Adaptive regularization of earthquake slip distribution inversion

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Abstract

Regularization is a routine approach used in earthquake slip distribution inversion to avoid numerically abnormal solutions. To date, most slip inversion studies have imposed uniform regularization on all the fault patches. However, adaptive regularization, where each retrieved parameter is regularized differently, has exhibited better performances in other research fields such as image restoration. In this paper, we implement an investigation into adaptive regularization for earthquake slip distribution inversion. It is found that adaptive regularization can achieve a significantly smaller mean square error (MSE) than uniform regularization, if it is set properly. We propose an adaptive regularization method based on weighted total least squares (WTLS). This approach assumes that errors exist in both the

regularization matrix and observation, and an iterative algorithm is used to solve the solution. A weight coefficient is used to balance the regularization matrix residual and the observation residual. An experiment using four slip patterns was carried out to validate the proposed method. The results show that the proposed regularization method can derive a smaller MSE than uniform regularization and resolution-based adaptive regularization, and the improvement in MSE is more significant for slip patterns with low-resolution slip patches. In this paper, we apply the proposed regularization method to study the slip distribution of the 2011 Mw 9.0 Tohoku earthquake. The retrieved slip distribution is less smooth and more detailed than the one retrieved with the uniform regularization method, and is closer to the existing slip model from joint inversion of the geodetic and seismic data.

Key words: adaptive regularization; weighted total least squares; earthquake; slip distribution; inversion

1. Introduction

Earthquake slip inversion with geodetic constraints is an important step in an earthquake study. Many earthquake research fields rely on its results, including earthquake mechanics (Dmowska *et al.*, 1996), recurrence estimation (Thatcher, 1990), earthquake triggering (Lin and Stein, 2004), and hazard assessment studies (Nishimura *et al.*, 2000). Modern geodesy, especially GPS and InSAR (Zhang *et al.*, 2014), have greatly increased the number of constraints on inversion (Wang *et al.*, 2012, Massonnet *et al.*, 1994, Johnson *et al.*, 2001), but using finite observations to invert the infinite fault slip on a continuous plane is always an ill-posed problem. Normally, discretization is first applied to the fault plane so that the infinite number of parameters can be reduced to a finite number of parameters.

However, even when the number of parameters is less than the number of observations, the inverse problem often remains ill-posed because there are always poorly constrained slip patches located at depth or in areas sparsely covered by measurements.

Therefore, regularization is necessary to avoid numerically abnormal solutions of the ill-posed slip inversion problem. The widely applied regularization methods, such as truncated singular value decomposition (TSVD) (Hansen, 1987) and the Tikhonov method (Tikhonov *et al.*, 1977), have also been applied to earthquake slip distribution in previous works (Fornaro *et al.*, 2012, Pritchard *et al.*, 2002). Please note that in the Bayesian inversion approaches (Jolivet *et al.*, 2014, Ide *et al.*, 1996, Minson *et al.*, 2013), the regularization normally corresponds to the prior probability. The application of the regularization in slip distribution inversion is mostly uniform-type regularization (Wright *et al.*, 2004, Jónsson *et al.*, 2002), where all the slip patches are treated uniformly with the same regularization. Little attention has been paid to adaptive regularization (Lohman, 2004), even though it gives better performances in many fields such as image restoration (Kang and Katsaggelos, 1995). Alternatively, some research interest has been paid to varying the grid size on the fault plane, where the fault is divided based on the resolution matrix (Simons *et al.*, 2002, Page *et al.*, 2009, Barnhart and Lohman, 2010, Atzori and Antonioli, 2011).

This paper aims to explore the adaptive regularization of slip distribution inversion. We start with an analysis of some widely accepted regularization methods and an investigation into the relationship between discretization and regularization. We then examine the performance of the existing non-uniform fault discretization method and some other kinds of adaptive regularization methods.

Finally, we propose a novel adaptive regularization method based on weighted total least squares (WTLS). To validate the performance of the new method, the results of a series of simulation experiments and a case study of the 2011 Tohoku earthquake are presented.

2. Regularization of slip distribution inversion

2.1 The ill-posed slip distribution inversion problem

Fault slip can occur in every point of a fault plane, and should therefore be described by a continuous function $m(\mathbf{x})$ with the coordinates $(\mathbf{x})$ as variables. To invert the fault slip from the geodetic constraints is a standard continuous inverse problem, which can be described by a Fredholm integral equation of the first kind (Menke, 2012):

$$d_i = \int_V G_i(\mathbf{x})m(\mathbf{x})d\mathbf{x}, (i = 1, 2, \dots, N) \quad (1)$$

where d_i is the i th observed displacement, G is the elastic response of the earth (Green's functions), m is the fault slip, and V is the volume of the coordinates on the fault plane.

Because the solution to equation (1) is not unique, we need to first parameterize the continuous function $m(\mathbf{x})$ by a finite number (M) of coefficients:

$$m(\mathbf{x}) \approx \sum_{j=1}^M m_j f_j(\mathbf{x}) \quad (2)$$

where $f_j(\mathbf{x})$ defines the a priori knowledge about the behavior of a fault slip (Menke, 2012). It is normally assumed that the fault slip is constant in a certain patch. Therefore, $f_j(\mathbf{x})$ is selected as a boxcar function, which is unity for the coordinates located on the j th patch and zero for those outside. The coefficient m_j then represents the slip on the j th patch. Other choices of $f_j(\mathbf{x})$, such as

polynomial approximation, a truncated Fourier series, and splines (Fukahata and Wright, 2008), can also be used to represent $m(\mathbf{x})$.

In combining equation (1) and equation (2), the continuous inverse problem turns out to be a discrete inverse problem:

$$d_i = \sum_{j=1}^M G_{ij} m_j \quad (3)$$

where $G_{ij} = \int_{V_j} G_i(\mathbf{x}) f_j(\mathbf{x}) d\mathbf{x}$. Here, V_j denotes the volume of the coordinates on the j th fault patch.

The size of the grid is controlled by prior assumption of the smoothness of the model (Menke, 2012). A smoother slip distribution means that the fault slips can be approximately constant in a larger grid.

There are three factors defining a well-posed discrete inverse problem: 1) the existence of a solution; 2) the uniqueness of the solution; and 3) the stability of the solution. With the development of synthetic aperture radar (SAR) satellites and GPS networks, the number of measurements is often larger than the number of parameters (slips on fault patches), and the solution can be uniquely determined using the ordinary least squares method. However, there will still be some poorly constrained fault slips located at depth or in areas sparsely covered by measurements. A small disturbance in the observations can lead to a great disturbance for these slips, and so the uniquely determined solution is unstable. Therefore, the discrete inverse problem shown in equation (3) is often ill-posed.

We can apply singular value decomposition (SVD) to matrix $\mathbf{G}$ to analyze how the solution instability comes out:

$$\mathbf{G}_{N \times M} = \mathbf{U}_{N \times N} \mathbf{S}_{N \times M} \mathbf{V}_{M \times M}^T \quad (4)$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\mathbf{S}$ is a diagonal matrix. Its elements λ_i ($i = 1, 2, \dots, M$) are defined as singular values. In using the column vector $\mathbf{u}_i \in R^{N \times 1}$ and $\mathbf{v}_i \in R^{M \times 1}$ to represent the matrices $\mathbf{U}$ and $\mathbf{V}$, the solution $\mathbf{m}$ from the ordinary least squares method is given by:

$$\mathbf{m} = \sum_{i=1}^M \frac{\mathbf{u}_i^T \mathbf{d}}{\lambda_i} \mathbf{v}_i \quad (5)$$

Assuming that the real observation $\tilde{\mathbf{d}}$ is perturbed from the real value $\mathbf{d}$ with noise $\mathbf{e}$, the solution $\tilde{\mathbf{m}}$ will take the form of:

$$\tilde{\mathbf{m}} = \mathbf{m} + \sum_{i=1}^M \frac{\mathbf{u}_i^T \mathbf{e}}{\lambda_i} \mathbf{v}_i \quad (6)$$

For an ill-posed inverse problem, λ_i decreases quickly from $i=1$ to $i=M$. Fig. 1 shows an example of the decreasing λ_i from an ill-posed inverse problem, which uses 2,939 InSAR observations to solve the slips on 120 fault patches. We can see that λ_i is almost zero when i is larger than 40. This indicates that the noise part $\frac{\mathbf{u}_i^T \mathbf{e}}{\lambda_i} \mathbf{v}_i$ in the solution $\tilde{\mathbf{m}}$ is very high.

Figure 1. Singular values from an ill-posed inverse problem

The purpose of regularization is to find a filter factor τ_i to suppress the instability caused by the small λ_i , and not to affect the large λ_i significantly at the same time. The solution with a filter factor τ_i is given by:

$$\tilde{\mathbf{m}} = \sum_{i=1}^M \tau_i \frac{\mathbf{u}_i^T \mathbf{d}}{\lambda_i} \mathbf{v}_i + \sum_{i=1}^M \tau_i \frac{\mathbf{u}_i^T \mathbf{e}}{\lambda_i} \mathbf{v}_i \quad (7)$$

The difference between regularization methods lies in the choice of filter factor τ_i .

2.2 Regularization methods

Before introducing the regularization methods, we firstly define the metric to evaluate the regularized solution. After regularization, the solution $\tilde{\mathbf{m}}$ becomes a biased solution, which means the expectation of $\tilde{\mathbf{m}}$ is not equal to the true solution $\mathbf{m}$. Three terms are involved in describing the accuracy of $\tilde{\mathbf{m}}$, including bias ($\mathbf{B}$), solution variance ($\mathbf{D}$) and mean square error (MSE). Bias ($\mathbf{B}$) denotes the discrepancy between the expectation of $\tilde{\mathbf{m}}$ and true value $\mathbf{m}$. Solution variance ($\mathbf{D}$) shows how far $\tilde{\mathbf{m}}$ spread out from its expectation $E(\tilde{\mathbf{m}})$. Mean square error (MSE) is composed by variance and bias, quantifying the difference between $\tilde{\mathbf{m}}$ and true value $\mathbf{m}$. Assuming $\mathbf{G}^{-g}$ is the regularized general inverse of $\mathbf{G}$ and σ_0^2 is the observation variance, the formulas for them are as follows:

$$\begin{aligned} \mathbf{B} &= E(\tilde{\mathbf{m}}) - \mathbf{m} = (\mathbf{G}^{-g} \mathbf{G} - \mathbf{I}) \mathbf{m} \\ \mathbf{D} &= E \left[(\tilde{\mathbf{m}} - E(\tilde{\mathbf{m}}))^2 \right] = \text{trace}(\sigma_0^2 \mathbf{G}^{-g} (\mathbf{G}^{-g})^T) \\ \text{MSE} &= E((\tilde{\mathbf{m}} - \mathbf{m})^2) = (E(\tilde{\mathbf{m}}) - \mathbf{m})^2 + E \left[(\tilde{\mathbf{m}} - E(\tilde{\mathbf{m}}))^2 \right] \\ &= \mathbf{B}^T \mathbf{B} + \mathbf{D} \end{aligned} \quad (8)$$

If the solution is not regularized, there is no bias but the solution variance is large, and therefore the MSE is very high. The principle of regularization is to cut down the solution variance at the cost of increasing the bias. As long as the bias increase is not larger than the decrease of solution variance, the MSE will drop correspondingly. We notice that the bias can also be written as $\mathbf{B} = (\mathbf{R} - \mathbf{I}) \mathbf{m}$, where $\mathbf{R} = \mathbf{G}^{-g} \mathbf{G}$ is termed as the model resolution matrix [Menke, 2012] in geophysical inversion. It implies the resolution matrix is another expression of the bias. When $\mathbf{R}$ increasingly looks like an identity matrix, the solution bias $\mathbf{B}$ then turns to be smaller.

As shown in equation (8), the MSE is the expectation of the square difference between the true solution and the estimated solution. It is composed of the bias ($\mathbf{B}$) and solution variance ($\mathbf{D}$). A good inversion

should let the estimated solution be close to the true one as much as possible, which means that the value of the MSE should be as small as possible. We therefore use the MSE as the metric to evaluate the derived earthquake slip solutions throughout this study. As the general form for calculating MSE is given in equation (8), we will only give the expression of the estimated solution ($\tilde{\mathbf{m}}$) and the general inverse ($\mathbf{G}^{-g}$) for the three regularization methods discussed below.

In the truncated SVD regularization method, the filter factor τ_i is simply defined as a mask operator. It suppresses the instability by directly removing the low λ_i , and takes the form of (Hansen, 1987):

$$\tau_i = \begin{cases} 1 & 1 \leq i \leq k \\ 0 & k < i \leq M \end{cases} \quad (9)$$

Here, the regularization parameter is the truncation number k , deciding the number of masked λ_i . The solution ($\tilde{\mathbf{m}}$), and the general inverse ($\mathbf{G}^{-g}$) are therefore given by:

$$\begin{aligned} \tilde{\mathbf{m}} &= \sum_{i=1}^k \frac{\mathbf{u}_i^T \mathbf{d}}{\lambda_i} \mathbf{v}_i + \sum_{i=1}^k \frac{\mathbf{u}_i^T \mathbf{e}}{\lambda_i} \mathbf{v}_i \\ \mathbf{G}^{-g} &= \sum_{i=k+1}^M \mathbf{v}_i \frac{\mathbf{u}_i^T}{\lambda_i} \end{aligned} \quad (10)$$

Given the general inverse ($\mathbf{G}^{-g}$), the bias ($\mathbf{B}$) and the solution variance ($\mathbf{D}$) can be derived from equation (8). If the truncated number k in equation (9) is set to be M , the inversion turns out to be ordinary least squares inversion, and there is no regularization of the solution, where the bias ($\mathbf{B}$) is zero and the solution variance ($\mathbf{D}$) is extremely high. After gradually imposing the regularization by reducing the truncated number k in TSVD, the bias ($\mathbf{B}$) increases and the solution variance ($\mathbf{D}$) decreases correspondingly.

Instead of directly removing the small singular values of λ_i in TSVD, Tikhonov regularization uses a gradually decreasing filter factor τ_i , which drops faster than λ_i . It takes the form of:

$$\tau_i = \frac{\lambda_i^2}{\lambda_i^2 + \alpha} \quad (11)$$

where α denotes the regularization parameter. The corresponding solution and general inverse are therefore given by:

$$\begin{aligned} \tilde{\mathbf{m}} &= \sum_{i=1}^M \frac{\lambda_i \mathbf{u}_i^T \mathbf{d}}{\lambda_i^2 + \alpha} \mathbf{v}_i + \sum_{i=1}^M \frac{\lambda_i \mathbf{u}_i^T \mathbf{e}}{\lambda_i^2 + \alpha} \mathbf{v}_i \\ \mathbf{G}^{-g} &= \sum_{i=1}^M \mathbf{v}_i \frac{\lambda_i \mathbf{u}_i^T}{\lambda_i^2 + \alpha} \end{aligned} \quad (12)$$

It should be noted that $\mathbf{G}^{-g}$ can also be expressed as $(\mathbf{G}^T \mathbf{G} + \alpha \mathbf{I})^{-1} \mathbf{G}^T$, which is the general inverse to the equation (Tikhonov *et al.*, 1977):

$$\begin{pmatrix} \mathbf{G}_{N \times M} \\ \sqrt{\alpha} \mathbf{I}_{M \times M} \end{pmatrix} \times \mathbf{m}_{M \times 1} = \begin{pmatrix} \tilde{\mathbf{d}}_{N \times 1} \\ \mathbf{0}_{M \times 1} \end{pmatrix} \quad (13)$$

The difference between equation (13) and equation (3) is that some extra terms ($\alpha \mathbf{I} \times \mathbf{m} = \mathbf{0}$) are added in the regularized equation. Tikhonov regularization can therefore be considered as introducing some virtual observations to constrain the unstable solution. The regularization parameter α determines the weights of these virtual observations.

However, since the expectations of slip solutions are not zero, the virtual observations do not fit exactly, and Tikhonov regularization may derive a highly biased solution. In order to make the virtual observations more realistic, most studies assume that the second-order differences on fault patches, rather than the slip, are zero. Therefore, we replace the identity matrix $\mathbf{I}$ by a full-rank smoothing matrix ($\mathbf{L}$), and the inverse equation turns to:

$$\begin{pmatrix} \mathbf{G} \\ \sqrt{\alpha} \mathbf{L} \end{pmatrix} \times \mathbf{m} = \begin{pmatrix} \tilde{\mathbf{d}} \\ \mathbf{0} \end{pmatrix} \quad (14)$$

By applying the generalized singular value decomposition (GSVD) of matrix pair $(\mathbf{G}, \mathbf{L})$, it can be seen that the virtual observations $\sqrt{\alpha}\mathbf{Lm} = \mathbf{0}$ also act as filter factors. The GSVD can be written as (Hansen, 1992):

$$\begin{aligned}\mathbf{G}_{N \times M} &= \mathbf{U}_{N \times M} \mathbf{\Sigma}_{M \times M} \mathbf{X}_{M \times M}^{-1} \\ \mathbf{L}_{M \times M} &= \mathbf{V}_{M \times M} \mathbf{M}_{M \times M} \mathbf{X}_{M \times M}^{-1}\end{aligned}\quad (15)$$

Here, $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices, and $\mathbf{\Sigma}$ and $\mathbf{M}$ are diagonal matrices with elements σ_i and μ_i ($i = 1, 2, \dots, M$), respectively. $\mathbf{X}$ is an $M \times M$ nonsingular matrix. The generalized singular value γ_i is defined as:

$$\gamma_i = \frac{\sigma_i}{\mu_i} \quad (16)$$

The solution and general inverse are given by:

$$\begin{aligned}\tilde{\mathbf{m}} &= \sum_{i=1}^M \frac{\gamma_i^2}{\gamma_i^2 + \alpha} \frac{\mathbf{u}_i^T \mathbf{d}}{\sigma_i} \mathbf{x}_i + \sum_{i=1}^M \frac{\gamma_i^2}{\gamma_i^2 + \alpha} \frac{\mathbf{u}_i^T \mathbf{e}}{\sigma_i} \mathbf{x}_i \\ \mathbf{G}^{-g} &= \sum_{i=1}^M \frac{\alpha}{\gamma_i^2 + \alpha} \mathbf{x}_i \frac{\mathbf{u}_i^T}{\sigma_i}\end{aligned}\quad (17)$$

Also, $\mathbf{G}^{-g}$ can be expressed as $(\mathbf{G}^T \mathbf{G} + \alpha \mathbf{L}^T \mathbf{L})^{-1} \mathbf{G}^T$ in a more general form.

For comparison, we derived slips from a simulated dataset using three regularization methods, i.e. TSVD, Tikhonov regularization using identity matrix $\mathbf{I}$ (TKI), and Tikhonov regularization using smoothing matrix $\mathbf{L}$ (TKL). To represent a physically desirable slip pattern, a smooth input slip model was used. A total of 4000 ground observations distributed around the fault trace were used to constrain the fault model. The synthetic displacement was calculated based on the Okada dislocation model (Okada, 1992). The simulated displacement was further perturbed by 1-cm Gaussian noise. The L-curve method was used to determine the regularization parameter. It was found that TSVD and TKI

give similar slip distributions, with the upper slip well resolved and the lower slip completely erased (Fig. 2). The TKL method, which utilizes the spatially distributed characteristic of the slips (smoothness), can derive more detailed slip characteristics capturing both upper and lower slips. The TSVD and TKI methods directly impose regularization on the fault slips. The influence of noise on the inverted slip can be suppressed; however, the constraint on the slip from the observation constraint is also greatly affected. This means that the deep fault patches (the least sensitive parts) are very difficult to solve. Unlike the TSVD and TKI methods, the TKL method instead regularizes the smoothness of the fault slips. Under the regularization, the smoothness of the deep fault patches will shrink to zero, so we can still retrieve the slips on deep fault patches.

Figure 2. Comparison between TSVD and Tikhonov regularizations. (a) Input slip. (b) TSVD. (c) Tikhonov regularization with identity matrix $\mathbf{I}$. (d) Tikhonov regularization with smoothing matrix $\mathbf{L}$.

2.3 Discretization and regularization

In the previous section, we noted that parameterization, in terms of discretization, can help convert continuous problems into discrete problems. As we know, continuous problems are more ill-posed than discrete problems. Therefore, discretization is somewhat similar to regularization. The Green's functions with infinite parameters in a continuous inverse problem are difficult to compute. Therefore,

we instead examined the Green's functions under two different discretizations on a 16×32 km fault plane, one with 4×8 fault patches (4×4 km) and another with 2×4 fault patches (8×8 km). Their singular values are shown in Fig. 3. We can see that the larger fault patches generate higher singular values, corresponding to less ill-posed problems. This implies that the coarse discretization plays the role of regularization on the dense discretization. If the fault patches within a given size from the dense discretization are constrained to have uniform slips, the solution is equal to the one from coarse discretization. Therefore, the solution space in coarse discretization can be considered to be a subspace of the solution space in dense discretization. This is similar to TSVD, which projects the solution space onto a subspace corresponding to the largest singular values. The change from dense grids (e.g. 4×8 fault patches) to coarse grids (e.g. 2×4 fault patches) can be represented by a projection process, which is expressed as:

$$\mathbf{Q}\mathbf{m}_d = \mathbf{m}_c \quad (18)$$

where $\mathbf{m}_d$ and $\mathbf{m}_c$ are fault slips in the dense grids and coarse grids, respectively, and $\mathbf{Q}$ is the projection matrix. Fig. 3 validates that the generalized inverse of the Green's functions in coarse discretization can be better conditioned than the Green's functions in dense discretization. Therefore, we can conclude that discretization is a form of regularization, and the discretization size determines the degree of regularization.

Figure 3. Comparison of singular values using a large fault patch and a small fault patch.

3. Adaptive regularization

TSVD and Tikhonov regularization treat all the slip patches uniformly with the same regularization. In coseismic slip inversion, some research interest has been paid to varying discretization on fault patches according to their resolutions (Simons *et al.*, 2002, Page *et al.*, 2009, Barnhart and Lohman, 2010, Atzori and Antonioli, 2011, Dettmer *et al.*, 2014). As denoted above, such adaptive discretization can also be treated as adaptive regularization. This section starts with an analysis of resolution-based adaptive regularization, and then discusses the selection of optimal adaptive regularization parameters in general ridge estimation and Tikhonov regularization.

3.1 Resolution-based adaptive discretization

Resolution-based adaptive discretization divides a fault plane into patches of different sizes according to their resolution (expressed as $\mathbf{R} = \mathbf{G}^{-g}\mathbf{G}$, where $\mathbf{G}^{-g}$ is the generalized inverse of $\mathbf{G}$). Based on this criterion, the fault patches located near observation points are divided more densely, because they have a higher resolution (i.e. less bias). The fault patches located at depth or in areas sparsely covered by measurements are poorly constrained, and are therefore divided coarsely. Early studies considering

adaptive discretization manually set patch sizes according to their depth, where the deeper patches have a larger size (Simons *et al.*, 2002, Page *et al.*, 2009). Recent studies have considered a smarter discretization algorithm which can automatically discretize fault patches according to the model resolution matrix (Barnhart and Lohman, 2010, Atzori and Antonioli, 2011). After discretization, the inverse problems are further regularized using TSVD or the Tikhonov regularization method with the identity matrix (Atzori and Antonioli, 2011) or the differential matrix (Barnhart and Lohman, 2010, Simons *et al.*, 2002) as the regularization matrix.

Figure 4. Example of slip inversion using resolution-based adaptive fault discretization. (a) Input slip, with the red circles denoting the observation points. (b) Inverted slip using uniform discretization. (c) Inverted slip using resolution-based adaptive discretization.

An example of inversion using adaptive discretization is shown in Fig. 4, where the fault patches are automatically divided according to their resolution. The input slip and observation points can be seen in Fig. 4. The other inversion settings (noise, dislocation mode, and regularization parameter selection) were the same as the above test in Fig. 2. The results show that the adaptive discretization shows no significant improvement over uniform discretization, and even produces poorer results in some areas (e.g. the upper slip artifacts).

261

262 Therefore, we need a more practical case to demonstrate whether resolution-based adaptive
263 discretization performs better than uniform regularization, and how to set a proper regularization on fault
264 patches. As adaptive discretization can be approximately treated as adaptive regularization, for
265 simplicity, we adopt only uniform discretization in the following part. The resolution-based adaptive
266 discretization is therefore replaced by resolution-based adaptive Tikhonov regularization.

267

268 **3.2 General ridge estimation**

269 General ridge estimation is a particular case of adaptive Tikhonov regularization, which requires an extra
270 linear transformation on the original parameters. The adaptive regularization is then implemented on the
271 new parameters.

272

273 Assuming $\mathbf{L}$ is reversible, we define a new kernel matrix $\mathbf{K}$ as:

$$\mathbf{K} = \mathbf{G}\mathbf{L}^{-1} \quad (19)$$

274 Applying SVD decomposition to $\mathbf{K}_{N \times M} = \mathbf{U}_{N \times N} \mathbf{S}_{N \times M} \mathbf{V}_{M \times M}^T$, we introduce new parameters $\mathbf{z}$, which
275 are linearly transformed from the fault slip $\mathbf{m}$:

$$\mathbf{z} = \mathbf{V}^T \mathbf{L} \mathbf{m} \quad (20)$$

276 With such transformation changes, the Tikhonov regularization problem is in a canonical form. We can
277 have a regularized inversion equation as:

$$\begin{pmatrix} \mathbf{U}\mathbf{S} \\ \sqrt{\alpha}\mathbf{I} \end{pmatrix} \times \mathbf{z} = \begin{pmatrix} \tilde{\mathbf{d}} \\ \mathbf{0} \end{pmatrix} \quad (21)$$

278 This equation is equal to applying TKL to parameters $\mathbf{z}$ when only one regularization parameter α is
279 used. Such a regularization approach is also called ridge estimation. If we consider assigning an

independent regularization parameter α_i to each unknown z_i , the method is then called general ridge estimation (Hoerl and Kennard, 1970). The regularized least squares solution and its bias and solution variance for general ridge estimation are therefore derived as:

$$\begin{aligned}\tilde{z}_i &= \frac{\lambda_i \mathbf{u}_i^T \tilde{\mathbf{d}}}{\lambda_i^2 + \alpha_i} \\ B_i &= \frac{\alpha_i z_i}{\lambda_i^2 + \alpha_i} \\ D_{i,i} &= \sigma_0^2 \frac{\lambda_i^2}{(\lambda_i^2 + \alpha_i)^2}\end{aligned}\tag{22}$$

The MSE is then given as:

$$MSE = \sum_{i=1}^M \frac{\alpha_i^2 z_i^2}{(\lambda_i^2 + \alpha_i)^2} + \sigma_0^2 \sum_{i=1}^M \frac{\lambda_i^2}{(\lambda_i^2 + \alpha_i)^2}\tag{23}$$

Making a derivative of the MSE and letting it be zero, we can derive the optimal α_i with the minimum MSE:

$$\begin{aligned}MSE' &= \sum_{i=1}^M \frac{2\lambda_i^2(\alpha_i z_i^2 - \sigma_0^2)}{(\lambda_i^2 + \alpha_i)^3} = 0 \\ \alpha_i &= \frac{\sigma_0^2}{z_i^2}\end{aligned}\tag{24}$$

Equation (24) implies that: 1) adaptive regularization can generate solutions with lower MSE than uniform regularization; and 2) the values of the optimal regularization parameters are determined by the data variance and the true values of parameters $\mathbf{z}$.

As mentioned above, previous studies have preferred to vary the regularization according to the parameter resolution. However, equation (24) suggests that the optimal regularization parameter α_i bears no relation to its resolution ($\mathbf{R} = \mathbf{G}^{-g}\mathbf{G}$). An example comparing uniform regularization, optimal regularization, and resolution-based regularization is shown in Fig. 5. The inversion setting was the same as the test in Fig. 2. For regularization by resolution, the weight was selected as $\sqrt{1 - R_{i,i}}$, as used

in a previous study by Lohman (2004). The regularization parameters for uniform regularization and resolution-based regularization were both determined by the L-curve method. In order to avoid improper selection of the regularization parameter affecting the performance of the resolution-based regularization, we also undertook an extra resolution-based regularization with regularization parameters manually adjusted to be lower than the determined ones (red line in Fig. 5). The results show that the optimal regularization can derive the best solution with the lowest MSE. However, the resolution-based regularization performs even worse than the uniform regularization, and it is only slightly better than the uniform regularization after the regularization parameters are manually adjusted. Therefore, it can be considered that the resolution-based adaptive regularization is not an ideal method in ridge estimation.

Figure 5. Regularization parameters used for the uniform regularization, optimal regularization, and resolution-based

regularization. The parameter number is the number i for regularization parameter α_i .

3.3 Adaptive Tikhonov regularization with multiple regularization parameters

We can derive an analytical expression of the best regularization parameters for general ridge estimation.

However, general ridge estimation is a particular case of adaptive Tikhonov regularization. In this

section, we discuss the general case of adaptive Tikhonov regularization with multiple regularization parameters. The general inverse matrix ($\mathbf{G}^{-g}$), bias ($\mathbf{B}$), solution variance ($\mathbf{D}$), and MSE are expressed as:

$$\begin{aligned}\mathbf{G}^{-g}_{M \times N} &= (\mathbf{G}^T \mathbf{G} + \alpha \mathbf{L}^T \mathbf{L})^{-1} \mathbf{G}^T \\ \mathbf{B} &= (\mathbf{G}^{-g} \mathbf{G} - \mathbf{I}) \mathbf{m} \\ \mathbf{D} &= \text{trace}(\sigma_0^2 \mathbf{G}^{-g} (\mathbf{G}^{-g})^T) \\ MSE &= \mathbf{B}^T \mathbf{B} + \mathbf{D}\end{aligned}\tag{25}$$

where α is a diagonal matrix with diagonal element α_{ii} denoting the regularization weight on the i th fault patch. However, the derivative $\partial MSE / \partial \alpha_{ii}$ cannot be explicitly given, as in general ridge estimation, so we cannot determine the analytical expression of α with the lowest MSE . Alternatively, we can use a nonlinear optimization method (e.g. Newton's method) to find a relatively optimal α with a local minimum MSE .

Here, we first apply uniform regularization and use the L-curve method to determine the regularization parameter α_0 . This is then considered as the starting value for the adaptive regularization parameters. Newton's method is implemented to find a local minimum of $MSE(\alpha)$, where the corresponding α are taken as the optimized regularization parameters. Fig. 6 is a test to find the optimized regularization parameters. In the test, the simulated displacement was perturbed by 1-cm Gaussian noise. As shown in Fig. 6d, the optimized regularization parameters are partly correlated to the roughness $\mathbf{L}$, where strong regularizations correspond to smooth patches, and vice versa. If we treat the regularization as virtual observations, it is somewhat similar to the situation in general ridge estimation, where the regularization degree ($\alpha_i = \sigma_0^2 / z_i^2$) is proportional to the fitness of the virtual observation ($z_i - 0$). After applying the

regularization using the optimal α , the results exhibit a much higher resolution, especially for the slips at depth. We calculated the MSE of the results and found that the optimal regularization performs significantly better than the uniform regularization (Table 1). For comparison, we also applied adaptive regularization based on model resolution ($\mathbf{R}$). This was in accordance with a previous study which used the resolution as the reference to regularize the fault patches (Lohman, 2004). The regularization weight of each patch was set as $\sqrt{1 - R_{ii}}$. We derived very similar results to uniform regularization, as shown in Fig. 6d, and the MSE values were very close, at 11.9413 m² and 11.9634 m², respectively. In conclusion, it is possible for adaptive regularization to improve on uniform regularization, but the optimal regularization parameters are partly related to the fitness of the regularization equations.

Figure 6. Test of adaptive Tikhonov regularization with multiple regularization parameters. (a) 3D view of the fault geometry and observation sites. (b) Resolution of the fault slip patches. (c) Roughness of the input slip. (d) Optimal regularization parameters. (e) Input slip and inversion results of the different methods.

343

344

Table 1 MSEs of the three regularization approaches for the test in Fig. 6

	Uniform	Resolution-based	Optimized
MSE (m ²)	11.9634	11.9413	2.4206

345

346 4 Adaptive regularization method based on WTLS

347 In the adaptive Tikhonov regularization experiment described in section 3.3, the optimized
 348 regularization parameters determined by Newton's method give significantly better results. However,
 349 the observation variance and the true values of the fault slips are required to calculate the *MSE*. In
 350 practice, the observation variance is sometimes difficult to estimate, and the true values are not available.
 351 The uniformly regularized solution can be used as a substitute, but it still differs a lot from the true
 352 values, and would cause the method to fail. Therefore, we cannot expect to improve the inversion
 353 results by using the nonlinear optimization method, as demonstrated in section 3.3.

354

355 In this study, we introduce WTLS to automatically adjust the uniform regularization matrix. Differing
 356 from the least squares method, which only assumes that there are errors in the observations, the total least
 357 squares (TLS) method assumes that errors exist in both the design matrix and the observations (Golub
 358 and Van Loan, 1980). WTLS is a version of TLS, which allows the setting of weights for elements in
 359 the design matrix (Schaffrin and Wieser, 2008). In earthquake slip inversion, the regularization matrix
 360 is designed based on the prior knowledge that fault slips show a smooth spatial distribution. However,
 361 the regularization matrix cannot exactly represent the smooth characteristic of the fault slips. Some
 362 fault patches may be very smooth and others may be less smooth, while the uniform regularization

treats all the patches equally. Furthermore, simply regularizing the fault slip roughness to zero is not the best choice, because the expectation of fault slip roughness is not equal to zero. The traditional earthquake slip inversion method using least squares does not take the inaccuracy of the regularization matrix into account, so the derived results are normally greatly biased, especially for the deep fault patches. WTLS introduces the consideration of uncertainty into the design matrix, and allows us to specify the weights for each row in the design matrix. It is expected that this can partly reduce the bias in the earthquake slip inversion. The errors-in-variables model for TLS can be defined as:

$$\mathbf{y} - \mathbf{e}_y = (\mathbf{A} - \mathbf{E}_A)\mathbf{x} \quad (26)$$

where $\mathbf{y}$ is the observation, $\mathbf{e}_y$ is the observation error, $\mathbf{A}$ is the design matrix, $\mathbf{E}_A$ is the error of the design matrix, and $\mathbf{x}$ is the parameter vector to be estimated.

In earthquake slip inversion with Tikhonov regularization, the parameters, observation, and design matrix separately take the form of:

$$\begin{aligned} \mathbf{x}_{M \times M} &= \mathbf{m}_{M \times M} \\ \mathbf{y}_{(N+M) \times 1} &= \begin{bmatrix} \mathbf{d}_{N \times 1} \\ \mathbf{0}_{M \times 1} \end{bmatrix} \\ \mathbf{A}_{(N+M) \times M} &= \begin{bmatrix} \mathbf{G}_{N \times M} \\ \sqrt{\alpha} \mathbf{L}_{M \times M} \end{bmatrix} \end{aligned} \quad (27)$$

We let the stochastic properties of the errors be characterized by:

$$\begin{bmatrix} \mathbf{e}_y \\ \mathbf{e}_A \end{bmatrix} := \begin{bmatrix} \mathbf{e}_y \\ \text{vec} \mathbf{E}_A \end{bmatrix} \sim \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \sigma^2 \begin{bmatrix} \mathbf{Q}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_A \end{bmatrix} \right) \quad (28)$$

Here, vec is an operator that stacks the columns of a $N \times M$ matrix into a $NM \times 1$ vector. σ^2 is an (unknown) variance component. The observation error ($\mathbf{e}_y$) and design matrix error ($\mathbf{e}_A$) are assumed to be Gaussian-distributed and their covariance are $\sigma^2 \mathbf{Q}_y$ and $\sigma^2 \mathbf{Q}_A$, respectively. The covariance

matrix $\mathbf{Q}_A$ is a $(N + M)M \times (N + M)M$ matrix composed by row vector covariance and column vector covariance, taking the form of $\mathbf{Q}_A = \mathbf{Q}_0 \otimes \mathbf{Q}_x$, where $\otimes$ is the “Kronecker-Zehfuss product”, $\mathbf{Q}_0$ is the $(N + M) \times (N + M)$ covariance matrix for the row vector of $\mathbf{A}$, and $\mathbf{Q}_x$ is the $M \times M$ covariance matrix for the column vector of $\mathbf{A}$.

The WTLS problem can be formulated as:

$$\mathbf{e}_y^T \mathbf{Q}_y^{-1} \mathbf{e}_y + \mathbf{e}_A^T \mathbf{Q}_A^{-1} \mathbf{e}_A = \min \quad (29)$$

subject to

$$\mathbf{y} - \mathbf{e}_y - \mathbf{A}\mathbf{x} + \mathbf{E}_A\mathbf{x} = \mathbf{0} \quad (30)$$

The covariance matrix $\mathbf{Q}_y$ and $\mathbf{Q}_A$ need to be fixed before solving WTLS problem. Because we only allow the regularization matrix to be adjusted in the earthquake slip inversion, the variance of the Green’s functions errors is set to be zero. The cofactor matrix $\mathbf{Q}_0$ therefore takes the form of:

$$\mathbf{Q}_0 = k \begin{bmatrix} \mathbf{0}_{N \times N} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M \times M} \end{bmatrix} \quad (31)$$

where k is a weight coefficient to balance the design matrix uncertainty and observation uncertainty. We assume that there is no difference in the uncertainty of the column elements, so the covariance matrix for $\mathbf{Q}_x$ is set as an identity matrix $\mathbf{I}_{(N+M) \times (N+M)}$. The covariance matrix $\mathbf{Q}_y$ for the observation is set according to the variance of the observations. In the case of the observations having equal variance, it will also be an identity matrix $\mathbf{I}_{(N+M) \times (N+M)}$.

When the covariance matrix $\mathbf{Q}_y$ and $\mathbf{Q}_A$ are available, an iterative algorithm can be used to solve the WTLS problem (Schaffrin and Wieser, 2008).

$$\text{step 1: } \hat{\mathbf{v}}^{(0)} = 0, \hat{\mathbf{x}}^{(0)} = (\mathbf{A}^T \mathbf{Q}_y^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Q}_y^{-1}$$

$$\hat{\mathbf{x}}^{(1)} = (\mathbf{A}^T (\mathbf{Q}_y + (\hat{\mathbf{x}}^{(0)T} \mathbf{Q}_0 \hat{\mathbf{x}}^{(0)}) \mathbf{Q}_x)^{-1} \mathbf{A})^{-1} \mathbf{A}^T (\mathbf{Q}_y + (\hat{\mathbf{x}}^{(0)T} \mathbf{Q}_0 \hat{\mathbf{x}}^{(0)}) \mathbf{Q}_x)^{-1} \mathbf{y}$$

$$\text{step 2: } \hat{\boldsymbol{\lambda}}^{(i)} = (\mathbf{Q}_y + (\hat{\mathbf{x}}^{(i)T} \mathbf{Q}_0 \hat{\mathbf{x}}^{(i)}) \mathbf{Q}_x)^{-1} (\mathbf{y} - \mathbf{A} \hat{\mathbf{x}}^{(i)}), \hat{\mathbf{v}}^{(i)} = \hat{\boldsymbol{\lambda}}^{(i)T} \mathbf{Q}_x \hat{\boldsymbol{\lambda}}^{(i)}$$

$$\hat{\mathbf{x}}^{(i+1)} = (\mathbf{A}^T (\mathbf{Q}_y + (\hat{\mathbf{x}}^{(i)T} \mathbf{Q}_0 \hat{\mathbf{x}}^{(i)}) \mathbf{Q}_x)^{-1} \mathbf{A} - \hat{\mathbf{v}}^{(i)} \mathbf{Q}_0)^{-1} \mathbf{A}^T (\mathbf{Q}_y + (\hat{\mathbf{x}}^{(i)T} \mathbf{Q}_0 \hat{\mathbf{x}}^{(i)}) \mathbf{Q}_x)^{-1} \mathbf{y}$$

$$\text{step 3: repeat step2 until } \|\hat{\mathbf{x}}^{(i+1)} - \hat{\mathbf{x}}^{(i)}\| < \varepsilon \text{ or iteration times} > n$$

399 In step 3, we set an additional stopping condition (iteration times $> n$), which differs from Schaffrin
 400 and Wieser (2008), in order to prevent the iteration from non-convergence. After obtaining the WTLS
 401 solution $\hat{\mathbf{x}}$, the observation residual ($\hat{\mathbf{e}}_y$) and design matrix residual ($\hat{\mathbf{e}}_A$) are given by (Schaffrin and
 402 Wieser, 2008):

$$\hat{\mathbf{e}}_y = \mathbf{Q}_y (\mathbf{Q}_y + (\hat{\mathbf{x}} \mathbf{Q}_0 \hat{\mathbf{x}}) \mathbf{Q}_x)^{-1} (\mathbf{y} - \mathbf{A} \hat{\mathbf{x}})$$

$$\hat{\mathbf{e}}_A = -\mathbf{Q}_x (\mathbf{Q}_y + (\hat{\mathbf{x}} \mathbf{Q}_0 \hat{\mathbf{x}}) \mathbf{Q}_x)^{-1} (\mathbf{y} - \mathbf{A} \hat{\mathbf{x}}) \hat{\mathbf{x}}^T \mathbf{Q}_0 \quad (32)$$

403

404 The general inverse matrix ($\mathbf{G}^{-g}$), bias ($\mathbf{B}$), variance ($\mathbf{D}$), and MSE for the WTLS solution are expressed
 405 as:

$$\mathbf{G}^{-g}_{M \times N} = ((\mathbf{A} - \hat{\mathbf{e}}_A)^T (\mathbf{A} - \hat{\mathbf{e}}_A))^{-1} \mathbf{G}^T$$

$$\mathbf{B} = (\mathbf{G}^{-g} \mathbf{G} - \mathbf{I}) \mathbf{m}$$

$$\mathbf{D} = \text{trace}(\sigma_0^2 \mathbf{G}^{-g} (\mathbf{G}^{-g})^T)$$

$$MSE = \mathbf{B}^T \mathbf{B} + \mathbf{D} \quad (33)$$

406

407 Before starting the WTLS estimation, there are two coefficients that need to be fixed, which are
 408 regularization parameter α and weight coefficient k , respectively. The proposed algorithm in this
 409 study starts from an ordinary slip distribution inversion, and the L-Curve method is used to determine

regularization parameter α for the uniform regularization. We then initialize a series of coefficients k . The WTLS estimation is run by using the determined α and the initialized k . For each trial, if the WTLS estimation converges, the residuals for observation ($\hat{\mathbf{e}}_y$) and design matrix ($\hat{\mathbf{e}}_A$) are recorded. After finishing the trials of all the initialized coefficients k , the recorded residuals for the observation ($\hat{\mathbf{e}}_y$) and design matrix ($\hat{\mathbf{e}}_A$) are plotted to form a tradeoff curve. Similar to the L-curve method, the slip model corresponding to the corner point on the tradeoff curve is selected as the preferred one. A detailed workflow for applying WTLS in earthquake slip inversion is shown in Fig. 7.

Figure 7. Workflow of the proposed method.

The main feature of the proposed method is that the regularization matrix is allowed to change in the inversion. In this method, the modifications of the regularization matrix and slip model are calculated through minimizing a cost function combining the residuals of the observation and regularization matrix. It is therefore expected that this approach can correct some of the unreasonable regularization and give a better regularized solution.

5. Results and discussions

5.1 Simulation study

To validate the performance of the proposed method, we simulated four kinds of slip distribution, as shown in Fig. 8: 1) one deep slip concentration and one shallow slip concentration; 2) two shallow slip concentrations; 3) one deep slip concentration; and 4) one shallow slip concentration. We undertook forward modeling of the 2,939 ground observations from the input slips using the Okada model (Okada, 1992), and then interrupted the displacements by Gaussian noise of $\sigma = 1\text{cm}$. The locations of the observation sites and the fault geometry were the same as the above test shown in Fig 6a. We implemented the inversion using uniform regularization, resolution-based adaptive regularization, and the proposed regularization method.

Figure 8. The four input slip patterns in the simulated experiment.

Fig. 9 shows the recovered slips from the three tested regularization methods. It was found that the uniform regularization and the resolution-based adaptive regularization give very similar results, which is consistent with the findings in the general ridge estimation and adaptive regularization experiments. Among the three tested methods, the proposed regularization method gives a slip distribution with the best resolution. For the slip patterns with both upper and deeper slips (pattern 1) in particular, the proposed method recovers more slips on deep patches. For the slip patterns with two shallow slip concentrations (pattern 2), the proposed method recovers more slips of the right slip concentration than the other methods (the maximum slip is larger and closer to the true one). For the slip pattern with one shallow slip concentration (pattern 3), the difference between the three methods is not very significant. This may be because the data can already constrain the fault slip very well, and the different regularization methods do not greatly affect the results in this case. For the slip pattern with only deep slips (pattern 4), none of the tested methods perform very well. However, the proposed regularization method gives slightly better results, where the slips are more clustered and closer to the input slips.

Figure 9. Slip distributions using: (a) uniform regularization; (b) resolution-based adaptive regularization; and (c) the proposed regularization method.

We calculated the MSE for a quantitative evaluation of the three regularization methods using equation (33). The MSEs for the four slip patterns are shown in Table 2. It can be seen that the proposed method gives the solution with the lowest MSE. The difference between the resolution-based adaptive regularization and the uniform regularization is not obvious. This quantitative assessment further verifies that allowing the regularization matrix to change is beneficial for earthquake slip inversion, and WTLS is an efficient way to adjust the regularization matrix. In particular, the improvements in the MSEs of the proposed method are more significant for the slip patterns with deep slip patches (pattern 1 and pattern 4). It is possible that because the resolution ($R_{i,i}$) for the deep slip patches is poor, this

causes them to be more sensitive to the regularization. A refined regularization method can therefore better show its advantage.

Table 2 MSEs of the three regularization methods for the test in Fig. 9

	MSE 1 (m ²)	MSE 2 (m ²)	MSE 3 (m ²)	MSE 4 (m ²)
Uniform	11.9634	8.0925	1.3028	11.2735
Resolution-based	11.9413	8.2817	1.2887	11.2948
Proposed method	8.6692	6.5026	1.2459	8.8964

In the proposed method, the weight coefficient k is very important because it is responsible for balancing the design matrix residual and the observation residual. A larger value of k represents a greater uncertainty in the design matrix, and therefore corresponds to a larger design matrix residual. If k is set to zero, the solution will then be the same as the results from the traditional least squares method. Therefore, we can expect the 2-norm of the design matrix residual $\hat{\mathbf{e}}_A$ to increase and the 2-norm of the observation residual $\hat{\mathbf{e}}_y$ to decrease with increasing k . However, this trend for $\|\hat{\mathbf{e}}_y\|$ and $\|\hat{\mathbf{e}}_A\|$ may change if the WTLS cannot converge in the allowed number of iterations.

In our experiment, we found that a large value of k will normally result in the WTLS not converging. Fig. 10a is the plot of $\|\hat{\mathbf{e}}_y\|$ in the function of $\|\hat{\mathbf{e}}_A\|$ for the above experiment with input slip pattern 4. Here, the i th weight coefficient k was set to be 2^{i-10} . It can be noted that the plotted points become unordered after k reaches a certain value ($k > 2^{15-10}$, i.e. $i > 15$). The reason for this is that the solution has not converged after a certain number of iterations. We also plotted the slip series with these weight coefficients, where it can be seen that the derived slip model becomes unstable after $i > 15$

(Fig. 10c). A coefficient leading to an un-converged solution has no value for the further analysis. In the proposed algorithm, we include a decision box to discard these unstable solutions. As shown in Fig. 10b, the plot of $\|\hat{\mathbf{e}}_y\|$ and $\|\hat{\mathbf{e}}_A\|$ for the converged solutions exhibits a continuous L-shaped curve. Similar to the L-curve method used for selecting the regularization parameter, we consider the solution corresponding to the corner of the curve (marked as the red star in Fig. 10b) to have a reasonable tradeoff between the design matrix residual and observation residual.

Figure 10. (a) Weight coefficient selection. (b) Enlarged map of the valid points in (a). (c) Slip model series using different

weight coefficients. The red frame denotes the slip model using the selected weight coefficient.

In the above experiment with four slip patterns, the inversions for patterns 1–3 have similar weight coefficients, while pattern 4 has an obviously larger one. The input slip patterns 1–3 all have upper slips, which can generate larger ground displacements than deeper slips. As the perturbed noise is the same for all the input slips, the signal-to-noise ratio (SNR) for the constrained data of patterns 1–3 is therefore much larger than for pattern 4. This implies that the selected value of k is related to the SNR, where a larger SNR corresponds to a smaller k . We simulated two different noise levels for input slip pattern 4, and plotted the curve for the design matrix residual and observation residual, as shown in Fig. 11. Here, it can be observed that the lower noise level prefers a smaller weight coefficient, which is consistent with the previous finding in the four tested slip patterns that a large SNR corresponds to a smaller k . A possible reason for this is that a larger SNR usually implies a stronger data constraint, leading to it being less necessary to modify the regularization matrix to improve the slip resolution.

Figure 11. Weight coefficient selections under different noise levels.

which normally locate at depth or in areas sparsely covered by observations. Adaptive regularization can partly reduce this kind of bias by a more reasonable regularization. Therefore, it may work better for slip patterns with low-resolution slip patches. Among the tested input slip patterns, slip patterns 1 and 4 have deep slip patches. Therefore, the improvements for these slip patterns are significantly better than for the other patterns.

5.2 Example: 2011 Mw 9.0 Tohoku earthquake

We further explored the proposed method with a case study of the 2011 Mw 9.0 Tohoku earthquake. This earthquake occurred off the Pacific coast of Tohoku, Honshu Island, Japan, on March 11th, 2011 (UTC 05:46, local time 14:46). The epicenter was located approximately 70 km east of the Oshika Peninsula of Tohoku. It was the largest earthquake ever recorded to have hit Japan, and caused immense economic and human losses. In the north of Japan, the Pacific plate subducts northwestwards into the North American plate along the Kuril and Japan Trenches at a rate of 80–90 mm/yr (DeMets *et al.*, 1994, Sella *et al.*, 2002). Most historic earthquakes in northeastern Japan have been thrust faulting events distributed NNE along the Japan Trench, which is associated with the subduction motion between the North American and the Pacific plates. This earthquake was a typical thrust faulting event, resulting from strong stress accumulation along the Japan Trench.

Japan has a dense nationwide GPS network called GEONET that provides a large volume of high-quality crustal deformation data (Sagiya *et al.*, 2001). With the GEONET RINEX data provided by the Geospatial Information Authority (GSI) of Japan, the ARIA team at JPL and Caltech produced GPS coseismic displacements for the Tohoku earthquake, which were estimated from 5 min interval

kinematic solutions, using JPL’s rapid orbit solution and GIPSY-OASIS software. This showed that all the stations in northern Japan moved eastwards, with the largest displacement being over 5 m (Fig. 12). In the offshore region, there were five sites that obtained coseismic observations by the use of the GPS/acoustic combination technique (Sato *et al.*, 2011). The largest displacement happened at the easternmost site, with a 24-m horizontal displacement and a 3-m vertical displacement.

Figure 12. GPS observations of the 2011 Mw 9.0 Tohoku earthquake and the fault geometry setting.

This event has been the focus of numerous studies of the coseismic slip distribution. We therefore took this event as an example to validate the proposed regularization method. The fault geometry was generally guided by the slab contours of Huang *et al.* (2011): the strike was 195° and the dimension was 700 km along strike and 231 km along depth, dipping northwest at 15° . We discretized the fault plane into 30×10 patches. Both dip- and strike-slip were allowed for each fault patch. The detailed fault geometry parameters are shown in Table 3.

544

Table 3. Parameter settings of the Tohoku earthquake inversion

Rake ($^{\circ}$)	Strike ($^{\circ}$)	Dip ($^{\circ}$)	East-UTM (km)	North-UTM (km)	Upper depth (km)	Lower depth (km)
Not fixed	195	15	749.4	4192	0	60

545

546 We applied both the traditional uniform regularization method and the proposed regularization method
547 to study this event. The regularization parameter for the traditional method was determined by the
548 L-curve method (Fig. 12a). The proposed method followed the procedure described in Fig. 7. The slip
549 models constrained by the GPS observations from the two different inversion methods are shown in Fig.

550 12.

551

552 Figure 12. Slip distribution results of the Tohoku earthquake. (a) L-curve used to select the regularization parameter for slip

553 inversion. Red star denotes the location of the selected parameter. (b) Slip distribution from the traditional method. (c) Slip

554 distribution from the proposed regularization method.

555

556 Both slip distribution results suggest a large slip concentration in the middle of the fault plane, with the
557 depths extending to about 48 km. The slip characteristic is dominated by thrust slips with some small
558 lateral components. Assuming a shear modulus of 40 GPa, which is a rough average of 29, 41, and
559 50 GPa for the upper crust, lower crust, and upper mantle in northeastern Japan (from seismic

observations) (Nakajima *et al.*, 2001), the estimated moments are 3.23×10^{22} Nm and 3.14×10^{22} Nm for the traditional method and the proposed method, equal to an event of Mw 9.006 and Mw 8.998. respectively (Table 4). The proposed method gives a slightly larger maximum slip of 27.35 m, compared with 24.34 m for the traditional method. As found by the synthetic experiments, the proposed method may not improve very significantly for the shallow slip case. The slip of the 2011 Mw 9.0 Tohoku earthquake mainly occur on the shallow patches of the fault plane, so it is not surprising to find the main slip features from two methods are quite similar.

However, we still can observe the slip patterns from two methods differ a bit, where the proposed method gives a relatively less smooth and more detailed slip distribution than the traditional method. Compared with previous studies of this event, the simpler and smoother slip pattern derived from the traditional method is similar to most of the published coseismic results obtained from only GPS measurements (Ozawa *et al.*, 2011, Iinuma *et al.*, 2011). However, the relatively sharp and more detailed slip pattern from the proposed method is closer to the joint inversion result with constraints from both geodetic and seismic data (Koketsu *et al.*, 2011). Considering that the GPS measurements are far from the epicenter, the joint inversion with seismic data may be more accurate. From this aspect, we believe that the proposed regularization method has partly improved the resolution of the slip model in this case, because it derives a slip distribution that is closer to the joint inversion results.

578

Table 4. Results of the Mw 9.0 Tohoku earthquake inversions

	GPS-RMSR	Maximum slip	Moment	Magnitude
Traditional	0.0329 m	24.34 m	3.23×10^{22} Nm	9.006
Proposed method	0.0314 m	27.35 m	3.14×10^{22} Nm	8.998

579

580 In order to further check the resolution power of each method, we undertook a checkerboard test, in
581 which we attempted to recover a given slip distribution. In the test, three separate slip concentrations
582 with a maximum value of 30 m were initialized on the fault plane. Displacements of the GPS data were
583 simulated by forward modeling. The simulated displacements were then perturbed by Gaussian noise
584 of $\sigma_0 = 1\text{cm}$. The results show that the traditional method is only able to find the general location of
585 three slip concentrations. The slip value is underestimated and the slip concentrations are confused.
586 Although the proposed method cannot fully recover the input slip, the results are significantly better
587 than those of the traditional method. The maximum slip is closer to the true value, and the slip
588 concentrations can be distinguished much more easily. The left concentration has been completely
589 separated from the middle one. In the above inversion with the real dataset, we also found that a small
590 slip concentration on the left (north in geographical coordinates) is recognized by the proposed method.
591 Considering the performance of the two methods shown in the resolution test, the slip concentration
592 recovered only by the proposed method is possibly a true slip concentration, which is erased due to
593 over-regularization in the traditional method.

Figure 14. Resolution test for the Tohoku earthquake inversion. (a) Input slip. (b) Results from the traditional method. (c) Results from the proposed method.

The traditional uniform regularization method imposes the same smoothing constraint on all the fault patches. The magnitude of the constraint is represented by the regularization parameter. The L-curve method is the most widely used method for determining this parameter. However, previous studies indicated that the L-curve method may over-smooth the ill-posed problem (Xu, 1998, Hansen, 1999). Thus, it is not surprising to find that the regularized slip is normally smoother than the input slip. However, reducing the smoothing constraint is not a good solution because the variance of the solution will increase accordingly. The only choice is to take an appropriate regularization matrix which can better represent the physical characteristics of the fault slip. As shown in Fig. 2, regularization using a smoothing matrix can recover more details than using the identity matrix. However, assuming equal smooth characteristics for all the slip patches is not good enough. With a uniform regularization parameter, some slip patches may be over-smoothed. The proposed method assumes there are also Gaussian errors in the predefined regularization matrix, and allows it to change during the inversion.

This makes it possible to partly avoid the over-smoothing problem caused by the simple assumption in uniform regularization. The above experiments and resolution test confirm that the new regularization method can recover more details, so it is believed that the slip distribution derived by the proposed method is more reliable than the uniformly regularized one.

6. Conclusions

In this work, we have explored the characteristics of regularization of slip distribution inversion, and proposed an adaptive regularization method based on weighted total least squares (WTLS). The main conclusions can be summarized as follows:

(1) After examining the performances of the different traditional regularization methods, it was found that the TKL method, which considers the spatially distributed characteristic of the slips (smoothness), can derive more detailed slip characteristics (capturing both upper and lower slips) than the TSVD and TKI methods. This suggests that an appropriate regularization matrix can better represent the physical characteristics of the fault slips.

(2) If the regularization is set properly, adaptive regularization can achieve a significantly smaller MSE than uniform regularization. However, the resolution-based adaptive regularization is not a guaranteed way to obtain a better result than uniform regularization. Experiments on general ridge estimation and adaptive Tikhonov regularization all suggested that the optimal regularization parameters bear more relation to the fitness of the regularization equations than the model resolution.

(3) We have proposed a new adaptive regularization method for earthquake slip inversion, which assumes that errors exist in the predefined regularization matrix. WTLS is used to estimate the slip distribution. Experiments using four slip patterns showed that the proposed regularization method can obtain a smaller MSE than uniform regularization and resolution-based adaptive regularization. This implies that allowing the regularization matrix to change is beneficial for earthquake slip inversion, and WTLS is an efficient way to adjust the regularization matrix.

(4) In the proposed method, a weight coefficient is defined to balance the residuals of the regularization matrix and the observation. The coefficient is determined by finding the corner point on the tradeoff curve plotted with the residuals of the regularization matrix and the observation. It was found that the value of the selected weight coefficient is related to the SNR of the observation, where a larger SNR would choose a smaller weight coefficient. As a smaller weight coefficient corresponds to less regularization matrix residual, this suggests that it is less necessary to modify the regularization matrix to improve the slip resolution for the cases with a strong data constraint (large SNR).

(5) The proposed method is applied to study the 2011 Mw 9.0 Tohoku earthquake. As this event has a relatively shallow and simple slip pattern, the proposed method does not differ very significant from the traditional method. However, we still can observe a larger maximum slip and less smooth slip pattern from the proposed method. These features are closer to the slip model from a joint inversion with both geodetic and seismic data (Koketsu *et al.*, 2011). A resolution test also showed that the proposed method can recover more details than the traditional method. It can therefore be considered

that the proposed method partly improves the results of the 2011 Mw 9.0 Tohoku earthquake slip distribution inversion.

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References:

- Atzori, S. & Antonioli, A., 2011. Optimal fault resolution in geodetic inversion of coseismic data, *Geophysical Journal International*, 185(1), 529-538.
- Barnhart, W.D. & Lohman, R.B., 2010. Automated fault model discretization for inversion for coseismic slip distributions, *Journal of Geophysical Research: Solid Earth*, 115(B10), B10419.
- DeMets, C., Gordon, R.G., Argus, D.F. & Stein, S., 1994. Effect of recent revisions to the geomagnetic reversal time scale on estimates of current plate motions, *Geophysical research letters*, 21(20), 2191-2194.
- Dettmer, J., Benavente, R., Cummins, P.R. & Sambridge, M., 2014. Trans-dimensional finite-fault inversion, *Geophysical Journal International*, 199(2), 735-751.
- Dmowska, R., Zheng, G. & Rice, J.R., 1996. Seismicity and deformation at convergent margins due to heterogeneous coupling, *Journal of Geophysical Research: Solid Earth (1978–2012)*, 101(B2), 3015-3029.
- Fornaro, G., Atzori, S., Calo, F., Reale, D. & Salvi, S., 2012. Inversion of Wrapped Differential Interferometric SAR Data for Fault Dislocation Modeling, *Geoscience and Remote Sensing, IEEE Transactions on*, 50(6), 2175-2184.
- Fukahata, Y. & Wright, T.J., 2008. A non-linear geodetic data inversion using ABIC for slip distribution on a fault with an unknown dip angle, *Geophys. J. Int.*, 173(2), 353-364.
- Golub, G.H. & Van Loan, C.F., 1980. An analysis of the total least squares problem, *SIAM Journal on Numerical Analysis*, 17(6), 883-893.

685 Hansen, P.C., 1987. The truncatedsvd as a method for regularization, *BIT Numerical Mathematics*,
686 27(4), 534-553.

687 Hansen, P.C., 1992. Analysis of discrete ill-posed problems by means of the L-curve, *SIAM review*,
688 34(4), 561-580.

689 Hansen, P.C., 1999. The L-curve and its use in the numerical treatment of inverse problems. *in*
690 *Computational inverse problems in electrocardiology*, pp. 119-142, ed. P., J. WIT Press,
691 Southampton.

692 Hoerl, A.E. & Kennard, R.W., 1970. Ridge regression: Biased estimation for nonorthogonal problems,
693 *Technometrics*, 12(1), 55-67.

694 Huang, Z., Zhao, D. & Wang, L., 2011. Seismic heterogeneity and anisotropy of the Honshu arc from
695 the Japan Trench to the Japan Sea, *Geophysical Journal International*, 184(3), 1428-1444.

696 Ide, S., Takeo, M. & Yoshida, Y., 1996. Source process of the 1995 Kobe earthquake: Determination of
697 spatio-temporal slip distribution by Bayesian modeling, *Bulletin of the Seismological society*
698 *of America*, 86(3), 547-566.

699 Inuma, T., Ohzono, M., Ohta, Y. & Miura, S., 2011. Coseismic slip distribution of the 2011 off the
700 Pacific coast of Tohoku Earthquake (M 9.0) estimated based on GPS data—Was the asperity
701 in Miyagi-oki ruptured?, *Earth, planets and space*, 63(7), 643-648.

702 Jónsson, S., Zebker, H., Segall, P. & Amelung, F., 2002. Fault Slip Distribution of the 1999 Mw7.1
703 Hector Mine, California Earthquake, Estimated from Satellite Radar and GPS Measurements,
704 *Bulletin of the Seismological Society of America*, 92(4), 1377-1389.

705 Johnson, K.M., Hsu, Y.J., Segall, P. & Yu, S.B., 2001. Fault geometry and slip distribution of the 1999
706 Chi - Chi, Taiwan Earthquake imaged from inversion of GPS data, *Geophysical Research*
707 *Letters*, 28(11), 2285-2288.

708 Jolivet, R., Duputel, Z., Riel, B., Simons, M., Rivera, L., Minson, S., Zhang, H., Aivazis, M., Ayoub, F.
709 & Leprince, S., 2014. The 2013 Mw 7.7 Balochistan earthquake: seismic potential of an
710 accretionary wedge, *Bulletin of the Seismological Society of America*, 104(2), 1020-1030.

711 Kang, M.G. & Katsaggelos, A.K., 1995. General choice of the regularization functional in regularized
712 image restoration, *Image Processing, IEEE Transactions on*, 4(5), 594-602.

713 Koketsu, K., Yokota, Y., Nishimura, N., Yagi, Y., Miyazaki, S.i., Satake, K., Fujii, Y., Miyake, H., Sakai,
714 S.i. & Yamanaka, Y., 2011. A unified source model for the 2011 Tohoku earthquake, *Earth and*
715 *Planetary Science Letters*, 310(3), 480-487.

716 Lin, J. & Stein, R.S., 2004. Stress triggering in thrust and subduction earthquakes and stress interaction
717 between the southern San Andreas and nearby thrust and strike-slip faults, *J. Geophys. Res.*,
718 109(B2), B02303.

719 Lohman, R.B., 2004. The inversion of geodetic data for earthquake parameters, California Institute of
720 Technology.

721 Massonnet, D., Feigl, K., Rossi, M. & Adragna, F., 1994. Radar Interferometric mapping of
722 deformation in the year after the Landers Earthquake, *Nature*, 369(6477), 227-230.

723 Menke, W., 2012. *Geophysical data analysis: discrete inverse theory*, 3 edn, Academic press.

724 Minson, S., Simons, M. & Beck, J., 2013. Bayesian inversion for finite fault earthquake source models
725 I—theory and algorithm, *Geophysical Journal International*, 194(3), 1701-1726.

726 Nakajima, J., Matsuzawa, T., Hasegawa, A. & Zhao, D., 2001. Seismic imaging of arc magma and
727 fluids under the central part of northeastern Japan, *Tectonophysics*, 341(1), 1-17.

728 Nishimura, T., Miura, S., Tachibana, K., Hashimoto, K., Sato, T., Hori, S., Murakami, E., Kono, T.,

729 Nida, K. & Mishina, M., 2000. Distribution of seismic coupling on the subducting plate
730 boundary in northeastern Japan inferred from GPS observations, *Tectonophysics*, 323(3),
731 217-238.

732 Okada, Y., 1992. Internal deformation due to shear and tensile faults in a half-space, *Bull. Seism. Soc.*
733 *Am.*, 85(2), 1018-1040.

734 Ozawa, S., Nishimura, T., Suito, H., Kobayashi, T., Tobita, M. & Imakiire, T., 2011. Coseismic and
735 postseismic slip of the 2011 magnitude- 9 Tohoku - Oki earthquake, *Nature*, 475(7356),
736 373-377.

737 Page, M.T., Custodio, S., Archuleta, R.J. & Carlso, J.M., 2009. Constraining earthquake source
738 inversions with GPS data: 1. Resolution-based removal of artifacts, *J. Geophys. Res.*, 114(B1),
739 B01314.

740 Pritchard, M., Simons, M., Rosen, P., Hensley, S. & Webb, F., 2002. Co - seismic slip from the 1995
741 July 30 Mw= 8.1 Antofagasta, Chile, earthquake as constrained by InSAR and GPS
742 observations, *Geophysical Journal International*, 150(2), 362-376.

743 Sagiya, T., Miyazaki, S.i. & Tada, T., 2001. Continuous GPS array and present-day crustal deformation
744 of Japan. in *Microscopic and Macroscopic Simulation: Towards Predictive Modelling of the*
745 *Earthquake Process*, pp. 2303-2322Springer.

746 Sato, M., Ishikawa, T., Ujihara, N., Yoshida, S., Fujita, M., Mochizuki, M. & Asada, A., 2011.
747 Displacement above the hypocenter of the 2011 Tohoku-Oki earthquake, *Science*, 332(6036),
748 1395-1395.

749 Schaffrin, B. & Wieser, A., 2008. On weighted total least-squares adjustment for linear regression,
750 *Journal of Geodesy*, 82(7), 415-421.

751 Sella, G.F., Dixon, T.H. & Mao, A., 2002. REVEL: A model for recent plate velocities from space
752 geodesy, *Journal of Geophysical Research: Solid Earth (1978–2012)*, 107(B4), ETG
753 11-11-ETG 11-30.

754 Simons, M., Fialko, Y. & Rivera, L., 2002. Coseismic Deformation from the 1999 Mw 7.1 Hector Mine,
755 California, Earthquake as Inferred from InSAR and GPS Observations, *Bulletin of the*
756 *Seismological Society of America*, 92(4), 1390-1402.

757 Thatcher, W., 1990. Order and diversity in the modes of circum - Pacific earthquake recurrence,
758 *Journal of Geophysical Research: Solid Earth (1978 - 2012)*, 95(B3), 2609-2623.

759 Tikhonov, A.N., Arsenin, V.I.A.k. & John, F., 1977. *Solutions of ill-posed problems*, edn, Winston
760 Washington, DC.

761 Wang, C., Ding, X., Shan, X., Zhang, L. & Jiang, M., 2012. Slip distribution of the 2011 Tohoku
762 earthquake derived from joint inversion of GPS, InSAR and seafloor GPS/acoustic
763 measurements, *Journal of Asian Earth Sciences*, 57(6), 128-136.

764 Wright, T.J., Lu, Z. & Wicks, C., 2004. Constraining the Slip Distribution and Fault Geometry of the
765 M_w 7.9, November 2002, Denali Fault earthquake with Interferometric Synthetic Aperture
766 Radar and Global Positioning System Data, *Bulletin of the Seismological Society of America*,
767 94(6B), S175-S189.

768 Xu, P., 1998. Truncated SVD methods for discrete linear ill - posed problems, *Geophysical Journal*
769 *International*, 135(2), 505-514.

770 Zhang, L., Ding, X., Lu, Z., Jung, H.-S., Hu, J. & Feng, G., 2014. A Novel Multitemporal InSAR
771 Model for Joint Estimation of Deformation Rates and Orbital Errors, *IEEE Transactions on*
772 *Geoscience and Remote Sensing*, 52(6), 3529-3540.