) the previous derivations and conclusions can be generalized to matrices $A(x)$ and $K(x)$, which are smooth functions of x .

Unfortunately, the wave propagation equations considered here do not follow precisely in the most general framework of Friedrichs' systems. In fact, condition 3 above holds only if $\bar{\sigma} = 0$, which makes the Friedrichs' system non-dissipative.

Remark. One important point of note, is that if we perform a change of variables using a time exponential map, namely, if we define $Y = Ve^{\beta t}$, we obtain:

$$\mathcal{L}_t Y = e^{\beta t} A_0 \partial_t V + e^{\beta t} A_i \partial_{x_i} V + \beta e^{\beta t} A_0 V = F \implies \mathcal{L}_t V + \beta A_0 V = e^{-\beta t} F = \tilde{F}. \quad (159)$$

In this new set of variables, $K = \beta A_0 > 0$ and the system, according to the Friedrichs' notation, acquires a dissipative nature. A small value of β can guarantee a very small discrepancy from the non-dissipative case. This approach was suggested by Johnson Navert and Piktartanta [43] to reconnect the non-dissipative wave propagation system to the context of Friedrichs systems. In practice, this means to reduce the time horizon to β , in order to have existence and uniqueness of the solution. In addition, we would obtain a much stronger stability statement than (84), in the following graph norm

$$B_{\text{STAB}}(V, V) = \|V\|_{\text{STAB}}^2 + \beta \|A_0^{1/2} V\|_{0,Q}^2. \quad (160)$$

Analogously, the estimate (100) would involve the graph norm (160).

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