B-spline based finite-element method for the stationary
quasi-geostrophic equations of the ocean

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Abstract

We present a B-spline based conforming finite-element method of the streamfunction formulation of the stationary one-layer stationary quasi-geostrophic equations for the study of the large scale wind-driven ocean circulation. The method encompasses standard simplifications of these equations, in particular the linear Stommel and Stommel–Munk models. A variational form of the method is developed and its consistency is established. In this formulation Dirichlet boundary conditions are enforced only weakly and stabilization is achieved via Nitsche’s method. Stability parameters are evaluated by solving a local generalized eigenvalue problem on the Dirichlet boundary and by monitoring the condition number of the resulting linear systems. Comparisons of the results from our simulations with previously published results and convergence studies lead us to conclude that our finite-element discretization is accurate.

Keywords: Conforming finite-element method, Fourth-order partial-differential equation, Geophysical fluid dynamics

1. Introduction

The large scale ocean currents are very important in understanding the dynamics of the global climate system (Dijkstra [1], Ghil et al. [2]). They are mainly driven by wind and buoyancy effects. Winds drive two recirculating gyres, the subtropical gyre and the subpolar gyre. Examples include the Gulf stream in the north Atlantic ocean, the Kuroshio current in the north Pacific region, and their counterparts in the southern hemisphere. These gyres display some of common features such as strong western boundary currents, weak eastern boundary currents, and weak interior flows.

A popular mathematical model employed to understand these features of the large scale wind-driven ocean circulation is based on the (one-layer) quasi-geostrophic equations (QGE) (Vallis [3], Cushman-Roisin and Beckers [4], Majda [5], Majda and

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Wang [6], Pedlosky [7], McWilliams [8]). The QGE are often used to model the dynamics of the climate (Dijkstra [1]). As a simplified model, the QGE allow efficient computational simulations while preserving many of the essential features of the underlying large scale ocean flows. Most finite-element (FE) methods of the QGE have been developed based on the streamfunction-vorticity formulation. This is because the streamfunction-vorticity formulation is a second-order partial-differential equation (PDE). Thus, its conforming FE discretization requires easy-to-implement low-order C^0 -elements. The streamfunction-vorticity formulation has two variables, the potential vorticity and the streamfunction. Moreover, to the best of our knowledge, its available error estimates are suboptimal (Fix [9]).

In contrast to the streamfunction-vorticity formulation, the streamfunction formulation is fourth-order and, thus, its conforming FE discretization requires higher-order C^1 -elements that make implementation challenging. The advantage of the streamfunction formulation over the streamfunction-vorticity formulation is that it involves only one variable, the streamfunction. Recently, Foster et al. [10, 11] presented an FE discretization of the streamfunction formulation based on the Argyris triangular element. They showed that optimal error estimates can be derived for their formulation. The purpose of the present study is to develop a FE method using B-spline basis functions to discretize the streamfunction formulation of the stationary (one-layer) quasi-geostrophic equations (SQGE) and two related simplifications, namely the linear Stommel and Stommel–Munk models.

Being smooth basis functions, B-splines can be readily used to construct C^1 -elements for conforming FE discretizations of fourth-order PDEs. B-splines can also represent curved geometric domains accurately, as in isogeometric analysis (Hughes et al. [16]). Their use in FE discretizations for fourth-order PDEs is therefore becoming increasingly popular. Recently, B-splines have been applied to study the Cahn–Hilliard and Canham–Helfrich–Evans theories for modeling chemical and mechanical contributions to the energies of giant-unilamellar vesicles (Ma and Klug [12], Feng and Klug [13], Embar et al. [14]) and the wrinkling of stretched thin sheets (Kim et al. [15]).

Despite these advantages, several challenges remain in using FE discretizations with B-splines. Whereas they can be easily constructed over rectilinear domains, their use with arbitrarily shaped domains often involves either a mapping (as with isogeometrics) or a ‘fictitious-domain’ type of approaches (Höllig [17]). More importantly, B-splines are non-interpolatory and, thus, imposing even simple Dirichlet boundary conditions can be problematic. In the present study, we employ Nitsche’s [18] method to weakly impose Dirichlet boundary conditions for the SQGE. Baker [19] first employed Nitsche’s method to develop a nonconforming FE formulation for a fourth-order elliptic problem. Recently, Nitsche’s method has been successfully applied for weakly imposing Dirichlet boundary conditions in standard FE methods for second- and fourth-order PDEs (Embar et al. [20]) and meshfree (Fernandez-Mendez and Huerta [21]) and embedded FE methods (Hansbo and Hansbo [22], Dolbow and Harari [23]). Moreover, this method has also been used with success for FE discretizations of a second-gradient theory based on low-order C^0 -elements (Kim et al. [24], Kim and Dolbow [25], and Kim et al. [26]).

The main goal of this paper is twofold. First, we derive the variational formulations for FE discretizations of the streamfunction formulation of the SQGE and two linear simplified models based on non-interpolatory basis functions — i.e., B-splines. The weak enforcement of Dirichlet boundary conditions is achieved using Nitsche’s method, which can be interpreted of as a stabilization method for imposing constraints on surfaces. Second, we perform the coercivity analyses for the two linear simplified models to determine the stabilization parameters involved in the variational formulations. Convergence studies are performed to support these analyses for these two linear models as well as the SQGE.

The remainder of the paper is organized as follows. In Section 2, we briefly present the SQGE. In Section 3, we develop variational formulations based on Nitsche’s method for the SQGE as well as linear Stommel and Stommel–Munk models, along with the demonstrations of their variational consistency. In Section 4, the FE discretization with B-splines is briefly explained. In Section 5, we prove coercivity of the variational formulations for two linear models and set up an local eigenvalue problem to identify stabilization parameters. In Section 6, numerical studies are performed for these Nitsche-type formulations. Finally, in Section 6, we provide conclusions and a summary of directions for future work.

2. Streamfunction formulation of the SQGE

We consider plane domain Ω with boundary Γ . For the study of the wind-driven circulation in an enclosed and midlatitude basin, the streamfunction formulation of the one-layer SQGE (Foster et al. [10]) is given by

$$Re^{-1}\Delta^2\psi + J(\psi, \Delta\psi) - Ro^{-1}\frac{\partial\psi}{\partial x} = Ro^{-1}F \quad \text{on } \Omega, \quad (1)$$

where ψ is the velocity streamfunction, F is the forcing, and

$$J(\psi, \Delta\psi) := \frac{\partial\psi}{\partial x} \frac{\partial\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y} \frac{\partial\Delta\psi}{\partial x} \quad (2)$$

is the Jacobian operator. As usual, the parameters Re and Ro appearing in (1) denote the Reynolds and Rossby numbers, defined here by

$$Re = \frac{UL}{A} \quad \text{and} \quad Ro = \frac{U}{\beta L^2}, \quad (3)$$

respectively, where β is the coefficient multiplying the y -coordinate in the β -plane approximation (see, for example, Vallis [3, Section 2.3.2] or Cushman-Roisin and Beckers [4]), A is the eddy viscosity parametrization, U is the characteristic velocity, and L is the characteristic length scale. In the one-layer SQGE, the flow is assumed to be homogeneous in the vertical direction. Stratification effects are therefore neglected in this model.

As in the two-dimensional Navier–Stokes equations, the Reynolds number Re is the coefficient of the diffusion term $\Delta^2\psi$. Therefore, with increasing Re , the magnitude of the diffusion term becomes smaller in comparison to the magnitude of

the nonlinear convective term $J(\psi, \Delta\psi)$. The Rossby number Ro , which is absent from the two-dimensional Navier–Stokes equations, quantifies the rotation effects in the QGE and SQGE. For small Ro , which corresponds to large rotation effects, the forcing term $Ro^{-1}F$ becomes large compared with the other terms. Moreover, as Ro decreases, the effect of the term $Ro^{-1}\partial\psi/\partial x$, which can be interpreted as a convective contribution to the governing equation, becomes significant. For large scale oceanic flows, Re is large and Ro is small (i.e., small diffusion and large rotation, respectively) and, thus, the SQGE are dominated by convective terms with large forcing.

To close the model, we supplement the fourth-order partial-differential equation (1) with the boundary conditions

$$\psi = 0 \quad \text{and} \quad \nabla\psi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (4)$$

where $\mathbf{n}$ represents the unit normal vector on the boundary Γ directed outward from Ω . These boundary conditions are commonly used for the numerical studies of the two-dimensional Navier–Stokes equations (Gunzburger [27], Fairag [28], Fairag and Almulla [29]) and the SQGE (Foster et al. [10, 11]). More careful discussions of boundary conditions are provided by Vallis [3] and Cummins [30].

3. Variational formulations employing Nitsche’s method

In this section, we derive variational formulations for the B-spline FE discretization of the SQGE and the associated linear Stommel and Stommel–Munk models. When adopting classical (Lagrangian or Hermitian) polynomial interpolants for the solution space, Dirichlet boundary conditions can be imposed pointwise if the mesh is fitted to the boundary. However, due to the non-interpolatory nature of B-splines, we use Nitsche’s method [18] to develop variational formulations that weakly impose Dirichlet boundary conditions.

Unless otherwise indicated, we assume that all admissible streamfunctions and test fields belong to the Sobolev space $\mathcal{S} = H^2(\Omega)$ of functions defined on Ω that are square-integrable and have square-integrable first and second derivatives.

3.1. Stationary quasi-geostrophic equations

Multiplying (1) by a test function χ in $\mathcal{S}$, integrating the resulting relation that results from over Ω , and using the divergence theorem yields a variational formulation of the SQGE based on Nitsche’s method, which can be stated as: Find a streamfunction ψ in $\mathcal{S}$ such that

$$a(\psi, \chi) = \ell(\chi) \quad (5)$$

for all test fields χ in $\mathcal{S}$, where $a(\cdot, \cdot)$ and ℓ are defined in accord with

$$\begin{aligned}
a(\psi, \chi) = & Re^{-1} \int_{\Omega} \Delta\psi \Delta\chi \, dv + \int_{\Omega} \Delta\psi \left(\frac{\partial\psi}{\partial y} \frac{\partial\chi}{\partial x} - \frac{\partial\psi}{\partial x} \frac{\partial\chi}{\partial y} \right) \, da - Ro^{-1} \int_{\Omega} \frac{\partial\psi}{\partial x} \chi \, dv \\
& + Re^{-1} \int_{\Gamma} \chi (\nabla(\Delta\psi) \cdot \mathbf{n}) \, da + Re^{-1} \int_{\Gamma} \psi (\nabla(\Delta\chi) \cdot \mathbf{n}) \, da + \alpha_1 \int_{\Gamma} \chi \psi \, da \\
& - Re^{-1} \int_{\Gamma} (\nabla\chi \cdot \mathbf{n}) \Delta\psi \, da - Re^{-1} \int_{\Gamma} (\nabla\psi \cdot \mathbf{n}) \Delta\chi \, da \\
& + \alpha_2 \int_{\Gamma} (\nabla\chi \cdot \mathbf{n}) (\nabla\psi \cdot \mathbf{n}) \, da + \int_{\Gamma} \frac{\partial\psi}{\partial x} \Delta\psi \chi \, dx - \int_{\Gamma} \frac{\partial\psi}{\partial y} \Delta\psi \chi \, dy \quad (6)
\end{aligned}$$

and

$$\ell(\chi) = Ro^{-1} \int_{\Omega} \chi F \, dv. \quad (7)$$

While the second terms on the second and third lines of (6) are introduced to weakly impose Dirichlet boundary conditions (4) on Γ , the first terms on the second and third lines of (6) are required to impart a variational consistency. Further, the last term on the second line and the first term on the fourth line represent stabilization terms involving the parameters α_1 and α_2 . Notice that (6) reduces to the standard Galerkin formulation of the SQGE if all lines other than its first are neglected.

3.1.1. Proof of consistency

In contrast to a standard penalty method, our formulation is variationally consistent in the sense that solutions to (5) can provide solutions to the SQGE (1).

Applying integration by parts twice to the first term on the right-hand side of (6) and using the divergence theorem yields

$$\int_{\Omega} \Delta\psi \Delta\chi \, dv = \int_{\Omega} \Delta^2\psi \chi \, dv + \int_{\Gamma} (\nabla\chi \cdot \mathbf{n}) \Delta\psi \, da - \int_{\Gamma} \chi (\nabla(\Delta\psi) \cdot \mathbf{n}) \, da. \quad (8)$$

By repeatedly applying integration by parts to the second term of (6), we obtain the identity

$$\begin{aligned}
\int_{\Omega} \Delta\psi \left(\frac{\partial\psi}{\partial y} \frac{\partial\chi}{\partial x} - \frac{\partial\psi}{\partial x} \frac{\partial\chi}{\partial y} \right) \, dv = & \int_{\Omega} \left(\frac{\partial\psi}{\partial x} \frac{\partial\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y} \frac{\partial\Delta\psi}{\partial x} \right) \chi \, dv \\
& + \int_{\Gamma} \Delta\psi \frac{\partial\psi}{\partial y} \chi \, dy - \int_{\Gamma} \Delta\psi \frac{\partial\psi}{\partial x} \chi \, dx. \quad (9)
\end{aligned}$$

Substituting (8) and (9) into (5) gives

$$\begin{aligned} & \int_{\Omega} \chi \left(Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} - Ro^{-1} F \right) dv \\ & + \int_{\Gamma} (Re^{-1} \nabla(\Delta \chi) \cdot \mathbf{n} + \alpha_1 \chi) \psi da - \int_{\Gamma} (Re^{-1} \Delta \chi - \alpha_2 \nabla \chi \cdot \mathbf{n}) (\nabla \psi \cdot \mathbf{n}) da = 0, \end{aligned} \quad (10)$$

which, since the test function χ can be chosen arbitrarily on Ω and Γ , results the partial-differential equation (1) and the boundary conditions (4) of the SQGE, namely

$$\left. \begin{aligned} Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} &= Ro^{-1} F \quad \text{in } \Omega, \\ \psi &= 0, \quad \nabla \psi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \end{aligned} \right\} \quad (11)$$

This completes the proof of consistency for the Nitsche-type formulation of the SQGE.

3.1.2. Algorithmic linearization of the Nitsche-type variational formulation

The Nitsche-type variational formulation (5) of the problem is nonlinear. A full Newton–Raphson iteration method is employed to resolve the relevant system of equations. This involves recasting the problem as one involving a residual $\mathbf{R}$ with the form

$$\mathbf{R}(\psi) = a(\psi, \chi) - \ell(\chi) = \mathbf{0} \quad \forall \chi \in \mathcal{S}, \quad (12)$$

where a and ℓ are given by (6) and (7), respectively. Given solution ψ_n at the n -th iteration, the linearization of (12) yields an equation

$$\mathbf{0} = \mathbf{R}(\psi_{n+1}, \chi) \simeq \mathbf{R}(\psi_n, \chi) + \left. \frac{\partial \mathbf{R}}{\partial \psi} \right|_n \delta \psi, \quad \forall \chi \in \mathcal{S} \quad (13)$$

for the increment $\delta \psi$ of the streamfunction ψ and the increment determines the solution ψ_{n+1} at the $(n+1)$ -th iteration via

$$\psi_{n+1} = \psi_n + \delta \psi. \quad (14)$$

Additionally, the increment of $\mathbf{R}$ in (13) is given by

$$\begin{aligned} \left. \frac{\partial \mathbf{R}}{\partial \psi} \right|_n \delta \psi &= Re^{-1} \int_{\Omega} \Delta \delta \psi \Delta \chi dv + \int_{\Omega} \Delta \delta \psi \left(\frac{\partial \psi}{\partial y} \frac{\partial \chi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \chi}{\partial y} \right) da - Ro^{-1} \int_{\Omega} \frac{\partial \delta \psi}{\partial x} \chi dv \\ &+ \int_{\Gamma} \frac{\partial \delta \psi}{\partial x} \Delta \psi \chi dx + \int_{\Gamma} \frac{\partial \psi}{\partial x} \Delta \delta \psi \chi dx - \int_{\Gamma} \frac{\partial \delta \psi}{\partial y} \Delta \psi \chi dy - \int_{\Gamma} \frac{\partial \psi}{\partial y} \Delta \delta \psi \chi dy \\ &+ \int_{\Omega} \Delta \psi \left(\frac{\partial \delta \psi}{\partial y} \frac{\partial \chi}{\partial x} - \frac{\partial \delta \psi}{\partial x} \frac{\partial \chi}{\partial y} \right) da + Re^{-1} \int_{\Gamma} \chi (\nabla(\Delta \delta \psi) \cdot \mathbf{n}) da \\ &+ Re^{-1} \int_{\Gamma} \delta \psi (\nabla(\Delta \chi) \cdot \mathbf{n}) da + \alpha_1 \int_{\Gamma} \chi \delta \psi da - Re^{-1} \int_{\Gamma} (\nabla \chi \cdot \mathbf{n}) \Delta \delta \psi da \\ &- Re^{-1} \int_{\Gamma} (\nabla \delta \psi \cdot \mathbf{n}) \Delta \chi da + \alpha_2 \int_{\Gamma} (\nabla \chi \cdot \mathbf{n}) (\nabla \delta \psi \cdot \mathbf{n}) da. \end{aligned} \quad (15)$$

Granted the FE discretization introduced in (31), substituting (15) into (13) results in a linear system of equations,

$$\mathbf{K}\mathbf{d} = -\mathbf{R}, \quad (16)$$

for the increment $\mathbf{d} = (\delta\psi)^\top$ of the streamfunction ψ . In (16), the matrix $\mathbf{K}$ originates from (15) and the residual vector $\mathbf{R}$ stems from (12). Our solution strategy is to determine $\delta\psi$ from (16) and perform updates with (14) until convergence is attained.

3.2. Linear Stommel–Munk model

In this section, we provide a Nitsche-type variational formulation for the linear Stommel–Munk model (Vallis [3, eqn. (14.42)]). Similar to the SQGE, the linear Stommel–Munk model also contains the biharmonic operator $\Delta^2\psi$, the rotational term $\partial\psi/\partial x$, and the forcing term F . However, whereas the SQGE are nonlinear due to the presence of the Jacobian term $J(\psi, \Delta\psi)$, the Stommel–Munk model involves instead a Laplacian term $\Delta\psi$.

The linear Stommel–Munk model consists of the partial-differential equation

$$-\epsilon_s \Delta\psi + \epsilon_m \Delta^2\psi - \frac{\partial\psi}{\partial x} = F \quad \text{in } \Omega \quad (17)$$

and the boundary conditions (4). The parameters ϵ_s and ϵ_m are the nondimensional Stommel and Munk numbers, respectively, which are defined by

$$\epsilon_s = \frac{\gamma}{\beta L} \quad \text{and} \quad \epsilon_m = \frac{A}{\beta L^3}, \quad (18)$$

where γ is the coefficient of the linear drag (or Rayleigh friction), as might be generated by a bottom Ekman layer, and where β , L , and A retain the meanings introduced in connection with the SQGE (1).

For the linear Stommel–Munk model, the counterpart of the Nitsche-type variational formulation (5) of the SQGE is: Find a streamfunction ψ in $\mathcal{S}$ such that

$$a(\psi, \chi) = \ell(\chi) \quad (19)$$

for all test fields χ in $\mathcal{S}$, where $a(\cdot, \cdot)$ and ℓ are now defined in accord with

$$\begin{aligned} a(\psi, \chi) = & \int_{\Omega} \epsilon_s \nabla\psi \cdot \nabla\chi \, dv + \int_{\Omega} \epsilon_m \Delta\psi \Delta\chi \, dv - \int_{\Gamma} \epsilon_s (\nabla\psi \cdot \mathbf{n}) \chi \, da - \int_{\Gamma} \epsilon_s (\nabla\chi \cdot \mathbf{n}) \psi \, da \\ & + \int_{\Gamma} \epsilon_m \chi (\nabla(\Delta\psi) \cdot \mathbf{n}) \, da + \int_{\Gamma} \epsilon_m \psi (\nabla(\Delta\chi) \cdot \mathbf{n}) \, da + \alpha_1 \int_{\Gamma} \psi \chi \, da \\ & - \int_{\Gamma} \epsilon_m (\nabla\chi \cdot \mathbf{n}) \Delta\psi \, da - \int_{\Gamma} \epsilon_m (\nabla\psi \cdot \mathbf{n}) \Delta\chi \, da \\ & + \alpha_2 \int_{\Gamma} (\nabla\chi \cdot \mathbf{n}) (\nabla\psi \cdot \mathbf{n}) \, da - \int_{\Omega} \frac{\partial\psi}{\partial x} \chi \, dv \quad (20) \end{aligned}$$

and

$$\ell(\chi) = \int_{\Omega} F\chi \, dv. \quad (21)$$

As with the SQGE, α_1 and α_2 represent stabilization parameters. Whereas the last term on the first line and the second term on the second line of (20) are introduced to weakly impose the Dirichlet boundary condition $\psi = 0$ on Γ , the third term on the first line and the first term on the second line of (20) impart variational consistency. Similarly, while the last term on the third line of (20) is needed to weakly impose the boundary condition $\nabla\psi \cdot \mathbf{n} = 0$ on Γ , the first term on the third line of (20) imparts variational consistency. To establish the variational consistency of (19), we first apply integration by parts to the first term of (20) to give

$$\int_{\Omega} \nabla\psi \cdot \nabla\chi \, dv = - \int_{\Omega} \Delta\psi\chi \, dv + \int_{\Gamma} (\nabla\psi \cdot \mathbf{n})\chi \, da. \quad (22)$$

On substituting (8) and (22) into (19), we next find that

$$\begin{aligned} \int_{\Omega} \left(-\epsilon_s \Delta\psi + \epsilon_m \Delta^2\psi - \frac{\partial\psi}{\partial x} - F \right) \chi \, dv + \int_{\Gamma} (\epsilon_m (\nabla(\Delta\chi) \cdot \mathbf{n}) - \epsilon_s (\nabla\chi \cdot \mathbf{n}) + \alpha_1 \chi) \psi \, da \\ + \int_{\Gamma} (-\epsilon_m \Delta\chi + \alpha_2 \nabla\chi \cdot \mathbf{n}) (\nabla\psi \cdot \mathbf{n}) \, da = 0, \end{aligned} \quad (23)$$

which, since the test function χ can be chosen arbitrarily on Ω and Γ , (23) results in the partial-differential equations (17) and boundary conditions (4) comprising the linear Stommel–Munk model, namely

$$\left. \begin{aligned} -\epsilon_s \Delta\psi + \epsilon_m \Delta^2\psi - \frac{\partial\psi}{\partial x} &= F \quad \text{in } \Omega, \\ \psi = 0, \quad \nabla\psi \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma. \end{aligned} \right\} \quad (24)$$

This completes the proof of consistency for the Nitsche-type variational formulation of the linear Stommel–Munk model.

3.3. Linear Stommel model

In this section, we derive a Nitsche-type variational formulation for the linear Stommel model (Vallis [3, eqn. (14.22)]). In contrast to the previously considered models, this one involves a second-order PDE involving a Laplacian term, a rotational term, and a forcing term. Moreover, only the Dirichlet condition is imposed on the boundary. Specifically, the linear Stommel model is given by

$$\left. \begin{aligned} -\epsilon_s \Delta\psi - \frac{\partial\psi}{\partial x} &= F \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \Gamma, \end{aligned} \right\} \quad (25)$$

where ϵ_s is the Stommel number defined in (18)₁. Notice that the Stommel model can be scaled by dividing through by ϵ_s . In this case, this model is similar to the SQGE. Furthermore, ϵ_s can be identified with Ro^{-1} .

Since the PDE entering (25) is of second-order, we assume that all admissible solution and test fields belong to the Sobolev space $\mathcal{S} = H^1(\Omega)$ of functions defined on Ω that are square-integrable and have square-integrable first derivatives.

The Nitsche-type variational formulation of the linear Stommel model can be stated as: Find a streamfunction ψ in $\mathcal{S}$ such that

$$a(\psi, \chi) = \ell(\chi) \quad (26)$$

for all test fields χ in $\mathcal{S}$, where $a(\cdot, \cdot)$ and ℓ are defined in accord with

$$\begin{aligned} a(\psi, \chi) = & \int_{\Omega} \epsilon_s \nabla \psi \cdot \nabla \chi \, dv - \int_{\Gamma} \epsilon_s (\nabla \psi \cdot \mathbf{n}) \chi \, da \\ & - \int_{\Gamma} \epsilon_s (\nabla \chi \cdot \mathbf{n}) \psi \, da + \alpha \int_{\Gamma} \chi \psi \, da - \int_{\Omega} \frac{\partial \psi}{\partial x} \chi \, dv \end{aligned} \quad (27)$$

and

$$\ell(\chi) = \int_{\Omega} F \chi \, dv, \quad (28)$$

with α being a stabilization parameter. While the third term of (27) is required to weakly impose the Dirichlet boundary condition $\psi = 0$ on Γ , the second term of (27) is needed to impart method variational consistency. To establish the consistency of (26), notice that on substituting (22) into (27), (26) reduces to

$$\int_{\Omega} \left(-\epsilon_s \Delta \psi - \frac{\partial \psi}{\partial x} - F \right) \chi \, dv + \int_{\Gamma} (-\epsilon_s (\nabla \chi \cdot \mathbf{n}) + \alpha \chi) \psi \, da = 0, \quad (29)$$

which, since the test function χ can be chosen arbitrarily on Ω and Γ , yields the linear Stommel model (25). This completes the proof of consistency for the Nitsche-type variational formulation of the linear Stommel model.

Substituting the FE discretization introduced in (31) into (19) for the Stommel–Munk model or (26) for the Stommel model yields a linear system of equations

$$\mathbf{M}\boldsymbol{\psi} = \mathbf{F} \quad (30)$$

where $\boldsymbol{\psi} = (\psi)^\top$ represents the numerical solution vector representing the approximation of the streamfunction ψ , $\mathbf{M}$ is the matrix originating from (20) for the Stommel–Munk model and (27) for the Stommel model, and $\mathbf{F}$ is the force vector.

4. Finite-element discretization with B-splines

The SQGE and the linear Stommel–Munk model involve fourth-order PDEs. For these models, a standard FE approximation requires the solution space $\mathcal{S}$ to be

globally H^2 conforming. In other words, the basis functions are required to be C^1 -continuous, thereby ensuring the continuity of the streamfunction and its first derivatives. In the present study, this requirement is satisfied by using globally C^2 -continuous cubic B-splines. Specifically, bicubic B-splines associated with a uniform set I of grid points are used to discretize the solution space. With a Galerkin approximation, spatial discretization of the Nitsche-type formulation is carried out such that $\psi^h \in \mathcal{S}^h \subset \mathcal{S}$, where $\mathcal{S}^h$ denotes the appropriate discrete space. (Whereas $\mathcal{S}^h$ is H^1 conforming for the second-order problem, it is H^2 conforming for the fourth-order problem.) The approximation to the streamfunction ψ takes the form

$$\psi^h(\mathbf{x}) = \sum_I N_I(\mathbf{x}(\boldsymbol{\xi}))\psi_I, \quad (31)$$

where ψ_I is the nodal value at node I . Here, N_I is the I -th B-spline basis function and $\boldsymbol{\xi}$ denotes the parametric coordinates used to construct the B-spline basis. The corresponding test field χ can be approximated analogously. A unified statement of the FE discretizations of the Nitsche-type variational formulations of the SQGE, linear Stommel–Munk model, and linear Stommel model is: find an element ψ^h of $\mathcal{S}^h$ such that

$$a(\psi^h, \chi^h) = \ell(\chi^h), \quad \forall \chi^h \in \mathcal{S}^h. \quad (32)$$

where $a(\psi^h, \chi^h)$ and $\ell(\chi^h)$ represent discrete versions of the appropriate continuous operators, namely (6) and (7) for the SQGE, (20) and (21) for the linear Stommel–Munk model, and (27) and (28) for the linear Stommel model.

A brief account of the B-spline basis functions in one dimension is provided next. Extension to multiple dimensions is easily accomplished using tensor product splines. A more detailed treatment of the subject can be found in Pieggl and Tiller [31]. Consider a knot vector Ξ , where

$$\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\} \quad (33)$$

with n denoting the number of knots and p the degree of the spline. B-spline basis functions are defined recursively starting with piecewise constants (degree $p = 0$):

$$N_{I,0}(\xi) = \begin{cases} 1 & \text{if } \xi_I \leq \xi < \xi_{I+1}, \\ 0 & \text{otherwise.} \end{cases} \quad (34)$$

For $p \geq 1$,

$$N_{I,p}(\xi) = \frac{\xi - \xi_I}{\xi_{I+p} - \xi_I} N_{I,p-1}(\xi) + \frac{\xi_{I+p+1} - \xi}{\xi_{I+p+1} - \xi_{I+1}} N_{I+1,p-1}(\xi). \quad (35)$$

In Figure 1, we illustrate one- and two-dimensional cubic B-spline basis functions.

5. Analysis of coercivity

The accuracies of the variational formulations derived in Section 3 are sensitive to the chosen values of the stabilization parameters. To determine suitable choices

Figure 1: C^2 -continuous cubic B-spline basis functions resulting from a uniform not vector: (a) One-dimensional cubic B-spline basis and (b) Two-dimensional cubic B-spline basis generated using a tensor product of the one-dimensional cubic B-spline indicated with red color in (a).

for those values, we invoke a strategy based on studying conditions of coercivity that was developed by Dolbow and Harari [23] and Embar et al. [20]. While we derive conditions of coercivity for the linear Stommel and Stommel–Munk models, we relegate the problem of deriving analogous conditions for the inherently nonlinear SQGE to a future work. Since the SQGE are nonlinear, we solve the linearized system of equation (16) based on the matrix $\mathbf{K}$ originating from (15). As Embar et al. [20] observe, stability parameters are thus required to ensure coercivity of the linearized Nitsche-type formulation (15).

The FE discretizations of the variational formulations derived in Section 3 read: Find an element ψ^h of $\mathcal{S}^h$ such that

$$a(\psi^h, \chi^h) = \ell(\chi^h), \quad \forall \chi^h \in \mathcal{S}^h, \quad (36)$$

where $a(\psi^h, \chi^h)$ and $\ell(\chi^h)$ represent discrete versions of the appropriate continuous operators, namely (6) and (7) for the SQGE, (20) and (21) for the linear Stommel–Munk model, and (27) and (28) for the linear Stommel model.

For a scalar or vector field w defined on Ω , let

$$\|w\| = \left(\int_{\Omega} |w|^2 dv \right)^{1/2} \quad \text{and} \quad \|w\|_{\Gamma} = \left(\int_{\Gamma} |w|^2 da \right)^{1/2} \quad (37)$$

denote its respective L_2 norms on Ω and the boundary Γ of Ω .

5.1. Linear Stommel model

To establish the coercivity of the linear Stommel model, we begin by replacing χ^h with ψ^h to give

$$a(\psi^h, \psi^h) = \epsilon_s \|\nabla \psi^h\|^2 - 2\epsilon_s \int_{\Gamma} \psi^h (\nabla \psi^h \cdot \mathbf{n}) da + \alpha \|\psi^h\|_{\Gamma}^2 - \int_{\Omega} \psi^h \frac{\partial \psi^h}{\partial x} dv. \quad (38)$$

By Green's theorem, the final term on the right-hand side of (38) (see, also. Foster et al. [10, eq. (25)]) obeys the inequality

$$\begin{aligned}
-\int_{\Omega} \psi^h \frac{\partial \psi^h}{\partial x} dx dy &= -\frac{1}{2} \int_{\Gamma} (\psi^h)^2 dy \\
&= -\frac{1}{2} \int_{t_0}^{t_1} \left(0, (\psi^h)^2 \right) \cdot \mathbf{c}'(t) dt \\
&\geq -\frac{1}{2} \int_{t_0}^{t_1} \left| \left(0, (\psi^h)^2 \right) \right| |\mathbf{c}'(t)| dt \\
&= -\frac{1}{2} \int_{\Gamma} |\psi^h|^2 da,
\end{aligned} \tag{39}$$

$\mathbf{c} : [t_0, t_1] \longrightarrow \mathbb{R}$ denotes an arbitrary parameterization of Γ . Invoking both Young's inequality and the inequality the inequality

$$\|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2 \leq C \|\nabla \psi^h\|^2, \tag{40}$$

in which C is a constant dependent on the mesh size h (see Figure 3), therefore yields the inequality

$$\begin{aligned}
a(\psi^h, \psi^h) &\geq \epsilon_s \|\nabla \psi^h\|^2 - \frac{\epsilon_s}{\delta} \|\psi^h\|_{\Gamma}^2 - \delta \epsilon_s \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2 + \alpha \|\psi^h\|_{\Gamma}^2 - \frac{1}{2} \|\psi^h\|_{\Gamma}^2 \\
&\geq (\epsilon_s - \epsilon_s \delta C) \|\nabla \psi^h\|^2 + \left(\alpha - \frac{1}{2} - \frac{\epsilon_s}{\delta} \right) \|\psi^h\|_{\Gamma}^2,
\end{aligned} \tag{41}$$

where δ is necessarily positive.

To ensure that the right-hand side of (41) is positive it suffices to assume that

$$\delta < \frac{1}{C} \quad \text{and} \quad \alpha > \frac{1}{2} + \frac{\epsilon_s}{\delta}. \tag{42}$$

Infinitely many choices of the positive constant δ satisfy the first inequality in (42). For the particular choice

$$\delta := \frac{1}{2C}, \tag{43}$$

the second inequality in (42) requires that α obeys

$$\alpha > 2C\epsilon_s + \frac{1}{2}. \tag{44}$$

Using the parameter choices (43) and (44) in (41) suffices to ensure that the linear Stommel model is coercive.

Notice that the choice made in (43) is arbitrary. Choosing a different scaling in (43) would lead to different scalings for the stability parameter α in (44). Importantly, this would yield different weights for the interior or boundary contributions of the norm in (41).

5.2. Linear Stommel–Munk model

To show that the linear Stommel–Munk model is coercive, we begin by replacing χ^h with ψ^h in (20) and proceed as in the derivation of (41) to obtain the inequality

$$\begin{aligned}
a(\psi^h, \psi^h) &= \epsilon_m \|\Delta \psi^h\|^2 + \epsilon_s \|\nabla \psi^h\|^2 - \int_{\Omega} \psi^h \frac{\partial \psi^h}{\partial x} dv \\
&\quad - 2\epsilon_s \int_{\Gamma} \psi^h (\nabla \psi^h \cdot \mathbf{n}) d\Gamma + 2\epsilon_m \int_{\Gamma} \psi^h (\nabla(\Delta \psi^h) \cdot \mathbf{n}) d\Gamma \\
&\quad - 2\epsilon_m \int_{\Gamma} (\nabla \psi^h \cdot \mathbf{n}) \Delta \psi^h d\Gamma + \alpha_1 \|\psi^h\|_{\Gamma}^2 + \alpha_2 \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2 \\
&\geq \epsilon_m \|\Delta \psi^h\|^2 + \epsilon_s \|\nabla \psi^h\|^2 - \frac{\epsilon_s}{\delta_1} \|\psi^h\|_{\Gamma}^2 - \epsilon_s \delta_1 \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2 \\
&\quad - \frac{\epsilon_m}{\delta_2} \|\psi^h\|_{\Gamma}^2 - \epsilon_m \delta_2 \|\nabla(\Delta \psi^h) \cdot \mathbf{n}\|_{\Gamma}^2 - \epsilon_m \delta_3 \|\Delta \psi^h\|_{\Gamma}^2 - \frac{\epsilon_m}{\delta_3} \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2 \\
&\quad + \left(\alpha_1 - \frac{1}{2} \right) \|\psi^h\|_{\Gamma}^2 + \alpha_2 \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2. \tag{45}
\end{aligned}$$

To complete the proof, we make the following assumption, which corresponds to invoking inequalities (25a) and (25b) of Embar et al. [20]: There exist two constants C_1 and C_2 , both depending on the mesh size h (see Figure 7), such that

$$\|\nabla(\Delta \psi^h) \cdot \mathbf{n}\|_{\Gamma}^2 \leq C_1 \|\Delta \psi^h\|^2 \quad \text{and} \quad \|\Delta \psi^h\|_{\Gamma}^2 \leq C_2 \|\Delta \psi^h\|^2. \tag{46}$$

To guarantee that the bilinear form $a(\cdot, \cdot)$ of the linear Stommel–Munk model is coercive, we must ensure that the *right-hand side* of (45) is positive. To this end, the negative terms on the right-hand side of (45) should be dominated by the positive terms. It is clear that assumptions (46)₁ and (46)₂ (together with appropriately chosen values for the constants δ_1 , δ_2 , and δ_3) should be used in connection with the terms $-\epsilon_m \delta_2 \|\nabla(\Delta \psi^h) \cdot \mathbf{n}\|_{\Gamma}^2$ and $-\epsilon_m \delta_3 \|\Delta \psi^h\|_{\Gamma}^2$, respectively. For the terms $-\epsilon_s \delta_1 \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2$ and $-\frac{\epsilon_m}{\delta_3} \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2$ it is not, however, clear whether assumption (40) is suitable. Our analysis therefore treats four distinct cases, depending on whether assumption (40) is applied.

Remark 5.1. *We emphasize that Embar et al. [20] consider the pure biharmonic problem. Since no Laplacian term appears in their equation, the term $\|\nabla \psi^h\|^2$ cannot be used to dominate either $-\epsilon_s \delta_1 \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2$ or $-\frac{\epsilon_m}{\delta_3} \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2$ (see Embar et al. [20, eqn. (26)] Assumption (40) therefore does not appear in the coercivity analysis of Embar et al. [20]. Our approach to the Stommel–Munk model, which allows the use of assumption (40), is substantially different.*

Case I. Referring to the right-hand side of (45), we use assumption (40) to bound the term $-\epsilon_s \delta_1 \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2$ but not the term $-\frac{\epsilon_m}{\delta_3} \|\nabla \psi^h \cdot \mathbf{n}\|_{\Gamma}^2$. We also use assumptions (46)₁ and (46)₂ for the terms $-\epsilon_m \delta_2 \|\nabla(\Delta \psi^h) \cdot \mathbf{n}\|_{\Gamma}^2$ and $-\epsilon_m \delta_3 \|\Delta \psi^h\|_{\Gamma}^2$, respectively. It thus

follows that

$$a(\psi^h, \psi^h) \geq (\epsilon_m - \epsilon_m \delta_2 C_1 - \epsilon_m \delta_3 C_2) \|\Delta \psi^h\|^2 + (\epsilon_s - \epsilon_s \delta_1 C) \|\nabla \psi^h\|^2 \\ + \left(\alpha_1 - \frac{\epsilon_s}{\delta_1} - \frac{\epsilon_m}{\delta_2} - \frac{1}{2} \right) \|\psi^h\|_\Gamma^2 + \left(\alpha_2 - \frac{\epsilon_m}{\delta_3} \right) \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2. \quad (47)$$

To ensure that the right-hand side of (47) is positive, it suffices to assume that

$$\left. \begin{aligned} 1 - \delta_2 C_1 - \delta_3 C_2 &> 0, & 1 - \delta_1 C &> 0, \\ \alpha_1 &> \frac{\epsilon_s}{\delta_1} + \frac{\epsilon_m}{\delta_2} + \frac{1}{2}, & \text{and} & \alpha_2 > \frac{\epsilon_m}{\delta_3}. \end{aligned} \right\} \quad (48)$$

Infinitely many choices of the positive constants δ_1 , δ_2 , and δ_3 satisfy the first two inequalities in (48). In particular, taking

$$\delta_1 := \frac{1}{2C}, \quad \delta_2 := \frac{1}{4C_1}, \quad \text{and} \quad \delta_3 := \frac{1}{4C_2} \quad (49)$$

yields

$$\alpha_1 > 2C\epsilon_s + 4C_1\epsilon_m + \frac{1}{2} \quad \text{and} \quad \alpha_2 > 4C_2\epsilon_m. \quad (50)$$

If the parameters α_1 and α_2 satisfy restriction (50)₂, then the right-hand side of (47) is positive for all $\psi^h \in \mathcal{S}^h$. This, in turn, shows that the bilinear form $a(\cdot, \cdot)$ of the linear Stommel–Munk model is coercive.

Case II. We use assumption (40) to bound terms $-\epsilon_s \delta_1 \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2$ and $-\frac{\epsilon_m}{\delta_3} \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2$ on the right-hand side of (45). We also use (46)₁ and (46)₂ to deal respectively with the terms $-\epsilon_m \delta_2 \|\nabla(\Delta \psi^h) \cdot \mathbf{n}\|_\Gamma^2$ and $-\epsilon_m \delta_3 \|\Delta \psi^h\|_\Gamma^2$:

$$a(\psi^h, \psi^h) \geq (\epsilon_m - \epsilon_m \delta_2 C_1 - \epsilon_m \delta_3 C_2) \|\Delta \psi^h\|^2 + \left(\epsilon_s - \frac{C\epsilon_m}{\delta_3} - \epsilon_s \delta_1 C \right) \|\nabla \psi^h\|^2 \\ + \left(\alpha_1 - \frac{\epsilon_s}{\delta_1} - \frac{\epsilon_m}{\delta_2} - \frac{1}{2} \right) \|\psi^h\|_\Gamma^2 + \alpha_2 \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2. \quad (51)$$

To ensure that the right-hand side of (51) is positive, it suffices to assume that

$$1 - \delta_2 C_1 - \delta_3 C_2 > 0, \quad \epsilon_s - \frac{C\epsilon_m}{\delta_3} - \epsilon_s \delta_1 C > 0, \quad \text{and} \quad \alpha_1 > \frac{\epsilon_s}{\delta_1} + \frac{\epsilon_m}{\delta_2} + \frac{1}{2}, \quad (52)$$

which imply that the positive constants δ_1 , δ_2 , and δ_3 must obey the restrictions

$$\delta_1 < \frac{1}{C}, \quad \delta_2 < \frac{1}{C_1}, \quad \delta_3 < \frac{1}{C_2}, \quad \text{and} \quad \delta_3 > \frac{C\epsilon_m}{\epsilon_s}. \quad (53)$$

The two restrictions on δ_3 in (53) require that the parameters satisfy

$$\frac{C\epsilon_m C_2}{\epsilon_s} < 1. \quad (54)$$

If inequality (54) does not hold, then the the right-hand side of (51) may not be positive. If, however, (54) is satisfied, we may proceed as follows: First, we may choose δ_3 to satisfy the last two inequalities in (53). Second, for δ_3 fixed, we may choose δ_1 and δ_2 to satisfy the first two inequalities in (52). Finally, for δ_1, δ_2 , and δ_3 fixed, we may choose α_1 to satisfy the third inequality in (52). Thus, if α_1 and α_2 satisfy (52), then the right-hand side of (51) is positive for all $\psi^h \in \mathcal{S}^h$ and the bilinear form a of the Stommel–Munk model is coercive.

Remark 5.2. *When inequality (54) is satisfied and the parameter α_1 is selected as described above, the bilinear form of the linear Stommel–Munk model is coercive even if the parameter α_2 is set to vanish. Our analysis therefore suggests that the term $\alpha_2 \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2$ in (45) can actually be discarded. As we explain in Remark 5.1, this is in clear contrast with the approach taken by Embar et al. [20].*

Case III. Referring again to the right-hand side of (45), we use assumption (40) to bound the term $-\frac{\epsilon_m}{\delta_3} \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2$ but not the term $-\epsilon_s \delta_1 \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2$. We also treat the treat terms $-\epsilon_m \delta_2 \|\nabla(\Delta \psi^h) \cdot \mathbf{n}\|_\Gamma^2$ and $-\epsilon_m \delta_3 \|\Delta \psi^h\|_\Gamma^2$ using assumptions (46)₁ and (46)₂, respectively. This yields

$$a(\psi^h, \psi^h) \geq (\epsilon_m - \epsilon_m \delta_2 C_1 - \epsilon_m \delta_3 C_2) \|\Delta \psi^h\|^2 + \left(\epsilon_s - \frac{C \epsilon_m}{\delta_3} \right) \|\nabla \psi^h\|^2 + \left(\alpha_1 - \frac{\epsilon_s}{\delta_1} - \frac{\epsilon_m}{\delta_2} - \frac{1}{2} \right) \|\psi^h\|_\Gamma^2 + (\alpha_2 - \epsilon_s \delta_1) \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2. \quad (55)$$

To ensure that the right-hand side of (55) is positive, it suffices to assume that

$$1 - \delta_2 C_1 - \delta_3 C_2 > 0, \quad \epsilon_s - \frac{C \epsilon_m}{\delta_3} > 0, \quad \alpha_1 > \frac{\epsilon_s}{\delta_1} + \frac{\epsilon_m}{\delta_2} + \frac{1}{2}, \quad \text{and} \quad \alpha_2 > \epsilon_s \delta_1. \quad (56)$$

The foregoing inequalities imply that the positive constants δ_1 , δ_2 , and δ_3 must obey the restrictions

$$\delta_2 < \frac{1}{C_1}, \quad \delta_3 < \frac{1}{C_2}, \quad \text{and} \quad \delta_3 > \frac{C \epsilon_m}{\epsilon_s}, \quad (57)$$

the second and third of which lead to a single inequality,

$$\frac{C \epsilon_m C_2}{\epsilon_s} < 1, \quad (58)$$

identical to that, namely (54), arising in Case I.

If inequality (58) is not satisfied, then the right-hand side of (55) need not be positive. If, however, (58) is satisfied, we may proceed as follows: First, we may choose δ_3 to satisfy the last two inequalities in (57). Second, for δ_3 fixed, we may choose δ_2 to satisfy the first inequality in (56). Finally, for arbitrary δ_1 , and for δ_2 and δ_3 fixed, we may choose α_1 to satisfy the third inequality in (56). Thus, if α_1 and α_2 satisfy (56), then the right-hand side of (55) is positive for all $\psi^h \in \mathcal{S}^h$ and the bilinear form a in (45) in the linear Stommel–Munk model is coercive.

Case IV. We do not use assumption (40) to bound the terms $-\epsilon_s \delta_1 \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2$ or $-\frac{\epsilon_m}{\delta_3} \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2$ on the right-hand side of (45). However, we apply assumptions (46)₁ and (46)₂ to the terms $-\epsilon_m \delta_2 \|\nabla(\Delta \psi^h) \cdot \mathbf{n}\|_\Gamma^2$ and $-\epsilon_m \delta_3 \|\Delta \psi^h\|_\Gamma^2$, respectively, leading to:

$$a(\psi^h, \psi^h) \geq (\epsilon_m - \epsilon_m \delta_2 C_1 - \epsilon_m \delta_3 C_2) \|\Delta \psi^h\|^2 + \epsilon_s \|\nabla \psi^h\|^2 + \left(\alpha_1 - \frac{\epsilon_s}{\delta_1} - \frac{\epsilon_m}{\delta_2} - \frac{1}{2} \right) \|\psi^h\|_\Gamma^2 + \left(\alpha_2 - \frac{\epsilon_m}{\delta_3} - \epsilon_s \delta_1 \right) \|\nabla \psi^h \cdot \mathbf{n}\|_\Gamma^2. \quad (59)$$

To ensure that the right-hand side of (59) is positive, it suffices to assume that

$$1 - \delta_2 C_1 - \delta_3 C_2 > 0, \quad \alpha_1 > \frac{\epsilon_s}{\delta_1} + \frac{\epsilon_m}{\delta_2} + \frac{1}{2}, \quad \text{and} \quad \alpha_2 > \frac{\epsilon_m}{\delta_3} + \epsilon_s \delta_1. \quad (60)$$

Infinitely many possible choices of the positive constants δ_2 and δ_3 satisfy the first inequality in (60). In particular, regardless of the value of δ_1 , taking

$$\delta_2 := \frac{1}{4 C_1} \quad \text{and} \quad \delta_3 := \frac{1}{4 C_2}, \quad (61)$$

yields

$$\alpha_1 > \frac{\epsilon_s}{\delta_1} + 4 C_1 \epsilon_m + \frac{1}{2} \quad \text{and} \quad \alpha_2 > 4 C_2 \epsilon_m + \epsilon_s \delta_1. \quad (62)$$

Thus, if α_1 and α_2 satisfy (62)₂, then the right-hand side of (59) is positive for all $\psi^h \in \mathcal{S}^h$ and the bilinear form $a(\cdot, \cdot)$ in (45) in the Stommel–Munk model is coercive.

Summary of Cases I–IV. To summarize, Cases I–IV yield the following conclusions regarding the parameter choices: All cases yield the same parameter value for α_1 , although these values might scale slightly differently with respect to the constants C , C_1 , and C_2 . The parameter values for α_2 , however, differ from case to case. In Case II, the choice of α_2 is unrestricted if the additional constraint (54) is satisfied. In each remaining case, α_2 cannot be freely chosen: In Case I, $\alpha_2 > 4 C_2 \epsilon_m$. In Case II, $\alpha_2 > 0$. In Case III, $\alpha_2 > \epsilon_s \delta_1$. In Case IV, $\alpha_2 > 4 C_2 \epsilon_m + \epsilon_s \delta_1$. Notice that in Case II and Case III the parameters of the model satisfy (54).

6. Numerical studies

To verify our variational formulations for the SQGE, the linear Stommel–Munk model, and the linear Stommel model, we perform numerical studies on several benchmark problems commonly used (see, for example, Vallis [3]) in geophysical fluid dynamics. Specifically, we provide convergence studies using cubic B-spline basis functions presented in Section 4. For this purpose, we define the errors $\|e\|_0$, $\|e\|_1$, and $\|e\|_2$ in the L^2 -norm, the H^1 -semi norm, and the H^2 -semi norm by

$$\|e\|_0 = \frac{\|\psi - \psi^h\|_0}{\|\psi\|_0}, \quad \|e\|_1 = \frac{\|\psi - \psi^h\|_1}{\|\psi\|_1}, \quad \text{and} \quad \|e\|_2 = \frac{\|\psi - \psi^h\|_2}{\|\psi\|_2}, \quad (63)$$

respectively, where ψ^h is the approximation of the exact solution ψ . While we do not however provide error estimates for our variational formulations, on the basis of an analysis of similar variational formulations obtained by Foster et al. [10] we expect that the optimal rates of convergence of a FE discretization using cubic B-splines are respectively quartic, cubic, and quadratic in the L^2 , H^1 , and H^2 norms of the error. Our numerical studies focuses on determining whether the optimal rates of convergence can be obtained with the variational formulations proposed in the previous sections. Further, in Section 5 we investigate the influence of the stabilization parameters on the rates of convergence and the accuracy of the solutions.

6.1. Linear Stommel model

In this section, we present a numerical investigation of the variational formulation (26) for the Stommel model. Our study is performed using the exact solution

$$\psi(x, y) = \frac{\sin(\pi y)}{\pi(1 + 4\pi^2\epsilon_s^2)} \{2\pi\epsilon_s \sin(\pi x) + \cos(\pi x) + [(1 + e^{R_2})e^{R_1 x} - (1 + e^{R_1})e^{R_2 x}]/(e^{R_1} - e^{R_2})\} \quad (64)$$

over the domain $\Omega = [0, 1] \times [0, 1]$. In (64), R_1 and R_2 are given by

$$R_1 = \frac{-1 + \sqrt{1 + 4\pi^2\epsilon_s^2}}{2\epsilon_s} \quad \text{and} \quad R_2 = \frac{-1 - \sqrt{1 + 4\pi^2\epsilon_s^2}}{2\epsilon_s}.$$

This solution was used by Myers and Weaver [32] and Foster et al. [10]. By taking the Stommel number to obey $\epsilon_s = 0.05$, we work in a setting identical to that considered in these references. The forcing term F is chosen to match that given by the exact solution (64). We consider a rectangular ocean as a computational domain, as shown in Figure 2. With the origin of a cartesian coordinate system at the southwest corner, the x - and y -axis point eastward and northward, respectively, and the boundaries of the computational domain are the shores of the ocean. Figure 2 shows the streamlines of the numerical solution obtained using a mesh size of $h = 1/64$. This solution is consistent with one obtained by Myers and Weaver [32] and Foster et al. [10]. In particular, as is evident from the figure, it exhibits a thin western boundary layer.

The variational formulation (26) involves the stabilization parameter α . As mentioned in Section 5.1, the accuracy of the numerical solutions is sensitive to this parameter. The coercivity analysis of the variational formulation for this model requires that α satisfies the inequality (44)—namely $\alpha > 2C\epsilon_s + \frac{1}{2}$, where C is the mesh-dependent constant in assumption (40). Emulating an idea applied to the Poisson equation by Embar et al. [20], this constant can be evaluated as the maximum eigenvalue $\max(\Lambda)$ of the local eigenvalue problem

$$\mathbf{A}\mathbf{x} = \Lambda\mathbf{B}\mathbf{x}, \quad (65)$$

where elements of $\mathbf{A}$ and $\mathbf{B}$ are given by

$$[\mathbf{A}]_{ij} = \int_{\Gamma} (\nabla N_i \cdot \mathbf{n})(\nabla N_j \cdot \mathbf{n}) \, da \quad \text{and} \quad [\mathbf{B}]_{ij} = \int_{\Omega} \nabla N_i \cdot \nabla N_j \, dv. \quad (66)$$

Figure 2: The Stommel model: Streamlines of the approximation to the solution on a mesh size of $h = 1/64$.

Figure 3: The Stommel model: Log-log plot of C versus $1/h$ obtained from the eigenvalue problem (65) using matrices (66). The constant C is the maximum value of the eigenvalues. The plot clearly indicates that C depends on h and grows linearly with $1/h$.

Here, N_i denotes a cubic B-spline basis function and, granted the representation (31) for ψ^h , the matrices $\mathbf{A}$ and $\mathbf{B}$ are respectively obtained from the left- and right-hand sides of the inequality (40).

Since the constant C in (40) is important in the coercivity analysis in Section 5.1, Figure 3 provides a log-log plot of C with respect to inverse h^{-1} of the mesh size. This result clearly confirms that C depends on h . In particular, it shows that C scales linearly with $1/h$.

Figure 4 displays the sensitivity of the numerical solutions with respect to α for different values of h . Figure 4(a) indicates that the accuracy of the numerical solutions depends on α . While the error highly increases for $\alpha > 10^{10}$, a relatively small error is observed in the range of $10^0 < \alpha < 10^{10}$.

We use two different approaches to determine a value of the parameter α that

suffices to ensure the accuracy and stability of our numerical solutions. The first approach, due to Embar et al. [20], invokes the coercivity analysis in Section 5.1 together with the estimate for the constant C provided by the eigenvalue problem (65), which led to the inequality $\alpha > 2C\epsilon_s + \frac{1}{2}$ in (44). Embar et al. [20] obtained optimal rates of convergence for the Poisson equation by choosing a slightly larger value of α than that provided by their coercivity analysis. We thus take $\alpha = 2C\epsilon_s + 1$. The red star in Figure 4(a) indicates the value α computed using the value of C obtained from solving the eigenvalue problem (65). We emphasize that these values are located in the range of the minimum L^2 -norm of the error for all values of h . The other approach for determining a value for the parameter α was suggested by Burman and Hansbo [34], who instead of solving an eigenvalue problem in the manner of Embar et al. [20] minimized the condition number of the resulting linear system. In Figure 4(b), we display the plot of the condition number for the matrix $\mathbf{M}$ of the system of equations (30) for various values of h and α . The value of α corresponding to the red star in Figure 4(b) is the same as that in Figure 4(a). Comparing the location of this red star and the value α for which the condition number is minimized in Figure 4(b), we notice that they belong to the same range. This leads us to conclude that the two approaches used to determine the values for the parameter α yield similar predictions.

Figure 4: The Stommel model: Log-log plots of (a) the L^2 -norm of the error $\|e\|_0$ versus α and (b) the condition number versus α for mesh sizes of $h = 1/8$ ($\circ$), $1/16$ ($\diamond$), $1/32$ ($\triangle$), $1/64$ ($\square$). The red stars represent the L^2 -norm of the error at the value of α computed using the value of C obtained from solving the eigenvalue problem (65).

Figure 5 shows the convergence rates in the L^2 -, H^1 -, and H^2 -norms for the values of the parameter α (indicated by red stars and determined by solving the local eigenvalue problem). Again, we note that quartic, cubic, and quadratic rates are optimal for the L^2 -, H^1 -, and H^2 -norms, respectively, for the FE discretization using cubic B-splines. The convergence rate in the L^2 -norm is optimal. However, the rates of convergence for the H^1 - and H^2 -norms are slightly suboptimal.

Figure 5: The Stommel model: Convergence rates in the L^2 -norm, the H^1 -norm, and the H^2 -norm.

6.2. Linear Stommel–Munk model

In contrast to the linear Stommel model, the linear Stommel–Munk model involves a fourth-order term and, thus, requires C^1 -continuous basis functions to achieve a conforming FE discretization. Again, we mention that cubic B-splines are C^2 -continuous so that they satisfy this condition. Numerical studies are performed for two test problems with exact solutions

$$\psi(x, y) = \sin^2(\pi x/3) \sin^2(\pi y) \quad \text{in } \Omega = [0, 3] \times [0, 1] \quad (67)$$

and

$$\psi(x, y) = [(1 - x/3)(1 - e^{-20x}) \sin(\pi y)]^2 \quad \text{in } \Omega = [0, 3] \times [0, 1]. \quad (68)$$

The forcing term F is chosen to match those determined by the above exact solutions. These solutions were used to test a FE algorithm for large-scale ocean circulation problems (Cascón et al. [33], Foster et al. [10]). Figure 6 displays the plots of the exact solutions for the two test problems. Notice that the test problem (68) (Figure 6(b))

Figure 6: The Stommel–Munk model: Plots of the exact solutions for the test problems (a) (67) without any thin boundary layer and (b) (68) with a strong western boundary layer.

exhibits a thin boundary layer, in the vicinity of $x = 0$, corresponding to a western

boundary layer; the test problem (67) (Figure 6(a)) does not display a boundary layer. We use conditions identical to those employed by Cascón et al. [33] and Foster et al. [10]. Unless otherwise specified, the Munk number $\epsilon_m = 6 \times 10^{-5}$ and the Stommel number $\epsilon_s = 0.05$ are used for the numerical tests.

The variational formulation (19) involves two stabilization parameters α_1 and α_2 . These parameters depend on mesh-dependent constants C_1 and C_2 defined in (46)₁ and (46)₂, respectively. Much as in Section 6.1, these constants are computed from the local eigenvalue problem (65) using the matrices

$$[\mathbf{A}]_{ij} = \int_{\Gamma} (\nabla(\Delta N_i) \cdot \mathbf{n})(\nabla(\Delta N_j) \cdot \mathbf{n}) da \quad \text{and} \quad [\mathbf{B}]_{ij} = \int_{\Omega} \Delta N_i \cdot \Delta N_j dv \quad (69)$$

for C_1 and

$$[\mathbf{A}]_{ij} = \int_{\Gamma} \Delta N_i \Delta N_j da \quad \text{and} \quad [\mathbf{B}]_{ij} = \int_{\Omega} \Delta N_i \cdot \Delta N_j dv \quad (70)$$

for C_2 . Notice that the matrices in (69) and (70) follow from the inequalities (46)₁ and (46)₂, respectively. Plots of the variations of C_1 and C_2 with mesh size h provided in Figure 7 confirm that the asserted mesh-dependency.

Figure 7: The Stommel–Munk model: Plots of (a) C_1 versus $1/h$ and (b) C_2 versus $1/h$ obtained from the eigenvalue problem (65) using matrices (69) and (70). The constants C_1 and C_2 are the maximum of the eigenvalues and the plots clearly indicate that C_1 and C_2 depend on h .

The remainder of this section is devoted to the investigation of the sensitivity of the numerical results with respect to the parameters α_1 and α_2 and the mesh size h . To determine values of parameters α_1 and α_2 sufficient to ensure the accuracy and stability of our numerical solutions, we employ an approach analogous used in Section 6.1. Recall that the coercivity analysis in Section 5.2 yields four cases for the values of α_1 and α_2 . To keep the length of the paper within reasonable limits,

we only consider Case I and Case II, which are representative of the remaining two cases. While Case I requires both α_1 and α_2 for the stabilization, only the first of these parameters is required Case II. To avoid repetition, the streamlines of the numerical approximations are no longer displayed. However, it is worth emphasizing that these streamlines are indistinguishable from those of Figure 6.

6.2.1. Numerical test for Case I

In this section, we perform a numerical investigation of Case I from Section 5.2. Figures 8 and 9 show the sensitivity of the numerical solutions with respect to the parameters α_1 and α_2 for different values of the mesh size h for the test problem (68) with a western boundary layer. Notice that similar results are observed for the test problem (67). In Figure 8, α_1 is varied and α_2 is fixed. In Figure 9, α_2 is varied and α_1 is fixed. In Figure 8(a) and Figure 9(a), we plot the L^2 -norm of the error. In Figure 8(b) and Figure 9(b), we plot the condition number for the matrix $\mathbf{M}$ of the system of the equation (30) for various values of h , α_1 , and α_2 . The red stars in Figure 8(a) and Figure 9(a) represent the values α_1 and α_2 computed from the local eigenvalue problem (65), respectively. The red stars in Figure 8(b) and Figure 9(b) are identical to those in Figure 8(a) and Figure 9(a).

The results in Figure 8 and Figure 9 yield the following conclusions:

- With respect to the numerical accuracy (displayed in Figures 8(a) and 9(a)), the minimum L^2 -norm of the error is achieved for relatively wide ranges of α_1 and α_2 . We emphasize that the parameter range varies with respect to the mesh size h , especially for α_1 (Figure 8(a)). Furthermore, we remark that α_1 and α_2 values (indicated by the red star) predicted by the local eigenvalue problem consistently yield numerical results that are in the range of the minimum L^2 -norm of the error.
- Regarding computational efficiency (i.e., the conditioning of the resulting linear system (30) displayed in Figures 8(b) and 9(b)), the values of α_1 and α_2 for which the condition number is minimized generally are in the range of the minimum L^2 -norm of the error.
- The values of α_1 and α_2 for which the condition number is minimized are not identical to those corresponding to the red stars computed from the local eigenvalue problems. Thus, in order to maximize both numerical accuracy and computational efficiency, it would suffice to choose α_1 and α_2 to fall between the condition number minimizers and the values corresponding to the red stars.

We next study the convergence rates of the cubic B-spline discretization of the variational formulation (19). In so doing, we monitor the convergence rates in three error norms: the L^2 -norm, the H^1 -norm, and the H^2 -norm. We consider the test problem (67) (which does not display a western boundary layer) and the test problem (68) (which displays a western boundary layer). To determine the values of parameters α_1 and α_2 , we use the coercivity analysis in Section 5.2 and, consistent with the inequalities in (50), take $\alpha_1 = 2.1C\epsilon_s + 4.1C_1\epsilon_m + 1$ and $\alpha_2 = 4.1C_2\epsilon_m$.

Figure 10(a) displays the rates of convergence for the test problems (67). We observe optimal convergence rates in all three norms. Figure 10(b) displays the rates of

Figure 8: Case I for the Stommel–Munk model: Log-log plots of (a) $\|e\|_0$ versus α_1 and (b) the condition number versus α_1 for fixed α_2 and various mesh sizes of $h = 1/8$ ($\circ$), $1/16$ ($\triangle$), $1/32$ ($*$), $1/64$ ($\diamond$), $1/128$ ($\square$) using the test problem (68). The red stars represent the L^2 -norm of the error and the condition number at the value of α_1 obtained from the eigenvalue problem (65).

Figure 9: Case I for the Stommel–Munk model: Log-log plots of (a) $\|e\|_0$ versus α_2 and (b) the condition number versus α_2 for fixed α_1 and various mesh sizes of $h = 1/8$ ($\circ$), $1/16$ ($\triangle$), $1/32$ ($*$), $1/64$ ($\diamond$), $1/128$ ($\square$) using the test problem (68). The red stars represent the L^2 -norm of the error and the condition number at the value of α_2 obtained from the eigenvalue problem (65).

convergence for the test problem (68). In contrast to the previous test problem (67), slightly lower convergence rates are observed in all three norms. Following the reasoning of Foster et al. [10], this can be attributed to the presence of the western boundary layer in test problem (68). In Table 1, we show the convergence rates for the test problem (68) for various values of h . We note that the convergence rates

Figure 10: Case I for the Stommel–Munk model: Convergence rates in the L^2 -norm, the H^1 -norm, and the H^2 -norm for the test problems: (a) (67) without the western boundary layer and (b) (68) with the western boundary layer.

h	Dofs	$\ e\ _0$	L_2 -order	$\ e\ _1$	H^1 -order	$\ e\ _2$	H^2 -order
1/8	297	9.7498e-2	–	4.6500e-1	–	8.0503e-1	–
1/16	969	1.1660e-2	3.0638	1.0746e-1	2.1135	3.8605e-1	1.0603
1/32	3,465	8.5410e-4	3.7710	1.5495e-2	2.7939	1.2040e-1	1.6810
1/64	13,065	4.8670e-5	4.1333	1.7944e-3	3.1102	3.0840e-2	1.9649
1/128	50,697	2.7130e-6	4.1651	2.0754e-4	3.1121	7.6481e-3	2.0116

Table 1: Case I-a for the Stommel–Munk model: Errors in the L^2 -norm, the H^1 -norm, and the H^2 -norm versus h . On coarse meshes, the orders of convergence are not optimal, but they are optimal on finer meshes.

are lower than optimal on coarser meshes, but optimal on finer meshes. We conclude that optimal convergence rates of the cubic B-spline discretization of the variational formulation (19) are obtained for Case I in Section 5.2.

6.2.2. Numerical test for Case II

In this section, we perform a numerical investigation of Case II from Section 5.2. As mentioned in the summary at the end of Section 5.2, Case II is different from all the other three cases in that α_2 is a *free parameter*. With this in mind, we explore the extent to which the coercivity analysis in Section 5.2 yields practical choices for the model parameters only for the variational formulation (19) with $\alpha_2 = 0$. Recall, from Remarks 5.1 and 5.2, that coercivity holds for this choice due to the presence of the Laplacian in the partial-differential equation of the linear Stommel–Munk model. This is in clear contrast with the problem considered by Embar et al. [20], where the absence of the Laplacian requires that $\alpha_2 > 0$. To investigate the role of the Laplacian, we consider a subcase in which $\alpha_2 = 0$ and $\epsilon_s = 0$. Since in this case coercivity cannot be guaranteed by the theoretical analysis in Section 5, we expect

the tests to reveal numerical instability. Thus, we consider the following two subcases:

- Case II-a: The parameters ϵ_s and ϵ_m satisfy the inequality (54).
- Case II-b: The parameter $\epsilon_s = 0$.

Case II-a. The analysis in Section 5.2 shows that inequality (54) must be satisfied to ensure the coercivity of the variational formulation in Case II. We emphasize, however, that the parameters that we used in Case I for test problem (68) (i.e., $\epsilon_s = 0.05$ and $\epsilon_m = 6 \times 10^{-5}$) do not satisfy this inequality. that previously used and use the same

Figure 11: Case II-a for the Stommel–Munk model: Plot of $C_{\epsilon_m} C_2 / \epsilon_s$ versus $1/h$ for $\epsilon_s = 0.05$ and $\epsilon_m = 6 \times 10^{-9}$. The inequality (54) is satisfied for various values of h .

value for ϵ_s . For this choice of the parameters, the inequality (54) is satisfied for all test cases of the mesh size h , as is evident from the plot of $C_{\epsilon_m} C_2 / \epsilon_s$ for various values of h in Figure 11.

Since the constants C , C_1 , and C_2 are known by solving the eigenvalue problem (65), we choose $\delta_3 = 0.5(C_{\epsilon_m}/\epsilon_s + 1/C_2)$ to satisfy the last two inequalities in (53). Then, for δ_3 fixed, we choose $\delta_1 = 0.5(\epsilon_s - C_{\epsilon_m}/\delta_2)/\epsilon_s C$ and $\delta_2 = 0.5(1 - \delta_3 C_2)/C_1$ to satisfy the first two inequalities in (52). Finally, for δ_1, δ_2 , and δ_3 fixed, we choose $\alpha_1 = 10(\epsilon_s/\delta_1 + \epsilon_m/\delta_2 + 1/2)$ to ensure that the third inequality in (52) holds.

Figure 12 shows the sensitivity of the numerical solutions with respect to the parameter α_1 for different values of the mesh size h for the test problem (68) with the western boundary layer. Notice that similar results are observed for the test problem (67). In Figure 12(a), we plot the L^2 -norm of the error for various values of h and α_1 . In Figure 12(b), we plot the condition number for the matrix $\mathbf{M}$ of the system of the equation (30) for various values of h and α_1 . The red stars represent the values α_1 computed from the local eigenvalue problem (65). The results in the figure are similar to the general conclusions described in the previous section. Of particular interest is the value of α_1 (indicated by the red star) predicted by the local eigenvalue problem, which yields numerical results that are in the range of the minimum L^2 -norm of the error. Furthermore, the values of α_1 for which the condition number is

Figure 12: Case II-a for the Stommel–Munk model: Log-log plots of (a) $\|e\|_0$ versus α_1 and (b) the condition number versus α_1 for various mesh sizes of $h = 1/8$ ($\circ$), $1/16$ ($\triangle$), $1/32$ ($*$), $1/64$ ($\diamond$), $1/128$ ($\square$) for the test problem (68). The red stars represent the L^2 -norm of the error and the condition number at the value of α_1 computed from the eigenvalue problem (65).

minimized are in the range of the minimum L^2 -norm of the error. Notice that similar results are observed for the test problem (67).

Figures 13(a) and 13(b) display the rates of convergence for both test problems (67) and (68), respectively. While we observe the optimal convergence rates in all three norms for the test problem (67), slightly lower convergence rates are observed for the test problem (68) due to the presence of the western boundary layer. We conclude that optimal convergence rates of the cubic B-spline discretization of the variational formulation (19) are obtained for Case II in Section 5.2.

Case II-b. In Figure 14, we display the convergence plots in the L^2 -norm, the H^1 -norm, and the H^2 -norm for the test problem (68) with the western boundary layer. The blow-up evident from these plots suggests the presence of a numerical instability for $\epsilon_s = 0$. The numerical results in this section and the theoretical analysis in Section 5 therefore show that the Laplacian term in the linear Stommel–Munk model is needed for stable numerical simulations with $\alpha_2 = 0$.

6.3. Stationary quasi-geostrophic equations

This section presents a numerical study for the variational formulation (5) of the SQGE using two test problems (67) and (68), corresponding to the solutions given in Tests 1 and 2 of Cascón et al. [33] and Tests 5 and 6 of Foster et al. [10]. We consider the same setting with these references by taking the Reynolds number $Re = 1.667$ and the Rossby number $Ro = 10^{-4}$. In contrast to the previous two linear models, the SQGE are nonlinear and, thus, requires a nonlinear solver. We employ the Newton–Raphson iteration method described in Section 3.1.2 and use $\psi = 0$ as an initial guess.

Figure 13: Case II-a for the Stommel–Munk model: Convergence rates in the L^2 -norm, the H^1 -norm, and the H^2 -norm for the test problems: (a) (67) without the western boundary layer and (b) (68) with the western boundary layer.

We use the maximum residual norm of 10^{-8} as the stopping criteria for the nonlinear solver, just as in the work of Foster et al. [10].

The streamlines of the numerical solutions for two examples are not displayed in the present study. However, we note that they are indistinguishable from the exact solutions shown in Figure 6 and are similar to numerical solutions obtained by Cascón et al. [33] and Foster et al. [10].

Figure 15 shows the sensitivity of the L^2 -norm of the error with respect to the parameters α_1 and α_2 for different values of the mesh size h . To keep the length of the paper within reasonable limits, we perform this sensitivity study only for the test problem (68), which displays a western boundary layer. In Figure 15(a), the parameter α_2 is fixed and the parameter α_1 is varied. In Figure 15(b), α_1 is fixed and α_2 is varied. Since we did not perform a coercivity analysis for the SQGE, we cannot invoke theoretical arguments to determine the values of α_1 and α_2 as we did for the linear Stommel model (Section 6.1) and the linear Stommel–Munk model (Section 6.2). Nevertheless, for the SQGE, we take the stabilization parameters $\alpha_1 = 2.1C_1$ and $\alpha_2 = 2.1C_2$, where the mesh-dependent constants C_1 and C_2 are computed from the eigenvalue problem (65) using matrices (69) and (70). Embar et al. [20] had success with this choice in their study of a boundary-value problem for the biharmonic equation. The red stars in Figure 15 represent the values α_1 and α_2 computed from the local eigenvalue problem (65).

The results in Figure 15 yield the following general conclusions, which are similar to those appearing in Section 6.1 and Section 6.2. Regarding numerical accuracy (displayed in Figure 15), the minimum L^2 -norm of the error is achieved for a relatively wide α_1 parameter range, which varies with respect to the mesh size h . Furthermore, the values of α_1 (indicated by the red stars and predicted by the local eigenvalue problem) consistently yield numerical results in the range of the minimum L^2 -norm

Figure 14: Case II-b for the Stommel–Munk model with $\epsilon_s = 0$: Convergence plots in the L^2 -norm, the H^1 -norm, and the H^2 -norm for the test problem (68).

h	Dofs	$\ e\ _0$	L^2 -order	$\ e\ _1$	H^1 -order	$\ e\ _2$	H^2 -order
1/8	297	0.0539	–	0.3418	–	0.7681	–
1/16	969	0.0076	2.8242	0.0904	1.9183	0.3816	1.0092
1/32	3,465	5.6230e-04	3.7574	0.0134	2.7591	0.1198	1.6709
1/64	13,065	3.3256e-05	4.0796	0.0016	3.0669	0.0307	1.9627
1/128	50,697	2.0443e-06	4.0240	1.9297e-04	3.0461	0.0076	2.0099

Table 2: SQGE: Errors in the L^2 -norm, the H^1 -norm, and the H^2 -norm, respectively, for various values of the mesh size h for the test problem (68). On coarse meshes, convergence rates are not optimal, but they are optimal on finer meshes.

of the error. We also note that the convergence of the Newton–Raphson iteration depends on α_1 and α_2 . If, in particular, these parameters take values that are either higher or lower than the values plotted in Figure 15, then the nonlinear solver fails to converge.

Next, we study the convergence rates of the cubic B-spline discretization of the variational formulation (5). We monitor the convergence rates in three error norms: the L^2 -norm, the H^1 -norm, and the H^2 -norm. We consider the test problem (67) (which does not display a western boundary layer) and the test problem (68) (which displays a western boundary layer). For the parameters α_1 and α_2 , we use the same values as above—i.e., $\alpha_1 = 2.1C_1$ and $\alpha_2 = 2.1C_2$, where C_1 and C_2 are the mesh-dependent constants that are computed from the eigenvalue problem (65) using matrices (69) and (70).

Figure 16(a) displays the rates of convergence for the test problem (67). We observe optimal convergence rates in all three norms. Figure 16(b) displays the rates of convergence for the test problem (68). In contrast to the previous test problem, slightly lower convergence rates are observed in all three norms. This is due to the presence of the western boundary layer in test problem (68), just as the work of Foster et al. [10]. In Table 2, we show the convergence rates for the test problem (68) for

Figure 15: SQGE: Log-log plots of (a) $\|e\|_0$ versus α_1 and (b) $\|e\|_0$ versus α_2 for various mesh sizes of $h = 1/8(\circ)$, $1/16(\triangle)$, $1/32(*)$, $1/64(\diamond)$ using the test problem (68). The red stars represent the L^2 -norm of the error at the values of α_1 and α_2 computed from the eigenvalue problem (65).

various values of h . Although the convergence rates are lower than optimal on coarser meshes, they are optimal on finer meshes. We conclude that optimal convergence rates of the cubic B-spline discretization of the variational formulation (5) are obtained.

Figure 16: SQGE: Convergence rates in the L^2 -norm $\|e\|_0$, the H^1 -norm $\|e\|_1$, and the H^2 -norm $\|e\|_2$ for the test problems: (a) (67) without the western boundary layer and (b) (68) with the western boundary layer.

7. Conclusion

In this paper, we introduced B-spline based FE discretizations for the streamfunction formulation of the SQGE. Due to the non-interpolatory nature of B-splines, we used Nitsche’s method to weakly impose Dirichlet boundary conditions. Variational consistency of the method was established. In addition to the SQGE, we provided Nitsche-type formulations for two standard simplified versions of the SQGE used in geophysical fluid dynamics, namely the linear Stommel–Munk and Stommel models. The proposed strategy involves stabilization parameters which influence both convergence rates and the numerical accuracy. For the linear Stommel and Stommel–Munk models, we use a coercivity analysis to determine values of the stabilization parameters.

Our strategy was verified using numerical examples with and without a thin boundary layer. Optimal rates of convergence were observed for the test problem without a thin boundary layer. However, slightly lower rates of convergence were obtained for the example with a thin western boundary layer. On the other hand, the results exhibit optimal rates of convergence as well as improved accuracy on fine meshes, indicating that the proposed FE method can capture a thin boundary layer with good accuracy. We verified that the fidelity of the method is sensitive to the values of the stabilization parameters. The values of these parameters, which yield highly accurate numerical approximations, can be evaluated numerically by formulating and solving general local eigenvalue problems.

In the future, we will perform a coercivity analysis for the SQGE and theoretical error estimates for our variational formulations of the SQGE and the linear Stommel–Munk and Stommel models. Furthermore, a FE discretization using T-splines and *a posteriori* error estimate will be developed to capture more efficiently and accurately a thin western boundary layer. We will also employ the isogeometric concept to treat nonrectangular domains, including, for instance, arbitrarily shaped coastal boundaries. Moreover, the accuracy and efficiency of our approach will be compared with those of Argyris element used by Foster et al. [10, 11]. Finally, we intend to extend this study to the time-dependent QGE and the two-layer QGE.

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