A global–local discontinuous Galerkin finite element for finite-deformation analysis of multilayered shells

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Abstract

We present a global–local finite element formulation for the analysis of doubly-curved laminated composite shells. The proposed formulation is applicable to a wide range of problems, including those involving finite strains/rotations, nonlinear constitutive behavior, and static or dynamic structural response. Moreover, in dynamic analysis applications, it can be combined with either explicit or implicit time integration schemes. The global–local framework is based on the superposition of a global displacement field, spanning the thickness of the entire laminate, and local (i.e. layerwise) displacement fields associated with each layer in the laminate. This approach affords highly-resolved representations in regions of critical interest, and allows a smooth transition from higher to lower resolution zones. Discontinuous Galerkin fluxes are used to enforce interlaminar continuity conditions in perfectly-bonded laminates, and the cohesive-zone methodology is used to represent interfacial delamination. Stresses and surface normals are referred to the initial configuration in the proposed total Lagrangian formulation. Assumed natural strain and enhanced assumed strain techniques are employed to alleviate shear locking and other pathologies that are known to afflict bilinear shell elements. The performance of the proposed finite element is examined with the aid of several numerical examples involving thick as well as thin shells.

Keywords: Laminated composite shells, global–local shell theory, dynamic structural analysis, discontinuous Galerkin method, cohesive-zone model, nonlinear finite element analysis, total Lagrangian formulation.

1. Introduction

Multi-layered composite plates and shells are light-weight, high-performance structural components utilized in a variety of application areas, ranging from sports equipment to aerospace and marine structures, where they are required to withstand demanding service conditions. For several decades, thin laminated structures of this type have been the focus of intensive research, aimed at the development of theoretical and computational formulations capable of predicting their performance characteristics and service life, both accurately and efficiently. Contributions to this field of research are surveyed and catalogued in several review articles [1–4] found in the literature; see also Ref. [5] for a more detailed exposition.

Classical laminated plate/shell theories may be divided into three broad categories [3, 5]: equivalent single layer (ESL), layerwise, and zig-zag theories. In the ESL framework, the variation of the displacement field across the thickness of the entire laminate is assumed to be known *a priori*. Due to this fundamental assumption, ESL-based finite element formulations are simple and efficient, with a number of unknowns that is independent of the number of layers constituting the laminate. However, typically, they can only predict the global response of undamaged, relatively thin structures, and are incapable of resolving local field distributions within the laminate. To remedy these shortcomings, layerwise theories [6–10] assume independent displacement fields for each layer, thereby introducing a

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larger number of unknown parameters (depending on the number of layers) and incurring a higher computational cost. Attempts to combine the advantages of ESL and layerwise theories lead to the development of the zigzag approach [11–16]. As in layerwise theories, a displacement field particular to each layer is assumed to exist. However, to avoid the accompanying proliferation of degrees of freedom, constitutive relations and continuity requirements at the interfaces between adjacent layers are combined and used *a priori* in zig-zag theories, limiting them to the analysis of elastic laminates.

Global–local theories [17–20] represent a more general framework for the analysis of laminated shells, where the displacement within a layer is represented as a superposition of a layerwise field, local to that particular layer, and a global field spanning the entire thickness of the laminate. Thus, the governing equations exhibit explicit coupling between effects associated with different (global and local) length scales. This multi-scale representation allows the accuracy of layerwise theories to be achieved at a reduced computational cost [21]. It also facilitates the use of highly-refined kinematic descriptions in regions of critical interest with reduced-order descriptions elsewhere in the structure, and affords a straight-forward means of transitioning between such descriptions. The distribution of available computational resources in this manner can improve both accuracy and efficiency, and can be carried out adaptively to maximize its potential benefits.

Moreover, the multi-scale representation at the center of global–local formulations endows them with a significant advantage in the treatment of laminated structures subjected to complex and/or extreme loading. Under such conditions, different deformation and damage mechanisms operate and interact across several different length scales; e.g. viscoplastic deformation and delamination processes are manifested at the microstructural and laminar scales, respectively, while the overall structural response is usually associated with a much larger length scale. Additionally, global–local theories can be used in the analysis of perfectly-bonded as well as debonded laminates, and in general, do not introduce *a priori* assumptions concerning the type and extent of damage in the structure, recognizing that such assumptions are inappropriate in real applications, where damage is expected to evolve over time. It should also be noted that the global–local theories of Williams [17, 19, 20] admit through-thickness expansions of arbitrary functional form for the global and local displacement fields. Therefore, they can be regarded as unified formulations from which all existing ESL, layerwise, and zig-zag theories (i.e. all single-scale theories) can be obtained through appropriate elimination of either the global or local fields and appropriate specialization of the remaining fields.

This paper describes the formulation of a 4-node shell finite element, suitable for finite-deformation analysis of doubly-curved shells. The proposed formulation extends and generalizes previous work by Versino et al. [22] and Mourad et al. [23], and is based on the global–local theory of Williams [17]. It is applicable to various static and/or dynamic problems, including problems involving curved structures, with perfectly-bonded or delaminated interfaces, exhibiting history-dependent material behaviors. The formulation is also compatible with both implicit and explicit time integrators, allowing it to be used in the treatment of dynamic problems spanning a wide range of deformation rates and associated time scales. The formulation adopts a total Lagrangian approach, obviating the need to frequently recompute the normal to the surface of the shell [24]. Assumed natural strain (ANS) interpolation [25–27] is used to alleviate shear and curvature locking, and the enhanced assumed strain (EAS) methodology is used to mitigate Poisson thickness locking [28–32] and improve the element’s performance in membrane-dominated problems [32, 33]. Interlaminar continuity conditions are enforced by means of discontinuous Galerkin (DG) fluxes [34–36], introduced into the variational equations of motion. This allows the onset of interlaminar debonding to be numerically represented in a consistent manner, characterized by a smooth transition from the perfectly-bonded to the delaminated state [37]. The DG method used here is an interior penalty method [38, 39], and is thus closely related to Nitsche’s method [40].

The remainder of this paper is organized as follows. The equilibrium equations for a multi-layered composite are introduced in the following section. Geometric and kinematic characteristics of the shell are presented in Section 3, employing a tensorial representation. In Section 4, trial and test functions are introduced in the context of the global–local framework. Subsequently, the weak form of the equations of motion is developed in Section 5, and the resulting semi-discrete formulation is presented in Section 6. Several numerical simulations, involving thick as well as thin shells, are carried out in Section 7 to assess the performance of the present formulation in the contexts of static and dynamic analyses. Finally, a summary and concluding remarks are presented in Section 8.

Figure 1: Shell geometry.

2. Problem definition and notation

A curved laminated shell of thickness h is considered. The Cartesian coordinates of every point belonging to the shell's reference surface, Ω , are given by its position vector $\mathbf{r}(\xi^1, \xi^2)$, where ξ^1, ξ^2 are curvilinear coordinates defined on the closed domain Ω (see Fig. 1). The ξ^3 -coordinate spans the range $[-1, 1]$, and is directed along the unit vector $\mathbf{a}_3(\xi^1, \xi^2)$, normal to the reference surface. The Cartesian coordinates of every point in the closed domain occupied by the shell in the initial undeformed configuration, $\mathcal{B}_0 = \Omega \times [-h/2, h/2]$, are given by its position vector,

$$\mathbf{R}(\xi^1, \xi^2, \xi^3) = \mathbf{r}(\xi^1, \xi^2) + \frac{h}{2} \xi^3 \mathbf{a}_3. \quad (1)$$

The domain's boundary, $\partial\mathcal{B}_0$, consists of two disjoint subsets, $\partial_u\mathcal{B}_0$ and $\partial_\sigma\mathcal{B}_0$, where essential and natural boundary conditions, respectively, are imposed. Accordingly, the boundary of the reference surface, $\partial\Omega$, is divided into two subsets $\partial_u\Omega = \partial_u\mathcal{B}_0 \cap \partial\Omega$ and $\partial_\sigma\Omega = \partial_\sigma\mathcal{B}_0 \cap \partial\Omega$. The laminated shell is composed of n layers, separated by $n - 1$ interfaces. Using the notation $(\bullet)^{(k)}$ to designate a quantity associated with the k -th layer or interface, we note that the k -th interface, situated between layers k and $k + 1$, coincides with the surface $\xi^3 = \xi^{3(k)}$. Accordingly, the thickness of the k -th layer is given by

$$h^{(k)} = \frac{h}{2} \left[\xi^{3(k)} - \xi^{3(k-1)} \right]. \quad (2)$$

The problem is governed by the principle of conservation of linear momentum, and can be stated as follows [41]:

$$\nabla_0 \cdot \mathbf{P} + \mathbf{b} = \rho_0 \ddot{\mathbf{u}} \quad \text{in } \mathcal{B}_0, \quad (3)$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \partial_u\mathcal{B}_0, \quad (4)$$

$$\mathbf{P} \cdot \mathbf{n} = \bar{\mathbf{h}} \quad \text{on } \partial_\sigma\mathcal{B}_0, \quad (5)$$

for all time $t \in (0, \tau]$. Here, $\mathbf{u}$ denotes the displacement, ρ_0 is the initial mass density, $\mathbf{b}$ is the body force per unit initial volume, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\bar{\mathbf{u}}$ and $\bar{\mathbf{h}}$ are the specified boundary displacements and tractions, respectively, and $\mathbf{n}$ is the outward unit normal to $\partial_\sigma\mathcal{B}_0$. Initial conditions are given by

$$\mathbf{u} = \mathbf{u}_0 \quad \text{in } \mathcal{B}_0, \text{ at } t = 0, \quad (6)$$

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_0 \quad \text{in } \mathcal{B}_0, \text{ at } t = 0, \quad (7)$$

where $\mathbf{u}_0$ and $\dot{\mathbf{u}}_0$ are known.

A quantity that undergoes a jump $[[\bullet]]^{(k)}$ across the k -th interface, has an average value at the interface denoted by $\langle \bullet \rangle^{(k)}$, such that

$$[[\bullet]]^{(k)} = \left[(\bullet)^{(k+1)} \Big|_{\xi^3(k)} - (\bullet)^{(k)} \Big|_{\xi^3(k)} \right], \quad (8)$$

$$\langle \bullet \rangle^{(k)} = \frac{1}{2} \left[(\bullet)^{(k+1)} \Big|_{\xi^3(k)} + (\bullet)^{(k)} \Big|_{\xi^3(k)} \right]. \quad (9)$$

Equilibrium considerations require traction continuity across every interface, i.e.

$$[[\mathbf{P} \cdot \mathbf{a}_3]]^{(k)} = \mathbf{0}, \quad (10)$$

for all $k = 1, \dots, n-1$. In addition, when a perfectly-bonded interface is considered, the displacement field must also be continuous:

$$[[\mathbf{u}]]^{(k)} = \mathbf{0}. \quad (11)$$

With a debonded interface, the interfacial traction field is assumed to be a known function of the displacement jump:

$$\mathbf{P}^{(k)} \Big|_{\xi^3(k)} \cdot \mathbf{a}_3 = \mathbf{f}([[\mathbf{u}]]^{(k)}). \quad (12)$$

Finally, it is noted that in the following, unless otherwise stated, Greek indices vary from 1 to 2, while their Latin counterparts vary from 1 to 3.

3. Kinematic description

It is recalled from the previous section that the midsurface of the shell is defined by the mapping, $\mathbf{r}$, of convected curvilinear coordinates, ξ^α , into Cartesian coordinates in $\mathbb{R}^3$. The covariant base vectors of the tangential plane are thus given by (cf. Ref. [42]):

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \xi^\alpha}, \quad (13)$$

and the unit normal vector is obtained from

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}. \quad (14)$$

In light of the mapping (1), three-dimensional covariant base vectors for the ξ -reference frame are given by

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial \xi^i}, \quad (15)$$

Substituting Eq. (1) into (15), the covariant base vectors $\mathbf{g}_i$ may be expressed as follows

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha + \frac{h}{2} \xi^3 \mathbf{a}_{3,\alpha}, \quad \mathbf{g}_3 = \frac{h}{2} \mathbf{a}_3. \quad (16)$$

Following Dvorkin and Bathe [26], $\mathbf{a}_3$ is considered here and henceforth to be an approximation to the reference surface normal given by Eq. (14). Accordingly, it is not treated as strictly orthogonal to $\mathbf{a}_\alpha$ in the above expressions. Quantities computed in the reference frame defined by the covariant base $\mathbf{g}_i$ are mapped to the reference surface using the shifter tensor [43, 44], given by

$$\mu_\beta^\alpha = \mathbf{a}_\beta \cdot \mathbf{g}^\alpha \quad (17)$$

where the contravariant base $\mathbf{g}^\alpha$ is obtained from $\mathbf{g}_\alpha$ by means of tensor transformations. Therefore, an element of area on the k -th interface is given by

$$d\Omega^{(k)} = \mu^{(k)} d\Omega, \quad (18)$$

where

$$\mu = \det(\mu_\beta^\alpha), \quad d\Omega = \|\mathbf{a}_1 \times \mathbf{a}_2\| d\xi^1 d\xi^2, \quad (19)$$

and

$$\mu^{(k)} = \mu|_{\xi^3(k)}. \quad (20)$$

For more details, see Versino et al. [22].

Denoting by $\hat{\mathbf{R}}$ the mapping between the initial and the current (deformed) configurations,

$$\hat{\mathbf{R}}(\xi^i) = \mathbf{R}(\xi^i) + \mathbf{u}(\xi^i), \quad (21)$$

allows the deformation gradient to be expressed as follows (cf. Ref. [45, p. 56]):

$$\mathbf{F}_i = \frac{\partial \hat{\mathbf{R}}}{\partial \xi^i} = \mathbf{g}_i + \frac{\partial \mathbf{u}}{\partial \xi^i}, \quad (22)$$

and accordingly, the right Cauchy-Green tensor and Green-Lagrange strain tensor are given, respectively, by

$$\mathbf{C}_{ij} = \mathbf{F}_i \cdot \mathbf{F}_j, \quad (23)$$

$$\mathbf{E}_{ij} = \frac{1}{2}(\mathbf{C}_{ij} - \mathbf{g}_i \cdot \mathbf{g}_j). \quad (24)$$

4. Global–local shell theory

In the context of the global–local theory of Williams [17], the displacement of the k -th layer, $\mathbf{u}^{(k)}$, is expressed as the sum of a global displacement, $\bar{\mathbf{u}}$, and a layerwise displacement, $\tilde{\mathbf{u}}^{(k)}$, particular to the k -th layer:

$$\mathbf{u}^{(k)} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}^{(k)}, \quad (25)$$

with

$$\bar{\mathbf{u}}(\xi^i, t) = \sum_{r \in \bar{P}} \bar{\phi}_r(\xi^3) \bar{\psi}_r(\xi^\alpha, t), \quad (26)$$

$$\tilde{\mathbf{u}}^{(k)}(\xi^i, t) = \sum_{s \in \bar{P}^{(k)}} \tilde{\phi}_s^{(k)}(\xi^3) \tilde{\psi}_s^{(k)}(\xi^\alpha, t), \quad (27)$$

where $\bar{\psi}_r$ and $\tilde{\psi}_s^{(k)}$ are the unknown global and local solution functions, respectively. Moreover, $\bar{\phi}_r$ is the r -th order global expansion function, and $\tilde{\phi}_s^{(k)}$ is the s -th order expansion function defined on the k -th layer only. Although the global–local framework does not place any restrictions on the choice of these expansion functions, in practice, good convergence and stability characteristics require these functions to be chosen judiciously. Moreover, in order to guarantee the well-posedness of the problem, it is crucial to not have redundant degrees of freedom [5, 18, 22, 23]. In this work, monomial expansions, $(\xi^3)^m$ with $m \in \mathbb{N}$, and Lagrangian polynomial expansions [46], $\mathbb{L}_m$ with $m \in \mathbb{N}$, are employed.

Combining Eqs. (22–25), the Green-Lagrange strain in the k -th layer may be written as a sum of linear and nonlinear terms, denoted by $\varepsilon_{ij}^{(k)}$ and $\eta_{ij}^{(k)}$, respectively:

$$\mathbf{E}_{ij}^{(k)} = \varepsilon_{ij}^{(k)} + \eta_{ij}^{(k)}, \quad (28)$$

with

$$\varepsilon_{ij}^{(k)} = \frac{1}{2} \left(\mathbf{g}_i \cdot \frac{\partial(\bar{\mathbf{u}} + \tilde{\mathbf{u}}^{(k)})}{\partial \xi^j} + \frac{\partial(\bar{\mathbf{u}} + \tilde{\mathbf{u}}^{(k)})}{\partial \xi^i} \cdot \mathbf{g}_j \right), \quad (29)$$

$$\eta_{ij}^{(k)} = \frac{1}{2} \left(\frac{\partial(\bar{\mathbf{u}} + \tilde{\mathbf{u}}^{(k)})}{\partial \xi^i} \cdot \frac{\partial(\bar{\mathbf{u}} + \tilde{\mathbf{u}}^{(k)})}{\partial \xi^j} \right). \quad (30)$$

It should be noted that the nonlinear part of the strain field, $\eta_{ij}^{(k)}$, introduces a strong coupling between global and local fields. Therefore, it is not possible to identify global and local strain fields as in the small-deformation case [22].

The displacement weighting functions, $\mathbf{w}^{(k)}$, are introduced to derive the weak form of the initial boundary value problem from Eqs. (3–5):

$$\mathbf{w}^{(k)} = \bar{\mathbf{w}} + \tilde{\mathbf{w}}^{(k)}. \quad (31)$$

Here, $\bar{\mathbf{w}}$ and $\tilde{\mathbf{w}}^{(k)}$ are the global and local components of the weighting function, respectively, and are given by:

$$\bar{\mathbf{w}}(\xi^i) = \sum_{p \in \bar{P}} \bar{\phi}_p(\xi^3) \bar{\omega}_p(\xi^\alpha), \quad (32)$$

$$\tilde{\mathbf{w}}^{(k)}(\xi^i) = \sum_{q \in P^{(k)}} \tilde{\phi}_q^{(k)}(\xi^3) \tilde{\omega}_q^{(k)}(\xi^\alpha), \quad (33)$$

where $\bar{\omega}_p$ and $\tilde{\omega}_q^{(k)}$ are arbitrary functions.

5. Weak form of the equations of motion

Multiplying Eq. (3) by the weighting function, integrating the product over the initial configuration of the shell, and making use of the divergence theorem, we obtain the weak form of the equations of motion, which can be written as follows:

$$\begin{aligned} \int_{\Omega} \int_h \mathbf{E}(\mathbf{w}) : \mathbf{S} \mu d\xi^3 d\Omega - \sum_{k=1}^{n-1} \int_{\Omega} \left(\mathbf{w}^{(k)} \cdot \mathbf{P}^{(k)} \cdot \mathbf{a}_3 - \mathbf{w}^{(k+1)} \cdot \mathbf{P}^{(k+1)} \cdot \mathbf{a}_3 \right) \Big|_{\xi^3=\xi^{3(k)}} \mu^{(k)} d\Omega \\ - \int_{\Omega} \mathbf{w}^{(1)} \Big|_{\xi^3=\xi^{3(0)}} \cdot \bar{\boldsymbol{\tau}}^{(0)} \mu^{(0)} + \mathbf{w}^{(n)} \Big|_{\xi^3=\xi^{3(n)}} \cdot \bar{\boldsymbol{\tau}}^{(n)} \mu^{(n)} d\Omega \\ - \int_{\partial_r \Omega} \int_h \mathbf{w} \cdot \bar{\mathbf{h}} \mu d\xi^3 d\Gamma - \int_{\Omega} \int_h \mathbf{w} \cdot \mathbf{b} \mu d\xi^3 d\Omega + \int_{\Omega} \int_h \rho \mathbf{w} \cdot \ddot{\mathbf{u}} \mu d\xi^3 d\Omega = 0, \end{aligned} \quad (34)$$

where the notation $\int_h(\bullet) d\xi^3$ signifies integration through the thickness of the shell, i.e.

$$\int_h(\bullet) d\xi^3 = \sum_{k=1}^n \int_{h^{(k)}}(\bullet) d\xi^3 = \frac{h}{2} \sum_{k=1}^n \int_{\xi^{3(k-1)}}^{\xi^{3(k)}}(\bullet) d\xi^3, \quad (35)$$

and $\mathbf{S} = \mathbf{P} \cdot \mathbf{F}^{-T}$ is the second Piola-Kirchhoff stress.

Making use of the enhanced assumed strain (EAS) methodology [47, 48], we introduce the incompatible enhanced strain, $\hat{\mathbf{E}}_u$, and the associated weighting function, $\hat{\mathbf{E}}_w$, and write:

$$\mathbf{E}(\mathbf{u}) = \mathbf{E}^c(\mathbf{u}) + \hat{\mathbf{E}}_u, \quad \mathbf{E}(\mathbf{w}) = \mathbf{E}^c(\mathbf{w}) + \hat{\mathbf{E}}_w, \quad (36)$$

where $\mathbf{E}^c(\mathbf{u})$ is the compatible strain field obtained from (24). Substituting (36) into (34) leads to a mixed formulation with displacement and enhanced strain as primary unknowns. Thus, introducing a tessellation Ω^h of the reference surface Ω , consisting of a set η_e of individual elements Ω^e with boundaries $\partial\Omega^e$, allows the weak form to be written in the following compact form:

$$\sum_{\Omega^e \in \eta_e} \left[\Pi_1^\sigma(\mathbf{w}, \hat{\mathbf{E}}_w, \mathbf{u}, \hat{\mathbf{E}}_u) + \Pi_1^\tau(\mathbf{w}, \mathbf{u}, \hat{\mathbf{E}}_u) - \Pi_E(\mathbf{w}, \mathbf{u}) - \Pi_E^\tau(\mathbf{w}, \mathbf{u}) + \Pi_\rho(\mathbf{w}, \mathbf{u}) \right] = 0, \quad (37)$$

in terms of the individual virtual work contributions, defined as follows:

$$\Pi_1^\sigma = \int_{\Omega^e} \int_h [\mathbf{E}^c(\mathbf{w}) + \hat{\mathbf{E}}_w] : \mathbf{S} \mu d\xi^3 d\Omega^e, \quad (38)$$

is the internal work, associated with deformation of the composite layers in the presence of stress. It is worth noting here that within the EAS framework, the enhanced strain field is orthogonal to the stress field, i.e.

$$\int_{\Omega^e} \int_h \hat{\mathbf{E}}_w : \mathbf{S} \mu d\xi^3 d\Omega^e = 0. \quad (39)$$

The contribution of interfacial tractions to the virtual work is given by:

$$\Pi_I^\tau = \sum_{k=1}^{n-1} \int_{\Omega^e} \llbracket \mathbf{w} \rrbracket^{(k)} \cdot \left[\boldsymbol{\alpha}^{(k)} \cdot \llbracket \mathbf{u} \rrbracket^{(k)} - \langle \mathbf{P} \cdot \mathbf{a}_3 \rangle^{(k)} \right] \mu^{(k)} d\Omega^e, \quad (40)$$

where the second term of the weak form (34) is expressed in terms of interfacial jump and average quantities [49]. Discontinuous Galerkin numerical fluxes of the non-symmetric interior penalty type [36] are also introduced here in order to impose interfacial displacement continuity, i.e. $\llbracket \mathbf{u} \rrbracket^{(k)} = 0$, with $k = 1, \dots, n-1$. The rank-2 tensor $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha}^{(k)} = \frac{\lambda^{(k)}}{h_s^{(k)}} \left(\mathbf{a}_3 \cdot \langle \mathbb{C} \rangle^{(k)} \cdot \mathbf{a}_3 \right), \quad (41)$$

where $\mathbb{C}^{(k)}$ is the tensor of tangent moduli of the k -th layer material, h_s is a characteristic length, and λ is an arbitrary scalar penalty parameter; see Ref. [36] for details.

Furthermore, the external work, associated with body forces and tractions applied on the lateral boundaries of the shell, is given by

$$\Pi_E = \int_{\Omega^e} \int_h \mathbf{w} \cdot \mathbf{b} \mu d\xi^3 d\Omega^e + \int_{\partial_{\sigma}\Omega^e} \int_h \mathbf{w} \cdot \bar{\mathbf{h}} \mu d\xi^3 d\Gamma^e, \quad (42)$$

where $\partial_{\sigma}\Omega^e = \partial\Omega^e \cap \partial_{\sigma}\Omega^h$, whereas the contribution of the traction forces $\bar{\boldsymbol{\tau}}^{(0)}$ and $\bar{\boldsymbol{\tau}}^{(n)}$, acting on the bottom and top surfaces of the shell, respectively, is given by

$$\Pi_E^\tau = \int_{\Omega^e} \mathbf{w}^{(1)} \Big|_{\xi^3=\xi^3(0)} \cdot \bar{\boldsymbol{\tau}}^{(0)} \mu^{(0)} + \mathbf{w}^{(n)} \Big|_{\xi^3=\xi^3(n)} \cdot \bar{\boldsymbol{\tau}}^{(n)} \mu^{(n)} d\Omega^e, \quad (43)$$

and the inertial contribution is given by

$$\Pi_p = \int_{\Omega^e} \int_h \rho \mathbf{w} \cdot \ddot{\mathbf{u}} \mu d\xi^3 d\Omega^e. \quad (44)$$

Finally, it is noted that the weak form (37) yields a non-symmetric tangent stiffness matrix. [However, this does not have a negative impact on the stability or convergence characteristics of the formulation, as shown by Noels and Radovitzky \[36\]. In fact, using the three-field Hu–Washizu variational principle \(generalized to the finite-deformation regime\) as a starting point, leads to a symmetric discontinuous Galerkin weak form, which loses its symmetry only after certain interfacial terms are discarded \[36\]. Here, we adopt this common practice \[49\] of ignoring the symmetrizing interfacial terms, and relying instead on the penalty term for enforcement of interfacial continuity conditions.](#)

6. Computational formulation and implementation

In this section, we present the formulation of the finite element based on the weak form derived in Section 5. Bilinear shape functions are employed and the ANS and EAS strategies are used to improve the element's performance. The EAS interpolation scheme is described and the semi-discrete form of the equations of motion is written in terms of residuals from which the stiffness matrix is derived via linearization. In addition, the constitutive update procedure is outlined and the solution strategy is summarized.

6.1. Spatial interpolation

A point on the reference surface is mapped from the reference bi-unit square, $[-1, 1]^2$, to physical space by the standard isoparametric bilinear shape functions, $\mathbf{N}(\xi^\alpha)$. The interpolation of global and local displacements is hence given by

$$\bar{\psi}_r(\xi^\alpha, t) = \mathbf{N}(\xi^\alpha) \bar{\mathbf{d}}_u^r(t), \quad (45)$$

$$\tilde{\psi}_s^{(k)}(\xi^\alpha, t) = \mathbf{N}(\xi^\alpha) \tilde{\mathbf{d}}_u^{s(k)}(t). \quad (46)$$

The same interpolation scheme is used for the global weighting function, $\bar{\omega}$, and its local counterpart, $\tilde{\omega}^{(k)}$.

In the computation of the deformation gradient using Eq. (22), the director (i.e. unit normal) and curvatures of the reference surface are required. When smooth surfaces are considered, the continuity of the director field along inter-element boundaries is ensured by first computing the director at each node, as an average between elements sharing the node [50]. The curvatures are obtained as the spatial derivatives of the director field, using the procedure detailed in [51, 52]. The isoparametric bilinear shape functions and their spatial derivatives are used, respectively, to interpolate the director and evaluate the curvatures at the Gauss quadrature points.

6.2. Treatment of mesh locking

Different mesh locking phenomena affect the performance of bilinear isoparametric shell elements that account for transverse strains, in problems involving thin structures under bending-dominated conditions. The current formulation employs assumed natural strain (ANS) interpolation [25, 53] to remedy shear and curvature locking. Namely, MITC4 interpolation [26] is used to alleviate transverse shear locking, and bilinear interpolation of the transverse normal strain, E_{33} , is used to mitigate curvature locking [27]. In addition, an enhanced assumed strain (EAS) approach [47, 54] is adopted to relieve Poisson thickness locking, which is known to occur due to the imbalance between the linear through-thickness variation of the in-plane strains, $E_{\alpha\beta}$, and the constant normal strain, E_{33} , giving rise to a *parasitic* transverse normal stress in the case of non-zero Poisson's ratio. Following Büchter et al. [28, 29], this imbalance is addressed by introducing a linearly-varying enhanced strain component, $\hat{E}_{33}$. Membrane locking has a relatively mild effect when bilinear in-plane interpolation is used, as in the current formulation [33]. Nevertheless, membrane locking may affect the behavior of highly distorted elements. Therefore, appealing to the same EAS concepts, the in-plane strain components are enhanced to improve the membrane behavior of the present element, as explained in the next section.

6.3. Enhanced strain interpolation

The enhanced strain field, $\hat{\mathbf{E}}_u$, and the associated weighting function, $\hat{\mathbf{E}}_w$, are obtained at the quadrature points via the following transformation:

$$\hat{\mathbf{E}} = \frac{J}{J_0} \hat{\mathbf{E}}_{ij}^0 \mathbf{g}_0^i \otimes \mathbf{g}^j \quad (47)$$

where $J = \det([\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3])$ and $J_0 = \det([\mathbf{g}_{01}, \mathbf{g}_{02}, \mathbf{g}_{03}])$, with $\mathbf{g}_{0i}$ and $\mathbf{g}_0^i$ denoting the covariant and contravariant base vectors at the element center, respectively. This transformation is necessary in order to satisfy patch test requirements [51]. Here, the enhanced strains, $\hat{\mathbf{E}}_u^0$, are defined as:

$$\hat{\mathbf{E}}_u^0(\xi^i, t) = \hat{\mathbf{N}}(\xi^i) \boldsymbol{\beta}_u(t), \quad (48)$$

in terms of the known interpolation function, $\hat{\mathbf{N}}$, and the unknown parameters, $\boldsymbol{\beta}_u$. A similar interpolation scheme is used for the weighting functions, i.e. $\hat{\mathbf{E}}_w^0 = \hat{\mathbf{N}} \boldsymbol{\beta}_w$. Numerous works in the literature focused on the optimal choice of EAS interpolation function, $\hat{\mathbf{N}}$, which can best address finite element pathological behaviors ranging from shear locking to volumetric locking, e.g. see Ref. [32]. In the present work, $\hat{\mathbf{N}}$ is defined via

$$\begin{Bmatrix} \hat{E}_{11}^0 \\ \hat{E}_{22}^0 \\ \hat{E}_{12}^0 \\ \hat{E}_{33}^0 \end{Bmatrix} = \hat{\mathbf{N}} \boldsymbol{\beta} = \begin{bmatrix} \xi^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^1 & \xi^2 & \xi^1 \xi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi^3 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{Bmatrix} \quad (49)$$

which includes five EAS parameters $(\beta_1, \dots, \beta_5)$ to improve the membrane behavior and one additional parameter (β_6) to alleviate Poisson thickness locking. It is noted that, when either the global or local displacement expansion includes quadratic or higher order terms, the element is not susceptible to Poisson thickness locking, and thus $\hat{\mathbf{E}}_{33}^0$ is not included in the formulation.

6.4. Semi-discrete weak form

For a simpler, more compact notation, we collect global and local nodal displacements, respectively, within the vectors $\tilde{\mathbf{d}}_u$ and $\tilde{\mathbf{d}}_u$; i.e.

$$\tilde{\mathbf{d}}_u = \left\{ \tilde{\mathbf{d}}_u^r, \forall r \in \bar{P} \right\}, \quad \tilde{\mathbf{d}}_u = \left\{ \left\{ \tilde{\mathbf{d}}_u^{s(k)}, \forall s \in \tilde{P}^{(k)} \right\}, \forall k \in [1, n] \right\}, \quad (50)$$

and similarly, the arbitrary coefficients of the global and local weighting functions are collected, respectively, within the vectors $\tilde{\mathbf{d}}_w$ and $\tilde{\mathbf{d}}_w$. The semi-discrete weak form of the equations of motion is obtained by introducing the approximation of the solution and weighting function, Eqs. (45)–(49), into the weak form (37). The element residuals corresponding to the global expansions, $\tilde{\mathbf{r}}$, local expansions, $\tilde{\mathbf{r}}$, and strain enhancements, $\hat{\mathbf{r}}$, obtained from the weak form, are defined as follows

$$\tilde{\mathbf{r}} := \frac{\partial}{\partial \tilde{\mathbf{d}}_w} \left(\Pi_\rho + \Pi_I^\sigma + \Pi_I^\tau - \Pi_E - \Pi_E^\tau \right), \quad (51)$$

$$\tilde{\mathbf{r}} := \frac{\partial}{\partial \tilde{\mathbf{d}}_w} \left(\Pi_\rho + \Pi_I^\sigma + \Pi_I^\tau - \Pi_E - \Pi_E^\tau \right), \quad (52)$$

$$\hat{\mathbf{r}} := \frac{\partial \Pi_I^\sigma}{\partial \beta_w}. \quad (53)$$

Finally, the element's consistent tangent matrix is given by

$$\mathbf{K} := \begin{bmatrix} \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{\mathbf{d}}_u} & \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{\mathbf{d}}_u} & \frac{\partial \tilde{\mathbf{r}}}{\partial \beta_u} \\ \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{\mathbf{d}}_u} & \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{\mathbf{d}}_u} & \frac{\partial \tilde{\mathbf{r}}}{\partial \beta_u} \\ \frac{\partial \hat{\mathbf{r}}}{\partial \tilde{\mathbf{d}}_u} & \frac{\partial \hat{\mathbf{r}}}{\partial \tilde{\mathbf{d}}_u} & \frac{\partial \hat{\mathbf{r}}}{\partial \beta_u} \end{bmatrix}. \quad (54)$$

6.5. Solution procedure

The solution procedure is divided into three steps: (i) evaluation of the EAS parameters at the element level, exploiting the locality of the strain enhancement, (ii) assembly of the resulting element residual vectors and tangent matrix, and (iii) solution of the finite element matrix equations. In the first step, EAS unknowns are computed using a Newton-Raphson scheme where the residual is given by Eq. (53) and the tangent matrix is $\mathbf{K}_{\beta\beta} = \partial \hat{\mathbf{r}} / \partial \beta_u$. It is worth observing that the computation of the interfacial fluxes is not necessary during this local iterative process since the tangent matrix $\mathbf{K}_{\beta\beta}$ and the residual vector $\hat{\mathbf{r}}$ depend on Π_I^σ only. Finally, static condensation is performed to eliminate the local strain enhancements from the element's residual vector and tangent matrix; see Refs. [32, 55] for more details. Then, step (ii) is performed using the standard finite element assembly procedure [56].

In dynamic problems, an implicit or explicit Newmark scheme is used to step the solution forward in time. In the implicit case, or in static analysis, Newton's method is used to iteratively solve the finite element equations of motion for the nodal displacements (step iii). A detailed description of this step is given in Ref. [22]. Moreover, when explicit time integration is employed, the size of the critical time step Δt_c depends on the smallest element size, h_s , and the highest speed of sound in the model, c , as well as on the penalty parameter λ ; see Refs. [22, 36] for details.

6.6. Material constitutive models

The proposed element formulation does not place any restrictions on the admissible material behavior, and is thus compatible with general three-dimensional constitutive models. In Section 7, we consider numerical examples involving either finite elasticity or elasto-plasticity. In the purely elastic case, we adopt the Saint Venant–Kirchhoff model. Accordingly, the second Piola–Kirchhoff stress is computed as follows:

$$\mathbf{S} = \mathbb{C}^e : \mathbf{E}, \quad (55)$$

where $\mathbb{C}^e$ is the (generally anisotropic) fourth-order tensor of constant elastic moduli, and $\mathbf{E}$ is the Green-Lagrange strain including ANS and EAS enhancements, as described above.

In the case of elasto-plastic behavior, we adopt the notion of an intermediate stress-free configuration and the associated multiplicative decomposition of the deformation gradient $\mathbf{F}$ into elastic and plastic parts, denoted by $\mathbf{F}^e$ and $\mathbf{F}^p$, respectively. As in the case of finite elasticity, the stress–strain relations derive from a hyper-elastic potential, and can be expressed in the following form:

$$\tilde{\mathbf{S}} = \mathbb{C}^e : \tilde{\mathbf{E}}^e, \quad (56)$$

where $\tilde{\mathbf{E}}^e$ is the elastic Green-Lagrange strain tensor defined on the intermediate configuration, and $\tilde{\mathbf{S}}$ is related to the second Piola–Kirchhoff stress through the push-forward operation, $\tilde{\mathbf{S}} = \mathbf{F}^p \cdot \mathbf{S} \cdot \mathbf{F}^{pT}$. Here, it is assumed that the elasticity tensor $\mathbb{C}^e$ is isotropic, and the plastic response is governed by J_2 flow theory with linear isotropic hardening. In the present computational setting, we utilize the radial return algorithm, described in detail in Ref. [57, chap. 9], to update the local stress and plastic state variables ($\mathbf{F}^p$ and the accumulated plastic strain) at each finite element integration point, starting from a known configuration ($\mathbf{F}$) and material state at time $t = t_n$, and given $\mathbf{F}$ at $t = t_n + \Delta t$. Here, in order to make use of the ANS and EAS enhancements included in the Green-Lagrange strain $\mathbf{E}$, we calculate $\mathbf{F}$ using the following expression, inspired by the work of Dvorkin et al. [58]:

$$\mathbf{F} = \mathbf{R} \cdot (2\mathbf{E} + \mathbb{1})^{\frac{1}{2}} \quad (57)$$

where the rotation tensor $\mathbf{R}$, obtained from the polar decomposition of the deformation gradient given by Eq. (22), is multiplied by the enhanced stretch tensor $\mathbf{U} = (2\mathbf{E} + \mathbb{1})^{\frac{1}{2}}$, and $\mathbb{1}$ is the second-order identity tensor.

6.7. Failure criterion and cohesive law

In order to model multi-layered structures undergoing *progressive failure*, we define a failure criterion and a cohesive traction–separation law. The failure criterion defines the conditions under which an interface transitions from the perfectly-bonded to the delaminated state. When this criterion is met, the interfacial DG fluxes are replaced by the cohesive model, in a manner similar to that described by Mergheim et al. [37, 59]. For simplicity, we consider Mode I failure only in the current work, and adopt a maximum stress failure criterion; i.e. a crack initiates at a given quadrature point on the k -th interface when $T_3^{(k)} > T_{\max}$, where $T_{\max}$ is the Mode I maximum interfacial strength, and $\mathbf{T}^{(k)}$ is a pseudo-traction vector, defined as follows:

$$\mathbf{T}^{(k)} = \langle \mathbf{P} \cdot \mathbf{a}_3 \rangle^{(k)} - \alpha^{(k)} \cdot \llbracket \mathbf{u} \rrbracket^{(k)}. \quad (58)$$

Note that the pseudo-traction vector coincides with the traction average, $\langle \mathbf{P} \cdot \mathbf{a}_3 \rangle^{(k)}$, when the displacement-continuity condition is satisfied at the interface, i.e. $\llbracket \mathbf{u} \rrbracket^{(k)} = 0$. The second term in Eq. (58) accounts for the fact that the computed displacement field may exhibit a non-negligible jump across the interface, in the proximity of the crack tip. Arbitrariness of the penalty parameter, $\alpha^{(k)}$, is avoided by computing the scalar parameter, $\lambda^{(k)}$, introduced in Eq. (41), according to the technique described in Ref. [60].

An *extrinsic* traction–separation law [61] is employed to model delamination behavior. According to this cohesive law, the transverse normal component of the second Piola-Kirchhoff stress, at the (delaminated) k -th interface, is given by

$$S_{33}^{(k)} = \begin{cases} T_{\max}^{(k)} \left(1 + \frac{\llbracket \mathbf{u} \cdot \mathbf{a}_3 \rrbracket^{(k)}}{\llbracket \mathbf{u} \rrbracket_{\max}^{(k)}} \right), & \text{if } 0 < \llbracket \mathbf{u} \cdot \mathbf{a}_3 \rrbracket^{(k)} < \llbracket \mathbf{u} \rrbracket_{\max}^{(k)}, \\ 0, & \text{if } \llbracket \mathbf{u} \cdot \mathbf{a}_3 \rrbracket^{(k)} > \llbracket \mathbf{u} \rrbracket_{\max}^{(k)}, \end{cases} \quad (59)$$

where $\llbracket u \rrbracket_{\max}^{(k)}$ is the separation at complete decohesion, defined as:

$$\llbracket u \rrbracket_{\max}^{(k)} = 2 \frac{G_{IC}^{(k)}}{T_{\max}^{(k)}}, \quad (60)$$

and $G_{IC}^{(k)}$ is the Mode I fracture toughness of the k -th interface. Furthermore, a large stiffness is used to penalize interpenetration ($\llbracket u \cdot a_3 \rrbracket^{(k)} < 0$); see Ref. [62] for details.

7. Numerical examples

In this section, the performance of the developed element is assessed with the aid of well established numerical experiments, and its capabilities are examined in static as well as dynamic analysis. The global–local expansions used throughout the current section are given, for each test problem, in Table 1 where the following notation is introduced: $\mathbb{L}_n$ denotes a complete Lagrange polynomial of order n , while an array of ones and zeros defines an expansion composed of monomials (e.g. an expansion with constant, linear and cubic monomials is indicated with $\{1, 1, 0, 1\}$). Unless otherwise specified, material parameters, geometric dimensions, loads, displacement and stresses are given in consistent units of measure.

The remainder of this section is organized as follows. The first two problems involve thin isotropic homogeneous shells (Sections 7.1–7.2). Although a single-layer representation, with a linear expansion and EAS, would suffice to enable spatial convergence in these two problems, the shell’s thickness is divided into two computational layers for validation and verification purposes. **The third and fourth problems involve a thin annular laminated plate (Section 7.3) and a thin laminated cylindrical shell (Section 7.4), respectively. In Section 7.5, buckling of a hinged cylindrical panel subjected to a concentrated load is considered.** In Section 7.6, we consider the response of a thick laminated cylinder in the small-deformation regime, and make use of an existing analytical solution to assess the accuracy of the computed displacements and interfacial tractions. Subsequently, a crack propagation analysis is carried out on a two-layer laminate undergoing Mode I failure (Section 7.7). Finally, in Section 7.8, we present an explicit dynamic analysis of a cylindrical shell subjected to impulse loading.

To facilitate visualization of the computational results, each shell element is replaced, in the post-processing phase, by a set of hexahedral (brick) elements stacked in the through-thickness direction. The number of brick elements used to represent each layer is equal to the number of through-thickness quadrature points in that layer. Each brick has the same in-plane dimensions as its parent shell element, and is placed such that its centroid has the same through-thickness coordinate as the corresponding quadrature point.

7.1. Hemi-spherical cap

In this example problem [55, 63, 64], we consider a hemi-spherical shell with radius $R = 10$, thickness $h = 0.04$ and a hole cut out from the top (Fig. 2a) with angle $\theta = 18^\circ$ measured from the cap revolution axis. The shell is made

Table 1: Through-thickness expansions and number of EAS parameters used in each example problem.

Section	No. of layers	Global expansion	Local expansions	No. of EAS parameters
7.1, 7.2	2	–	$\{1, 1\}$ for all layers	6
7.3, 7.4	3	$\{1, 0\}$	$\{0, 1\}$ for all layers	6
7.5	4	$\{1, 0\}$	$\{0, 1\}$ for all layers	6
7.6	3	$\{1, 1, 1, 1\}$	$\{1, 1, 1, 1\}$ for layers 1,3 $\{0, 0, 0, 0\}$ for layer 2	5
7.7	2	$\{0, 0, 1\}$	$\{1, 1, 0\}$ for all layers	5
7.8	2	–	$\mathbb{L}_1$ for all layers	6

of an isotropic material with Young's modulus $E = 6.825 \times 10^7$ and Poisson's ratio $\nu = 0.3$. Making use of symmetry, we model one quarter of the cap only, using a mesh comprising 16×16 elements, as shown in Fig. 2a. Keeping its top and bottom edges traction-free, the shell is pinched by alternating radial point loads, acting along the radial direction at points A and B (Fig. 2a), leading to the deformed shape shown in Fig. 2b. The pinching loads are plotted in Fig. 3 against the radial deflections at A and B, showing excellent agreement with the reference solution obtained by Sze et al. [65] using the finite-strain quadrilateral (S4R) shell elements in ABAQUS [66].

7.2. Elasto-plastic cylinder

The problem considered in this section involves a homogeneous cylindrical shell made of an elasto-plastic isotropic material with Young's modulus $E = 3000$, Poisson's ratio $\nu = 0.3$, yield strength $\sigma_Y = 24.3$ and linear isotropic hardening modulus $K_{iso} = 300$. The cylinder's radius, length and thickness are $R = 300$, $L = 300$ and $h = 3$, respectively. Exploiting symmetry, only one quarter of the cylinder is modeled using $n_e \times n_e$ shell elements, as shown in Fig. 4. Two meshes corresponding to $n_e = 16, 32$ are considered. One of the cylinder's circular sides (side C–D) is clamped while the other (A–B) is free, and a pinching point-load is applied at point A. The load is plotted against the radial displacement, u_r , at point A in Fig. 5. Results computed with the present element, employing the expansion defined in Table 1, are compared to other solutions found in the literature [67–69]. Our results ($n_e = 32$) are in very close agreement with those of Miehe [68], especially for $|u_r| \leq 200$. Good agreement is also observed with the results of Valente et al. [69]. Moreover, wiggles in the load–deflection curve may be noticed in the neighborhood of $|u_r| = 150$. These oscillations seem to indicate snap-through behavior, accompanied by quick expansion of the plastic zone at $|u_r| = 157.5$ and $|u_r| = 171.3$, especially in the highly-curved regions that develop as the shell is deformed.

The deformed shape, at a radial displacement $|u_r| = 150$, is shown in Fig. 6. Regions in the structure that have undergone plastic deformation (i.e. plastic zones) are shaded in red, whereas regions that have remained elastic are shaded in blue. This simulation was carried out using a mesh characterized by $n_e = 32$. The results shown in Figs. 6a and 6b were obtained using a single-layer representation and a linear expansion with three quadrature points along the through-thickness direction. Moreover, results obtained with the two-layer expansion described in Table 1, using a total of seven integration points along the through-thickness direction (three points for each computational layer and one point for the shared interface) are presented in Figs. 6c and 6d.

Figure 7 shows the plastic zones in the final deformed configuration, corresponding to $|u_r| = 250$. At this stage of deformation, we obtain equivalent plastic strain values of 0.406 and 0.440, at point A, using the single-layer and two-layer expansions described above, respectively. In comparison, Wagner et al. [70] report a value of 0.3016, computed using a coarser mesh ($n_e = 16$). It can also be seen from Figs. 6 and 7 that the two-layer model (bottom row) predicts more widespread plastic deformation, i.e. larger plastic zones, compared to the single-layer model (top row).

7.3. Slit annular plate subjected to an end shear force

Next, we consider a laminated composite slit annular plate [43, 71] with inner radius $R_i = 6$, outer radius $R_o = 10$ and thickness $h = 0.03$. A shear load acting along the normal to the plate is applied to the free edge, while the opposite edge (A–B) is clamped; see Fig. 8a. The plate consists of three layers of equal thickness. Each layer is made of an orthotropic elastic material with Young's moduli $E_1 = 20.0 \times 10^6$ and $E_2 = E_3 = 6.0 \times 10^6$, shear moduli $G_{23} = 2.4 \times 10^6$ and $G_{13} = G_{12} = 3.0 \times 10^6$, and Poisson's ratios $\nu_{23} = 0.25$ and $\nu_{13} = \nu_{12} = 0.3$. During the simulation, the distributed shear load is gradually increased to a maximum value $P_{\max} = 0.45$ units per unit length. The two laminations (0/90/0) and (90/0/90) are considered, and the reference solutions are graphically acquired from Ref. [71]. The load–deflection curves for both lay-ups are shown in Fig. 8b and are in very good agreement with the reference solutions.

7.4. Laminated semi-cylinder

A semi-cylindrical shell with radius $R = 101.6$, length $L = 304.8$ and thickness $h = 3$ is subjected to a pinching point load applied at point A (Fig. 9a). The shell consists of three lamina, made of a transversely isotropic elastic material with $E_L = 2068.5$, $E_T = 517.125$, $G_{LT} = 795.6$ and $\nu_{LT} = \nu_{TT} = 0.3$, where the subscripts L and T signify, respectively, the longitudinal and transverse directions, E denotes Young's modulus, G is the shear modulus, and ν_{LT} is Poisson's ratio relating transverse strain to longitudinal stress. The two lay-ups considered here can be designated by [0/90/0] and [90/0/90], in terms of the angle (in degrees) between the axis of the semi-cylindrical shell and the

Figure 2: Hemi-spherical cap, 16×16 mesh.

Figure 3: Hemi-spherical cap: load–deflection curves computed using the present global–local shell element (GLT) and the reference solution of Sze et al. [65].

Figure 4: Elasto-plastic cylinder: undeformed geometry.

Figure 5: Elasto-plastic cylinder: load-deflection curves.

Figure 6: Elasto-plastic cylinder: deformed shape and plastic zones at radial displacement $|u_r| = 150$.

longitudinal direction of the material in each layer. Making use of symmetry, we model only half the structure using a mesh of $n_e \times n_e$ elements, and apply symmetry boundary conditions on side A–D (Fig. 9a). Furthermore, motion in the vertical direction is constrained on side B–C, and side C–D side is clamped. A downward point-load, P , is applied at point A, leading to the deformed configuration shown in Fig. 9b, with the lay-up $[0/90/0]$, $P = 2000$, and $n_e = 32$. The load–deflection curves for both lay-ups are shown in Fig. 10 and are clearly in excellent agreement with the results of Sze et al. [65], obtained using a similar mesh ($n_e = 32$).

7.5. Cylindrical panel under point load

A cylindrical panel with radius $R = 2540$, length $L = 508$, angle $\Phi = 0.1$ radians, and thickness $h = 12.7$, is subjected to a point load of intensity P applied at point C (Fig. 11a). The straight edges of the panel are hinged, whereas the curved edges are free. The load intensity P is increased gradually from zero to $P_{max} = 3000$. The shell is divided into four layers of equal thickness, made of an orthotropic material with Young’s moduli $E_1 = 3300$ and $E_2 = E_3 = 1100$, shear moduli $G_{23} = 440$ and $G_{12} = G_{13} = 660$, and Poisson’s ratios $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. The two layups $(-45/45/-45/45)$ and $(30/-60/-60/30)$ are considered, as in Refs. [43, 71], and the reference solution is acquired graphically from [71]. The load–deflection curves for both lay-ups are plotted in Fig. 11b. Excellent agreement with the reference solution is observed.

Figure 7: Elasto-plastic cylinder: deformed shape and plastic zones at radial displacement $|u_r| = 250$.

7.6. Simply-supported laminated cylindrical shell

In this section, we consider the laminated cylindrical shell shown in Fig. 11a, with dimensions $L = 10$, $R = 5$ and $\Phi = \pi/2$. The shell is subject to a bi-sinusoidal load of amplitude $q_0 = 0.01$, applied to its top surface. This small load amplitude allows the use of analytical reference solutions based on the small-deformation hypothesis. Each edge is subdivided into n_e equal segments, defining an $n_e \times n_e$ mesh. In both cases considered (perfectly-bonded and delaminated), the shell consists of three layers of equal thickness, made of a transversely isotropic material with $E_L = 25.0 \times 10^3$, $E_T = 1.0 \times 10^3$, $G_{LT} = 0.5 \times 10^3$ and $\nu_{LT} = \nu_{TT} = 0.25$, where the subscripts L and T signify, respectively, the longitudinal and transverse directions, E denotes Young's modulus, G is the shear modulus, and ν_{LT} is Poisson's ratio relating transverse strain to longitudinal stress. The stacking sequence is $[0/90/0]$, with the angles given in degrees between the shell's axial direction and the material's longitudinal direction. In the perfectly-bonded and delaminated cases (Sections 7.6.1 and 7.6.2, respectively), we compute the maximum deflection and evaluate the relative error with respect to the reference solution for several values of the slenderness ratio $S = R/h$. We also examine the variation of displacements and stresses over the thickness of the shell.

7.6.1. Perfectly-bonded case

Using Ren's analysis [72] as reference solution, we examine the element's convergence behavior in simulations of perfectly-bonded laminates. The relative error in the top-surface radial displacement, computed at the mid-point of the shell (i.e. point C in Fig. 11a), is presented in Table 2 for laminates with slenderness ratios of 2, 10 and 100,

(a) Initial and final configuration

(b) Load-deflection curves for the considered laminations

Figure 8: Slit annular ring, 10×80 elements.

and reference radial displacement values of 1.173×10^{-7} , 1.212×10^{-6} , and 1.789×10^{-5} , respectively. Evidently, the computed displacements converge to the analytical solution, exhibiting slightly better accuracy for smaller slenderness ratios. The through-thickness distributions of σ_{xr} and $\sigma_{\theta r}$ are shown in Figs. 12a and 12b, respectively. It can be seen from these figures that both transverse shear stress components converge to the reference solution, varying continuously across interfaces as required. More generally, it can be seen that stresses are predicted accurately even with the coarsest mesh used ($n_e = 4$).

7.6.2. Delaminated case

Keeping our attention focused on the three-layer cylindrical shell shown in Fig. 11a, we assume that all interfaces ($k = 1, 2$) are *fully delaminated*. Interfacial behavior is represented using a linear traction–separation law:

$$\langle \mathbf{S} \rangle^{(k)} \cdot \mathbf{a}_3 = \mathcal{I}^{(k)} \cdot \llbracket \mathbf{u} \rrbracket^{(k)}, \quad (61)$$

where

$$\mathcal{I}^{(k)} = \begin{bmatrix} R_s^{(k)} & 0 & 0 \\ 0 & R_s^{(k)} & 0 \\ 0 & 0 & R_n^{(k)} \end{bmatrix}, \quad (62)$$

with normal and shear interfacial stiffnesses of $R_n^{(k)} = 1.089 \times 10^5$ and $R_s^{(k)} = 1.651 \times 10^5$, respectively. Top-surface radial displacement values, computed at point C (Fig. 11a), are compared to the following reference values, obtained

Table 2: Laminated cylindrical shell (perfectly-bonded): Relative error on radial displacement.

n_e	$1 - u_r/u_r^{ref}, (\%)$		
	$\mathcal{S} = 2$	$\mathcal{S} = 10$	$\mathcal{S} = 100$
2	5.73	8.55	8.86
4	1.57	2.23	2.31
8	0.43	0.56	0.58
16	0.13	0.14	0.15

Figure 9: Laminated semi-cylindrical shell, 32×32 mesh.

Figure 10: Laminated semi-cylindrical shell: load-deflection curves.

Figure 11: Cylindrical panel under point load, $n_e \times n_e$ elements with $n_e = 8$.

using Ren's analysis, extended to the delaminated case in Ref. [22]: $u_r^{ref} = 1.173 \times 10^{-7}$ for slenderness ratio $S = 2$, $u_r^{ref} = 1.212 \times 10^{-6}$ for $S = 10$ and $u_r^{ref} = 1.789 \times 10^{-5}$ for $S = 100$. The relative error with respect to these reference values is given in Table 3. Very good convergence behavior is observed for the three slenderness ratios considered. The through-thickness variation of the displacement component u_θ is presented in Fig. 13. Clearly, the displacement field is discontinuous at the interfaces, due to delamination. Conversely, it can be seen from Fig. 14 that both transverse shear stress components, σ_{xr} and $\sigma_{\theta r}$, are continuous as expected. It can also be seen from Figs. 13 and 14 that all three distributions produced using the most refined mesh are in very close agreement with the analytical solution.

7.7. Double cantilever beam

A beam of length $L = 150$ mm, width $b = 20$ mm and thickness $h = 3.96$ mm is composed of two layers of equal thickness. Each layer is made of an orthotropic material with mechanical properties given in Table 4. One end of the beam is clamped, while the other end is subjected to equal and opposite loads, $P/2$, as shown in Fig. 15. This causes a pre-existing inter-laminar crack of initial length $a_0 = 55$ mm to open and propagate toward the fixed end of the beam. This interfacial failure process is treated in the manner described in Section 6.7. Interfacial properties are given in Table 4. Three levels of mesh refinement are considered, corresponding to element size $l = 1.667$ mm, $l = 1.0$ mm, and $l = 0.5$ mm, respectively. Figure 15 shows the transverse normal stress field in the final deformed configuration, corresponding to a crack opening of $2\delta w = 16$ mm. As expected, the stress is concentrated in the vicinity of the crack tip. It can also be seen that the positions of the crack tip computed with $l = 1$ mm (top) and $l = 0.5$ mm (bottom) are in agreement. Moreover, plotting the load, P , against the crack opening displacement, in Fig. 16, shows that numerical results obtained using the present element converge to the analytical solution of Reeder et al. [73] as the mesh is refined.

Table 3: Laminated cylindrical shell (delaminated): Relative error on radial displacement.

n_e	$1 - u_r/u_r^{ref}, (\%)$		
	$S = 2$	$S = 10$	$S = 100$
2	5.48	8.29	8.83
4	1.50	2.15	2.30
8	0.40	0.55	0.59
16	0.11	0.13	0.15

Figure 12: Laminated cylindrical shell (perfectly-bonded), $S = 10$: Through-thickness stress distributions.

7.8. Impulsively loaded cylindrical panel

The 120° cylindrical panel, shown in Fig. 17a, has radius $R = 2.94$ and length $L = 12.56$, and is subjected to impulsive loading. The curved sides of the panel are simply supported whereas the straight sides are fixed. An initial radial velocity of -5650 is applied to an area of the panel's surface with dimensions 10.205×3.08 (shaded patch in Fig. 17a). The panel is made of an isotropic elastic-perfectly plastic material with Young's modulus $E = 10.5 \times 10^6$, Poisson's ratio $\nu = 0.33$, yield strength $\sigma_Y = 4.4 \times 10^4$ and density $\rho = 2.5 \times 10^4$. Making use of symmetry, only one half of the panel is modeled using a mesh of 12×16 elements. Two computational layers are used with linear local expansions and three integration points along the through-thickness direction in each layer. No global expansion is used; however, it is noted that the enhanced strain field spans the entire thickness of the structure, and thus can be regarded as a global field. The final configuration and contours of equivalent plastic strain are shown in Fig. 17b. As expected, high-curvature regions of the panel exhibit more accumulated plastic strain. Figure 18 shows the variation of radial displacement with time at points A and B, situated along the panel's axis of symmetry at axial coordinates $z_A = L/2$ and $z_B = 3L/4$, respectively. These computed results are compared to the experimental observations of Morino et al. [74] and the finite element results of Belytschko et al. [75], showing good agreement. It is also noted that the proposed element over-predicts the radial displacement after $t=0.8$ (msec), compared to the experimental data. The numerical results of Ref. [75] exhibit a similar trend, especially at point A. The simplicity of the rate-independent perfect-plasticity model, used in both computational studies, offers a possible explanation of this observation.

8. Summary and concluding remarks

The present paper describes the development of a four-node finite element for finite deformation analysis of doubly-curved shell structures. The global-local theoretical framework hinges on the superposition of displacement fields pertaining to different length scales, thus providing a highly adaptable formulation for static and dynamic analysis of perfectly-bonded and delaminated multi-layered composites. In contrast with the small strain formulation, a strong coupling exists between global and local scales, due to the non-linear part of the Green-Lagrange strain. Thus, the strain tensor cannot be expressed as a sum of global and local contributions. Continuity at each perfectly bonded interface is enforced by discontinuous-Galerkin fluxes, whereas arbitrary traction-separation laws can be employed for delaminated interfaces. To efficiently model progressive interface failure, we introduce a pseudo-traction vector which depends on the stress average and displacement jump across the interface, and employ an extrinsic cohesive zone model. This formulation is compatible with explicit as well as implicit dynamic analysis.

Convergence and accuracy of the proposed shell element are demonstrated with the aid of several numerical examples involving different geometries, laminations, loading conditions and material models. The results obtained

Figure 13: Laminated cylindrical shell (delaminated), $\mathcal{S} = 10$: Through-thickness distribution of u_θ at point B.

Figure 14: Laminated cylindrical shell (delaminated), $\mathcal{S} = 10$: Through-thickness stress distributions.

Table 4: Double cantilever beam: material and interface properties.

E_{11}	$E_{22}=E_{33}$	$G_{12}=G_{13}$	G_{23}	$\nu_{12} = \nu_{13}$	ν_{23}	G_{1C}	$T_{\max}$
150.0 GPa	11.0 GPa	6.0 GPa	3.7 GPa	0.25	0.45	0.268 N/mm	45 MPa

Figure 15: Double cantilever beam, through-thickness stress and crack tip position for $2\delta w = 16$.

Figure 16: Double cantilever beam: Computational results (GLT) compared to the analytical solution of Ref. [73].

with the current element are compared with analytical, numerical and experimental reference solutions. Excellent agreement with the reference solution is observed in each case. Moreover, we demonstrate that the global–local formulation can take advantage of several concepts developed over the years to improve the efficiency and accuracy of finite element analysis of thin structures. ANS interpolation and EAS methods are employed to alleviate parasitic mesh locking and improve the performance of the element.

Future work directions include the extension of the current formulation to allow the treatment of mixed-mode progressive failure problems, and re-parametrization of displacement variables belonging to expansion orders higher than zero (i.e. constant) with rotational degrees of freedom, using a multiplicative split of the normal director field into extensible and inextensible parts, to avoid ill-conditioning in the thin shell limit [76].

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Figure 17: Impulsively loaded cylindrical panel.

Figure 18: Impulsively loaded cylindrical panel: deflection vs. time.

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