A SPECTRAL MUTISCALE HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR SECOND ORDER ELLIPTIC PROBLEMS

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Dedicated to Mary Wheeler on the occasion of her 75-th anniversary

ABSTRACT. We design a multiscale model reduction framework within the hybridizable discontinuous Galerkin finite element method. Our approach uses local snapshot spaces and local spectral decomposition following the concept of Generalized Multiscale Finite Element Methods. We propose several multiscale finite element spaces on the coarse faces that provide a reduced dimensional approximation for numerical traces within the HDG framework. We provide a general framework for systematic construction of multiscale trace spaces. Using local snapshots, we avoid high dimensional representation of trace spaces and use some local features of the solution space in constructing a low dimensional trace space. We investigate the solvability and numerically study the performance of the proposed method on a representative number of numerical examples.

1. INTRODUCTION

In this paper, we propose a novel hybridized discontinuous Galerkin multiscale finite element method for flows in highly heterogeneous porous media. We consider the second-order elliptic differential equation

$$(1) \quad -\nabla \cdot (\kappa(x) \nabla u) = f(x) \text{ in } \Omega,$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$. We assume that $f \in L^2(\Omega)$ and the coefficient $\kappa(x) \geq \kappa_0 > 0$ represents the permeability of a highly heterogeneous porous media with multiple scales and high contrast (the ratio between the maximum and minimum values of κ). Such problems appear in many industrial, scientific, and environmental applications. Resolving all scales requires very fine meshes and this makes the corresponding algebraic problem (arising from the finite element, finite volume, or mixed FEM approximations) a challenging and often prohibitively expensive task. This necessitates the use of reduced-order methods or specialized techniques for solving the algebraic system, e.g. multigrid (e.g., [7, 15]), multilevel methods (e.g., [31, 32]), and/or domain decomposition techniques (e.g., [29, 18, 22, 23, 25, 30]).

More recently, new classes of reduced-order methods have been introduced and used in various applications. These include Galerkin multiscale finite element method (e.g., [3, 11, 14, 19, 20, 21]), mixed multiscale finite element methods (e.g., [1, 2, 5, 27]), the multiscale finite volume method (see, e.g., [28]), mortar multiscale methods (see e.g., [8, 33]), and variational multiscale methods (see e.g., [26]). In this paper, we use the Generalized Multiscale Finite

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Element [16] concept to construct a local reduced-order approximation for the solution space. In particular, the local snapshot functions and spectral decomposition of the snapshot space are constructed to approximate the solution in each coarse patch. The multiscale approach allows bringing the local properties of the solution into a global coarse-grid problem in a systematic way that substantially reduces the number of coupled degrees of freedom in the algebraic system. The local snapshot solutions are computed separately (possibly in parallel or off-line) on each coarse block. In general, these solutions are discontinuous along the coarse grid interfaces and, consequently, robust discontinuous Galerkin approaches are needed.

One of the earlier efficient mixed finite element methods on multi-block grids were devised by Mary Wheeler and her co-authors in [6] by using the framework of domain decomposition method and mortaring technique. Mortaring technique (see, e.g. [10]) was introduced to accommodate approximations on subdomains that could be meshed independently and use non-conforming finite element spaces. This technique introduces an auxiliary space on each subdomain boundary associated with the Lagrange multiplier that imposes the continuity constraint on the global approximation. The classical mortaring, devised for the needs of domain decomposition methods, [10], has been adapted recently as multiscale discretization by Mary Wheeler and her collaborators (e.g. [8, 9]). In a two-scale (two-grid, fine and coarse) method, the aim is to resolve the local heterogeneities on the fine grid introduced on each coarse block and then glue these approximations together via mortar spaces. The functions in these spaces, which play the role of Lagrange multipliers, are defined on the boundaries of the coarse partition. In order to design a stable method, the mortar spaces have to satisfy a proper inf-sup condition. This approach is shown to be well suited for problems with heterogeneous media and a number of efficient methods and implementations have been proposed, studied, and used for solving a variety of applied problems (e.g. [3, 4, 8]).

In this paper, we study multiscale model reduction techniques within the framework of the Hybridizable Discontinuous Galerkin method (HDG). The hybridization of the finite element methods, as outlined in [12], provides ample possibilities for "gluing" together various finite element approximations. The mechanism of this "glue" is based on the notions of *numerical trace* and *numerical flux*. Numerical trace is a single valued function on the finite element interfaces and belongs to a certain Lagrange multiplier space which is used to solve the global problem. The well-posedness and accuracy are ensured by a proper choice of the numerical flux, that involves a stabilization parameter τ (see, e.g. [12]). Standard approaches for selecting numerical traces involve the use of piecewise polynomials. However, the solution along the boundaries of a coarse grid can not have an arbitrary form and can be represented by a much lower dimensional space. In this paper, we seek a low dimensional representation of trace spaces. Our approach relies on local basis functions that are constructed independently (and in an offline stage) on each coarse-grid cell. The coupling of these local functions is provided by the HDG technology, where the role of the "glue" plays the space of the *numerical trace* whose construction is the main goal of this paper. In this paper, we construct several multiscale spaces for the *numerical trace* that provide reasonable approximations of the solution and results in a stable multi-scale method. The efficiency of the proposed methods is demonstrated on a set of numerical experiments with flows in high-contrast media.

The paper is organized in the following way. In Section 2, we present our model problem by using two grids, fine (that resolves all scales of the heterogeneity in κ) and coarse (where the solution will be sought) and hybridizable discontinuous Galerkin method. In Section 3, we

FIGURE 1. Fine and coarse grids: the coarse-grid blocks are squares while the fine grid finite elements are triangles.

describe the multiscale FEM based on HDG framework and present several different choices of spectral basis functions. In Section 4, we present some numerical experiments and report the error of HDG with the constructed spectral basis spaces and in Section 5, we discuss the numerical results.

2. PRELIMINARY

2.1. Fine and coarse grids. The multiscale finite element method for the equation (1) is based on the hybridizable discontinuous Galerkin method (HDG) and uses the concept of two-grid approximation.

We split the domain Ω into disjoint polygonal subdomains $\{\Omega_i\}_{i=1}^N$ of diameter $O(H_i)$ so that $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$, see e.g. Figure 1. We denote by F_H an face (edge) of the subdomain Ω_i , if F_H is either shared by Ω_i and its neighboring subdomain Ω_j , i.e. $F_H = \bar{\Omega}_i \cap \bar{\Omega}_j$ or $F_H = \bar{\Omega}_i \cap \partial\Omega$. Then $\mathcal{E}_H(\Omega_i)$ denotes the set of all coarse edges of a subdomain Ω_i and $\mathcal{E}_H = \cup_{i=1}^N \mathcal{E}_H(\Omega_i)$.

In each subdomain Ω_i we introduce a shape regular triangulation $\mathcal{T}_h(\Omega_i)$ with triangular elements and maximum mesh-size h_i . Faces (edges) of this triangulation are denoted by F_h . Let $\mathcal{T}_h = \cup_{i=1}^N \mathcal{T}_h(\Omega_i)$, $\mathcal{E}_h(\Omega_i)$ be the set of all faces of the triangulation $\mathcal{T}_h(\Omega_i)$ and $\mathcal{E}_h^0(\Omega_i)$ be the set of all interior edges of the triangulation $\mathcal{T}_h(\Omega_i)$ and set $\mathcal{E}_h = \cup_{i=1}^N \mathcal{E}_h(\Omega_i)$. Further $\mathbf{x}_i$, $i = 1, \dots, N_v$, denote the nodes in the partition $\mathcal{T}_h$. Note that the scale H_i is associated only with the subdomains Ω_i . Similar construction is assumed in 3-D with Ω_i being polyhedra while the fine-grid elements are simplices or hexahedra.

2.2. Fine-scale hybridizable discontinuous Galerkin method. In this subsection, we present HDG method for the problem (1) on the fine-grid $\mathcal{T}_h$. We first write the equation (1) with homogeneous Dirichlet boundary conditions in a mixed form:

$$(2) \quad \begin{aligned} \alpha \mathbf{q} + \nabla u &= 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{q} &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

with $\alpha(x) = \kappa(x)^{-1}$ and further introduce the finite element spaces that are used in the HDG method:

$$\begin{aligned}
 (3) \quad W_h &:= \{w \in L^2(\mathcal{T}_h) : w|_T \in W(T), T \in \mathcal{T}_h\}, \\
 \mathbf{V}_h &:= \{\mathbf{r} \in \mathbf{L}^2(\mathcal{T}_h) : \mathbf{r}|_T \in \mathbf{V}(T), T \in \mathcal{T}_h\}, \\
 M_h &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_{F_h} \in M(F_h), F_h \in \mathcal{E}_h, \mu|_{\partial\Omega} = 0\}, \\
 M_h^0 &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_{F_h} \in M(F_h), F_h \in \mathcal{E}_h, \mu|_{\partial T} = 0\}, \\
 M_H &:= \{\mu \in L^2(\mathcal{E}_H) : \mu|_{F_H} \in M(F_H), F_H \in \mathcal{E}_H, \mu|_{\partial\Omega} = 0\}, \\
 M_{h,H} &:= M_h^0 \oplus M_H.
 \end{aligned}$$

Here for $T \in \mathcal{T}_h$ the spaces $W(T)$, $\mathbf{V}(T)$, and $M(F_h)$ are suitable finite element spaces (see, e.g. [12]). For example, all three space can consist of linear functions. The focus in this paper will be the choice of the space M_H .

Further, we assume that the specified above spaces have "nodal" bases so that the coefficients in the representation of a function with respect to the nodal bases form column vectors in an Euclidean space which dimension is equal to the dimension of the finite elements space. Often we shall use the same notation for the function and for its vector representation (with respect to the nodal bases) when this does not lead to a confusion.

The multiscale HDG method is: Find $(u_h, \mathbf{q}_h, \hat{u}_{h,H}) \in W_h \times \mathbf{V}_h \times M_{h,H}$ such that

$$\begin{aligned}
 (4) \quad & (\alpha \mathbf{q}_h, \mathbf{r})_{\mathcal{T}_h} - (u_h, \nabla \cdot \mathbf{r})_{\mathcal{T}_h} + \langle \hat{u}_{h,H}, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \mathbf{r} \in \mathbf{V}_h, \\
 & - (\mathbf{q}_h, \nabla w)_{\mathcal{T}_h} + \langle \hat{\mathbf{q}}_{h,H} \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} = (f, w)_{\mathcal{T}_h} \quad \forall w \in W_h, \\
 & \langle \hat{\mathbf{q}}_{h,H} \cdot \mathbf{n}, \mu \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \mu \in M_{h,H}, \\
 & \hat{u}_{h,H} = 0 \quad \text{on } \partial\Omega.
 \end{aligned}$$

Here, $\hat{u}_{h,H}$ and $\hat{\mathbf{q}}_{h,H}$ are the *numerical trace* and the *numerical flux*, correspondingly. We write $(\eta, \zeta)_{\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} (\eta, \zeta)_T$, where $(\eta, \zeta)_D$ denotes the integral of $\eta\zeta$ over the domain $D \in \mathbb{R}^n$ and $\langle \eta, \zeta \rangle_{\partial \mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} \langle \eta, \zeta \rangle_{\partial T}$, where $\langle \eta, \zeta \rangle_{\partial D}$ denotes the integral of $\eta\zeta$ over the boundary of the domain $\partial D \subset \mathbb{R}^{n-1}$. Consequently, $\langle \eta, \zeta \rangle_{\partial \Omega_i} := \int_{\partial \Omega_i} \eta \zeta ds$. The HDG method formulation is completed with the definition of the normal component of the numerical trace :

$$(5) \quad \hat{\mathbf{q}}_{h,H} \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \hat{u}_{h,H}),$$

where τ is a non-negative stabilization parameter. We remark that if there is only one coarse grid (i.e. $\Omega_i = \Omega, i = 1$) then this is the standard HDG method [12].

3. THE MUTISCALE HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD

In this section, we will construct local mutiscale basis functions for M_H . For this, we propose the structure of mutiscale HDG method and local spectral problems are designed and solved to compute multiscale basis functions. To keep our presentation simple, we first show the structure of the mutiscale HDG with given multiscale functions. Next, we will discuss how these functions can be generated.

3.1. The mutiscale HDG framework. The main feature of this method is that it could be implemented in such a way that we need to solve global system on the coarse mesh only. To do this, we split the third equation of (4) into two equations by testing separately with $\mu \in M_h^0$ and $\mu \in M_H$ so that

$$(6) \quad \langle \hat{\mathbf{q}}_{h,H} \cdot \mathbf{n}, \mu \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \mu \in M_h^0$$

and

$$(7) \quad \langle \hat{\mathbf{q}}_{h,H} \cdot \mathbf{n}, \mu \rangle_{\partial\Omega_i} = 0 \quad \forall \mu \in M_H \text{ and } i = 1, \dots, N.$$

Now because of the structure of the space M_h^0 we can implement the solution of equation (6) independently on each subdomain Ω_i . Indeed, for a particular subdomain Ω_i , let $\hat{u}_{h,H} = \xi_H$, where $\xi_H \in M_H$. Then on Ω_i , we can find the solution $(u_h, \hat{\mathbf{q}}_h, \hat{u}_{h,H})|_{\Omega_i}$ by restricting the equation (4) to Ω_i :

$$(8) \quad \begin{aligned} (\alpha \mathbf{q}_h, \mathbf{r})_{\mathcal{T}_h(\Omega_i)} - (u_h, \nabla \cdot \mathbf{r})_{\mathcal{T}_h(\Omega_i)} &+ \langle \hat{u}_{h,H}, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h(\Omega_i)} = 0, \\ - (\mathbf{q}_h, \nabla w)_{\mathcal{T}_h(\Omega_i)} &+ \langle \hat{\mathbf{q}}_{h,H} \cdot \mathbf{n}, w \rangle_{\partial\mathcal{T}_h(\Omega_i)} = (f, w)_{\mathcal{T}_h(\Omega_i)}, \\ \langle \hat{\mathbf{q}}_{h,H} \cdot \mathbf{n}, \mu \rangle_{\partial\mathcal{T}_h(\Omega_i)} &= 0, \\ \hat{u}_{h,H} &= \xi_H \text{ on } \partial\Omega_i \end{aligned}$$

for all $(w, \mathbf{r}, \mu) \in W_h|_{\Omega_i} \times \mathbf{V}_h|_{\Omega_i} \times M_h^0|_{\mathcal{E}_h^0(\Omega_i)}$. Here, we have used the relation (5) for the "numerical flux" $\hat{\mathbf{q}}_{h,H}$. In fact, the above system is the regular HDG methods defined on Ω_i . From [12], we already know that under proper choice of the spaces $W(T)$, $\mathbf{V}(T)$, and $M(F_h)$ this problem is stable. Using superposition principle the solution of equation (8) can be further split into two parts, namely,

$$(9) \quad (\mathbf{q}_h, u_h, \hat{u}_{h,H}) = (\mathbf{q}_h(f), u_h(f), \hat{u}_{h,H}(f)) + (\mathbf{q}_h(\xi_H), u_h(\xi_H), \hat{u}_{h,H}(\xi_H)),$$

where $(\mathbf{q}_h(f), u_h(f), \hat{u}_{h,H}(f))$ satisfies

$$\begin{aligned} (\alpha \mathbf{q}_h(f), \mathbf{r})_{\mathcal{T}_h(\Omega_i)} - (u_h(f), \nabla \cdot \mathbf{r})_{\mathcal{T}_h(\Omega_i)} &+ \langle \hat{u}_{h,H}(f), \mathbf{r} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h(\Omega_i)} = 0, \\ - (\mathbf{q}_h(f), \nabla w)_{\mathcal{T}_h(\Omega_i)} &+ \langle \hat{\mathbf{q}}_{h,H}(f) \cdot \mathbf{n}, w \rangle_{\partial\mathcal{T}_h(\Omega_i)} = (f, w)_{\mathcal{T}_h(\Omega_i)}, \\ \langle \hat{\mathbf{q}}_{h,H}(f) \cdot \mathbf{n}, \mu \rangle_{\partial\mathcal{T}_h(\Omega_i)} &= 0, \\ \hat{u}_{h,H}(f) &= 0 \text{ on } \partial\Omega_i, \end{aligned}$$

for all $(w, \mathbf{r}, \mu) \in W_h|_{\Omega_i} \times \mathbf{V}_h|_{\Omega_i} \times M_h^0|_{\mathcal{E}_h^0(\Omega_i)}$ and $(\mathbf{q}_h(\xi_H), u_h(\xi_H), \hat{u}_{h,H}(\xi_H))$ satisfies

$$\begin{aligned} (\alpha \mathbf{q}_h(\xi_H), \mathbf{r})_{\mathcal{T}_h(\Omega_i)} - (u_h(\xi_H), \nabla \cdot \mathbf{r})_{\mathcal{T}_h(\Omega_i)} &+ \langle \hat{u}_{h,H}(\xi_H), \mathbf{r} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h(\Omega_i)} = 0, \\ - (\mathbf{q}_h(\xi_H), \nabla w)_{\mathcal{T}_h(\Omega_i)} &+ \langle \hat{\mathbf{q}}_{h,H}(\xi_H) \cdot \mathbf{n}, w \rangle_{\partial\mathcal{T}_h(\Omega_i)} = 0, \\ \langle \hat{\mathbf{q}}_{h,H}(\xi_H) \cdot \mathbf{n}, \mu \rangle_{\partial\mathcal{T}_h(\Omega_i)} &= 0, \\ \hat{u}_{h,H}(\xi_H) &= \xi_H \text{ on } \partial\Omega_i, \end{aligned}$$

for all $(w, \mathbf{r}, \mu) \in W_h|_{\Omega_i} \times \mathbf{V}_h|_{\Omega_i} \times M_h^0|_{\mathcal{E}_h^0(\Omega_i)}$.

Then the equation (7) reduces to finding $\xi_H \in M_H$ such that

$$(10) \quad a(\xi_H, \mu) = l(\mu) \quad \text{for all } \mu \in M_H,$$

where the bilinear form $a(\xi_H, \mu) : M_H \times M_H \rightarrow \mathbb{R}$ and the linear form $l(\mu) : M_H \rightarrow \mathbb{R}$ are defined as

$$(11) \quad a(\xi_H, \mu) := \sum_{i=1}^N \langle \hat{\mathbf{q}}_{h,H}(\xi_H), \mu \rangle_{\partial\Omega_i} \quad \text{and} \quad l(\mu) := - \sum_{i=1}^N \langle \hat{\mathbf{q}}_{h,H}(f), \mu \rangle_{\partial\Omega_i}.$$

We note that the same procedure can be applied for the case of non-homogeneous boundary data $u = g$ on $\partial\Omega$ as well.

To motivate the need for multiscale spaces for M_H , we first present numerical examples using polynomial functions for M_H . In Figure 2, the high-contrast coefficient $\kappa(x)$, fine-scale

FIGURE 2. Numerical results for multiscale HDG framework with $M_H(F) = \mathcal{P}_2(F)$, $h = 1/32$, $H = 1/2$, and $\eta = 10^4$.

solution, and multiscale solution with polynomial M_H space are shown. In Figure 2-(A), the black region indicates the coefficient $\kappa(x) = \eta$ and the white region the coefficient $\kappa(x) = 1$. The error between the fine-scale solution and the multiscale solution with polynomial space M_H is 37.4%. This representative example shows a need to go beyond piece-wise polynomial functions for M_H . In particular, when using spectral basis introduced in next sections, one can reduce the error below 5%. Next, we will discuss the construction of new multiscale spaces.

3.2. A framework for constructing the multiscale space M_H . Although the multiscale space M_H lives only on the coarse skeleton $\mathcal{E}_H$, it needs to involve some information of the permeability field within coarse blocks. A reasonable approach is to construct a number of snapshots using local solutions defined on a subdomain around a face F (or set of faces) and then take the traces of those functions on $\mathcal{E}_H$ to form the space M_H . This will resemble traces of the global solution and avoid a very high-dimensional representation of the traces. In what follows, we present a general strategy of the construction of M_H :

Step 1: Partition of skeleton. First, we decompose the skeleton $\mathcal{E}_H$ by defining a covering $\mathcal{E}_H = \cup \mathcal{E}_i$, where the summation is over all partitions $\mathcal{E}_i$. Here each $\mathcal{E}_i$ can be a single coarse edge or a union of several coarse edges. We allow overlaps between $\mathcal{E}_i, \mathcal{E}_j$, for some i and j , $\mathcal{E}_i \cap \mathcal{E}_j \neq \emptyset$. For each $\mathcal{E}_i$, we associate a support domain ω_i such that $\mathcal{E}_i \subset \bar{\omega}_i$. For instance, if $\mathcal{E}_i$ is a single coarse face then ω_i can be any one of the neighboring coarse blocks sharing this face or the union of the neighboring blocks that share the face.

Step 2: Local snapshot spaces $V^{\text{snap}}(\omega_i)$. For each ω_i , we introduce the local snapshots $\{\phi_j\}_{j=1}^{N_i}$ that are defined on $\bar{\omega}_i$ and take

$$(12) \quad V^{\text{snap}}(\omega_i) = \text{span}\{\phi_j\}_{j=1}^{N_i}.$$

The snapshots can be given explicitly or computed via solutions of local boundary value or local spectral problems in ω_i . The snapshots can represent the fine-scale features of the solution and thus the choice of the snapshot space impacts the dimension of the local spectral problems.

Now, we present two examples for computing the snapshot space. These examples will be used in later sections.

The simplest choice would be a snapshot space $V^{\text{snap}}(\omega_i)$ that consists of all fine-grid basis functions (see Section 3.3.1) defined on ω_i . Such choice substantially reduces the computational cost associated with calculating the snapshot vectors.

A second approach is based on local solutions of the problem. Let $\{\mathbf{x}_l\}_{l=1}^{N_i}$ be the set of all fine nodes on $\partial\omega_i$. For each ω_i we define the function ϕ_l as the finite element solution of the following local problem (in a discrete setting):

$$(13) \quad \nabla \cdot (\kappa \nabla \phi_l) = 0 \quad \text{in } \omega_i, \quad \phi_l = \delta_l \quad \text{on } \partial\omega_i.$$

Here δ_l is a continuous piecewise linear function defined on $\partial\omega_i$ such that $\delta_l(x) = 1$ on $\mathbf{x}_l$ and vanishes on all other nodes.

Step 3: Form the multiscale space M_H . For each ω_i , we perform a spectral decomposition of $V^{\text{snap}}(\omega_i)$ via some generalized eigenvalue problem

$$(14) \quad \mathbf{A}\psi = \lambda \mathbf{M}\psi.$$

Choices of the (stiffness) matrix $\mathbf{A}$ and the (mass) matrix $\mathbf{M}$ will be discussed in the following section. If the snapshot space consists of all fine-grid basis functions in ω_i , the eigenvalue problem (14) reduces to solving the spectral problem with local stiffness and mass matrices defined on ω_i . We order the eigenvalues as $0 \leq \lambda_{i,1} \leq \lambda_{i,2} \leq \dots \leq \lambda_{i,N_i}$ and select the first L_i eigenfunctions $\psi_{i,1}, \dots, \psi_{i,L_i}$ to form a smaller subspace $V^{\text{off}}(\omega_i)$:

$$V^{\text{off}}(\omega_i) = \text{span}\{\psi_{i,j}\}_{j=1}^{L_i}.$$

For each ω_i , we form the local multiscale space $M_H(F)$ by taking the traces of the functions in V^{off} on each F and form the set of trace of $\psi_{i,j}$ on the face F :

$$\bigcup_{F \in \mathcal{E}_i} \{\psi_{i,j}|_F : \psi_{i,j} \in V^{\text{off}}(\omega_i), j = 1, \dots, L_i\}.$$

Note that, in general, these functions will not be linearly independent so we need to perform a *Proper Orthogonal Decomposition* (POD) in order to eliminate the linearly dependent modes (e.g., [24]) or to select most important modes in order to further reduce the dimension to L_F . In this way, we produce the space $M_H(F)$. Finally, the multiscale space M_H is defined as:

$$M_H := \bigcup_{F \in \mathcal{E}_H} M_H(F).$$

Remark 3.1. Notice that in general M_H is nonconforming in the sense that the functions in M_H will have discontinuities on the coarse grid nodes. If it is desired to have a conforming space, in **Step 1**, we need to ensure that $\{\omega_i\}_{i=1}^N$ is a covering of the domain Ω . Let $\{\chi_i\}_{i=1}^N$ be a set of partition of unity functions associated with the covering $\{\omega_i\}_{i=1}^N$. In **Step 3**, we modify the definition of V^{off} by multiplying the space with the partition of unity function χ_i (so the functions are globally continuous and hence M_H is conforming) to get

$$(15) \quad V^{\text{off}}(\omega_i) = \text{span}\{\chi_i \psi_{i,j}\}_{j=1}^{L_i}.$$

3.3. Examples of multiscale spaces M_H . In this section, we present examples of the construction of multiscale spaces. The Mutiscale Spaces I (see Section 3.3.1 and 3.3.2) will use the traces of the multiscale functions constructed locally on each coarse block, where these multiscale functions are a low dimensional representation of the solution in each Ω_i and thus they provide a low dimensional trace space M_H . Furthermore, we remove linearly dependent components.

For the Mutiscale Spaces II, defined in Section 3.3.3 and 3.3.4, we employ local eigenvalue problem defined on coarse blocks that strictly contain the face F . Compared to the first

approach, the second approach uses oversampling technique in computing M_H . The method proposed in Section 3.3.4 uses a partition of unity and provides a conforming M_H .

Due to the *stabilization parameter* τ , the solvability of the system is independent of the choice of the multiscale space M_H . A brief discussion on this issue is in Section 3.4.

3.3.1. The Multiscale Space I-a. For this choice, $\mathcal{E}_i$ refers to the boundary of a coarse-grid element Ω_i and $\omega_i = \Omega_i$ (see Figure 3). The snapshot space consists of all fine-grid functions in Ω_i .

We generate the $V^{\text{off}}(\omega_i)$ by solving the local spectral problem described below. Let $X_h(\Omega_i)$ be the conforming finite element space of continuous piece-wise linear functions in $\mathcal{T}_h(\Omega_i)$. For any given subdomain Ω_i , we find λ and $\psi \in X_h(\Omega_i)$ such that

$$(16) \quad \int_{\Omega_i} \kappa(x) \nabla \psi \nabla z dx = \lambda \int_{\Omega_i} \tilde{\kappa}(x) \psi z dx \quad \forall z \in X_h(\Omega_i).$$

$\tilde{\kappa}$ is a properly selected weight. For a scalar permeability, we select $\tilde{\kappa} = \kappa$. For tensor permeability, for the choice of $\tilde{\kappa}$, we refer to [18]. We order the eigenvalues so that $0 = \lambda_{i,1} \leq \lambda_{i,2} \leq \dots \leq \lambda_{i,N_i}$, where N_i is the number of vertices of $\mathcal{T}_h(\Omega_i)$. Then $V^{\text{off}}(\Omega_i)$ is the space spanned by the eigenfunctions $\psi_{i,l}$, $l = 1, \dots, L_i$ corresponding to the first L_i eigenvalues.

As we see from (3), the space M_H is defined face-wise. Therefore, for a face $F \in \mathcal{E}_H$ common for the coarse blocks Ω_i and Ω_j we collect all traces of the constructed above eigenfunctions, namely

$$(17) \quad F \in \mathcal{E}_H(\Omega_i) \cap \mathcal{E}_H(\Omega_j) : \quad \psi_{i,l,F} = \psi_{i,l}|_F, \quad l = 1, \dots, L_i, \quad \psi_{j,l,F} = \psi_{j,l}|_F, \quad l = 1, \dots, L_j.$$

The selected traces are in general linearly dependent, so we need to eliminate the linearly dependent ones. We use POD method and select the first L_F linearly independent functions denoted by $\psi_{j,F}$, $j = 1, \dots, L_F$. Thus, for any $F \in \mathcal{E}_H$ we define

$$(18) \quad M_H^{I_a}(F) := \text{span}\{\psi_{j,F}, 0 \leq j \leq L_F\}, \quad \text{and} \quad M_H^{I_a} := \bigcup_{F \in \mathcal{E}_H} M_H^{I_a}(F).$$

3.3.2. The Multiscale Space I-b. The definition of ω_i and $\mathcal{E}_i$ are the same as in the case of the Multiscale Space I-a. Let $m_{\mathcal{E}_i}$ be the number of nodal basis function corresponding to the fine-grid nodes on $\mathcal{E}_i$. Then the snapshot space is defined by the finite element approximations (we keep the same notation for discrete solution as the continuous solution in (13)) ϕ_l , $l = 1, \dots, m_{\mathcal{E}_i}$ of the problem (13) on $X_h(\omega_i)$:

$$(19) \quad V^{\text{snap}}(\Omega_i) = \text{span}\{\phi_l, 1 \leq l \leq m_{\mathcal{E}_i}\}.$$

To compute local spectral problem, we define the matrices

$$A_i^{\text{snap}} = \left[\int_{\Omega_i} \kappa \nabla \phi_l \nabla \phi_{l'} dx \right]_{l,l'=1}^{m_{\mathcal{E}_i}}, \quad M_i^{\text{snap}} = \left[\frac{1}{H} \int_{\mathcal{E}_i} \kappa \phi_l \phi_{l'} ds \right]_{l,l'=1}^{m_{\mathcal{E}_i}}.$$

Using these matrices, we solve the following algebraic eigenvalue problem:

$$(20) \quad A_i^{\text{snap}} \mathbf{\Sigma}_i = \Lambda_i M_i^{\text{snap}} \mathbf{\Sigma}_i,$$

where Λ_i is the diagonal matrix with the eigenvalues on the diagonal, $\mathbf{\Sigma}_i$ is a square matrix with columns that are the corresponding eigenvectors. We define the functions along the edge $F \in \mathcal{E}_i$

$$\psi_{j,F} = \psi_{j,\mathcal{E}_i}|_F,$$

where $\psi_{j,\mathcal{E}_i}$ is the linear combination of snapshot vectors with the coefficients being the coordinates of the j th eigenvector.

Further, if F is a common face of Ω_i and Ω_j , we need to apply to POD procedure in order to eliminate linearly dependent functions. After POD process, we get L_F functions $\psi_{1,F}, \dots, \psi_{L_F,F}$. Then for $F \in \mathcal{E}_H$

$$(21) \quad M_H^{I_b}(F) = \text{span} \{ \psi_{j,F} : 0 \leq j \leq L_F \} \quad \text{and} \quad M_H^{I_b} := \bigcup_{F \in \mathcal{E}_H} M_H^{I_b}(F).$$

3.3.3. The Mutiscale Space II-a. We define $\mathcal{E}_i$ as a single coarse face and ω_i as the union of two coarse elements, Ω_1 and Ω_2 , that share this face $\mathcal{E}_i$, see Figure 3(A). The snapshot space will consist of all fine-grid functions and thus the local spectral problem is solved directly on a fine grid (locally). In this case, we solve the following eigenvalue problem in a coarse neighborhood ω_i (see Figure 3). Let $X_h(\omega_i)$ be the conforming finite element space of continuous piece-wise

FIGURE 3. Illustration of a coarse neighborhood and coarse element.

linear functions in $\mathcal{T}_h(\omega_i)$. For any given coarse neighborhood ω_i and the common face $\mathcal{E}_i$, we find λ and $\psi \in X_h(\omega_i)$ such that

$$\int_{\omega_i} \kappa(x) \nabla \psi \nabla z dx = \frac{\lambda}{H} \int_{\mathcal{E}_i} \bar{\kappa} \psi z ds, \quad \forall z \in X_h(\omega_i),$$

with $\bar{\kappa} = 2\kappa_1\kappa_2/(\kappa_1 + \kappa_2)$ and $\kappa_1 = \kappa|_{\Omega_1}$ and $\kappa_2 = \kappa|_{\Omega_2}$.

We order the eigenvalues so that $0 \leq \lambda_{i,1} \leq \lambda_{i,2} \leq \dots \leq \lambda_{i,N_i}$, where N_i is the number of vertices of $\mathcal{T}_h(\omega_i)$ and denote the corresponding eigenfunctions by $\psi_{i,1}, \psi_{i,2}, \dots, \psi_{i,N_i}$, which form the space $V^{\text{off}}(\omega_i)$. Since the space M_H is defined face-wise, we consider the traces of the constructed above eigenfunctions, denoted by $\psi_{l,F}$, $l = 1, \dots, N_i$, where F is the common coarse face of ω_i . These functions may be linearly dependent, so we choose linearly independent ones and further reduce their number by taking those that correspond to the smallest L_F eigenvalues. These are the functions we use as a basis for the space M_H :

$$(22) \quad M_H^{II_a}(F) := \text{span} \{ \psi_{j,F}, 1 \leq j \leq L_F \} \quad \text{so that} \quad M_H^{II_a} := \bigcup_{F \in \mathcal{E}_H} M_H^{II_a}(F).$$

3.3.4. *The Mutiscale Space II-b.* For each node $\mathbf{x}_i$ we define $\mathcal{E}_i$ as the union of all coarse faces (edges) that share the node $\mathbf{x}_i$. Similarly, ω_i is the union of coarse elements that share a common node $\mathbf{x}_i$ (see Figure 3, where the coarse grid elements that form ω_i are $\Omega_1, \dots, \Omega_4$).

We choose for the snapshot space the set of all fine-grid functions. Then the local spectral problem (14) is a finite element approximation of the following homogeneous Neumann eigenvalue problem: find $\psi^{\text{aux}} \in X_h(\omega_i)$ such that

$$(23) \quad \int_{\omega_i} \kappa(x) \nabla \psi^{\text{aux}} \nabla z dx = \lambda \int_{\omega_i} \tilde{\kappa} \psi^{\text{aux}} z ds, \quad \forall z \in X_h(\omega_i) \quad \text{with} \quad \tilde{\kappa} := \kappa \sum_{l=1}^{N_v} |\nabla \chi_l|^2,$$

where χ_l form partition of unity, e.g., piece-wise bilinear functions (see [22]).

Then, we order the eigenvalues as $0 = \lambda_{i,1} \leq \lambda_{i,2} \leq \dots$ and choose the L_i eigenfunctions $\psi_{i,1}^{\text{aux}}, \dots, \psi_{i,L_i}^{\text{aux}}$, associated to the smallest eigenvalues and form basis $\psi_{i,j} = \chi_i \psi_{i,j}^{\text{aux}}$. Due to the properties of χ_i , $\psi_{i,j} \in H_0^1(\Omega)$. Thus, we form $V^{\text{off}}(\omega_i) = \text{span}\{\psi_{i,j}\}_{j=1}^{L_i}$.

Finally, we define

$$M_H^{II_b}(F) = \bigcup_{F \in \mathcal{E}_i} V^{\text{off}}(\omega_i)|_F \quad \text{and} \quad M_H^{II_b} := \bigcup_{F \in \mathcal{E}_H} M_H^{II_b}(F),$$

where $V^{\text{off}}(\omega_i)|_F$ denotes the traces on F of the functions in $V^{\text{off}}(\omega_i)$.

3.4. Solvability of the spectral mutiscale HDG method. Next, we discuss the stability and solvability of HDG method with multiscale spaces constructed in the previous section. To proceed, we need the following two assumptions on the local fine element spaces.

Assumption 3.2. For any $K \in \mathcal{T}_h$, F^* an arbitrary edge of K , and $\mu \in M_h(F)$, $F \in \partial K$, there exists a element $\mathbf{Z} \in \mathbf{V}(K)$ such that

$$\begin{aligned} (\mathbf{Z}, \nabla w) &= 0, \quad \text{for all } w \in W(K), \\ \mathbf{Z} \cdot \mathbf{n}|_F &= \mu, \quad \text{for all } F \in \partial K \setminus F^*. \end{aligned}$$

This assumption is trivially satisfied by all classical mixed finite elements as well as HDG elements (e.g. see [12, 13] for more details). The next assumption is for the stabilization parameter τ .

Assumption 3.3. On each $F_H \in \mathcal{T}_E$, for any T adjacent to F_H , i.e. $\bar{T} \cap F_H \neq \emptyset$, there exists at least one element $K \in \mathcal{T}_T$ adjacent to F_H , such that the stabilization operator $\tau > 0$ on $F^* = F_H \cap \partial K$.

Now we state the solvability result of the method that is established by using approach developed in [13].

Theorem 3.4. If Assumptions 3.2 and 3.3 are satisfied, then for any f , the system (8) has unique solution.

We end this section by some discussions on the motivation of the construction of a coarse space M_H . In an ongoing research on the topic we have established some error estimates for the multiscale HDG method. Roughly speaking, for the primary unknown u , we have the bound:

$$(24) \quad \|u - u_h\|_{\Omega} \leq C \|u - P_M u\|_{\frac{1}{2}, \partial \mathcal{T}_H} + C h^{\frac{1}{2}} \|u - P_M u\|_{0, \partial \mathcal{T}_H} + \mathcal{C}(h, \tau).$$

Here $P_M u$ denotes the orthogonal L^2 projection onto the space M_H . The norms are defined as $\|u - P_M u\|_{s, \partial \mathcal{T}_H} := \sum_{i=1}^N \|u - P_M u\|_{s, \partial \Omega_i}$. The last term, $\mathcal{C}(h, \tau)$, in this estimate involves several terms that depend on the mesh size h and the stabilization parameter τ and converges to zero if the fine-grid size h approaches to zero and $\tau = \mathcal{O}(1)$. As one can see, $\|u - P_M u\|_{\frac{1}{2}, \partial \mathcal{T}_H}$ is crucial in the estimate. With a further refined analysis, this term could be bounded by

$$(25) \quad \|u - P_M u\|_{\frac{1}{2}, \partial \mathcal{T}_H} \leq \inf_{v \in M_H} \|u - v\|_{\frac{1}{2}, \partial \mathcal{T}_H} + \mathcal{C}(\tau, h),$$

where $\mathcal{C}(h, \tau)$ has the same meaning as above. The above estimate is essential for choosing an appropriate space M_H and a projection of the solution to M_H . From this estimate, we can see that one should construct the coarse space M_H in such a way that it can provide a good approximation of u in the $H^{\frac{1}{2}}$ -norm along the coarse edges. In particular, we will need a majorizing norm which will define a local spectral problem and corresponding snapshot functions that will allow us to bound the error. This is a guideline for us in the above construction. Using (24) and (25), we can obtain error bounds similar to those in [19] for the MS Space II-b where the error scales as the inverse of the smallest eigenvalue that the corresponding eigenvector is not included in the coarse space. Because of the oversampling, we expect the approach for the MS Space II-b will have a better convergence rate compared to [17, 19].

4. NUMERICAL RESULTS

In this section we discuss representative numerical results. We compute the coarse-grid solution and study the error with respect to the reference solution (or the fine-grid solution of (4)). We note that the solution of (4) depends on both fine-scale and coarse-scale parameters, h and H . We are mainly interested on the convergence (to the reference solution) when we add more spectral basis functions. We study the error behavior due to the addition of coarse basis functions for fixed value of h and H .

We consider the domain $\Omega = (0, 1)^2$ and divide Ω into $N = M \times M$ square coarse blocks, $\{\Omega_i\}_{i=1}^N$, which are unions of fine elements. In this case $H = 1/M$ is the coarse mesh parameter. Inside each subdomain Ω_i , we generate a structured triangulation with m subintervals in each coordinate direction (and thus $h = 1/(Mm)$ is the fine mesh parameter). We consider the solution of Equation (4) with $f = 1$ and the stabilization parameter $\tau = 1$ on each edge (coarse and fine edge). We test our methods on two different distributions of the coefficient $\kappa(x)$ shown in Figure 4. $\kappa(x) = 1$ are for the points in the white background and $\kappa(x) = \eta$ are for the points in the black regions. Example 1 represents the permeability of a media with highly permeable inclusions and relatively short isolated channels (not interconnected). Example 2 has one global channel. In our experiments, we consider η to be 10^4 .

We consider the following finite element spaces which satisfy the assumptions of Subsection 3.4 in order to apply the multiscale HDG method. W_h and V_h consist of piece-wise linear, discontinuous functions on $\mathcal{T}_h$, M_h of piece-wise linear, discontinuous functions in $\mathcal{E}_h$, and M_H is the multiscale finite element space, which is constructed in the previous section. To compute the error of $\mathbf{q}$, we define the weighted L^2 -norm as following :

$$\|\mathbf{q}\|_{\alpha, L^2(\Omega)}^2 := \int_{\Omega} \alpha |\mathbf{q}|^2 dx.$$

FIGURE 4. Topology of heterogeneous coefficient κ .

We compute the relative L^2 -norm of the error for the solution $\|u_h^* - u_H\|_{L^2(\Omega)}$ and the relative weighted L^2 -norm of the error for the velocity $\|\mathbf{q}_h^* - \mathbf{q}_H\|_{\alpha, L^2(\Omega)}$ - between the fine-scale solutions $(u_h^*, \mathbf{q}_h^*)$ and the coarse-grid solutions: (I) $(u_H^I, \mathbf{q}_H^I)$ introduced in the Subsection 3.3.1 and 3.3.2 or (II) $(u_H^{II}, \mathbf{q}_H^{II})$ in Subsection 3.3.3 and 3.3.4.

4.1. Numerical results for Multiscale Spaces I applied to Example 1. In this subsection, we present the numerical experiments for the spaces introduced in Subsection 3.3.1 and 3.3.2.

TABLE 1. Numerical results for Example 1 when using Multiscale Spaces I (Subsection 3.3.1 and 3.3.2) with increasing dimension of the coarse space, $h = 1/100, H = 1/10, \eta = 10^4$. "Dim." stands for the dimension of the coarse space.

Multiscale Space I-a			Mutiscale Space I-b		
Dim.	$\ u_h^* - u_H\ _{L^2(\Omega)}$	$\ \mathbf{q}_h^* - \mathbf{q}_H\ _{\alpha, L^2(\Omega)}$	Dim.	$\ u_h^* - u_H\ _{L^2(\Omega)}$	$\ \mathbf{q}_h^* - \mathbf{q}_H\ _{\alpha, L^2(\Omega)}$
180	0.6412	0.8028	180	0.6412	0.8028
469	0.1208	0.3547	514	0.0393	0.2321
682	0.0362	0.2181	738	0.0312	0.1395
888	0.0204	0.1659	848	0.0266	0.1247

In these numerical experiments, we add multiscale basis functions and study the error decay as the coarse space dimension increases. The coefficient $\kappa(x)$ are depicted on Figure 4. The results for the errors are presented in Table 1 for $h = 1/100$ and $H = 1/10$. We report the dimension of the coarse spaces (designated by "Dim."). We remind that for n -th row of the table, we add n eigenfunctions from both sides of the edge and remove linearly dependent vectors. For this reason, the dimensions of the coarse space do not form a geometric progression.

The results show a convergence to the reference solution (fine-grid solution). We observe a fast error decay when adding more coarse basis functions. In particular, for $H = 1/10$, we only have the dimensions of coarse spaces 682 and 738 to get the solution error 3.6% and 3.1%

for the MS Space I-a and I-b, respectively. This error is computed with respect to the fine-grid solution with the fine-grid mesh size $h = 1/100$ and the dimension of fine-scale system is 60,000. We observe from this table that the MS Space I-b performs slightly better compared to the MS Space I-a.

FIGURE 5. Comparison of the coarse-scale solutions for MS Space II-a with the reference (fine-scale) solution.

4.2. Numerical results for Mutiscale Spaces II applied to Example 1. Next, we consider the spaces introduced in Section 3.3.3 and Section 3.3.4. We note that these spaces use an oversampling technique by solving local problems around faces and the numerical results corresponding to the MS Spaces II are better compared to MS Spaces I.

Figure 5 illustrates the effect of increasing the dimension of the coarse space in the MS Space II-a. We show the fine-scale solution and coarse-scale solutions computed with three different coarse space dimensions in Figure 5. We repeat the detailed numerical study described in Section 4.1. The results for the computation of errors with the MS Space II-a and II-b are presented in Table 2 and 3 for $h = 1/100$ and $H = 1/10$, respectively. When we use the coarse space with the dimension of 720, we observe that the solution errors for the MS Space II-a

TABLE 2. Numerical results for Multiscale Space II-a (Subsection 3.3.3) with increasing dimension of the coarse space, $h = 1/100$, $H = 1/10$, $\eta = 10^4$. "Dim." stands for the dimension of the coarse space.

Dim.	$\ u_h^* - u_H\ _{L^2(\Omega)}$	$\ \mathbf{q}_h^* - \mathbf{q}_H\ _{\alpha, L^2(\Omega)}$
180	0.6412	0.8028
360	0.0561	0.2843
540	0.0274	0.2306
720	0.0251	0.1627

TABLE 3. Numerical results for Multiscale Space II-b (Subsection 3.3.4) with increasing dimension of the coarse space, $h = 1/100$, $H = 1/10$, $\eta = 10^4$. "Dim." stands for the dimension of the coarse space.

Dim.	$\ u_h^* - u_H\ _{L^2(\Omega)}$	$\ \mathbf{q}_h^* - \mathbf{q}_H\ _{\alpha, L^2(\Omega)}$
360	0.1078	0.3290
720	0.0384	0.0922
1080	0.0381	0.0629

and II-b are 2.5% and 3.8% and the velocity error are 16.3% and 9.2%, respectively. Here, the dimension of the fine-grid system is 60,000.

4.3. Numerical results for Example 2. Next, we consider a different permeability field for Example 2 (see Figure 4) with the contrast $\eta = 10^4$. For this permeability field, a long channel in the middle of the domain is added to introduce a long-range effect in the solution. The numerical results are presented in Figure 6 and 7 where solution and velocity errors for these spaces are depicted. We note that the convergence behavior of these methods is similar to the previous permeability case with the errors that are slightly smaller.

We have also tested all the presented approaches for $\eta = 10^6$ and observed similar accuracy. This shows that the method is robust with respect to the contrast.

FIGURE 6. Numerical results for MS Spaces I and II applied to permeability field of Example 2 of Fig. 4 with $h = 1/100$, $H = 1/10$, and $\eta = 10^4$ and increasing dimension of the coarse space. "Dim." stands for the dimension of the coarse space. The graphs show the relative error $\|u_h^* - u_H\|_{L^2(\Omega)}$ vs. the dimension of the coarse space.

FIGURE 7. Numerical results for MS Spaces I and II applied to permeability field of Example 2 of Fig. 4 with $h = 1/100$, $H = 1/10$, and $\eta = 10^4$ and increasing dimension of the coarse space. "Dim." stands for the dimension of the coarse space. The graphs show the relative error $\|\mathbf{q}_h^* - \mathbf{q}_H\|_{\alpha, L^2(\Omega)}$ vs. the dimension of the coarse space.

5. CONCLUSION

In this paper, we propose spectral multiscale finite element methods for second order elliptic equations in the framework of the hybridizable discontinuous Galerkin finite element method. We propose several finite element spaces for the numerical traces. The main idea of the proposed method is to construct a low dimensional trace space M_H . In the paper, we present a general framework for defining trace spaces, which consists of three steps: (1) a partition of the coarse skeleton; (2) a construction of a local snapshot space; (3) a construction of the offline space and consequently a construction of the numerical trace space. Within this framework, we propose and test two classes of coarse spaces. The first class uses the boundaries of the coarse-grid subdomain to construct trace space, while the second class constructs the traces on the faces that are strictly within coarse blocks. In this regard, the second approach uses an oversampling technique where the information in larger domains are used in constructing multiscale basis functions. The presented numerical results for both methods show that one can achieve a good accuracy with a few degrees of freedom along each edge and the approaches that use oversampling provide better accuracy. In our future work, we plan to present detailed error analysis.

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